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Globally stable cylinders for hyperbolic CAT(0) cube complexes

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Abstract. Rips and Sela (1995) introduced the notion of globally stable cylinders and asked if all Gromov hyperbolic groups admit such. We prove that hyperbolic cubulated groups admit globally stable cylinders.

Globally stable cylinders. Rips and Sela [6] introduced the notion of globally stable cylinders in their work on solutions of equations over groups. In the context of a δ-hyperbolic group, the idea is as follows. Given two points x and y in a δ-hyperbolic space X, one can choose a geodesic $[x, y]$ joining them. Then given three vertices x, y and z, one has that the two geodesics $[x, y]$ and $[x, z]$ "fellow travel" (i.e., are within δ of each other) up to some median point from which they diverge. The idea of stable cylinders is to thicken the geodesics into "cylinders" which are not just close to one another, but actually agree for most of the time they fellow travel.

More precisely, let $\theta \geq 0$. A θ-cylinder $C(x, y)$ of $x, y \in X$ is a subset that satisfies $[x, y] \subseteq C(x, y) \subseteq N_\theta([x, y])$ for every geodesic $[x, y]$ connecting x, y.

A choice of θ-cylinders $C\colon X \times X \to 2^X$ for every $x, y \in X$ is called globally stable if there exist $k, R \geq 0$ such that

(1) *inversion invariance:* $C(x, y) = C(y, x)$ for all $x, y \in X$,

(2) (k, R)-*stability:* for all $x, y, z \in X$ there exist k R-balls $B_1, \dots, B_k$ in X such that

$$(C(x, y) \cap B(x, \rho)) - \bigcup_{i=1}^{k} B_i = (C(x, z) \cap B(x, \rho)) - \bigcup_{i=1}^{k} B_i, \qquad (*)$$

where $\rho = (y.z)_x = \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$ is the Gromov product (see Figure 1).

Let G be a hyperbolic group. The group G *admits globally stable cylinders* if some geodesic hyperbolic space X on which G acts properly cocompactly admits globally stable cylinders which are G-*invariant*, i.e., $C(gx, gy) = gC(x, y)$ for all $x, y \in X$ and

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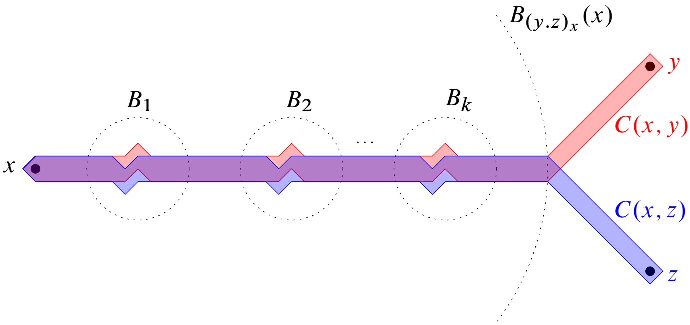

Figure 1. (k, R)-uniformly stable cylinders.

$g \in G$. For completeness, we show in Proposition 11 that this is a group property, and does not depend on the space X.

Rips and Sela [6] showed that hyperbolic $C'(1/8)$-small cancellation groups have globally stable cylinders. In fact, they show that such groups are $(1, R)$-stable. They also asked if all hyperbolic groups admit globally stable cylinders. Recently, Kharlampovich and Sklinos [4] used the globally stable cylinders of $C'(1/8)$-small cancellation groups to study the first-order theory of random groups. Globally stable cylinders are also used by the first author [5] to study the connection between complexity and volume of hyperbolic groups.

In this paper, we prove the existence of globally stable cylinders for hyperbolic cubulated groups.

Theorem. *Let X be a hyperbolic d-dimensional* CAT(0) *cube complex. Then, X admits globally stable cylinders which are* Aut(X)*-invariant. In particular, every hyperbolic cubulated group admits globally stable cylinders.*

Remark 1. (1) Note that in the theorem, X is not required to be locally finite. Moreover, the parameters k and R in $(*)$ depend only on the hyperbolicity constant δ and the dimension d of X.

(2) Since all δ-hyperbolic $C'(1/8)$-small cancellation groups are cubulated [7], the theorem above provides a new proof of the result of Rips and Sela [6]. While their proof constructs 2δ-cylinders which are $(2, R)$-stable by considering points which lie on certain quasigeodesics, the proof below makes use of the hyperplane structure of CAT(0) cube complexes and produces (k, R)-stable cylinders with $k > 2$.

Preliminaries on CAT(0) cube complexes. We briefly recall standard material on CAT(0) cube complexes, as well as introducing some new definition which we will employ. For a more extensive discussion on CAT(0) cube complexes see, e.g., [1]. Let X be a hyperbolic d-dimensional CAT(0) cube complex. Let $\widehat{\mathcal{H}}(X)$ be its set of hyperplanes. Every hyperplane $\hat{h}$ separates X into two components. In each component there is a unique maximal subcomplex h of X which we call a halfspace bounded by $\widehat{h}$. The halfspaces of X are convex, and the collection of all of them is denoted by $\mathcal{H}(X)$. The set $\mathcal{H}(X)$ has a natural involution $h \mapsto h^*$ mapping a halfspace to its complementary halfspace. Moreover, we denote by $\widehat{\ }\colon \mathcal{H}(X) \to \widehat{\mathcal{H}}(X)$ the map $h \mapsto \widehat{h}$ that assigns to a halfspace its bounding hyperplane. For $x, y \in X^{(0)}$, define the set of halfspaces separating x from y as

$$\vec{\mathcal{H}}(x, y) = \{k \in \mathcal{H}(X) \mid x \notin k \ni y\},$$

and its corresponding set of hyperplanes by

$$\widehat{\mathcal{H}}(x, y) = \{\widehat{k} \mid k \in \vec{\mathcal{H}}(x, y)\}.$$

All other hyperplanes are called *peripheral* to x, y, and the set of *peripheral halfspaces* is defined as

$$\breve{\mathcal{H}}(x, y) = \{h \in \mathcal{H}(X) \mid x, y \in h^*\}.$$

The interval between x and y is the union of all ℓ_1 geodesics between x and y. It turns out that the interval is also the intersection of all the halfspaces that contain both x and y. That is, it is the set

$$I(x, y) = \{z \in X \mid d(x, y) = d(x, z) + d(z, y)\} = \bigcap_{h \in \breve{\mathcal{H}}(x,y)} h^*,$$

where d denotes the ℓ_1-metric on X. The interval $I(x, y)$ is a convex subcomplex of X, and is isomorphic to the dual cube complex to $\mathcal{H}(I(x, y))$, the pocset of halfspaces associated to the set of hyperplanes in $\vec{\mathcal{H}}(x, y)$. The nearest-point projection $\pi_{I(x,y)}\colon X \to I(x, y)$ can be expressed using the language of ultrafilters as $\pi_{I(x,y)}(v) = v \cap \mathcal{H}(I(x, y))$.

Observation 2. Note that in the case that X is δ-hyperbolic, any two ℓ_1-geodesics between x and y are contained in δ-neighborhoods of one another. Thus the interval $I(x, y)$ is contained in the δ-neighborhood of any ℓ_1-geodesic joining x and y.

The median $m(x, y, z)$ of x, y, z in X is the unique point in $I(x, y) \cap I(y, z) \cap I(x, z)$. Alternatively, $m(x, y, z)$ is the vertex $\pi_{I(x,y)}(z)$.

We write $\widehat{h} \pitchfork \widehat{k}$ if the hyperplanes $\widehat{h}, \widehat{k} \in \widehat{\mathcal{H}}(X)$ intersect. A nested sequence of halfspaces $h_1 < \cdots < h_n$ is called a *pencil of halfspaces* and the corresponding sequence of bounding hyperplanes will be called a *pencil of hyperplanes*.

We now want to refine the notion of peripheral hyperplanes and halfspaces to those that do not run too long parallel to a given interval. For a subset $\mathcal{K} \subset \mathcal{H}$ and a halfspace $h \in \mathcal{H}$, define their intersection number as the maximal size of a pencil of hyperplanes in $\mathcal{K}$ that intersect $\widehat{h}$, i.e.,

$$i(h, \mathcal{K}) = \max\{n \mid \exists k_1 < \cdots < k_n \in \mathcal{K} \text{ such that } \widehat{h} \pitchfork \widehat{k_i}\ \forall 1 \le i \le n\}.$$

Note that $i(h, \mathcal{K})$ may be infinite. We then define the following subset of $\breve{\mathcal{H}}(x, y)$:

$$\breve{\mathcal{H}}_D(x, y) = \{h \in \breve{\mathcal{H}}(x, y) \mid i(h, \vec{\mathcal{H}}(x, y)) \le D\},$$

We call these the *D-peripheral halfspaces* of $[x, y]$, and correspondingly,

$$I_D(x, y) = \bigcap_{h \in \breve{\mathcal{H}}_D(x,y)} h^*.$$

Observation 3. If $h \in \breve{\mathcal{H}}_D(x, y)$, then $\mathrm{diam}(\pi_{I(x,y)} h) \leq Dd$. Otherwise, there are points $u, v \in \pi_{I(x,y)}(h)$ at distance $> Dd$. By the pigeonhole principle, out of the $Dd + 1$ hyperplanes separating u, v, there are $D + 1$ hyperplanes satisfying $k_1 < \cdots < k_{D+1} \in \vec{\mathcal{H}}(x, y)$. But, by the properties of the nearest point projection, $\hat{k}_1, \ldots, \hat{k}_{D+1}$ are transverse to $\hat{h}$, contradicting the assumption that $h \in \breve{\mathcal{H}}_D(x, y)$.

Henceforth, we will assume that X is a δ-hyperbolic CAT(0) cube complex of dimension d. A *grid of size m* is a pair of pencils of hyperplanes $\hat{h}_1, \ldots, \hat{h}_m$ and $\hat{k}_1, \ldots, \hat{k}_m$ such that $\hat{h}_i \pitchfork \hat{k}_j$ for all $1 \leq i, j \leq m$.

Lemma 4. *There exists a maximal number D such that X contains a grid of size D. Moreover, $D \leq 2\delta$.*

Proof. Assume $h_1 < \cdots < h_D \in \mathcal{H}$ and $k_1 < \cdots < k_D \in \mathcal{H}$ satisfy $\hat{h}_i \pitchfork \hat{k}_j$ for all $1 \leq i, j \leq D$. Let v_1, v_2, v_3, v_4 be vertices of X in $h_D \cap k_D$, $h_1^* \cap k_D$, $h_1^* \cap k_1^*$, $h_D \cap k_1^*$. Consider a geodesic quadrilateral $v_1 v_2 v_3 v_4$. Both pairs of opposite sides are separated by at least D hyperplanes. Since X is δ-hyperbolic, every quadrilateral has to be 2δ-slim. Thus, $D \leq 2\delta$. ■

Remark 5. In fact, a CAT(0) cube complex X is hyperbolic if and only if it is finite-dimensional and there is a global bound on the size of grids.

Henceforth, let D be as in the Lemma 4. Our goal is to prove that the choice $C(x, y) = I_D(x, y)$ for all $x, y \in X$ is a (k, R)-uniformly stable choice of θ-cylinders for some fixed k, R, θ that depend only on X (in fact, only on d and D).

We begin by showing that $I_D(x, y)$ are θ-cylinders. First note, that by Observation 2, $I(x, y) \subset N_\delta([x, y])$ for any geodesic $[x, y]$. Thus, to prove that $I_D(x, y)$ are θ-cylinders, it suffices to prove the following lemma.

Lemma 6. *Let $\theta = Dd$. Then $I(x, y) \subset I_D(x, y) \subseteq N_\theta(I(x, y))$.*

Proof. By definition, $I(x, y) \subset I_D(x, y)$. To prove the second inclusion, assume for contradiction that there exists a vertex $w \in I_D(x, y)$ for which the distance $d(w, I(x, y)) > \theta$. Since $I(x, y)$ is a convex subcomplex, this means that w is separated by more than $\theta = Dd$ hyperplanes from $I(x, y)$. By the pigeonhole principle, there are $D + 1$ halfspaces $h_1 < \cdots < h_{D+1}$ such that $I(x, y) \subseteq h_1$ and $w \notin h_{D+1}$. By definition of $I_D(x, y)$, the hyperplane h_{D+1} is not in $\breve{\mathcal{H}}_D(x, y)$, and thus there exist $k_1 < \cdots < k_{D+1} \in \vec{\mathcal{H}}(x, y)$ such that $\hat{h}_{D+1} \pitchfork \hat{k}_j$ for all $1 \leq j \leq D$. Since each $\hat{k}_i$ intersects $I(x, y)$, and $\hat{h}_i$ separates $I(x, y)$ from $\hat{h}_{D+1}$, it follows that $\hat{h}_i \pitchfork \hat{k}_j$ for all $1 \leq i, j \leq D + 1$. A contradiction to the choice of D. ■

Let

$$\breve{\mathcal{H}}_{D,\rho}(x, y) = \{h \in \breve{\mathcal{H}}_D(x, y) \mid d_{\max}(x, \pi_{I(x,y)}(h)) < \rho\},$$

where $d_{\max}(x, A) = \max\{d(x, a) \mid a \in A\}$. Roughly speaking, this is the collection of D-peripheral halfspaces whose projection to $I(x, y)$ is ρ-close to x.

The key step in proving the existence of globally stable cylinders is to bound the number of projections of D-peripheral halfspaces for $[x, y]$ which are not D-peripheral for $[x, z]$. More precisely, we have the following.

Proposition 7. *There exists $M = M(\delta, d) \geq 0$ such that for all $x, y, z \in X$,*

$$|\{\pi_{I(x,y)}(h) \mid h \in \breve{\mathcal{H}}_{D,\rho-Dd}(x, y) - \breve{\mathcal{H}}_D(x, z)\}| \leq M,$$

where $\rho = (y.z)_x$.

Proof. First, we show that $\breve{\mathcal{H}}_{D,\rho-Dd}(x, y)$ is contained in the full peripheral set of hyperplanes for the pair (x, z).

Claim 8. $\breve{\mathcal{H}}_{D,\rho-Dd}(x, y) \subset \breve{\mathcal{H}}(x, z)$.

Proof. Note that $\rho = d(x, m)$, where $m = m(x, y, z)$ is the median of x, y, z in X. Assume for contradiction that $h \in \breve{\mathcal{H}}_{D,\rho-Dd}(x, y) - \breve{\mathcal{H}}(x, z)$. It follows that $x \in h^*$ but the hyperplane $\widehat{h}$ separates x, z, so $z \in h$. Whence

$$m = \pi_{I(x,y)}(z) \in \pi_{I(x,y)}(h).$$

But then

$$\rho = d(x, m) < d_{\max}(x, \pi_{I(x,y)}(h)) < \rho - Dd,$$

a contradiction. ∎

To prove the proposition, we will assume that the number of projections $\pi_{I(x,y)}(h)$ of halfspaces in $h \in \breve{\mathcal{H}}_{D,\rho-Dd}(x, y) - \breve{\mathcal{H}}_D(x, z)$ is large. Our strategy will be to construct a grid of hyperplanes of size greater than D which contradicts the hyperbolicity of X. In carrying out this strategy, we will need to make use of the following two technical "pigeonhole claims", which ensure that our objects of interest are disjoint. We abuse notation and let $\mathbb{R}^d$ denote the standard cubulation of $\mathbb{R}^d$.

Claim 9. *For all m, R, d, there exists $N_1 = N_1(m, R, d)$ such that for all N_1 distinct subcomplexes of $\mathbb{R}^d$ of diameter $\leq R$, there is a subcollection of m subcomplexes $A_1, \ldots, A_m$ and a pencil of $m-1$ halfspaces $s_1 < \cdots < s_{m-1}$ such that $A_i \subset s_{i-1}^* \cap s_i$ for all $1 \leq i < m$.*

Proof. Let $K = K(d, R)$ be the number of subcomplexes in $[0, R]^d$. Set

$$N_1 = m(R + 1)^2 dK.$$

Let $\mathcal{A}$ be a collection of N_1 distinct subcomplexes of $\mathbb{R}^d$ of diameter $\leq R$. Projecting the subcomplexes to each of the standard axes , we get intervals of length at most R. Thus, at most K subcomplexes in $\mathcal{A}$ can have the same projections on all axes. By the pigeonhole principle, there exists a standard axis ℓ such that $\frac{N_1}{dK} = m(R+1)^2$ of them have distinct projections to ℓ. Since each projection is an interval of length $\leq R$, at least $\frac{m(R+1)^2}{(R+1)^2} = m$ of $\mathcal{A}$ have disjoint projections to ℓ. These projections are separated by $m-1$ midpoints of edges in ℓ, and the subcomplexes are thus separated by the pencil of $m-1$ hyperplanes perpendicular to ℓ at those midpoints, as required. ∎

Claim 10. *Let $(P, \leq)$ be a partially ordered set such that at most d elements are pairwise incomparable. For all m, there exists $N_2 = N_2(m, d)$ such that every sequence of length N_2 of elements in P has a subsequence of length m which is strictly monotonic or constant.*

Proof. Set $N_2 = dm^3$. By Dilworth's theorem [3], P can be partitioned to d linearly ordered subsets. By the pigeonhole principle, there is a subsequence of length m^3 whose elements are totally ordered. If at least m of them are equal, we are done. Thus, we can pass to a further subsequence of length m^2 of distinct elements. A standard application of the pigeonhole principle shows that one can now find a strictly monotonic subsequence. ∎

Returning to the proof of Proposition 7, recall that we will need to apply the pigeonhole claims to a collection of M halfspaces to obtain a disjointness property for large enough M. Specifically, let $L = 2D + 2$, let

$$T = N_2(N_2(\ldots(N_2(L, d)\ldots, d), d)$$

be the $D+1$ fold application of N_2, and finally, let $M = N_1(T, Dd, d)$.

Let $h^1, \ldots, h^M$ be halfspaces in $\breve{\mathcal{H}}_{D,\rho-Dd}(x, y) - \breve{\mathcal{H}}_D(x, z)$ with distinct projections $\pi_{I(x,y)}(h^i)$. By [2], the interval $I(x, y)$ is isomorphic to a subcomplex of $\mathbb{R}^d$. By Observation 3, the projections $\pi_{I(x,y)}(h^i)$ to $I(x, y)$ have diameter $\leq Dd$. Thus, by Claim 9, we can find T halfspaces among $h^1, \ldots, h^M$, without loss of generality $h^1, \ldots, h^T$, which have pairwise disjoint projections to $I(x, y)$ and which are separated by a pencil of hyperplanes. That is, there exist $s^1 < \cdots < s^{T-1}$ in $\vec{\mathcal{H}}(x, y) \cap \vec{\mathcal{H}}(x, z)$ such that $\pi_{I(x,y)}(h^i) \subset s^i \cap (s^{i-1})^*$. This is equivalent to $(h^i)^* < s^i < h^{i+1}$ for all $1 \leq i < T$.

For each $1 \leq i \leq T$, the halfspace h^i is in $\breve{\mathcal{H}}(x, z)$ by Claim 8, but not in $\breve{\mathcal{H}}_D(x, z)$ by assumption. Hence there exist $k_1^i < \cdots < k_{D+1}^i \in \vec{\mathcal{H}}(x, z)$ such that $\hat{h}^i \pitchfork \hat{k}_j^i$ for all $1 \leq j \leq D+1$. But since $h^i \in \breve{\mathcal{H}}_D(x, y)$, at least one of $k_1^i < \cdots < k_{D+1}^i$ is not in $\vec{\mathcal{H}}(x, y)$. Namely, $k_1^i \in \mathcal{H}(y, z)$ for all $1 \leq i \leq T$.

By $D+1$ applications of Claim 10, we can pass to a subsequence of length L, which by abuse of notation we will simply denote $h^1, \ldots, h^L$, for which the sequences of halfspaces k_j^i are constant or strictly monotonic in i. That is, for each $1 \leq j \leq D+1$, either

$$k_j^1 < \cdots < k_j^L, \quad k_j^1 > \cdots > k_j^L, \quad \text{or} \quad k_j^1 = \cdots = k_j^L.$$

Let us show by induction on j that (up to changing the halfspaces k_j^i if necessary) we may assume that $k_j^1 = \cdots = k_j^L$ for all $1 \leq j \leq D+1$ in addition to the properties $k_1^i < \cdots < k_{D+1}^i$ and $\hat{h}^i \pitchfork \hat{k}_j^i$.

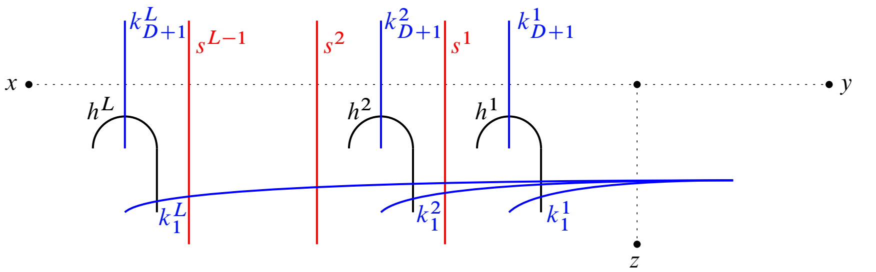

Figure 2. The hyperplane arrangement for the proof of Proposition 7.

For the base of the induction, let us show that $k_1^1 = \cdots = k_1^L$. Assume for contradiction that $k_1^1, \ldots, k_1^L$ are distinct. Since each k_1^i intersects h^i on the one hand, and separates y and z on the other, it must intersect $s^{i'-1}$ for all $i' < i$. Since $L > 2D+1$, this gives a grid of hyperplanes of size $D+1$ which contradicts the definition of D. Thus, $k_1^1 = \cdots = k_1^L$.

For the induction step, let $1 \leq j \leq D$ and assume that $k_j^1 = \cdots = k_j^L$. Let us prove that we may choose k_{j+1}^i such that $k_{j+1}^1 = \cdots = k_{j+1}^L$. Assume that $k_{j+1}^1 < \cdots < k_{j+1}^L$. Then $k_j^i = k_j^1 < k_{j+1}^1 < k_{j+1}^i$. It follows that

$$k_1^i < \cdots < k_j^i < k_{j+1}^1 < k_{j+2}^i < \cdots < k_{D+1}^i$$

and $\widehat{h}^i \pitchfork \widehat{k}_{j+1}^1$ (since $k_j^i < k_{j+1}^1 < k_{j+1}^i$ and both $\widehat{k}_j^i \pitchfork \widehat{h}^i \pitchfork \widehat{k}_{j+1}^i$). This means that we can replace k_{j+1}^i for all $1 \leq i \leq L$ by k_{j+1}^1, and get $k_{j+1}^1 = \cdots = k_{j+1}^L$ as desired. Similarly, if $k_{j+1}^1 > \cdots > k_{j+1}^L$, we can replace k_{j+1}^i for all $1 \leq i \leq L$ by k_{j+1}^L and get $k_{j+1}^1 = \cdots = k_{j+1}^L$ as desired.

This shows that we may assume $k_j^1 = \cdots = k_j^L =: k_j$ for all $1 \leq j \leq D+1$. However, since $\widehat{k}_1, \ldots, \widehat{k}_{D+1}$ intersect all the hyperplanes $\widehat{h}^1, \ldots, \widehat{h}^t$, they must intersect also the separating hyperplanes $\widehat{s}^1, \ldots, \widehat{s}^{t-1}$. As $L > D+2$ the sets $k_1 < \cdots < k_{D+1}$ and $s^1 < \cdots < s^{t-1}$ form a grid of size $D+1$ which contradicts the definition of D. ∎

Proof of Theorem. Define $C\colon X \times X \to 2^X$ by $C(x,y) := I_D(x,y)$.

By Lemma 6 and Observation 2, $C(x,y)$ are θ-cylinders for $\theta = Dd + \delta$. It is clear from the definition that this choice of cylinders is Aut(X)-invariant and inversion invariant. It remains to show that it is (k,R)-uniformly stable.

Set $R = 5Dd$. Let

$$P = \{m\} \cup (\cup \pi_{I(x,y)}(h)),$$

where $m = m(x,y,z)$, and the union ranges over all $h \in \breve{\mathcal{H}}_{D,\rho-Dd}(x,y) - \breve{\mathcal{H}}_D(x,z)$. By Observation 3 and Proposition 7, the set P has at most $k = k(d,\delta)$ points. Let $\mathcal{B} = \bigcup_{p \in P} B(p,R)$ be the union of the R-balls with centers in P. By symmetry, it suffices to show that $C(x,z) - C(x,y) \cap B(x,\rho) \subseteq \mathcal{B}$.

Assume $w \in (C(x,z) - C(x,y)) \cap B(x,\rho) - B$, where $B = B(m,R)$ is the ball of radius R around m. Let $[x,z]$ be a geodesic passing through m. Since $C(x,z)$ is a θ-cylinder, there exists a point $w' \in [x,z]$ such that $d(w,w') \leq \theta$. Since $w \in B(x,\rho)$ and $w \notin B(m,R)$, we have that $d(x,w') \leq \rho - R + \theta$, and in particular, w' belongs to $C(x,y)$.

Let $w'' = \pi_{I(x,y)}(w)$. Since $w' \in C(x,y)$, we have $d(w,w'') \leq d(w,w') \leq \theta$, and it follows that $d(x,w'') \leq d(x,w') + d(w',w) + d(w,w'') \leq \rho - R + 3\theta$.

Now, there exists $h \in \breve{\mathscr{H}}_D(x,y) - \breve{\mathscr{H}}_D(x,z)$ such that $w \in h$. By Observation 3, $\operatorname{diam}(\pi_{I(x,y)}(h)) \leq Dd$. Since $w'' \in \pi_{I(x,y)}(h)$,

$$d_{\max}(x, \pi_{I(x,y)}(h)) \leq d(x,w'') + Dd \leq \rho - R + 3\theta + Dd \leq \rho - Dd.$$

Hence $h \in \breve{\mathscr{H}}_{D,\rho-R}(x,y) - \breve{\mathscr{H}}_{D,\rho}(x,z)$ and $w'' \in P$. Since $d(w,w'') \leq \theta = Dd \leq R$, we get that $w \in B(w'',R) \subseteq \mathscr{B}$ as desired. ∎

Stable cylinders for cubulated groups. As discussed in the introduction, stability of cylinders does not depend on the space on which G acts properly and cocompactly.

Proposition 11. *Let G be a hyperbolic group. Let X, X' be graphs on which G acts properly and cocompactly. The space X admits G-globally stable cylinders if and only if X' does.*

Proof. Assume X has G-globally stable cylinders.

Let F be a finite fundamental domain for $G \curvearrowright X$. Define the map $\varphi\colon X \to 2^{X'}$ by sending each $x \in F$ to a non-empty set $\varphi(x)$ of diameter $\operatorname{diam}(\varphi(x)) \leq D$ stabilized by the finite group $\operatorname{Stab}_G(x)$, and extend to X G-equivariantly. By the Švarc–Milnor lemma, the coarse map φ is a quasi-isometry. By replacing $\varphi(x)$ by $N_r(\varphi(x))$ for some large enough r, we assume that φ is "surjective" on X' in the sense that for all $x' \in X'$ there exists $x \in X$ such that $x' \in \varphi(x)$.

By the Morse lemma for X', let R be the constant such that $N_R(\varphi([x,y]))$ contains any geodesic between $x' \in \varphi(x)$ and $y' \in \varphi(y)$. Define the cylinder $C'(x',y')$ for $x', y' \in X'$ by

$$C'(x',y') = N_R(\cup\{\varphi(z) \in X' \mid \forall x,y,z,\ x' \in \varphi(x),\ y' \in \varphi(y),\ z \in C(x,y)\}).$$

It is straightforward to verify that $C'(x',y')$ are G-invariant globally stable cylinders for X'. ∎

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References

[1] M. Bestvina, M. Sageev, and K. Vogtmann, *Geometric group theory*. IAS/Park City Math. Ser. 21, American Mathematical Society, Providence, RI, 2014 Zbl 1306.20002 MR 3307635

[2] J. Brodzki, S. J. Campbell, E. Guentner, G. A. Niblo, and N. J. Wright, Property A and CAT(0) cube complexes. *J. Funct. Anal.* **256** (2009), no. 5, 1408–1431 Zbl 1233.20036 MR 2490224
[3] R. P. Dilworth, A decomposition theorem for partially ordered sets. *Ann. of Math. (2)* **51** (1950), no. 1, 161–166 Zbl 0038.02003 MR 32578
[4] O. Kharlampovich and R. Sklinos, First-order sentences in random groups I: Universal sentences. 2022, arXiv:2106.05461,
[5] N. Lazarovich, Volume vs. complexity of hyperbolic groups. 2021, arXiv:2107.13250,
[6] E. Rips and Z. Sela, Canonical representatives and equations in hyperbolic groups. *Invent. Math.* **120** (1995), no. 3, 489–512 Zbl 0845.57002 MR 1334482
[7] D. T. Wise, Cubulating small cancellation groups. *Geom. Funct. Anal.* **14** (2004), no. 1, 150–214 Zbl 1071.20038 MR 2053602

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