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On virtual indicability and property (T) for outer automorphism groups of RAAGs

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Abstract. We give a condition on the defining graph of a right-angled Artin group, which implies that its automorphism group is virtually indicable, that is, it has a finite index subgroup that admits a homomorphism onto $\mathbb{Z}$. We use this as a part of a criterion that determines precisely when the outer automorphism group of a right-angled Artin group defined on a graph with no separating intersection of links has property (T). As a consequence, we also obtain a similar criterion for graphs in which each equivalence class under the domination relation of Servatius generates an abelian group.

1. Introduction

Kazhdan's property (T) is a rigidity property for groups with many wide-ranging applications, as discussed in the introduction to [2]. A group is said to have property (T) if every unitary representation with almost invariant vectors has invariant vectors. It is notoriously challenging to prove that a group has property (T), however there are certain obstructions to it that can more readily be demonstrated. One such obstruction is that of *virtual indicability*, which occurs for a group when it has a finite index subgroup that admits a surjection onto $\mathbb{Z}$.

Recall that a right-angled Artin group (RAAG) is defined by a presentation usually determined by a simplicial graph Γ. The vertex set of Γ provides the generating set for this presentation, while the defining relators come from commutators between all pairs of adjacent vertices in Γ. The RAAG associated to the graph Γ is denoted by A_Γ.

In the universe of finitely presented groups, the outer automorphism groups of RAAGs provide a bridge between the groups $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Out}(F_n)$, the outer automorphism groups of the non-abelian free groups. At each head of the bridge we understand the groups' behaviours with regards to property (T). For $n \geq 3$, it is well known that $\mathrm{GL}(n, \mathbb{Z})$ has property (T), while $\mathrm{GL}(2, \mathbb{Z})$ does not; at the other end, for $n \geq 5$, computer-aided proofs of Kaluba, Kielak, Nowak and Ozawa [19, 20] tell us $\mathrm{Aut}(F_n)$ (and hence $\mathrm{Out}(F_n)$ since property (T) is inherited by quotients) have property (T), while $\mathrm{Out}(F_2)$ and $\mathrm{Out}(F_3)$ do not [14]. A recent preprint of Nitsche has extended the proof of property (T) for $\mathrm{Aut}(F_n)$ to $n = 4$ [22].

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Besides property (T), there are many characteristics that are shared by both $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Out}(F_n)$, such as finite presentability, residual finiteness, finite virtual cohomological dimension, satisfying a Tits alternative, or being of type VF. All these mentioned properties hold in fact for $\mathrm{Out}(A_\Gamma)$ for all RAAGs A_Γ [7–9,12,17]. Other properties hold at both ends of the bridge, but somewhere in between they may fail. For example, $\mathrm{Out}(F_n)$ and $\mathrm{GL}(n, \mathbb{Z})$ are of course infinite groups (for $n \geq 2$), and both contain non-abelian free subgroups, but there are non-cyclic RAAGs A_Γ for which $\mathrm{Out}(A_\Gamma)$ is finite (for example, take the defining graph Γ to be a pentagon), or which are infinite but do not contain free subgroups [11]. Another example is having a finite outer automorphism group of $\mathrm{Out}(A_\Gamma)$. Both $\mathrm{Out}(\mathrm{Out}(F_n))$ and $\mathrm{Out}(\mathrm{GL}(n, \mathbb{Z}))$ are finite [4, 18], but this is not always true for $\mathrm{Out}(\mathrm{Out}(A_\Gamma))$ [3].

In this paper, we address the following.

Question 1. For which Γ does $\mathrm{Out}(A_\Gamma)$ satisfy Kazhdan's property (T)?

Beyond the above results for $\mathrm{GL}(n, \mathbb{Z})$ and $\mathrm{Out}(F_n)$, there are a number of partial answers to this question. For example, Aramayona and Martínez-Pérez give conditions (see Theorem 2 below) that imply virtual indicability, and hence deny property (T) [1]. We can also describe precisely when $\mathrm{Out}(A_\Gamma)$ is finite (and so has property (T)), and when it is infinite virtually nilpotent (and so it does not have property (T)) [11]. See Section 1.1 for more details on what is known and what is open.

The objective of this paper is to definitively answer Question 1 whenever Γ contains no *separating intersection of links* (SILs). These occur when all paths from some vertex z of Γ to either of a non-adjacent pair x, y must pass through a vertex that is adjacent to both x and y, i.e., the path hits the intersection of the links of x and y, $\mathrm{lk}(x) \cap \mathrm{lk}(y)$ (see Definition 2.3).

Having a SIL is precisely the condition required for $\mathrm{Out}(A_\Gamma)$ to contain two non-commuting partial conjugations (automorphisms defined by conjugating certain vertices in Γ by a given vertex, see Section 2.2). One way to interpret the meaning of a SIL is that its absence pushes $\mathrm{Out}(A_\Gamma)$ to behave more like $\mathrm{GL}(n, \mathbb{Z})$. If $A_\Gamma = F_n$ for $n \geq 3$, then any three vertices of Γ form a SIL, while if $A_\Gamma = \mathbb{Z}^n$, then Γ has no SIL. For the latter, all partial conjugations are clearly trivial, while for the former they play a crucial role in the structure of $\mathrm{Out}(F_n)$. This interpretation is strengthened with the following two observations. With no SIL, a finite index subgroup of $\mathrm{Out}(A_\Gamma)$ admits a quotient by a finite-rank free abelian subgroup that is a block-triangular matrix group with integer entries. This quotient is from the standard representation of $\mathrm{Out}(A_\Gamma)$, obtained by acting on the abelianisation of A_Γ. The second observation is that, again with no SIL, a finite index subgroup of $\mathrm{Out}(A_\Gamma)$ admits a quotient by a finitely generated nilpotent group that is a direct product of groups $\mathrm{SL}(n_i, \mathbb{Z})$, for various integers n_i [15, Theorem 2].

We highlight also a second feature of Γ that affects the automorphism group of the corresponding RAAG. The *domination relation* of Servatius [24] determines when a transvection R_u^v, defined for vertices u, v of Γ by sending u to uv and fixing all other vertices, is an automorphism of A_Γ. We say u is dominated by v, and write $u \leq v$, if any vertex adjacent to u is also adjacent or equal to v (i.e., the link of u is contained in the star of v: $\mathrm{lk}(u) \subseteq \mathrm{st}(v)$). We have that $u \leq v$ if and only if $R_u^v \in \mathrm{Aut}(A_\Gamma)$. The relation of domination is a preorder and therefore determines equivalence classes of vertices.

The following is the aforementioned criteria of Aramayona and Martínez-Pérez to deny property (T).

Theorem 2 (Aramayona–Martínez-Pérez [1]). *Consider the following properties of a graph Γ:*

(A1) *If u, v are distinct vertices of Γ such that $u \leq v$, then there exists a third vertex w such that $u \leq w \leq v$.*

(A2) *If v is a vertex such that $\Gamma \setminus \mathrm{st}(v)$ has more than one connected component, then there exists a vertex u such that $u \leq v$.*

Let Γ be any simplicial graph. If either property (A1) or (A2) fails, then $\mathrm{Aut}(A_\Gamma)$ has a finite index subgroup that maps onto $\mathbb{Z}$.

(In [1], condition (A1) is referred to by (B2), while condition (A2) is the hypothesis of Theorem 1.6.)

When condition (A1) fails, there are vertices u, v in Γ such that there is no w satisfying $u \leq w \leq v$. A homomorphism from a finite index subgroup of $\mathrm{Aut}(A_\Gamma)$ can be constructed with image $\mathbb{Z}$ so that the transvection R_u^v has an infinite order image. The failure of condition (A2) is used to give a surjection onto $\mathbb{Z}$ by exploiting a partial conjugation with multiplier v whose star separates Γ into two or more connected components, but which does not dominate any other vertex.

The first main task of this paper is to modify condition (A2), defining a condition (A2′) to obtain more graphs Γ for which $\mathrm{Aut}(A_\Gamma)$ is virtually indicable. The key, as with (A2), is to exploit certain partial conjugations.

We say a graph Γ satisfies condition (A2′) if for every vertex x and component C of $\Gamma \setminus \mathrm{st}(x)$ some non-zero power of the partial conjugation by x on C can be expressed as a product in $\mathrm{Out}(A_\Gamma)$ of partial conjugations (or their inverses) by x with supports that are components of $\Gamma \setminus \mathrm{st}(y)$ for some y dominated by, but not equal to, x. See the start of Section 5 for another definition, and Lemma 5.1 for equivalence with the one given here. One can observe that if (A2′) holds, then necessarily (A2) holds also.

Our first main result is the following.

Theorem 3. *If a simplicial graph fails property (A2′), then $\mathrm{Aut}(A_\Gamma)$ has a finite index subgroup that admits a surjection onto $\mathbb{Z}$.*

We use this to answer Question 1 entirely when the defining graph has no SIL.

Theorem 4. *Suppose Γ has no SIL. Then $\mathrm{Out}(A_\Gamma)$ has property (T) if and only if both properties (A1) and (A2′) hold in Γ.*

This also allows us to give an answer in any case when all equivalence classes in Γ are abelian.

Corollary 5. *Suppose all equivalence classes in Γ are abelian. Then* $\mathrm{Out}(A_\Gamma)$ *has property (T) if and only if* Γ *has no SIL and both properties (A1) and (A2′) hold.*

Proof. If Γ does contain a SIL, then $\mathrm{Out}(A_\Gamma)$ is large by [15, Theorem 2], and hence does not have property (T). If Γ does not contain a SIL, then we apply Theorem 4. ∎

The proof of Theorem 3 involves a composition of restriction and projection maps (see Section 2.3) to focus our attention on a smaller portion of Γ, ultimately mapping into the automorphism group of a free product of free abelian groups. We then apply a homological representation by acting on a certain cover of the Salvetti complex of the free product. The image of the composition of all these maps can be seen to admit a surjection onto $\mathbb{Z}$.

Once Theorem 3 is established, to prove Theorem 4 we use the standard representation of $\mathrm{Out}(A_\Gamma)$, obtained by acting on the abelianisation of A_Γ. This gives a short exact sequence such that, when Γ contains no SIL, the kernel IA_Γ, sometimes called the Torelli subgroup, is free abelian. To fully exploit this structure of $\mathrm{Out}(A_\Gamma)$, we need this sequence to be split, which we show is (virtually) the case.

In the following, we denote by $\mathrm{SOut}^0(A_\Gamma)$ the subgroup of $\mathrm{Out}(A_\Gamma)$ of finite index that is generated by the set of all transvections and partial conjugations of A_Γ.

Proposition 6. *The standard representation of* $\mathrm{SOut}^0(A_\Gamma)$ *gives the short exact sequence*

$$1 \to \mathrm{IA}_\Gamma \to \mathrm{SOut}^0(A_\Gamma) \to Q \to 1,$$

which splits if Γ *has no SIL.*

We note that the short exact sequence may be split even if there is a SIL. This is discussed in more detail in Remark 6.2, where weaker sufficient conditions are given for this to occur.

Compare Proposition 6 with [16, Theorem 1.2] and [25], where automorphism groups of certain graph products are expressed as a semidirect product in a manner similar to Proposition 6.

Finally, we comment on the fact we show in Theorem 4 that $\mathrm{Out}(A_\Gamma)$ has property (T), but prove nothing about $\mathrm{Aut}(A_\Gamma)$. We use the fact that when there is no SIL, partial conjugations commute in $\mathrm{Out}(A_\Gamma)$. When dealing with automorphisms (not outer), this fact fails. We use a decomposition of IA_Γ in Section 7 to prove Theorem 4, which involves subgroups, denoted by A_C^X, that are normal in $\mathrm{SOut}^0(A_\Gamma)$. Since they are subgroups of IA_Γ, they are free abelian. This enables us to build $\mathrm{SOut}^0(A_\Gamma)$ out of block-triangular groups, for each of which we can verify property (T) via a criterion of Aramayona and Martínez-Pérez [1]. When dealing with this situation for $\mathrm{SAut}^0(A_\Gamma)$, the groups A_C^X need to include some inner automorphisms, and they no longer remain free abelian in general.

1.1. The (un)resolved cases

We finish the introduction by summarising what is known, and what is left unknown with regard to property (T) for outer automorphism groups of RAAGs. Example A below gives a simple case where the literature does not determine property (T).

Properties of Γ that deny property (T) in Out(A_Γ).

- Condition (A1) fails (so Out(A_Γ) is virtually indicable) [1, Corollary 1.4].
- Condition (A2′) fails (so Out(A_Γ) is virtually indicable) (Theorem 3).
- If there is a non-abelian equivalence class of size three (so Out(A_Γ) is large) [15, Theorem 6].
- If there is a "special SIL" (so Out(A_Γ) is large) [15, Proposition 3.15]. A special SIL is a SIL $(x_1, x_2 \mid x_3)$ such that
  - each x_i is in an abelian equivalence class,
  - if $x_i \leq u \leq x_j$, then $u \in [x_1] \cup [x_2] \cup [x_3]$,
  - if $u \leq x_i$ for any i, then there is a connected component C of $\Gamma \setminus \mathrm{st}(u)$ so that $x_1, x_2, x_3 \in C \cup \mathrm{st}(u)$.

There is a handful of cases that are covered by other means, but are contained within one of the above situations. We explain them now.

Firstly, if there is an equivalence class of size two, then Out(A_Γ) is large and we do not have property (T) [15, Theorem 6]. In this case, we would also have the failure of condition (A1).

We also know that Out(A_Γ) is virtually nilpotent if and only if there is no SIL, and all equivalence classes are of size one [11, Theorem 1.3]. If, furthermore, there is either at least one vertex v whose star separates Γ, or at least one pair of vertices u, v satisfying $u \leq v$, then Out(A_Γ) must be infinite. Finitely generated virtually nilpotent groups have property (T) if and only if they are finite, so this gives a class of graphs where Out(A_Γ) does not have property (T). However, if all equivalence classes are of size one and there is some pair of vertices u, v with $u \leq v$, then condition (A1) necessarily fails. Meanwhile, if there is no such pair u, v, but there is some vertex x whose star separates Γ, then x is minimal and so condition (A2) fails (and hence also (A2′)).

A third situation worth remarking upon is that if all equivalence classes are abelian and Γ contains a SIL, then it contains a special SIL [15, Proposition 3.9]. This is used for Corollary 5.

Properties of Γ that imply property (T) in Out(A_Γ).

- If Γ has no SIL and satisfies conditions (A1) and (A2′) (Theorem 4).
- If Γ has no edges and at least four vertices (i.e., Out(A_Γ) $=$ Out(F_n) for $n \geq 4$) [19, 20, 22].

The first point here covers both the case when Out(A_Γ) $=$ GL($n, \mathbb{Z}$), for $n \geq 3$, as well as when Out(A_Γ) is finite. The latter occurs if and only if there are no vertices x, y in Γ so that $x \leq y$, and there is no vertex whose star separates Γ into two or more components (see, for example, [6, §6]).

It is tempting to speculate that the no SIL condition could be removed from Theorem 4, to answer Question 1 completely. However, if we allow Γ to contain a SIL, and assume

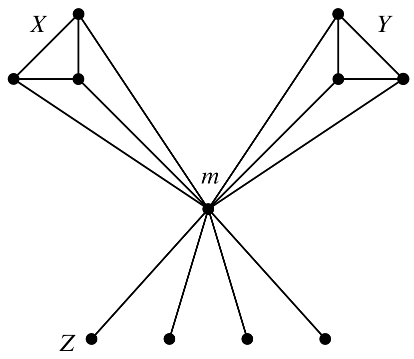

Figure 1. The graph Γ for Example A.

both (A1) and (A2′) hold, then there is nothing to stop us from having a free equivalence class of size three, or all equivalence classes be abelian. In both cases, we do not have property (T) by [15, Theorem 6].

Example A. Let Γ be the graph in Figure 1. This is an example where the existing literature does not tell us whether $\mathrm{Out}(A_\Gamma)$ has property (T) or not.

The graph Γ has been constructed so that it has a non-special SIL, but also satisfies both conditions (A1) and (A2′). The trick to making an example that satisfies both (A1) and (A2′) is to make sure the equivalence classes are sufficiently big and at least one is not abelian.

It has four equivalence classes of vertices: X, Y, Z and $\{m\}$. All vertices are dominated by m, while Z is also dominated by X and Y.

A SIL can be described with three vertices, with the intersection of the links of two separating them from the third. The SILs in Γ involve at most one vertex from each of X and Y and the remainder coming from Z. None of these SILs are special, because in a special SIL the three vertices must come from distinct abelian equivalence classes, which is not possible here.

Since each equivalence class X, Y and Z has more than two elements, condition (A1) is satisfied.

Condition (A2′) is a bit more cumbersome to verify. We can see that for any $x \in X$, the components of $\Gamma \setminus \mathrm{st}(x)$ are Y, and then one for each $z \in Z$. The components are the same for any other x' in X, so the condition on the partial conjugations with multiplier in X holds. For those in Y, it is similar. For any $z \in Z$, if C is a component of $\Gamma \setminus \mathrm{st}(z)$, then either $C = X$, $C = Y$, or $C = \{z'\}$ for some other $z' \in Z$. In any case, we can take $z'' \in Z$ distinct from both z and (if necessary) z'. Then C is also a component of $\Gamma \setminus \mathrm{st}(z'')$, and we see that the condition holds. Finally, for m the condition is vacuous.

The last thing to note is that we cannot use the equivalence classes, as in [15, Proposition 2.1], because the equivalence classes are too large.

Paper structure

We begin with some preliminary material in Section 2. In Section 3, we establish a homological representation for a free product of free abelian groups. This representation is used in obtaining surjections onto $\mathbb{Z}$ later. A new notion is introduced in Section 4 that describes which partial conjugations are necessary to virtually generate the Torelli subgroup IA_Γ when there is no SIL. Theorem 3 is proved in Section 5. The aim of this section is to construct homomorphisms onto groups on which we can then apply the homological representations of Section 3. In Section 6, we establish that when there is no SIL, we can express $\mathrm{SOut}^0(A_\Gamma)$ as a semidirect product $Q \ltimes \mathrm{IA}_\Gamma$, proving Proposition 6. The proof of Theorem 4 is given in Section 7.

2. Generators and SILs

We begin with some background material on RAAGs and their automorphism groups.

RAAGs are defined via a simplicial graph Γ. A generating set of the group A_Γ is the vertex set $V(\Gamma)$, and presentation is given by

$$A_\Gamma = \langle V(\Gamma) \mid [u, v] = 1 \text{ if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle.$$

Given a simplicial graph Γ and a vertex v of Γ, the *link* of v is the induced subgraph on the set of all vertices connected to v by an edge. It is denoted by $\mathrm{lk}(v)$. The *star* of v, written $\mathrm{st}(v)$ is the join of $\mathrm{lk}(v)$ with the vertex v.

2.1. Domination and equivalence classes of vertices

This section deals with transvections on a RAAG. We denote a transvection by R_u^v for vertices $u, v \in \Gamma$, where $R_u^v(u) = uv$ and all other vertices are fixed. Servatius made the following definition that determines when transvections are automorphisms [24].

Definition 2.1. The Servatius *domination* relation $\leq$ on vertices of Γ is given by $u \leq v$ if and only if $\mathrm{lk}(u) \subseteq \mathrm{st}(v)$.

We leave as an exercise to the reader to prove that a transvection R_u^v is an automorphism if and only if $u \leq v$.

Remark 2.2. This definition can be tightened up as follows: if u, v are adjacent, then $u \leq v$ if and only if $\mathrm{st}(u) \subseteq \mathrm{st}(v)$; if u, v are not adjacent, then $u \leq v$ if and only if $\mathrm{lk}(u) \subseteq \mathrm{lk}(v)$.

The domination relation is a preorder and determines *equivalence classes* of vertices. We denote the equivalence class containing a vertex x by $[x]$. These classes come in two flavours. They are either *abelian* if all vertices in the class are pairwise adjacent, or *non-abelian* (also referred to as *free*) otherwise. It is a straightforward exercise to see that the vertices of a non-abelian equivalence class share no edges and so generate a non-abelian free group (see [7, Lemma 2.3]).

For subsets X, Y in Γ, we write $X \leq Y$ whenever $x \leq y$ for each $x \in X$ and $y \in Y$. When X and Y are equivalence classes, if $x \leq y$ for any $x \in X$ and $y \in Y$, then $X \leq Y$.

2.2. Generating $\mathrm{Aut}(A_\Gamma)$

The Laurence–Servatius generating set of $\mathrm{Aut}(A_\Gamma)$ is a finite generating set that consists of the automorphisms of A_Γ of the following four types [21]:

- *Involutions*: automorphisms that send one generator v to its inverse v^{-1} and fix all others.
- *Graph symmetries*: automorphisms that permute the set of vertices of Γ according to a symmetry of the graph.
- *Transvections*: R_u^v for distinct vertices satisfying $u \leq v$.
- *Partial conjugations*: for a vertex v and a connected component C of $\Gamma \setminus \mathrm{st}(v)$, the partial conjugation π_C^v sends every vertex u of C to $v^{-1}uv$ and fixes all others.

For transvections R_u^v and partial conjugations π_C^v, we call the vertex v the *multiplier*, and u or C respectively the *support*. When dealing with partial conjugations, we will also let the support C denote a union of connected components of $\Gamma \setminus \mathrm{st}(v)$.

We denote by $\mathrm{SAut}^0(A_\Gamma)$ (resp. $\mathrm{SOut}^0(A_\Gamma)$) the finite index subgroup of $\mathrm{Aut}(A_\Gamma)$ (resp. $\mathrm{Out}(A_\Gamma)$) that is generated by partial conjugations and transvections.

2.3. Restriction and projection maps

In constructing virtual surjections to $\mathbb{Z}$, we use a composition of homomorphisms, most of which are of one of two types, restriction maps and projection (or factor) maps. These have been exploited in the study of automorphisms of RAAGs and other graph products before, for example [7, 8, 12, 15, 23]. We give a brief description of them here.

Let Λ be a subgraph of Γ. We can define a *restriction map* to $\mathrm{Out}(A_\Lambda)$ from the relative outer automorphism group $\mathrm{Out}(A_\Gamma; A_\Lambda)$ – that is the subgroup of $\mathrm{Out}(A_\Gamma)$ consisting of those automorphisms that preserve the subgroup A_Λ up to conjugacy

$$r\colon \mathrm{Out}(A_\Gamma; A_\Lambda) \to \mathrm{Out}(A_\Lambda).$$

Each outer automorphism Φ in $\mathrm{Out}(A_\Gamma; A_\Lambda)$ restricts to an outer automorphism $r(\Phi)$ of A_Λ.

We wish to use this when the relative outer automorphism group contains certain subgroups. Let X be a set consisting of partial conjugations and transvections. If X contains all of these, then

$$\langle X \rangle = \mathrm{SOut}^0(A_\Gamma),$$

but in general this may not be the case.

The following two tests can be performed to check if $\langle X \rangle$ is contained in $\mathrm{Out}(A_\Gamma; A_\Lambda)$, and hence whether r may be defined on $\langle X \rangle$:

- If $R_u^v \in X$, and $u \in \Lambda$, then $v \in \Lambda$.
- If $\pi_C^w \in X$ and $w \notin \Lambda$, then either $\Lambda \subseteq C \cup \mathrm{st}(w)$ or $\Lambda \cap C = \emptyset$.

Next we discuss *projection maps* (also called factor maps). These are homomorphisms

$$p\colon \langle X \rangle \to \mathrm{Out}(A_\Lambda)$$

obtained by killing vertices of Γ that are not in Λ. To be more explicit, let $\kappa\colon A_\Gamma \to A_\Lambda$ be the quotient map obtained by deleting the vertices in $\Gamma \setminus \Lambda$. Let φ be an automorphism in the outer automorphism class Φ. Then Φ is sent to the outer automorphism class of the map $\kappa(g) \mapsto \kappa \circ \varphi(g)$ for $g \in A_\Gamma$. Thus, for p to be well defined, we need the kernel of κ to be preserved, up to conjugacy by $\langle X \rangle$. This is true for each partial conjugation, so one just needs to check it for transvections in X:

- If $R_u^v \in X$ and $v \in \Lambda$, then $u \in \Lambda$.

2.4. Separating intersection of links

Separating intersections of links (SILs) are an important feature of Γ in relation to the properties of $\mathrm{Out}(A_\Gamma)$. In particular, with no SIL, the subgroup of $\mathrm{Out}(A_\Gamma)$ generated by partial conjugations is abelian. Whereas, with a SIL, this subgroup contains a non-abelian free subgroup.

Definition 2.3. A *separating intersection of links* (SIL) is a triple of vertices $(x, y \mid z)$ in Γ that are not pairwise adjacent, and such that the connected component of $\Gamma \setminus (\mathrm{lk}(x) \cap \mathrm{lk}(y))$ containing z does not contain either x or y.

The key consequence of Γ admitting a SIL $(x, y \mid z)$ is that, if Z is the connected component of $\Gamma \setminus (\mathrm{lk}(x) \cap \mathrm{lk}(y))$ containing z, then π_Z^x and π_Z^y generate a non-abelian free subgroup of $\mathrm{Out}(A_\Gamma)$.

There is a relationship between SILs and equivalence classes. Notably, any three vertices in a non-abelian equivalence class determine a SIL. Thus when considering graphs without a SIL, we immediately remove the possibility of admitting non-abelian equivalence classes of size at least 3. Furthermore, if a graph Γ satisfies condition (A1) of Theorem 4 and has no SIL, then it cannot have any equivalence class of size 2 (this is immediate from condition (A1)), and so all its equivalence classes must be abelian.

We end this section with some preliminary lemmas concerning SILs, particularly how they behave with respect to the domination relation.

Lemma 2.4. *Suppose u, v, w are distinct vertices of Γ such that $u \le v, w$ and $(v, w \mid u)$ is not a SIL. Then $[v, w] = 1$.*

Proof. Suppose $[v, w] \neq 1$. If u and v are adjacent, then $v \in \mathrm{lk}(u) \subseteq \mathrm{st}(w)$, contradicting $[v, w] \neq 1$. Thus $[u, v] \neq 1$, and similarly $[u, w] \neq 1$. Then we have $\mathrm{lk}(u) \subseteq \mathrm{lk}(v) \cap \mathrm{lk}(w)$ by Remark 2.2, implying that $(v, w \mid u)$ is a SIL. ∎

We want to understand how partial conjugations behave under conjugation by transvections. When there is no SIL, either the partial conjugation and transvection commute, or the conjugation action itself behaves like a transvection – see Lemma 2.6. Before stating this, we quickly note the following.

Lemma 2.5. *Suppose $x \leq y$ and C is a connected component of $\Gamma \setminus \mathrm{st}(x)$. Then $C' = C \setminus \mathrm{st}(y)$ is a (possibly empty) union of connected components of $\Gamma \setminus \mathrm{st}(y)$.*

Proof. Suppose z is any vertex of Γ not in C or $\mathrm{st}(y)$. If there is a path between z and a vertex of C, then it must pass through $\mathrm{st}(x)$, and hence through $\mathrm{lk}(x)$. Since $x \leq y$, the path therefore intersects $\mathrm{st}(y)$. ■

Note that if C is empty, then by convention we understand π_C^x to mean the identity map.

Lemma 2.6. *Let $v, x, y \in \Gamma$ be such that $x \leq y$, and let C be a connected component of $\Gamma \setminus \mathrm{st}(v)$. Then, for $\varepsilon, \delta \in \{1, -1\}$, in $\mathrm{Out}(A_\Gamma)$,*

- (i) *π_C^v and R_x^y commute if $v \neq x$ and either $(v, y \mid x)$ is not a SIL or $x, y \notin C$ or $x, y \in C \cup \mathrm{st}(v)$,*
- (ii) *$(\pi_C^v)^\varepsilon R_x^y (\pi_C^v)^{-\varepsilon} = (R_x^v)^{-\varepsilon} R_x^y (R_x^v)^\varepsilon$ if $v \neq x$, $(v, y \mid x)$ is a SIL, and exactly one of x or y is in C,*
- (iii) *$(R_x^y)^\delta \pi_C^x (R_x^y)^{-\delta} = \pi_C^x (\pi_{C'}^y)^\delta$ if $v = x$, where $C' = C \setminus \mathrm{st}(y)$.*

Proof. For part (i), the conclusion when $x, y \notin C$ is immediate as the supports and multipliers are disjoint. Meanwhile, if $x, y \in C \cup \mathrm{st}(v)$, we can multiply by an inner automorphism, effectively replacing C by $\Gamma \setminus (C \cup \mathrm{st}(v))$, to get $x, y \notin C$.

Now suppose that $v \neq x$ and $(v, y \mid x)$ is not a SIL. Up to multiplication by an inner automorphism, we may assume that $x \notin C$. If $v = y$, then (i) holds since the supports of the transvection and partial conjugation are disjoint and the multipliers of each are fixed by one-another. So assume $v \neq y$. We claim that $y \notin C$ also, which implies the automorphisms commute as before. To prove the claim, if $x \in \mathrm{st}(v)$, then $x \leq y$ implies $y \in \mathrm{st}(v)$ too. Assume $x \notin \mathrm{st}(v)$ and $y \in C$. Since $x \notin C$, any path from x to y passes through $\mathrm{st}(v)$, and in particular, using the fact that $x \leq y$, we must therefore have $\mathrm{lk}(x) \subseteq \mathrm{lk}(v)$. Hence $x \leq v, y$. Since $(v, y \mid x)$ is not a SIL, Lemma 2.4 implies that $[v, y] = 1$, contradicting $y \in C$.

Now assume the hypotheses for (ii) hold. As $(v, y \mid x)$ is a SIL, we must have $x \leq v$, as $\mathrm{lk}(x)$ must be contained in $\mathrm{lk}(v) \cap \mathrm{lk}(y)$. As C contains exactly one of x or y, up to multiplication by an inner automorphism, we may assume $y \in C$ and $x \notin C$. Direct calculation then yields

$$\pi_C^v R_x^y (\pi_C^v)^{-1}(x) = x v^{-1} y v = (R_x^v)^{-1} R_x^y R_x^v(x),$$

while all other vertices, including those in C, are fixed. This confirms the claimed identity in (ii) when $\varepsilon = 1$. The case when $\varepsilon = -1$ is similar.

Now assume that $v = x$. Again, up to an inner automorphism, we may assume that $y \notin C$. Then direct calculation (left to the reader) verifies relation (iii), with Lemma 2.5 ensuring $\pi^y_{C'}$ makes sense. ■

2.5. The standard representation

The standard representation of $\mathrm{Out}(A_\Gamma)$ is obtained by acting on the abelianisation of A_Γ. We denote it by

$$\rho\colon \mathrm{Out}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z}),$$

where n is equal to the number of vertices in Γ.

The image of ρ is described in more detail in Section 6. Here we focus on the kernel. It is denoted by IA_Γ, and is sometimes referred to as the Torelli subgroup of $\mathrm{Out}(A_\Gamma)$. Day and Wade independently proved that IA_Γ is generated by the set of partial conjugations and commutator transvections $R_u^{[v,w]} = [R_u^v, R_u^w]$ (see [10, §3] and [26, §4.1]). It follows that IA_Γ is a subgroup of $\mathrm{SOut}^0(A_\Gamma)$, so the kernel is unchanged when taking the restriction of ρ to $\mathrm{SOut}^0(A_\Gamma)$. We will abuse notation by also calling this restriction ρ.

The no SIL condition implies that we have no commutator transvections – this is a consequence of Lemma 2.4. It is not hard to see that with no SILs all partial conjugations commute up to an inner automorphism (it is proved in, or follows immediately from results in, each of [5, 11, 15, 16]). In particular, we have the following.

Proposition 2.7. *Suppose Γ has no SIL. Then IA_Γ is free abelian and is generated by the set of all partial conjugations.*

3. A homological representation for a free product of abelian groups

In this section, we describe a homological representation for a finite index subgroup of $\mathrm{Aut}(G)$, where

$$G = \mathbb{Z}^{c_0} * \cdots * \mathbb{Z}^{c_s} * \mathbb{Z}^d.$$

This representation will be used in the proof of Theorem 3 to obtain a virtual surjection onto $\mathbb{Z}$ when condition (A2′) fails. It is obtained by acting on a subspace of the homology of a certain cover of the Salvetti complex associated to G.

For each $\mathbb{Z}^{c_i}$ factor, let Z_i be a basis set. For the $\mathbb{Z}^d$ factor, write a basis as $\mathcal{Y} = \{y_1, \ldots, y_k, x\}$, so that $d = k + 1$. The reason for distinguishing the element x from the y_i's will become clear in Section 5 when the homological representation is applied.

We now describe the cover of the Salvetti complex on which we act to get the representation. Take the elements of $\mathbb{Z}_2$ to be 1 (the identity) and g (the non-identity element). Let $\pi\colon G \to \mathbb{Z}_2$ be defined by

$$\pi(v) = \begin{cases} 1 & \text{if } v \in \mathcal{Y}, \\ g & \text{otherwise.} \end{cases}$$

The Salvetti complex S of G is a wedge of tori, of dimensions $c_0, c_1, \ldots, c_s, d$. Let T be the double cover of S on which $\mathbb{Z}_2$ acts by deck transformations. The 1-skeleton of T can be thought of as a variation on the Cayley graph of $\mathbb{Z}_2$, where instead of using a generating set to construct the edges, we use $Z_0 \cup \cdots \cup Z_s \cup \mathcal{Y}$. Specifically, there will be edges connecting the vertex labelled 1 to the vertex g for each $a \in Z_i$. We denote these edges by e_a, and for each one there is a corresponding edge ge_a from g to 1. When a is in $\mathcal{Y}$, we have two single-edge loops, e_a at the vertex 1, and its image ge_a at g. The edges e_a and ge_a for $a \in Z_i$ form the 1-skeleton for a c_i-dimensional torus, a two-sheeted cover of the corresponding torus in S. The e_a edges for $a \in \mathcal{Y}$ are the 1-skeleton for a d-dimensional torus, while the edges ge_a form the 1-skeleton for a copy of this torus under g.

Let $\mathrm{Aut}_\pi(G)$ be the finite index subgroup of $\mathrm{Aut}(G)$ of automorphisms φ such that $\pi \circ \varphi = \pi$. These automorphisms are induced by homotopy equivalences of S which lift to homotopy equivalences of T, fixing the two vertices 1 and g. We therefore have an action of $\mathrm{Aut}_\pi(G)$ on the homology $H_1(T;\mathbb{Q})$, which preserves $H_1(T;\mathbb{Z})$ and commutes with the action of $\mathbb{Z}_2$.

The action we desire is on a subspace of $H_1(T;\mathbb{Q})$, namely the eigenspace corresponding to the eigenvalue -1 for the action of $\mathbb{Z}_2$. We will denote this eigenspace by V_{-1}. In the special case when $c_0 = c_1 = \cdots = c_s = d = 1$, the group G is the free group F_{s+2}, and by Gäschutz [13] the homology $H_1(T;\mathbb{Q})$ decomposes as $\mathbb{Q} \oplus \mathbb{Q}[\mathbb{Z}_2]^{s+1}$, and $V_{-1} \cong \mathbb{Q}^{s+1}$. In general, we show the following.

Lemma 3.1. *Let T be the two-sheeted cover of the Salvetti complex of G associated to the map $\pi: G \to \mathbb{Z}_2$ defined above, and let V_{-1} be the -1-eigenspace for the action by $\mathbb{Z}_2$. Then*

(I) $H_1(T;\mathbb{Q}) \cong \mathbb{Q}^{(\sum c_i)-s} \oplus \mathbb{Q}[\mathbb{Z}_2]^{d+s}$,

(II) $V_{-1} \cong \mathbb{Q}^{d+s}$.

Proof. Let A_i denote the $\mathbb{Q}$-vector space of i-dimensional chains in T. Each of these decomposes into the sum of $+1$ and -1-eigenspaces under the $\mathbb{Z}_2$ action: $A_i = A_i^{(+1)} \oplus A_i^{(-1)}$. The boundary maps $\partial_i: A_i \to A_{i-1}$ commute with the action of $\mathbb{Z}_2$, so we get restrictions of these to the eigenspaces

$$\partial_i^{(+1)}\colon A_i^{(+1)} \to A_i^{(+1)} \quad \text{and} \quad \partial_i^{(-1)}\colon A_i^{(-1)} \to A_i^{(-1)}.$$

The eigenspace V_{-1} of $H_1(T;\mathbb{Q})$ is the quotient $\ker \partial_1^{(-1)} / \operatorname{im} \partial_2^{(-1)}$.

Fix a vertex $z_0 \in Z_0$. The space of all 1-cycles has dimension $2d + 2(\sum c_i) - 1$ and a basis given by the following vectors:

- $e_x, e_{y_1}, \ldots, e_{y_k}$ and $ge_x, ge_{y_1}, \ldots, ge_{y_k}$,
- $e_{z_0} - e_a$ and $g(e_{z_0} - e_a)$ for $a \in Z_0 \cup \cdots \cup Z_s$, $a \neq z_0$,
- $e_{z_0} + ge_{z_0}$.

We can describe its structure as

$$\ker \partial_1 \cong \mathbb{Q} \oplus \mathbb{Q}[\mathbb{Z}_2]^{d+(\sum c_i)-1}.$$

When moving onto the -1-eigenspace, the dimension of $\ker \partial_1^{(-1)}$ is $k + (\sum c_i)$ and a basis is given by

- $(1-g)e_x, (1-g)e_{y_1}, \ldots, (1-g)e_{y_k}$,
- $(1-g)(e_{z_0} - e_a)$ for $a \in Z_0 \cup \cdots \cup Z_s$, $a \neq z_0$.

There are two types of 2-cells found in T. The first are those coming from the commutator relation between two vertices in Z_i, and the second are those from the commutator relation between two vertices in $\mathcal{Y}$. The latter have the boundary equal to zero in A_1. The former have the boundary $\pm(e_a + ge_b - ge_a - e_b)$, for $a, b \in Z_i$. In particular, $\operatorname{im} \partial_2$ is generated by $(1-g)(e_a - e_b)$ for $a, b \in Z_i$, and $i = 0, \ldots, s$. We can fix a vertex z_i in each of the remaining classes Z_i. Then a basis for $\operatorname{im} \partial_2$ is given by

$$\{(1-g)(e_{z_i} - e_a) \mid a \in Z_i \setminus \{z_i\},\ i = 0, \ldots, s\}.$$

We therefore have $\dim(\operatorname{im} \partial_2) = \sum(c_i - 1)$. Furthermore, these are all cycles in the -1-eigenspace of A_1, so $\operatorname{im} \partial_2 = \operatorname{im} \partial_2^{(-1)}$. We conclude that V_{-1} has dimension

$$k + \sum_{i=0}^{s} c_i - \sum_{i=0}^{s} (c_i - 1) = k + s + 1 = d + s$$

as required.

Finally, the claimed structure of $H_1(T; \mathbb{Q})$ follows by the fact that the following list of elements form a basis

- $e_x, e_{y_1}, \ldots, e_{y_k}$ and $ge_x, ge_{y_1}, \ldots, ge_{y_k}$,
- $e_{z_0} - e_{z_i}$ and $g(e_{z_0} - e_{z_i})$ for $i = 1, \ldots, s$,
- $e_{z_i} - e_a = g(e_{z_i} - e_a)$ for $a \in Z_i$ and $i = 0, \ldots, s$,
- $e_{z_0} + ge_{z_0}$. ■

We note that, following the proof of Lemma 3.1, we can write down a basis for V_{-1} as follows:

- $\mathbf{x} = (1-g)e_x$,
- $\mathbf{y}_j = (1-g)e_{y_j}$ for $j = 1, \ldots, k$,
- $\mathbf{z}_i = (1-g)(e_{z_0} - e_{z_i})$ for $i = 1, \ldots, s$.

We thus have a representation of $\operatorname{Aut}_\pi(G)$ obtained by acting on V_{-1},

$$\operatorname{Aut}_\pi(G) \to \operatorname{PGL}(V_{-1}).$$

We observe that inner automorphisms act on V_{-1} as -1, so are in the kernel of this representation. This means that it factors through the finite index subgroup $\operatorname{Out}_\pi(G)$ of $\operatorname{Out}(G)$ that is the quotient of $\operatorname{Aut}_\pi(G)$ by the inner automorphisms. Thus we define the representation ρ_π to be the representation on $\operatorname{Out}_\pi(G)$,

$$\rho_\pi \colon \operatorname{Out}_\pi(G) \to \operatorname{PGL}(V_{-1}).$$

3.1. The action of partial conjugations

We now take time to look at how partial conjugations behave under the representation ρ_π. Recall that we are acting on the projective space associated to V_{-1}.

We first look at how the partial conjugations $\pi^x_{Z_i}$ act on the vectors $\mathbf{z}_j$. First assume $i = 0$. Then $\pi^x_{Z_0}$ sends z_0 to $x^{-1}z_0x$ in G. This means that e_{z_0} is sent to $-e_x + e_{z_0} + ge_x = e_{z_0} - (1-g)e_x$, and we get for each $j = 1, \dots, r$,

$$\rho_\pi(\pi^x_{Z_0})(\mathbf{z}_j) = (1-g)(e_{z_0} - (1-g)e_x - e_{z_j}) = \mathbf{z}_j - (1-g)^2 e_x = \mathbf{z}_j - 2\mathbf{x}.$$

All other basis vectors are fixed by $\pi^x_{Z_0}$. Thinking of its matrix representation, if we order the basis elements as $\mathbf{z}_1, \dots, \mathbf{z}_r, \mathbf{y}_1, \dots, \mathbf{y}_k, \mathbf{x}$, then the matrix for $\pi^x_{Z_0}$ is as follows, with the first block marking off the basis vectors $\mathbf{z}_i$, the second block for $\mathbf{y}_i$ and the final block for $\mathbf{x}$,

$$\rho_\pi(\pi^x_{Z_0}) = \left(\begin{array}{cccc|cccc|c} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & & & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & & & \vdots & & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \\ \hline -2 & -2 & \cdots & -2 & 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Similar calculations yield that $\rho_\pi(\pi^x_{Z_i})$, for $i = 1, \dots, s$, will be a transvection:

$$\bar{\rho_\pi}(\pi^x_{Z_i})(\mathbf{z}_j) = \begin{cases} \mathbf{z}_j + 2\mathbf{x} & \text{if } i = j, \\ \mathbf{z}_j & \text{otherwise.} \end{cases}$$

The matrix representation for this will be an elementary matrix differing from the identity by a 2 in the appropriate entry.

Consider a partial conjugation π^a_D for $a \neq z$. Up to an inner automorphism, we can assume that $Z_0 \not\subseteq D$. Then vectors $\mathbf{z}_i$ are fixed whenever $Z_i \not\subseteq D$. First let us assume $a = y_j$ for some j. When $Z_i \subseteq D$, we have $\rho_\pi(\pi^{y_j}_D)(e_{z_i}) = -e_{y_j} + e_{z_i} + ge_{y_j}$. We therefore get

$$\rho_\pi(\pi^{y_j}_D)(\mathbf{z}_i) = \begin{cases} \mathbf{z}_i + 2\mathbf{y}_j & \text{if } Z_i \subseteq D, \\ \mathbf{z}_i & \text{if } Z_i \not\subseteq D. \end{cases}$$

If instead $a = z_j$ for some j, then $\rho_\pi(\pi^{z_j}_D)(e_{z_i}) = -ge_{z_j} + ge_{z_i} + e_{z_j}$ and

$$\rho_\pi(\pi^{z_j}_D)(\mathbf{z}_i) = \begin{cases} \mathbf{z}_i - 2\mathbf{z}_j & \text{if } Z_i \subseteq D, \\ \mathbf{z}_i & \text{if } Z_i \not\subseteq D. \end{cases}$$

Another way that we can write this is to use the standard inner product on $W = \mathrm{Span}_{\mathbb{Q}}(\mathbf{z}_1, \dots, \mathbf{z}_s) \cong \mathbb{Q}^s$. Define $\mathbf{w}_D \in W$, so that the i-th entry is equal to 1 if $Z_i \subseteq D$, and 0 otherwise (this is how it is also defined later, in Lemma 4.4). Then

$$\rho_\pi(\pi_D^{y_j})(\mathbf{z}_i) = \mathbf{z}_i + 2\langle \mathbf{z}_i, \mathbf{w}_D\rangle \mathbf{y}_j \quad \text{and} \quad \rho_\pi(\pi_D^{z_j})(\mathbf{z}_i) = \mathbf{z}_i - 2\langle \mathbf{z}_i, \mathbf{w}_D\rangle \mathbf{z}_j .$$

This extends linearly over W, as per the following lemma.

Lemma 3.2. *For $\mathbf{v} \in W$, and partial conjugations $\pi_D^{y_j}$ with $Z_0 \nsubseteq D$ and $j = 1, \dots, k$, we have*

$$\rho_\pi(\pi_D^{y_j})(\mathbf{v}) = \mathbf{v} + 2\langle \mathbf{v}, \mathbf{w}_D\rangle \mathbf{y}_j ,$$
$$\rho_\pi(\pi_D^{y_j})(\mathbf{x}) = \mathbf{x}, \quad \rho_\pi(\pi_D^{y_j})(\mathbf{y}_l) = \mathbf{y}_l \quad \textit{for } l = 1, \dots, k.$$

For partial conjugations π_D^{z} with $Z_0 \nsubseteq D$ and $z \in Z_i$, for $i = 1, \dots, s$, we have

$$\rho_\pi(\pi_D^{z})(\mathbf{v}) = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{w}_D\rangle \mathbf{z}_i ,$$
$$\rho_\pi(\pi_D^{z})(\mathbf{x}) = \pm\mathbf{x}, \quad \rho_\pi(\pi_D^{z})(\mathbf{y}_l) = \pm\mathbf{y}_l \quad \textit{for } l = 1, \dots, k.$$

Meanwhile, any partial conjugation with multiplier z in Z_0 acts as -1 on $\mathbf{z}_i$ if $Z_i \subseteq D$. Hence

$$\rho_\pi(\pi_D^{z}) = \mathrm{Id} .$$

Proof. The statements regarding partial conjugations with multipliers $y = y_j$ or $z \in Z_i$ for $i > 0$ follow from the discussion preceding the lemma, except when acting on $\mathbf{x}$ or $\mathbf{y}_j$. With multiplier y_j the action on $\mathbf{x}$ and $\mathbf{y}_l$ is trivial since x and y_l commute with y_j. With multiplier z, either $\mathcal{Y}$ is not in D and the vectors $\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_k$ are fixed, or e_x (resp. e_{y_l}) is sent to ge_x (resp. ge_{y_l}), meaning $\mathbf{x} \mapsto -\mathbf{x}$ (resp. $\mathbf{y}_j \mapsto -\mathbf{y}_l$).

The final statement concerning those with multiplier $z \in Z_0$ follows from direct calculation. If $z_i \in D$, then

$$\pi_D^{z}(z_i) = z^{-1} z_i z \Rightarrow e_{z_i} \mapsto -ge_{z_0} + ge_{z_i} + e_{z_0} \Rightarrow \mathbf{z}_i \mapsto -\mathbf{z}_i ;$$

if $x \in D$, then

$$\pi_D^{z}(x) = z^{-1} x z \Rightarrow e_x \mapsto -ge_{z_0} + ge_x + ge_{z_0} = ge_x \Rightarrow \mathbf{x} \mapsto -\mathbf{x}.$$

Calculations for any $y_i \in D$ are the same as for x. ■

4. Principal equivalence class-component pairs

We will introduce a new notion that concerns partial conjugations, with the objective being to refine the set of partial conjugations necessary to virtually generate the Torelli subgroup IA_Γ when Γ has no SIL (see Lemma 4.5).

Before we do so, we introduce some terminology. Deleting the star of a vertex x can divide Γ up into multiple pieces, each being the support of a partial conjugation with multiplier x. We use the following to refer to these pieces.

Definition 4.1. Let x be a vertex of Γ. We use the phrase *x-component* to mean a connected component of $\Gamma \setminus \mathrm{st}(x)$.

Let X be a subset of Γ. An *X-component* is an x-component C for some $x \in X$.

Note that the terminology is similar, but different, to that found in [8], where a $\hat{v}$-component consists of a set of vertices in $\Gamma \setminus \mathrm{st}(v)$ that can be connected by a path that has no edge in $\mathrm{st}(v)$.

We will be focussing on X-components when X is an equivalence class. There are two possibilities here depending on whether X is abelian or not. If it is, then each X-component is also an x-component for each $x \in X$. If X is a free equivalence class and C is an X-component that does not intersect X, then C is again an x-component for each $x \in X$. However, this is not always the case, the (only) exception to this rule being the following. If $x, x' \in X$, then $\{x'\}$ is an x-component, though clearly not an x'-component. In particular, each vertex in a free equivalence class X forms an X-component.

If $x \geq y$, then by Lemma 2.5, the x-components are each a union of y-components, with vertices from $\mathrm{st}(x)$ removed. We are interested in when a partial conjugation π_C^x can be expressed as a product in $\mathrm{Out}(A_\Gamma)$ of partial conjugations whose supports are y-components, for $y \leq x$.

Definition 4.2. Let x be a vertex of Γ and C an x-component. We say the partial conjugation π_C^x is *virtually obtained from dominated components* if there exists $n \neq 0$ so that in $\mathrm{Out}(A_\Gamma)$

$$(\pi_C^x)^n = (\pi_{D_1}^x)^{\varepsilon_1} \cdots (\pi_{D_m}^x)^{\varepsilon_m},$$

where each ε_i is an integer, and the set D_i is a y_i-component for some y_i dominated by, but not equal to x.

If furthermore each D_i can be taken to be a y_i-component for some y_i dominated by, but not *equivalent* to x, then we say π_C^x is *virtually obtained from dominated non-equivalent components*.

The product in Definition 4.2 is considered within $\mathrm{Out}(A_\Gamma)$. We note that in certain cases it is equivalent to look at products within $\mathrm{Aut}(A_\Gamma)$, however this is not always the case. Indeed, provided there is some vertex y (resp. equivalence class Y) dominated by, and distinct from, the vertex x (resp. equivalence class X), the inner automorphism by x is itself virtually obtained from dominated components (resp. from dominated non-equivalent components). However, if $\Gamma \setminus \mathrm{st}(x)$ is connected and x is not dominated by any other vertex, then π_C^x is inner and so is virtually obtained from dominated components (in $\mathrm{Out}(A_\Gamma)$, with the empty word).

Note that the product being in $\mathrm{Out}(A_\Gamma)$ means π_C^x (or its power) can be constructed using the complement of C in $\Gamma \setminus \mathrm{st}(x)$.

Definition 4.3. Let X be an equivalence class in Γ and C an X-component. We say the pair (X, C) is *non-principal* if π_C^x is virtually obtained from dominated non-equivalent components for any $x \in X$. Otherwise, we say (X, C) is *principal*.

We introduce some notation here. Let Y be an equivalence class of Γ, and let C be a Y-component. Define the set P_C^Y to be

$$P_C^Y = \{\pi_{C'}^x \mid x \geq Y,\ C' = C \setminus \mathrm{st}(x)\}.$$

Then π_C^x is virtually obtained from dominated non-equivalent components if there is a non-zero integer n such that

$$(\pi_C^x)^n \in \langle P_D^Y \mid Y \leq X,\ Y \neq X, \text{ and } D \text{ is a } Y\text{-component}\rangle$$

for each $x \in X$.

Observe that if for any $x \in X$ there is only one x-component, then π_C^x is inner and hence is virtually obtained from dominated non-equivalent components. Thus, if (X, C) is principal, then $\Gamma \setminus \mathrm{st}(x)$ is not connected for any $x \in X$.

An immediate example of principal pairs (X, C) occurs when X is a domination-minimal equivalence class such that for any $x \in X$ the star of x separates Γ. A simple example of a non-principal pair is given in Example B below. An example where we have to take $n > 1$ is given in Example C. Further examples are given in Section 5 below, in Examples D, E, and F.

Example B. We consider the graph in Figure 2. In this graph, the vertex x dominates vertices y, z_1, z_2 and z_3, and forms its own equivalence class $X = \{x\}$. The X-components are $C_1 = \{z_1\}$, $C_2 = \{z_2\}$, and $C_3 = \{z_3\}$. The pairs (X, C_1) and (X, C_2) are non-principal since C_1 and C_2 are also z_3-components, meaning that $\pi_{C_1}^x \in P_{C_1}^{[z_3]}$ and $\pi_{C_2}^x \in P_{C_2}^{[z_3]}$. Since

$$(\pi_{C_3}^x)^{-1} = \pi_{C_1}^x \pi_{C_2}^x \quad \text{in } \mathrm{Out}(A_\Gamma),$$

we also deduce that (X, C_3) is non-principal as $\pi_{C_3}^x \in \langle P_{C_1}^{[z_3]}, P_{C_2}^{[z_3]}\rangle$.

Example C. We consider the graph Γ constructed using Figure 3. There are four x-components, C_0, C_1, C_2, and C_3, labelled so that $z_i \in C_i$. The only vertices dominated by x are those added to the diagram: y_1, y_2, and y_3. The y_1-components are $C_0 \cup C_3$ and

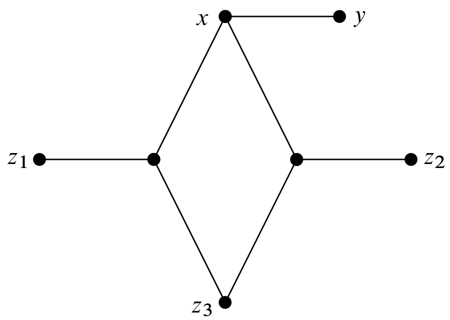

Figure 2. The graph Γ for Example B.

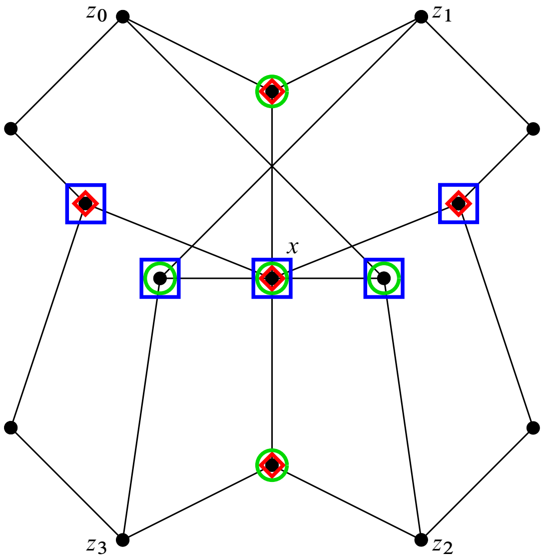

Figure 3. To construct Γ, add three vertices to the graph shown here. Add a vertex y_1 whose link consists of the vertices with a green circle ○; add a vertex y_2 whose link consists of the vertices with a blue square □; add a vertex y_3 whose link consists of the vertices with a red diamond ◇.

$C_1 \cup C_2$; the y_2-components are $C_0 \cup C_1$ and $C_2 \cup C_3$; the y_3-components are $C_0 \cup C_2$ and $C_1 \cup C_3$. Then

$$\pi^x_{C_0 \cup C_3} \pi^x_{C_0 \cup C_1} \pi^x_{C_0 \cup C_2} = (\pi^x_{C_0})^2 \operatorname{ad}_x = (\pi^x_{C_0})^2$$

implying

$$(\pi^x_{C_0})^2 \in \langle P^{[y_1]}_{C_0 \cup C_3}, P^{[y_2]}_{C_0 \cup C_1}, P^{[y_3]}_{C_0 \cup C_2} \rangle.$$

The following lemma translates the problem of determining principality into a linear algebra question. We will exploit this approach in Section 5.6.

Lemma 4.4. *Let X be an equivalence class, and let $C_0, C_1, \dots, C_r$ be the X-components. Let W be an r-dimensional vector space over $\mathbb{Q}$, and let $\{\mathbf{z}_1, \dots, \mathbf{z}_r\}$ be a basis. For each vertex y dominated by, but not in, X, and each y-component D, let $\mathbf{w}_D$ denote the vector*

$$\mathbf{w}_D = w_1 \mathbf{z}_1 + \cdots + w_r \mathbf{z}_r,$$

where

$$w_i = \begin{cases} 1 & \text{if } C_i \subset D, \\ 0 & \text{otherwise.} \end{cases}$$

Let Δ denote the set of all vectors $\mathbf{w}_D$ as D varies among all y-components, for all vertices y dominated by, but not in, X.

Then the vector $\langle 1, 1, \dots, 1 \rangle$ is in $\operatorname{Span}_{\mathbb{Q}}(\Delta)$ if and only if (X, C_0) is non-principal.

Proof. Consider a product of partial conjugations with multiplier $x \in X$ in the sets P_D^Y, for equivalence classes Y that are dominated by, but not equal to, X, and Y-components D. We write the product as $(\pi_{D_1}^x)^{\varepsilon_1} \cdots (\pi_{D_l}^x)^{\varepsilon_l}$, where each D_i is equal to a Y-component with $\mathrm{st}(x)$ removed, and $\varepsilon_i = \pm 1$. Up to an inner automorphism, and flipping the sign of ε_i, we may assume $C_0 \not\subset D_i$, and we then take the vector $\mathbf{w}_{D_i} \in \Delta$ defined above.

The vector $\mathbf{w} = \sum \varepsilon_i \mathbf{w}_{D_i}$ records the action of x on the vertices of each component C_j in the following way. For $z \in C_1 \cup \cdots \cup C_r$, we have

$$(\pi_{D_1}^x)^{\varepsilon_1} \cdots (\pi_{D_l}^x)^{\varepsilon_l}(z) = x^{-\alpha_j} z x^{\alpha_j},$$

where

$$\alpha_j = \sum_{\{i \mid C_j \subset D_i\}} \varepsilon_i \quad \text{and} \quad \langle \alpha_1, \ldots, \alpha_r \rangle = \sum_{i=1}^{l} \varepsilon_i \mathbf{w}_{D_i}.$$

If (X, C_0) is not principal, then $(\pi_{C_0}^x)^n$, for some integer n, can be written as a product as above. In particular, we would have $\alpha_i = n$ for each i, implying $\langle 1, 1, \ldots, 1 \rangle$ is in the span of Δ.

On the other hand, if we can write $\langle n, n, \ldots, n \rangle$ as a $\mathbb{Z}$-linear combination of the vectors Δ, then by the above we can translate that into a product of partial conjugations from sets P_D^Y, with $Y \leq X$ and $Y \neq X$ that equal $(\pi_{C_0}^x)^n$. ∎

The following lemma explains why we refer to the pairs as principal or non-principal. This is important in Section 7 when we deal with graphs with no SIL. The Torelli subgroup is the kernel of the standard representation, obtained by acting on the abelianisation of A_Γ. For this lemma, all we need to know about the Torelli subgroup is that when there is no SIL it is abelian and is generated by the set of partial conjugations (see also Proposition 2.7).

Lemma 4.5. *When Γ contains no SIL, the union of the sets P_C^X for which (X, C) is principal generates a finite index subgroup of the Torelli subgroup IA_Γ.*

Proof. Seeing that these sets generate a finite index subgroup is done by first noting that some power of every partial conjugation π_D^y is in the subgroup generated by the sets P_C^X with (X, C) principal. This follows from the definition of principal/non-principal pairs. Together with Proposition 2.7, which states that IA_Γ is abelian and generated by the set of all partial conjugations, this proves the lemma. ∎

5. Virtual indicability from partial conjugations

We begin by generalising condition (A2), defining the following property of a graph Γ. Note that Lemma 5.1 below gives an alternate definition with as much jargon removed as possible:

(A2′) If an equivalence class X admits an X-component C for which (X, C) is principal, then $|X| > 1$.

The aim of this section is to prove Theorem 3, which states that if Γ fails condition (A2′), then $\mathrm{Aut}(A_\Gamma)$ is virtually indicable.

Recall, $([x], C)$ being principal means that π_C^x cannot be virtually obtained from dominated non-equivalent components. That is, no non-trivial power of π_C^x can be expressed as a product of partial conjugations π_D^x, where D is a y-component for some y dominated by, but not equivalent to, x (see Definition 4.2). The product takes place in $\mathrm{Out}(A_\Gamma)$.

Condition (A2) of Aramayona–Martínez-Pérez [1] implies that the minimal star-separating equivalence classes have size greater than one. These will of course be principal, thus, as mentioned in the introduction, (A2′) implies (A2).

We remark that the definition of (A2′) is a peculiar mix of graphical and algebraic conditions. We can ask whether it is possible to express the condition in purely graphical language. However, we feel somewhat pessimistic about hopes to do this in a meaningful way. Indeed, Example C gives a situation where some naïve criterion to deny principality – either C is a y-component for some $y \leq X$, $y \notin X$, or the complement of C in $\Gamma \setminus (\mathrm{st}(x) \cup X)$ can be expressed as a disjoint union of such components – may apply. Instead, we suspect that using something like Lemma 4.4 is as close as we may get. We can remove the linear algebra language from Lemma 4.4, for example, by describing a "game" on an associated graph, as described below, and looking for solutions to this game.

The graph G for the game is defined as follows. For each X-component except C, add a vertex. Then for each vertex u of Γ that is dominated by, but not in, X, add edges between vertices of G that are in common u-components, but not for those that are in the same u-component as C. Label each edge by the corresponding u. The pair (X, C) is then non-principal if and only if the following game has a winning strategy. Each vertex of G is given a label in $\mathbb{Z}$, which is initially set to zero. A move in the game involves choosing some u dominated by X, some vertex D of G, and either increasing or decreasing by 1 the integer label on D and on every other vertex of G that is connected to D by an edge labelled by u. The player wins the game if they can simultaneously relabel each vertex of G by the same non-zero integer. (Note that if there are no such u, then there is no possible move to make, and the player automatically loses.)

The following is the reformulation of condition (A2′) given in the introduction, which removes the language of equivalence classes and principality.

Lemma 5.1. *A graph Γ satisfies (A2′) if and only if for every vertex x in Γ and every x-component C, the partial conjugation π_C^x is virtually obtained from dominated components.*

Proof. We prove the contrapositive statement. We have failure of (A2′) if and only if there is some equivalence class X of size one and an X-component C so that (X, C) is principal. That is to say, $X = \{x\}$, and π_C^x is not virtually obtained from dominated non-equivalent components. Since no vertex is equivalent to x, this gives the "if" direction.

Now suppose we have x and an x-component C, so that π_C^x is not virtually obtained from dominated components. We want to show that $[x] = \{x\}$ and $([x], C)$ is principal, so (A2′) fails. The latter follows immediately once we have shown the former. So suppose $x \sim x'$. Then we necessarily have $x' \in C$ since otherwise C would be an x'-component, contradicting the hypothesis on π^x_C. Since C is an x-component, we must furthermore have $C = \{x'\}$ and $[x] = \{x, x'\}$. Suppose $C_1, \dots, C_r$ are the remaining x-components. Then each C_i is also an x'-component and

$$(\pi^x_C)^{-1} = \pi^x_{C_1} \cdots \pi^x_{C_r}$$

again contradicts our assumption on π^x_C. Hence no vertex is equivalent to x, and the lemma follows. ∎

5.1. Outline of proof

The objective is to prove Theorem 3, asserting that if condition (A2′) fails, then $\mathrm{Aut}(A_\Gamma)$ is virtually indicable. Thus we assume (A2′) fails, giving us a vertex x and x-component C satisfying the following conditions:

- $X = [x] = \{x\}$,
- (X, C) is principal,
- x is domination-minimal among vertices satisfying the first two conditions.

We hereby fix x and C throughout Section 5.

We will construct a homomorphism from a finite index subgroup of $\mathrm{Out}(A_\Gamma)$ onto $\mathbb{Z}$. The process to construct such a virtual map onto $\mathbb{Z}$ involves refining the graph Γ through projection and restriction maps. Note that the domination relation that we will be referring to throughout always refers to domination in Γ.

Step 1: Decluttering Γ. We aim to exploit the partial conjugation π^x_C to obtain a surjection onto $\mathbb{Z}$. The x-components therefore play a crucial role. The first step, completed in Section 5.3, is to remove much of Γ by using a projection map, but being sure to leave something in each x-component for the partial conjugations by x to act on.

If $C = C_0, C_1, \dots, C_r$ are the x-components, we explain in Definition 5.2 how to choose a subset Z_i of C_i, for each i, to create a subgraph $\widehat{\Lambda}$ of Γ that admits a projection map

$$p_1\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(A_{\widehat{\Lambda}}).$$

The construction of $\widehat{\Lambda}$ ensures it contains x and all vertices dominated by x.

The graph $\widehat{\Lambda}$ is disconnected. We can abelianise each component to obtain a graph Λ_0, so that

$$A_{\Lambda_0} = \mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \cdots * \mathbb{Z}^{c_r} * \mathbb{Z}^{k+1}$$

for some integers $c_0, \dots, c_r, k$, and which furthermore admits a homomorphism (Lemma 5.5) from the image of p_1 to $\mathrm{Out}(A_{\Lambda_0})$. Composing this with p_1, we get a homomorphism

$$p\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(A_{\Lambda_0}).$$

Step 2: Dividing into cases. The arrangement of the subsets Z_i with regards to x and the vertices it dominates can cause different issues to arise. We use these to split into two cases in Section 5.4. The first case ultimately yields a homomorphism

$$q\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(\mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \mathbb{Z})$$

for some positive integers c_0 and c_1. In the second case, we get a similar map, but we may have more than three free factors in the target group. We get

$$q\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(\mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \cdots * \mathbb{Z}^{c_s} * \mathbb{Z}^d)$$

for positive integers $c_0, \ldots, c_s, d$.

Step 3: Employing the homological representation of Section 3. Readers familiar with homological representations will appreciate that in the first case, if the image of q in $\mathrm{Out}(\mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \mathbb{Z})$ is sufficiently rich (even just containing a non-trivial partial conjugation), then obtaining a virtual surjection onto $\mathbb{Z}$ is entirely plausible. In fact, if there are enough partial conjugations in the image, we can even obtain largeness of $\mathrm{Out}(A_\Gamma)$ (see Remark 5.11).

The second case, though, is more subtle. The homological representation we use initially yields an action on a vector space of dimension $d + s$. But by studying the action of partial conjugations and transvections on this space, we are able to restrict to a subspace V so that the image of a finite index subgroup of $\mathrm{Aut}(A_\Gamma)$ in $\mathrm{PGL}(V)$ is free abelian.

To summarise, Figure 4 shows the maps that are constructed in the process of obtaining a virtual epimorphism to $\mathbb{Z}$.

5.2. Examples

We describe these three steps in three examples, each with slight differences, before describing the full process.

Note that these are not "new" examples, in the sense that $\mathrm{Aut}(A_\Gamma)$ is already known to be virtually indicable by earlier results. For example, we can apply Theorem 2 (a result of Aramayona and Martínez-Pérez [1]) to obtain a virtual surjection onto $\mathbb{Z}$. We use these examples though to highlight our method. Some tricks, such as replacing vertices by equivalence classes, or introducing benign vertices, can be used to prevent the application of earlier results, but doing so over-complicates matters and defeats the purpose of including some, hopefully, enlightening examples.

In each example (and, indeed, the rest of this section), we refer to a set U which consists of those vertices in $\widehat{\Lambda}$ that are dominated by, but not equal to, x.

Example D. The graph Γ given in Figure 5 satisfies the conditions of case (1) of Section 5.4.

There are five x-components, $C_0, \ldots, C_4$, and in each a domination-minimal vertex z_i has been selected. In Γ, the vertex z_0 dominates x; meanwhile, z_2 and z_3 are dominated by x, and form the set U.

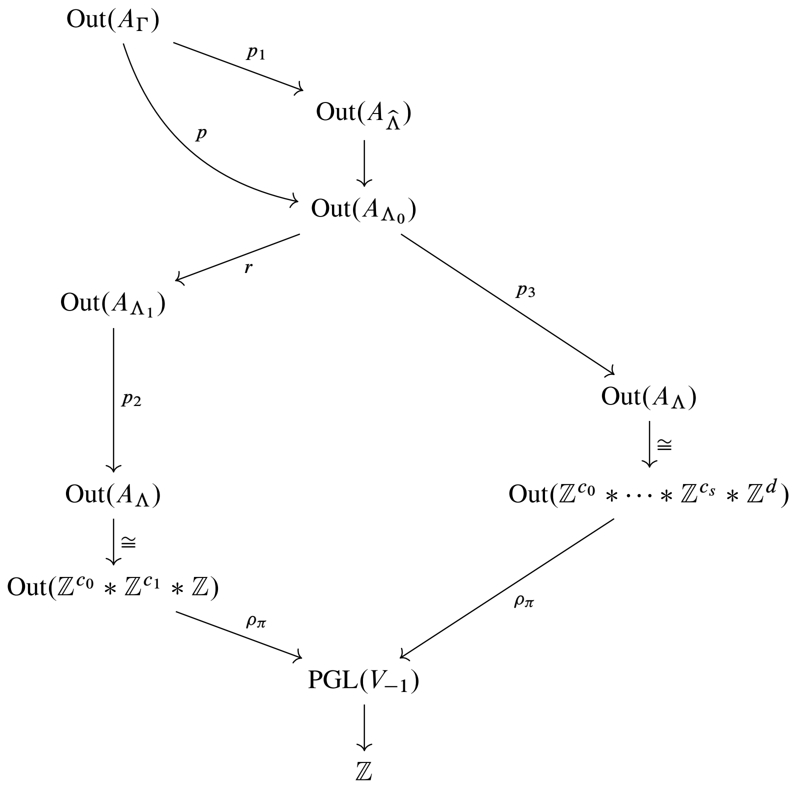

Figure 4. The maps used in the projection to $\mathbb{Z}$. Note that maps are defined on finite index subgroups of the groups shown, but we suppress this notation to reduce clutter.

We claim that (X, C_0) and (X, C_1) are both principal. There is only one z_2-component. There are two z_3-components, C_4 being one, and the other containing $C_0 \cup C_1 \cup C_2 \cup \{x\}$ and some unlabelled vertices. As there is no way to separate C_0 and C_1 by the star of either z_2 or z_3, both (X, C_0) and (X, C_1) must be principal.

The graph $\widehat{\Lambda}$ is the graph consisting of vertices $x, z_0, z_1, \dots, x_4$. It has no edges, and so is equal to Λ_0.

In Γ we could replace some z_i by an equivalence class Z_i of vertices, either free or abelian (it must be abelian for any vertex of U). Then $\widehat{\Lambda}$ would include these equivalence classes, and Λ_0 would abelianise any which were free.

This example yields a map into $\mathrm{Out}(F_3)$ (having $c_0 = c_1 = 1$, though these could differ if z_0 or z_1 were replaced by a larger equivalence class). The classes z_0 and z_1 are not separated by any star from U, while z_4 is separated from z_0 by $\mathrm{st}(z_3)$. We can take a restriction map r to $\mathrm{Out}(A_{\Lambda_1})$, with Λ_1 given by vertices $\{z_0, z_1, x\}$. The map p_2 is just the identity map in this example.

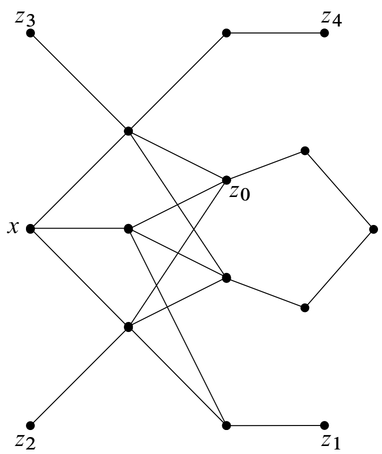

Figure 5. The graph Γ of Example D satisfying case (1).

The image of all these maps will include $\pi^x_{\{z_0\}}$, which will have infinite order image under the homological representation ρ_π. Since V_{-1} has dimension 2 in this case, the image will be an infinite subgroup of $\mathrm{PGL}(V_{-1})$, which is virtually free, implying the image is also virtually free (including of course, possibly virtually $\mathbb{Z}$).

Example E. The graph Γ given in Figure 6 satisfies the conditions of case (2) in Section 5.4.

There are four x-components, which we can denote C_i so that $z_i \in C_i$ for $i = 0, 1, 2, 3$. We set $Z_i = \{z_i\}$ for each i, and we can enlarge each Z_i to be an equivalence class, free or abelian, of any size.

We claim that (X, C_0) is principal, and to show this we use Lemma 4.4. Consider the vector space W over $\mathbb{Q}$ with basis given by $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$. The only vertices dominated by x are y_1 and y_2, so $U = \{y_1, y_2\}$. For each partial conjugation $\pi^{y_i}_D$, we can, up to an inner automorphism, assume that $C_0 \cap D = \emptyset$. Thus we have partial conjugations with supports $C_1 \cup C_2$ and $C_2 \cup C_3$. These translate into vectors $\langle 1, 1, 0\rangle$ and $\langle 0, 1, 1\rangle$ as described in Lemma 4.4. It is clear that $\langle 1, 1, 1\rangle$ is not in the $\mathbb{Q}$-span of these two vectors, so (X, C_0) is principal.

In our first step, we take a projection map p_1 so that the image lies in $\mathrm{Out}(A_{\widehat{\Lambda}})$, where $\widehat{\Lambda}$ is the induced graph with vertex set $\{x, y_1, y_2\} \cup Z_0 \cup Z_1 \cup Z_2 \cup Z_3$. If we enlarged some Z_i to be free, then we should abelianise these now to get Λ_0, otherwise $\widehat{\Lambda} = \Lambda_0$. In this example, we do not need to use the next projection map p_3, so we take this to be the identity map and $\Lambda = \Lambda_0$. (The reason we may take p_3 as the identity is because this graph gives no partial conjugations in the set B, defined in Section 5.5.2.) We therefore have

$$A_\Lambda \cong \mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \mathbb{Z}^{c_2} * \mathbb{Z}^{c_3} * \mathbb{Z}^3,$$

where we use c_i for the size of Z_i, which is 1 unless it was enlarged.

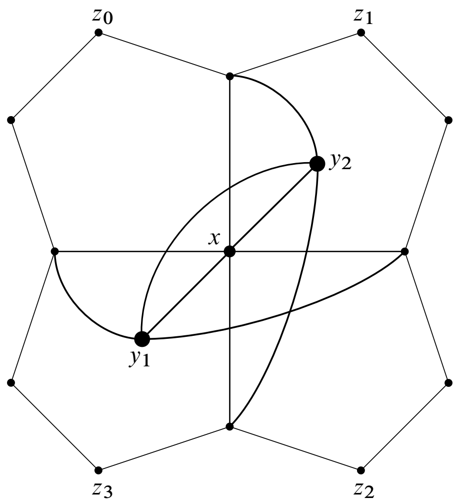

Figure 6. The graph Γ of Example E satisfying case (2).

In this example, V_{-1} will have dimension 6 (with basis $\{\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$). We will restrict our representation to a subspace of V_{-1} of dimension 2. Since our only transvections are $R^x_{y_i}$, we can try to restrict to some vectors $\mathbf{z}_i$ and $\mathbf{x}$. However, we have partial conjugations $\pi^{y_1}_{C_1\cup C_2}$ and $\pi^{y_2}_{C_2\cup C_3}$ which prevent us from naively doing this. But with a little thought, we can make a restriction.

The partial conjugations act as follows:

$$\varphi(\pi^{y_1}_{C_1\cup C_2})\colon \begin{cases} \mathbf{z}_1 \mapsto \mathbf{z}_1 + 2\mathbf{y}_1, \\ \mathbf{z}_2 \mapsto \mathbf{z}_2 + 2\mathbf{y}_1, \\ \mathbf{z}_3 \mapsto \mathbf{z}_3, \end{cases} \qquad \varphi(\pi^{y_2}_{C_2\cup C_3})\colon \begin{cases} \mathbf{z}_1 \mapsto \mathbf{z}_1, \\ \mathbf{z}_2 \mapsto \mathbf{z}_2 + 2\mathbf{y}_2, \\ \mathbf{z}_3 \mapsto \mathbf{z}_3 + 2\mathbf{y}_2. \end{cases}$$

Notice though that $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2 + \mathbf{z}_3$ is fixed by both partial conjugations. In this example, we can restrict the action to the subspace V spanned by $\mathbf{z}$ and $\mathbf{x}$. The partial conjugation $\pi^x_{C_0}$ acts as a transvection $\mathbf{z} \mapsto \mathbf{z} - 2\mathbf{x}$, so the image of the representation in $\mathrm{PGL}(V) \cong \mathrm{PGL}(2, \mathbb{Q})$ has infinite order in $\mathrm{PGL}(2, \mathbb{Z})$. This is sufficient to imply virtual indicability of $\mathrm{Out}(A_\Gamma)$.

Example F. The graph Γ given in Figure 7 satisfies the conditions of case (2) in Section 5.4.

There are six x-components, $C_0, \dots, C_5$, with a domination minimal vertex z_i chosen in each C_i. As in the previous example, we set $Z_i = \{z_i\}$, and each Z_i could be replaced by a larger equivalence class that is either free or abelian. Again, $U = \{y_1, y_2\}$. We note that $\mathrm{st}(y_1)$ separates z_1, z_2, and z_3 from z_0, while $\mathrm{st}(y_2)$ separates z_3, z_4, and z_5 from z_0. (This confirms we are in case (2).)

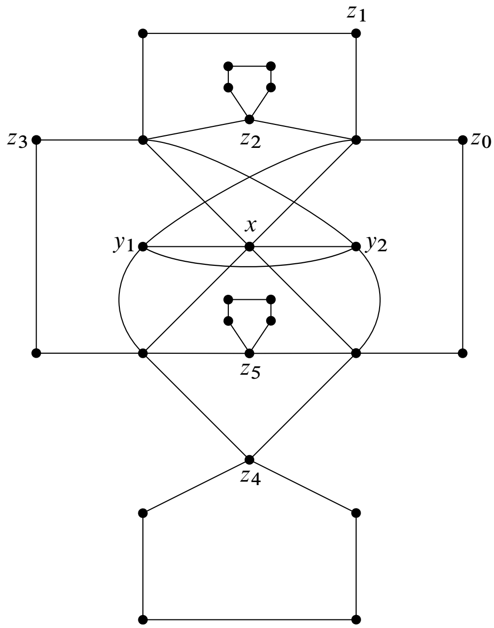

Figure 7. The graph Γ of Example F satisfying case (2).

To see that (X, C_0) is principal, using the notation of Lemma 4.4, Δ consists of two vectors, $\langle 1,1,1,0,0\rangle$ from y_1, and $\langle 0,0,1,1,1\rangle$ from y_2. Evidently, there is no way to get $\langle 1,1,1,1,1\rangle$ in the $\mathbb{Q}$-span of Δ, implying (X, C_0) is principal.

Our first step is to declutter Γ by taking a projection map p_1 with $\widehat{\Lambda}$ being the induced subgraph on vertex set $\{x, y_1, y_2\} \cup Z_0 \cup \cdots \cup Z_5$. We then abelianise any of the Z_i classes that are not already abelian to get Λ_0, composing this with p_1 gives us the map p.

In this example, we have some partial conjugations that can cause us problems, and dealing with these is the purpose of the projection map p_3. In Section 5.5.2, we define a set B that consists of these "bad" partial conjugations. Here, B consists of

- $\pi^{z_2}_{C_1}$,
- $\pi^{z_2}_{D}$, where D contains C_0, C_3, C_4, C_5 and some vertices from $\mathrm{st}(x)$,
- $\pi^{z_4}_{C_5}$,
- $\pi^{z_4}_{E}$, where E contains C_0, C_1, C_2, C_3 and some vertices from $\mathrm{st}(x)$,
- $\pi^{z_5}_{C_4}$, and
- $\pi^{z_5}_{E}$.

Note that there are also partial conjugations by z_2 that appear to meet the definition of B given in Section 5.5.2, specifically those that act on the vertices of $C_2 \setminus \mathrm{st}(z_2)$, and similarly for z_4 and z_5. However, none of these are in the image of p.

The following explains how we determine which vertices to delete to define the projection map p_3. (We follow the method and notation of the proof of Lemma 5.13.) We first define ψ_1 by killing z_2. This will leave $B_1 = B \setminus \ker \psi_1$ to contain the partial conjugations with multiplier z_4 or z_5. Then we define ψ_2 by killing both z_2 and z_4. Immediately, we see that the partial conjugations with multiplier z_4 are in the kernel, but it also contains $\pi^{z_5}_{C_4}$. Finally, $\pi^{z_5}_{E}$ has become inner. Thus $B_2 = B \setminus \ker \psi_2$ is empty – we have taken a projection map that kills our "bad" partial conjugations, but it still leaves enough information for our other partial conjugations to act non-trivially, in particular, $\pi^{x}_{C_0}$. We set $p_3 = \psi_2$.

We are left therefore with Λ that consists of the vertex set $\{x, y_1, y_2\} \cup Z_0 \cup Z_1 \cup Z_3 \cup Z_5$, and

$$A_\Lambda \cong \mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \mathbb{Z}^{c_3} * \mathbb{Z}^{c_5} * \mathbb{Z}^3,$$

where we use c_i for the size of an equivalence class optionally put in place of z_i.

We are now ready to apply the homological representation. We deleted vertices z_2 and z_4, so we define W to be the 3-dimensional subspace of V_{-1} with basis $\mathbf{z}_1$, $\mathbf{z}_3$, $\mathbf{z}_5$. We also define a set Π that consists of vectors

$$\mathbf{z}_1 + \mathbf{z}_3 = \langle 1, 1, 0\rangle \quad \text{and} \quad \mathbf{z}_3 + \mathbf{z}_5 = \langle 0, 1, 1\rangle.$$

These are the vectors that correspond to the y_1 and y_2-components that do not contain C_0. The vector

$$\mathbf{v} = \mathbf{z}_1 - \mathbf{z}_3 + \mathbf{z}_5 = \langle 1, -1, 1\rangle$$

is in $\Pi^\perp$, and indeed $\Pi^\perp = \mathrm{Span}_{\mathbb{Q}}(\mathbf{v})$.

Suppose that $V = \mathrm{Span}_{\mathbb{Q}}(\mathbf{v}, \mathbf{x})$. By direct calculations, or using Lemma 3.2, we see that $\pi^{y_1}_{C_1 \cup C_2 \cup C_3}$ acts trivially on $\mathbf{v}$ (and $\mathbf{x}$), so preserves V. The same is true for $\pi^{y_2}_{C_3 \cup C_4 \cup C_5}$. The partial conjugations by z_2 and z_4 were killed by the map p_3 since those vertices were killed, while the partial conjugation by z_5 was also killed because its support was C_4. The remaining partial conjugations are those with multiplier x, which preserve V,

$$\varphi(\pi^x_{C_1})(\mathbf{v}) = \mathbf{v} + 2\mathbf{x}, \quad \varphi(\pi^x_{C_3})(\mathbf{v}) = \mathbf{v} - 2\mathbf{x},$$
$$\varphi(\pi^x_{C_5})(\mathbf{v}) = \mathbf{v} + 2\mathbf{x}, \quad \varphi(\pi^x_{C_0})(\mathbf{v}) = \mathbf{v} + 2\mathbf{x}.$$

The other partial conjugations act trivially. The only transvections are by x on y_i, both of which act trivially on V. We conclude that the image of the homological representation restricted to V is $\mathbb{Z}$.

5.3. Projecting to a simpler RAAG

In this section, we do some spring cleaning on Γ, removing as much as we can while leaving enough so that the partial conjugations with multiplier x still act non-trivially. To do this, we pick out a minimal equivalence class in each x-component to keep, and kill the rest of the component through a projection map, as described in Section 2.3.

Recall that x and $C = C_0$ have been chosen so that $X = [x] = \{x\}$ and (X, C) is principal, with x domination-minimal among such vertices.

Definition 5.2. The following describes how to construct the graph $\widehat{\Lambda}$:

- Let $C = C_0, C_1, \dots, C_r$ be the x-components. If C_i consists of one vertex, set $Z_i = C_i$. Otherwise, let Z_i be a domination-minimal equivalence class in C_i.
- Let Y be the set of vertices in $\mathrm{st}(x)$ that are dominated by x.

Define $\widehat{\Lambda}$ to be the induced subgraph of Λ with vertex set $Z_0 \cup \cdots \cup Z_r \cup Y \cup \{x\}$.

To see that the sets Z_i are well defined, we observe that if C_i contains more than one vertex, then the equivalence class of each vertex in C_i is contained in C_i. Indeed, if $z \in C_i$, then $|C_i| > 1$ implies $\mathrm{lk}(z) \cap C_i$ is nonempty. Suppose z' is equivalent to z. Firstly, this implies z' cannot be in $\mathrm{st}(x)$. Next, if z' is adjacent to z, then z' is immediately in C_i. Otherwise, $\mathrm{lk}(z) = \mathrm{lk}(z')$, so the fact that $\mathrm{lk}(z) \cap C_i$ is nonempty gives a two-edge path in C_i connecting z to z'.

This observation also implies that a set Z_i can fail to be a complete equivalence class only if $Z_i = C_i$ consists of one vertex that is dominated by x.

Lemma 5.3. *Suppose $v \in \widehat{\Lambda}$ and $u \leq v$. Then*

(I) *if $u \in C_i$, $v \in Z_j$ and $i \neq j$, then $u \leq x$ and $\{u\} = Z_i$ is a v-component;*

(II) *$u \in \widehat{\Lambda}$.*

In particular, part (II) implies that the projection map

$$p_1\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}^0(A_{\widehat{\Lambda}})$$

is well defined.

Proof. First, we prove part (I). Any path from u to v must hit $\mathrm{st}(x)$. Hence $\mathrm{lk}(u) = \mathrm{lk}(u) \cap \mathrm{lk}(v) \subseteq \mathrm{st}(x)$, so $u \leq x$ and $C_i = Z_i = \{u\}$. Also, $\mathrm{lk}(u) \subseteq \mathrm{lk}(v)$ implies that Z_i is a v-component.

For part (II), first suppose $v \in Y \cup \{x\}$. Then $u \leq x$. Either $u \in Y$ or $\mathrm{lk}(u) \subset \mathrm{lk}(x)$, which implies u forms its own x-component, so there is some i so that $\{u\} = C_i = Z_i$. In particular, $u \in \widehat{\Lambda}$.

Now suppose $v \in Z_j \subset C_j$ for some j. Note that this implies $u \notin \mathrm{st}(x)$ since otherwise $x \in \mathrm{lk}(u) \subseteq \mathrm{st}(v)$. If u is also in C_j, then by minimality of the equivalence class Z_j in C_j, we must have that $u \in Z_j$ too, and hence in $\widehat{\Lambda}$. If on the other hand $u \in C_i$ with $i \neq j$, then we are in the situation for part (I), from which it follows that u is in $\widehat{\Lambda}$. ∎

The structure of $A_{\widehat{\Lambda}}$ as a group is as follows. The subgroup generated by Y could be any RAAG, denote it by A_Y. The Z_i that are equivalence classes could be free or free abelian. They cannot be adjacent to any $y \in Y$ or x, since the link of y is contained in the star of x. Thus the group generated by $\widehat{\Lambda}$ has the form

$$A_{\widehat{\Lambda}} = G_0 * G_1 * \cdots * G_r * (A_Y \times \mathbb{Z}),$$

where each G_i is either $\mathbb{Z}^{c_i}$ or F_{c_i} according to whether Z_i is abelian or not, and where $c_i = |Z_i|$. The final $\mathbb{Z}$ factor has generator x.

Following this projection p_1, we compose it with another homomorphism obtained by abelianising each free factor in the above decomposition of $A_{\widehat{\Lambda}}$.

Definition 5.4. Let Λ_0 be the graph obtained from $\widehat{\Lambda}$ by adding edges, so that A_Y and each equivalence class Z_i become abelian. The structure of A_{Λ_0} is therefore

$$A_{\Lambda_0} = \mathbb{Z}^{c_0} * \mathbb{Z}^{c_1} * \cdots * \mathbb{Z}^{c_r} * \mathbb{Z}^{k+1},$$

where $k = |Y|$.

Lemma 5.5. *The projection p_1 composes with a homomorphism* $\operatorname{im} p_1 \to \operatorname{Out}(A_{\Lambda_0})$ *to give a homomorphism*

$$p\colon \operatorname{SOut}^0(A_\Gamma) \to \operatorname{Out}(A_{\Lambda_0}).$$

Proof. The homomorphism sends an outer automorphism Φ in $\operatorname{im} p_1$ to the automorphism defined by the action of Φ on the vertices of Λ_0. To see that Φ is indeed sent to an automorphism, one just needs to verify that the kernel of the map $A_{\widehat{\Lambda}} \to A_{\Lambda_0}$, which is normally generated by the commutators $[a, b]$ for $a, b \in Z_i$ and $i = 0, 1, \dots, r$, is preserved by each generator in the image of p_1. Indeed, any transvection R_u^v preserves the kernel as either u and v are in the same factor G_i, or otherwise the factor containing u must be cyclic (since $\operatorname{lk}(u)$ must therefore be contained in $\operatorname{st}(x)$, regardless of what v is). Meanwhile, a partial conjugation π_D^v with $v \in \widehat{\Lambda}$ preserves the kernel since each factor in $A_{\widehat{\Lambda}}$ either is contained in D, does not intersect D, or both D and v are contained in Z_i. ■

We finish this subsection with a couple of useful observations about the behaviour of the sets Z_i. We define U to be the subset of vertices in $\widehat{\Lambda}$ dominated by (and not equal to) x,

$$U = \{u \mid u \leq x,\ u \neq x\}.$$

Remember, the domination relation that we refer to (and hence the equivalence classes of vertices) always comes from the relation in Γ.

Lemma 5.6. *Let $u \in U$, and let D be a u-component. Then either*

- *$D = C_i$ for some i, or*
- *$x \in D$.*

Proof. Let $x \notin D$, so $\operatorname{st}(u)$ separates D and x. We claim this means that $D \subset \Gamma \setminus \operatorname{st}(x)$. Indeed, otherwise x would be adjacent to a vertex of D, and since $x \notin \operatorname{st}(u)$, we would have $x \in D$. Thus, since D is connected, it must be contained in an x-component C_i. Finally, since $u \leq x$ no vertex of C_i is in $\operatorname{st}(u)$, and it therefore follows that $D = C_i$. ■

The next lemma gives us useful properties when we work under the assumption that condition (A1) of [1] holds. Recall that when (A1) fails, Theorem 2 from [1] tells us that we get virtual indicability. This lemma also highlights the role of the distinguished x-component $C = C_0$.

Lemma 5.7. *There exists some $i \neq 0$ so that both C_0 and C_i are not u-components for any $u \in U$. In particular, if condition (A1) holds, then x does not dominate Z_0 or Z_i.*

Proof. The fact that C_0 is not a u-component for any $u \in U$ follows immediately from principality of (X, C_0). Meanwhile, if for each C_i there is a $u_i \in U$ so that C_i are a u_i-component, then $\pi^x_{C_i} \in P^{[u_i]}_{C_i}$. In particular, $(\pi^x_{C_0})^{-1} = \pi^x_{C_1} \cdots \pi^x_{C_r}$ is in the subgroup generated by the sets $P^{[u_i]}_{C_i}$ for $i = 1, \dots, r$, contradicting principality of (X, C_0).

Suppose that C_i is not a u-component for any u (we allow $i = 0$ here). If Z_i is dominated by x, then $C_i = Z_i = \{z_i\}$. If condition (A1) holds, then there is some other vertex u such that $z_i \leq u \leq x$. In particular, $u \in U$ and since C_i cannot be a u-component, we must have u adjacent to z_i. But $u \leq x$ then implies $z_i \in \mathrm{st}(x)$, a contradiction. ∎

5.4. Two cases

We divide into two cases according to the possible arrangements of components Z_i. Recall that $U = \{u \mid u \leq x,\ u \neq x\}$. The two cases are

(1) There is a class Z_i, with $i \neq 0$ and not dominated by x, so that Z_0 and Z_i are in the same u-component for every $u \in U$.

(2) For every class Z_i, for $i \neq 0$ and which is not dominated by x, there is some $u \in U$ so that Z_0 and Z_i are in separate u-components.

We make the following observation.

Lemma 5.8. *If there is some i (possibly zero) such that Z_i dominates x, then*

(1) *Z_i is not dominated by x,*

(2) *there is $j \neq i$ such that $0 \in \{i, j\}$ and so that Z_i and Z_j are in the same u-component for every $u \in U$.*

In particular, we are in case (1).

Proof. Assume Z_i dominates x. Since x is not equivalent to any other vertex of Γ, we cannot have x dominating Z_i. By Lemma 5.7, we know there is some $j \neq i$ so that C_j is not a u-component for any $u \in U$, and furthermore that j can be chosen so that $0 \in \{i, j\}$. Since for any $u \in U$, we know that C_j is not a u-component, there must be some vertex $y \in \mathrm{lk}(x)$ that is in the same u-component as C_j, and is, in fact, adjacent to a vertex of C_j. Since Z_i dominates x, we also have that y is adjacent to Z_i. We can therefore construct a path from Z_j to Z_i that is entirely in $Z_i \cup C_j \cup \{y\}$, and hence disjoint from $\mathrm{st}(u)$. ∎

5.5. Refining the graph further

We will describe two different ways to cut down Λ_0 to a smaller graph Λ, depending on which case we are in. Ultimately, in each case we claim there are integers $c_0, c_1, \dots, c_s, d$ and a homomorphism

$$q\colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(\mathbb{Z}^{c_0} * \cdots * \mathbb{Z}^{c_s} * \mathbb{Z}^d)$$

such that the image is sufficiently rich so that we can use a homological representation to get a virtual map to $\mathbb{Z}$.

Note that we will work under the assumption that conditions (A1) and (A2) hold in Γ, so Theorem 2 does not already yield virtual indicability. In particular, Lemma 5.7 tells us that there is some $i > 0$ so that Z_0 and Z_i are not dominated by x.

5.5.1. Case (1). Relabel the classes Z_i for $i > 0$, if necessary, so that $Z_1, \dots, Z_t$ are all the classes that are not dominated by x and that are not separated from Z_0 by $\mathrm{st}(u)$ for any $u \in U$. We first refine the graph Λ_0 by removing all classes Z_i for $i > t$, and all vertices $y \in Y$, leaving a graph Λ_1 that has vertex set $Z_0 \cup \cdots \cup Z_t \cup \{x\}$.

Lemma 5.9. *The restriction map* $r\colon \mathrm{im}\, p \to \mathrm{Out}(A_{\Lambda_1})$ *is well defined.*

Proof. First consider transvections in the image of p. Suppose $z \in Z_i$ for some i with $0 \le i \le t$. By assumption, $Z_1, \dots, Z_t$ are not dominated by x, while Z_0 is not dominated by x by virtue of Lemma 5.7. Thus we cannot have $z \le z'$ with $z' \in Z_{i'}$ and $i' \neq i$, since then Lemma 5.3 (II) implies Z_i would be dominated by x. Similarly, we cannot have $z \le y \in Y$. Thus any transvection R_z^w must have $w \in \Lambda_1$, and thus preserve A_{Λ_1}.

The remaining transvections to consider are those of the form R_x^w. But then by Lemma 5.8, w must be a vertex of some $Z_i \subset \Lambda_1$, and so A_{Λ_1} is again preserved. All other transvections in $\mathrm{im}\, p$ act on a vertex not in Λ_1 and so fix A_{Λ_1}.

Next consider partial conjugations. By construction, those of the form π_D^y with $y \in Y$ act trivially on Λ_1. Indeed, $x \in \mathrm{st}(y)$ and the remaining vertices of Λ_1, namely $Z_0 \cup Z_1 \cup \cdots \cup Z_s$, are all in the same y-component.

That leaves us to consider the partial conjugations π_D^z for $z \in Z_i$ for some $i > t$. We want to show that $\mathrm{st}(z)$ cannot separate Λ_1 into more than one component.

For contradiction, first suppose $\mathrm{st}(z)$ separates Z_0 and Z_j for some $j \le t$. Fix $u \in U$. Let p be any path from Z_0 to Z_j that avoids $\mathrm{st}(u)$ (which is possible by choice of Z_j). Since $\mathrm{st}(z)$ separates Z_0 and Z_j, there must be a vertex v on p that is adjacent to z. See Figure 8 (a). We can therefore define a path p' that follows p from Z_0 to v, then jumps to z. Since $\mathrm{st}(z)$ separates Z_0 and Z_j, we have $z \notin U$. Furthermore, $\mathrm{lk}(u) \subseteq \mathrm{st}(x)$ implies $z \notin \mathrm{st}(u)$. Hence the path p' avoids $\mathrm{st}(u)$. We have shown that Z_0 and Z_i are not separated by $\mathrm{st}(u)$ for any $u \in U$, and also that Z_i is not dominated by x. This contradicts the choice of i.

On the other hand, suppose $\mathrm{st}(z)$ separates some Z_j in Λ_1 and x. By the previous paragraph, we can assume $j = 0$, since otherwise we saw Z_0 and Z_j are not separated by $\mathrm{st}(z)$. Let p be a path from Z_0 to Z_1 that avoids $\mathrm{st}(z)$. Then p passes through $\mathrm{st}(x)$. See Figure 8 (b). Let v be the first vertex of p in $\mathrm{st}(x)$, and define p' to be the path that follows p from Z_0 to v and then hops across to x. Then p' is a path from Z_0 to x that avoids $\mathrm{st}(z)$.

Thus every transvection and partial conjugation in $\mathrm{im}\, p$ preserves A_{Λ_1} and, since they generate the image, the restriction map is well defined. ∎

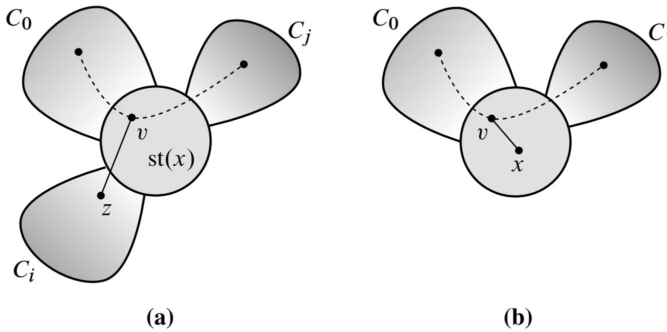

Figure 8. Showing that st(z) does not separate Λ_1. The dashed paths are the paths p, the solid line from v to either z or x represents an edge of Γ.

We observe that when Z_i is not dominated by x, as is the case for each $Z_1, \dots, Z_t$, we can use a projection map to eliminate it.

Lemma 5.10. *Suppose that $i_1, \dots, i_n \in \{1, \dots, t\}$. If Λ is the graph obtained by removing $Z_{i_1}, \dots, Z_{i_n}$ from Λ_1, then the projection map*

$$p_2 \colon \operatorname{im} r \to \operatorname{Out}(A_\Lambda)$$

is well defined.

Proof. Recall that for any projection map, partial conjugations always map to (possibly trivial) partial conjugations. So, to see that p_2 is well defined, we need only to check the image of transvections (in $\operatorname{im} r$) map to automorphisms under p_2. In particular, the transvection R_v^w will fail to map to an automorphism only if v is not in Λ, whereas w is. We may therefore assume that v is in some Z_{i_j} that is deleted to make Λ. Since $v \not\leq x$, Lemma 5.3 (II) tells us that v is not dominated by any $z \in Z_i$ for $i \neq i_j$. So for w to dominate v, it must therefore be in Z_{i_j}, hence not in Λ. ∎

We can therefore choose our favourite $i \in \{1, \dots, t\}$ and use p_2 from Lemma 5.10 to get

$$q = p_2 \circ r \circ p \colon \operatorname{SOut}^0(A_\Gamma) \to \operatorname{Out}(\mathbb{Z}^{c_0} * \mathbb{Z}^{c_i} * \mathbb{Z}).$$

We have Λ being the graph on vertex set $Z_0 \cup Z_i \cup \{x\}$.

Remark 5.11. If we can choose Z_i here so that st(Z_i) separates Z_0 and x in Γ, or so that st(Z_0) separates Z_i and x, then, in fact, $\operatorname{Out}(A_\Gamma)$ is large. This is the case, for example, if Z_0 or Z_i dominates x. To obtain largeness, observe that the vertices of Λ form a special SIL, as defined in [15], which implies largeness by [15, Proposition 3.15]. The special SIL will be either $(x, z_0 \mid z_i)$ or $(x, z_i \mid z_0)$, where $z_i \in Z_i$ and $z_0 \in Z_0$. There is a technical note here though. Special SILs require the equivalence class of each vertex involved to be abelian. This will be so once we reach Λ_0, but is not necessarily the case in Γ itself. Indeed, the only way to have an equivalence class that remains free in Λ_0 is if it is the union of singleton sets Z_i. Then Lemma 5.3 (I) implies that the class is dominated by x and hence not in Λ_0. Finally, to obtain largeness from the special SIL, one needs to verify that the required SIL automorphisms survive the map p.

5.5.2. Case (2). In this case, we use a projection map to kill the partial conjugations π^z_D, which would otherwise later cause trouble in the image of the homological representation. We gather these partial conjugations up in a set B, defined to consist of partial conjugations $\pi^z_D \in \operatorname{im} p$ such that

- $z \in Z_i$ for some $i \neq 0$,
- $z \not\leq x$,
- D is not a union of u-components for any $u \in U$.

The aim is to kill enough of the Z_i so that the partial conjugations in B act trivially on what remains, but not to kill too many so that the action of $\pi^x_{C_0}$ becomes trivial, and furthermore, so that we can still track the action of partial conjugations by $u \in U$ on each component C_i, whether it has been deleted or not.

The following lemma is a crucial tool in accomplishing this.

Lemma 5.12. *Suppose $\pi^z_D \in B$. Then either*

- *D and z are in the same u-component for every $u \in U$, or*
- *$x \in D$.*

Proof. Suppose neither condition holds. Let $u \in U$ be such that D and z are separated by $\operatorname{st}(u)$. Since $\operatorname{lk}(u) \subseteq \operatorname{st}(x)$, and $u \notin \operatorname{st}(x)$ only if $\{u\} = Z_i$ for some i, we must have that D does not intersect the x-component containing z. Since we assume $x \notin D$, we have $D \cap \operatorname{st}(x) = \emptyset$. Hence D is a union of x-components.

Let L be the set of vertices in $\operatorname{st}(x)$ that are adjacent to a vertex of D. Any vertex in L must be in $\operatorname{st}(z)$ as $\operatorname{st}(z)$ separates D and x by assumption. Meanwhile, L must also be contained in $\operatorname{st}(u)$ as otherwise it gives a two-edge path from D to z avoiding $\operatorname{st}(u)$. In particular, this implies that D must be a union of u-components, contradicting $\pi^z_D \in B$. ∎

The next lemma says we can delete some Z_i to make all partial conjugations in B trivial, while keeping some other $Z_{i'}$ so that the action of the surviving partial conjugations π^u_E on Z_i is tracked via its action on $Z_{i'}$.

Lemma 5.13. *We can reorder the sets Z_i so that we can apply a projection map*

$$p_3\colon \operatorname{im} p \to \operatorname{Out}(A_\Lambda),$$

where Λ is the subgraph of Λ_0 induced by the vertices

$$Z_0 \cup \cdots \cup Z_s \cup Y \cup \{x\}$$

for some $s \leq r$, so that the following hold:

(a) *For each i such that $s < i \leq r$, there is some j with $1 \leq j \leq s$ such that Z_i and Z_j lie in the same u-component for each $u \in U$.*

(b) *All partial conjugations $\pi_D^z \in B$ are sent to the identity.*

(c) *The image of $\pi_{C_0}^x$ under p_3 is non-trivial.*

Proof. The process to choose the Z_i that are deleted is iterative, and we define p_3 as a composition of projection maps, each one obtained by killing some Z_i.

Begin by setting $B_0 = B$, and $\psi_0 \colon \mathrm{im}\, p \to \mathrm{Out}(A_{\Lambda_0})$ the identity map. We suppose Λ_n has been obtained from Λ_0 by deleting some of the sets Z_i, and the projection map $\psi_n \colon \mathrm{im}\, p \to \mathrm{Out}(A_{\Lambda_n})$ is well defined. Let B_n be the set of partial conjugations $\pi_D^z \in B$ that are not in the kernel of ψ_n. We will define ψ_{n+1} so that the corresponding set B_{n+1} is strictly smaller than B_n.

Choose any Z_i so that $z \in Z_i$ admits a partial conjugation π_D^z in B_n. Since $\psi_n(\pi_D^z)$ is non-trivial, Z_i is in Λ_n and, up to an inner automorphism, we may assume $x \notin D$. Furthermore, since π_D^z acts non-trivially on Λ_n, D must contain some set $Z_{i'}$ that is in Λ_n. By Lemma 5.12, Z_i and $Z_{i'}$ are in the same u-component for each $u \in U$. Since we are in case (2), we must have $i' \neq 0$.

Since Z_i is not dominated by x, it cannot be dominated by any Z_j by Lemma 5.3 (II), or by any $u \in U$. Hence we may delete Z_i from Λ_n to get a new graph Λ_{n+1} and define a projection map $\mathrm{im}\, \psi_n \to \mathrm{Out}(A_{\Lambda_{n+1}})$ (compare with Lemma 5.10). Compose this projection map with ψ_n to get $\psi_{n+1} \colon \mathrm{im}\, p \to \mathrm{Out}(A_{\Lambda_{n+1}})$. In particular, the partial conjugation π_D^z is in the kernel of ψ_{n+1} so is not included in B_{n+1}.

Stop this process when we reach n with B_n empty. Then we set $p_3 = \psi_n$. Clearly, item (b) holds by construction.

For item (a), the construction yields for each $i > s$ some i', so that Z_i and $Z_{i'}$ are in the same u-component for each $u \in U$. It may be that $i' > s$ (and so deleted to form Λ) but then we have some i'' so that Z_i and $Z_{i''}$ are in the same u-component for each $u \in U$. We can repeat this until, after finitely many steps, we find the required set Z_j that is in Λ.

Finally, for point (c), the image of $\pi_{C_0}^x$ would be trivial only if all components Z_i have been killed except Z_0. This cannot happen because after each Z_i is killed, there must be some $Z_{i'}$ left surviving, with $i' \neq 0$. This implies $s > 0$. ■

To conclude, in case (2) we define the homomorphism q to be the composition

$$q = p_3 \circ p \colon \mathrm{SOut}^0(A_\Gamma) \to \mathrm{Out}(A_\Lambda).$$

5.6. Applying the homological representation

We compose the map q from Section 5.5 with the representation ρ_π of Section 3 to get

$$\varphi = \rho_\pi \circ q \colon \mathrm{Out}_\pi(A_\Gamma) \to \mathrm{PGL}(V_{-1}).$$

More specifically, we use the restriction of q to the finite index subgroup $\mathrm{Out}_\pi(A_\Gamma)$ that is the pre-image of $\mathrm{Out}_\pi(A_\Lambda)$.

The action of $\mathrm{Aut}_\pi(A_\Gamma)$ on $H_1(T;\mathbb{Q})$ preserves the lattice $H_1(T;\mathbb{Z})$, so a finite index subgroup $\mathrm{Aut}_{\pi,\mathbb{Z}}(A_\Gamma)$ of $\mathrm{Aut}_\pi(A_\Gamma)$ preserves the lattice $\mathbb{Z}\mathbf{x} \oplus \mathbb{Z}\mathbf{y}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{y}_k \oplus \mathbb{Z}\mathbf{z}_1 \oplus \cdots \oplus \mathbb{Z}\mathbf{z}_s$ in V_{-1}. Restricting φ to the image of this subgroup in $\mathrm{Out}(A_\Gamma)$, we therefore get

$$\varphi\colon \mathrm{Out}_{\pi,\mathbb{Z}}(A_\Gamma) \to \mathrm{PGL}(d+s,\mathbb{Z}),$$

where $\mathrm{Out}_{\pi,\mathbb{Z}}(A_\Gamma)$ has a finite index in $\mathrm{Out}(A_\Gamma)$. We now verify that in each case the image of φ is virtually indicable, implying the same of $\mathrm{Aut}(A_\Gamma)$.

5.6.1. Case (1). We have done nearly all the work needed to obtain a map onto $\mathbb{Z}$ from a finite index subgroup of $\mathrm{Out}(A_\Gamma)$ in this case. Indeed, V_{-1} has dimension 2, and the image of the partial conjugation $\pi^x_{C_0}$ is an infinite order element. It follows that the image of $\mathrm{Out}_{\pi,\mathbb{Z}}(A_\Gamma)$ under φ in this case is an infinite subgroup of a virtually free group. Hence it is virtually free itself (possibly virtually $\mathbb{Z}$) and, in particular, we can obtain a map onto $\mathbb{Z}$ from a finite index subgroup of $\mathrm{Out}(A_\Gamma)$. This proves Theorem 3 in case (1).

5.6.2. Case (2). Since the dimension of V_{-1} is not necessarily 2 in this case, we need to use a more involved strategy to get a map onto $\mathbb{Z}$. We claim that we can restrict everything in the image of φ to a subspace of V_{-1} so that the image is free abelian.

Consider the subspace W spanned by the vectors $\mathbf{z}_i$ for $i = 1,\dots,s$. As in Lemma 4.4, we associate to each partial conjugation π^u_D, with $u \in U$, a vector $\mathbf{w}_D$ in W. First, after multiplying by an inner automorphism, we may assume that $C_0 \not\subseteq D$. We define the vector $\mathbf{w}_D = \langle w_1,\dots,w_s\rangle$ in W by

$$w_i = \begin{cases} 1 & \text{if } C_i \subseteq D, \\ 0 & \text{otherwise.} \end{cases}$$

This gives us a set of vectors Π defined as

$$\Pi = \{\mathbf{w}_D \mid D \text{ is a } u\text{-component for some } u \in U\}.$$

As in Lemma 4.4, we can determine whether (X, C_0) is principal by inspecting the span of the set Π. This requires more than a simple application of Lemma 4.4 since we can only use W to (immediately) see the effect on components $C_1,\dots,C_s$. We need to ensure that the effect on the "lost" components $C_{s+1},\dots,C_r$ can still be discerned.

Lemma 5.14. *The vector $\langle 1,1,\dots,1\rangle$ is in $\mathrm{Span}_{\mathbb{Q}}(\Pi)$ if and only if (X, C_0) is not principal.*

Proof. The "if" direction follows from Lemma 4.4.

As in the proof of Lemma 4.4, consider a product of partial conjugations with multiplier x and supports D_i that are u-components for some $u \in U$. We write the product as

$$(\pi^x_{D_1})^{\varepsilon_1} \cdots (\pi^x_{D_l})^{\varepsilon_l}.$$

Up to an inner automorphism, and flipping the sign of ε_i, we may assume $C_0 \not\subset D_i$, and we then take the vector $\mathbf{w}_{D_i} \in \Pi$ defined above. For $z \in C_1 \cup \cdots \cup C_s$, we have

$$(\pi^x_{D_1})^{\varepsilon_1} \cdots (\pi^x_{D_l})^{\varepsilon_l}(z) = x^{-\alpha_j} z x^{\alpha_j}, \tag{1}$$

where

$$\alpha_j = \sum_{\{i | C_j \subset D_i\}} \varepsilon_i \quad \text{and} \quad \langle \alpha_1, \ldots, \alpha_s \rangle = \sum_{i=1}^{l} \varepsilon_i \mathbf{w}_{D_i}.$$

The entry α_j records the effect on the component C_j, however we have not recorded the effect on each component C_i for $i > s$. The key here is that we can use what is happening inside Λ to keep track of what is happening outside of Λ too. Indeed, by Lemma 5.13, each C_i for $s < i \leq r$ can be paired up with some C_j with $1 \leq j \leq s$, so that Z_i and Z_j are in the same u-component for every $u \in U$. The action of each partial conjugation $\pi^x_{D_i}$ is therefore the same on Z_i as it is on Z_j, and we can extend the labelling α_j from C_j to C_i. We set $\alpha_i = \alpha_j$, and equation (1) extends to $z \in C_j$ for $1 \leq j \leq r$. Note that the choice of j does not matter: if $Z_{j'}$ is another component in Λ which is in the same u-component as Z_i (and hence Z_j) for every u, then necessarily $\alpha_j = \alpha_{j'}$.

Hence, if $\langle 1, 1, \ldots, 1 \rangle \in \mathrm{Span}_{\mathbb{Q}}(\Pi)$, then we have a product of such partial conjugations that is equal to

$$(\pi^x_{C_1 \cup \cdots \cup C_r})^n = (\pi^x_{C_0})^{-n}$$

for some integer n, implying (X, C_0) is not principal. ∎

We now complete the proof of Theorem 3.

Consider an automorphism in $\mathrm{Aut}_\pi(A_\Gamma)$. It can be written as $R\pi$, where π is a product of partial conjugations and R a product of transvections. This follows from the relations in Lemma 2.6, by means of writing the automorphism as a product of partial conjugations, transvections, and their inverses, and then by shuffling the partial conjugations (and their inverses) to the right. Each partial conjugation is in $\mathrm{Aut}_\pi(A_\Gamma)$, so we can assume R is too.

Since (X, C_0) is principal, Lemma 5.14 tells us that the span of Π does not include $\langle 1, 1, \ldots, 1 \rangle$. Its orthogonal complement is therefore non-trivial, and we can define V to be the subspace

$$V = \mathrm{Span}_{\mathbb{Q}}(\Pi^\perp \cup \{\mathbf{x}\}).$$

We claim that V (more accurately, the associated projective space) is fixed by all partial conjugations except those with multiplier x, and is fixed by any product of transvections in $\mathrm{Aut}_\pi(A_\Gamma)$ (in particular, by R).

For any $\mathbf{v} \in \Pi^\perp$, by Lemma 3.2, we get $\varphi(\pi^a_D)(\mathbf{v}) = \mathbf{v}$ for any $a \leq x$, $a \neq x$ and corresponding components D. The vector $\mathbf{x}$ is sent to $-\mathbf{x}$, so these partial conjugations fix the projective space associated to V. The remaining partial conjugations are those with multiplier in $z \in Z_i$ for some i. If π^z_D is in the set B, then it is in the kernel of φ by construction – either it is in the kernel of p_3 or it is mapped to an inner automorphism by p_3, which is then in the kernel of ρ_π. If it is not in B, or not otherwise killed by q, then $1 \leq i \leq s$ and D is a union of u-components for some $u \in U$. In particular, $\mathbf{w}_D \in \mathrm{Span}_{\mathbb{Q}}(\Pi)$, so $\varphi(\pi^a_D)(\mathbf{v}) = \mathbf{v}$ for any $\mathbf{v} \in \Pi^\perp$. As previously, if $x \in D$, or not, the vector $\mathbf{x}$ is fixed.

As for transvections, we claim that R acts trivially on V. Consider R^b_a. By Lemma 5.8, we have $a \in Y \cup Z_0 \cup \cdots \cup Z_r$. If $a \in Y$, then R^b_a acts trivially on V. So we assume $a \in Z_0 \cup \cdots \cup Z_r$.

We will make use of the following observation.

Lemma 5.15. *Suppose $Z_i = \{z_i\}$ is dominated by x. If $i > 0$, then either*

(1) *$\mathbf{z}_i$ is in Π, or*

(2) *there is no vertex u such that $z_i \leq u \leq x$.*

In particular, if either $i = 0$ or $i > 0$ and $\mathbf{z}_i \notin \Pi$, then we get a finite index subgroup of $\mathrm{Aut}(A_\Gamma)$ *mapping onto $\mathbb{Z}$.*

Proof. Firstly, if $i > 0$ and Z_i is dominated by some $u \in U$, then Z_i is a u-component, so $\mathbf{z}_i \in \Pi$. This proves the first part of the lemma.

For the consequence regarding a virtual surjection onto $\mathbb{Z}$, both of the given possibilities imply that condition (A1) fails. Indeed, if $i = 0$, then Lemma 5.7 applies, while we have case (2) when $i > 0$. ∎

In light of Lemma 5.15, whenever Z_i is dominated by x, we can assume that $i > 0$ and $\mathbf{z}_i \in \Pi$. We know from Lemma 5.3 (II) that if R^b_a is a transvection with $a \in Z_i$ and $b \in Z_j$ with $j \neq i$, then $a \leq x$. Thus, unless a and b are both in Z_i, we must have that a is dominated by x. Then $\{a\} = Z_i$ for some i and $\mathbf{z}_i \in \Pi$. Thus if we take a vector $\mathbf{v}$ in $\Pi^\perp$, the $\mathbf{z}_i$-entry is zero for any i for which there is a transvection R^b_a with $a \in Z_i$, $b \notin Z_i$. We therefore focus on the effect of R on the entries of $\mathbf{v}$ that correspond to basis vectors $\mathbf{z}_i$, where Z_i is not dominated by x. The only way R can act on $a \in Z_i$ is by sending it to an element of $\langle Z_i \rangle$, and furthermore, in order to be in $\mathrm{Aut}_\pi(A_\Gamma)$, $R(a)$ must be representable by a word of odd length on $Z_i \cup Z_i^{-1}$. Thus $\varphi(R)$ sends $(1-g)e_{z_i}$ to

$$(1-g)(\kappa(1+g)e_{z_i} + e_{z_i}) = (1-g)e_{z_i}$$

for some integer κ, and so $\mathbf{z}_i$ is fixed. It follows that R fixes each $\mathbf{v} \in \Pi^\perp$ and acts trivially on V.

To conclude, we can restrict to V and get a new representation

$$\widehat{\varphi}\colon \mathrm{Out}_{\pi,\mathbb{Z}}(A_\Gamma) \to \mathrm{PGL}(V).$$

The image of $\widehat{\varphi}$ will be generated by the image of the partial conjugations $\pi^x_{C_i}$. The image will therefore be a free abelian group of rank equal to $\dim(\Pi^\perp)$, and this completes the proof of Theorem 3.

6. Splitting the standard representation when there is no SIL

As defined in Section 2.5, the standard representation of $\mathrm{Out}(A_\Gamma)$ is obtained by acting on the abelianisation of A_Γ, and we denote it by

$$\rho\colon \mathrm{Out}(A_\Gamma) \to \mathrm{GL}(n, \mathbb{Z}),$$

where n is equal to the number of vertices in Γ. We have a short exact sequence

$$1 \to \mathrm{IA}_\Gamma \to \mathrm{SOut}^0(A_\Gamma) \to Q \to 1, \tag{2}$$

where Q is a subgroup of $\mathrm{SL}(n, \mathbb{Z})$. The objective of this section is to show that under the assumption of no SIL, this short exact sequence splits, proving Proposition 6.

The structure of Q is well understood: it is (after conjugating Q by a suitable permutation matrix) a block triangular matrix group. This can be seen as follows.

Convention 6.1. Enumerate the vertices of Γ as $v_1, v_2, \dots, v_n$ in such a way so that if $v_i \leq v_j$, then either $i \leq j$ or v_i is equivalent to v_j, and so that equivalence classes of vertices are adjacent in this ordering.

Under ρ, transvections map to elementary matrices. We denote the image of R_u^v by E_u^v; if $u = v_i$ and $v = v_j$, then $E_u^v = E_{ji}$, the matrix that differs from the identity by a 1 in the (j, i)-entry.

Since the equivalence classes form clusters in this order, we obtain a block structure in Q. The blocks correspond to equivalence classes, and each diagonal block consists of matrices from $\mathrm{SL}(k, \mathbb{Z})$, where k is equal to the number of vertices in the corresponding equivalence class. Matrices in Q are lower block triangular by choice of the ordering on the vertices from Convention 6.1 and the fact that Q is generated by the set of elementary matrices E_{ji} when $v_i \leq v_j$.

We now prove Proposition 6, determining that the short exact sequence (2) splits when there is no SIL in Γ.

Proof of Proposition 6. Since $\mathrm{SOut}^0(A_\Gamma)$ is generated by partial conjugations and transvections and the kernel IA_Γ is generated by the partial conjugations, we know that Q is generated by the image of the transvections, namely

$$\{E_u^v \mid u \leq v\}.$$

Using the matrix structure of Q, we get the following set of defining relators (see [26, Proposition 4.11] for details) for $u \leq v \leq w$ and $x \leq y$:

(A) $[E_u^v, E_v^w] = E_u^w$ if $u \neq w$,

(B) $[E_u^v, E_x^y] = 1$ if $u \neq y$ and $v \neq x$,

(C) $(E_u^v (E_v^u)^{-1} E_u^v)^4 = 1$ if $u \neq v$ and $u \sim v$,

(D) $E_u^v (E_v^u)^{-1} E_u^v E_v^u (E_u^v)^{-1} E_v^u$ if $\{u, v\}$ is an equivalence class of size 2.

To see the short exact sequence splits, define $\sigma\colon Q \to \mathrm{SOut}^0(A_\Gamma)$ by sending each E_u^v to R_u^v. We need to check the four relators hold in the image of σ.

Since $u \leq v \leq w$, Lemma 2.4 implies $[v, w] = 1$. Then direct calculation, left to the reader, verifies the relation $[R_u^v, R_v^w] = R_u^w$, so (A) holds.

For (B), if $u = x$, then $[v, y] = 1$ by Lemma 2.4, and so R_u^v and R_u^y commute, as required. If $u \neq x$, then since also $u \neq y$ and $v \neq x$, the supports of R_u^v and R_x^y are disjoint and do not contain the multipliers. It follows that the transvections again commute.

Finally, for both (C) and (D), u and v are in the same equivalence class in Γ. Either this class has size at least 3, and so is abelian (since a non-abelian equivalence class of size at least 3 gives a SIL), or the equivalence class has size 2. We claim that in either case the subgroup $\langle R_u^v, R_v^u\rangle$ embeds into a copy of $\mathrm{SL}(n, \mathbb{Z})$, where n is the number of vertices in the equivalence class.

Let A denote the subgroup of A_Γ generated by the equivalence class containing u and v. Since R_u^v and R_v^u preserve the kernel of the projection map $\kappa\colon A_\Gamma \to A$ obtained by killing all vertices of Γ not in this class, we can define the factor map

$$f\colon \langle R_u^v, R_v^u\rangle \to \mathrm{SOut}^0(A)$$

so that $f(\Phi)(g) = \kappa(\Phi(g))$ for $\Phi \in \langle R_u^v, R_v^u\rangle$ and $g \in A$. It is clear that $\Phi(g)$ is in A, up to conjugacy, and thus

$$f(\Phi)(g) = \Phi(g).$$

Thus Φ is in the kernel of f only if Φ acts as an inner automorphism on A. If A is abelian, this is not possible for non-trivial $\Phi \in \langle R_u^v, R_v^u\rangle$, so f is an embedding into

$$\mathrm{SOut}^0(A) \cong \mathrm{SL}(n, \mathbb{Z})$$

as claimed. On the other hand, if A is not abelian, it must be non-abelian free of rank 2. In this case, using the fact that Γ has no SIL, we must have that A_Γ splits as a direct product $A \times B$. Indeed, B is generated by the vertices of Γ different to u or v, and if any such vertex x was not adjacent to u and v, then we would obtain a SIL $(u, v \mid x)$. In particular, if $f(\Phi)$ is inner on A, then Φ must have been inner on A_Γ. Thus f is an embedding into $\mathrm{SOut}^0(A) \cong \mathrm{SL}(2, \mathbb{Z})$.

With this claim, and since f maps R_u^v and R_v^u to elementary matrices in $\mathrm{SL}(n, \mathbb{Z})$, the relations (C) and (D) (the latter when $n = 2$) hold in $\mathrm{SL}(n, \mathbb{Z})$ and hence also in $\langle R_u^v, R_v^u\rangle$ as required. ■

Remark 6.2. We note that the short exact sequence may still split in cases when Γ does have a SIL. For example, as long as Γ has no SIL of the form $(x, y \mid z)$ with $z \leq x, y$, then if all its equivalence classes are abelian, the sequence will still be split. This can be seen from the above proof, since the no SIL condition was used for three reasons. One was in application of Lemma 2.4, which just requires the absence of the above type of SIL; a second was in deducing that equivalence classes of size at least three are abelian; and thirdly in the situation when we had a non-abelian equivalence class generating A.

7. Property (T) when there is no SIL

In this section, we show that, for a graph Γ with no SIL, if Theorems 2 and 3 do not imply virtual indicability, then the outer automorphism group $\mathrm{Out}(A_\Gamma)$ has property (T). This results in Theorem 4.

For background material concerning property (T), we refer the reader to the book by Bekka, de la Harpe, and Valette [2]. Some key facts regarding property (T) that we rely on are the following:

- it passes to and from finite index subgroups [2, Theorem 1.7.1];
- it is passed to quotients [2, Theorem 1.3.4];
- it is stable under short-exact sequences [2, Proposition 1.7.6].

The following is central to our method.

Lemma 7.1. *Suppose $H_1, \dots, H_s$ are normal subgroups of G and each has property (T). Then $\langle H_1, \dots, H_s \rangle$ has property (T).*

Proof. We use induction on s, with the case $s = 1$ trivial. Since $\langle H_1, \dots, H_s \rangle / H_1$ is a quotient of $\langle H_2, \dots, H_s \rangle$, it has property (T) by induction and the fact that property (T) passes to quotients. Stability of property (T) under short-exact sequences then implies $\langle H_1, \dots, H_s \rangle$ has property (T). ∎

The method to prove Theorem 4 is then as follows. We will decompose a finite index subgroup of IA_Γ into a direct product of subgroups $A_1, \dots, A_s$, each generated by a subset of partial conjugations. As Γ has no SIL, each A_i is abelian, and furthermore we will see that it is invariant under the action of $Q = \mathrm{SOut}^0(A_\Gamma)/\,\mathrm{IA}_\Gamma$. In particular, $Q \ltimes A_i$ is normal in $Q \ltimes A$, where $A = \langle A_1, \dots, A_s \rangle$ has finite index in IA_Γ. We will show that for each i, the group $Q \ltimes A_i$ has property (T), allowing us to apply Lemma 7.1. Then $\mathrm{Out}(A_\Gamma)$ inherits property (T) from its finite index subgroup $Q \ltimes A$.

We now describe the decomposition of IA_Γ (up to a finite index subgroup). Let X be an equivalence class in Γ and C an X-component. Recall that the set P_C^X is defined as

$$P_C^X = \{\pi_{C'}^y \mid y \geq X,\ C' = C \setminus \mathrm{st}(y)\}.$$

The set P_C^X is a basis for a free abelian group, which we denote by A_C^X.

It is clear that the union of all subsets P_C^X as X and C vary over all equivalence classes, and corresponding components will contain all partial conjugations and therefore generate IA_Γ by Proposition 2.7. However, we restrict ourselves to consider only those sets P_C^X for which (X, C) is principal. By Lemma 4.5, these will generate a finite index subgroup of IA_Γ.

Lemma 7.2. *Let X be an equivalence class in Γ and C an X-component. The subgroup $A_C^X = \langle P_C^X \rangle$ is normal in $\mathrm{SOut}^0(A_\Gamma)$.*

Proof. This follows from the relations in Lemma 2.6 under the assumption that there is no SIL. ∎

We aim to show that $Q \ltimes A_C^X$ has property (T) when (X, C) is principal. This is done by showing that $Q \ltimes A_C^X$ is itself a block triangular matrix group, and, in particular, one of those covered by criteria set out in [1, Section 4] that determine when such groups have property (T). We now introduce the relevant notation (which differs slightly from that of [1]).

Fix integers m_1, m_2 so that $m_1 < m_2$. It may help when first reading this to assume $m_1 = 1$; in practice, we will have either $m_1 = 1$ or $m_1 = 0$. Let $V_1, \dots, V_r$ be a partition of $I = [m_1, m_2] \cap \mathbb{Z}$ so that for each $x \in V_i$ and each $y \in V_j$, if $i < j$, then $x < y$. Let Λ be a directed graph with r vertices labelled by $V_1, \dots, V_r$. Assume that there is an edge from the vertex labelled V_i to the vertex labelled V_j only if $i \leq j$, and that the edge relation is transitive: if there are an edge from V_i to V_j and another from V_j to V_k, then there is an edge from V_i to V_k. Let n_i be the size of V_i and $n = m_2 - m_1 + 1$.

We let E_a^b, for $a, b \in I$, denote the $n \times n$ elementary matrix E_{ij}, where $i = b + 1 - m_1$ and $j = a + 1 - m_1$. (Thus if $m_1 = 1$, then we have $E_a^b = E_{ba}$.) Define the group $\mathcal{H}_\Lambda$ to be the block triangular matrix generated by

$$\{E_a^b \mid a \in V_i,\ b \in V_j \text{ and there is an edge from } V_i \text{ to } V_j\}.$$

The group $\mathcal{H}_\Lambda$ is a block lower-triangular matrix, with i-th diagonal block corresponding to $\mathrm{SL}(n_i, \mathbb{Z})$, and the (i, j)-th block, for $i \neq j$, being non-trivial if and only if there is an edge from V_j to V_i.

We can identify $Q = \mathrm{Out}(A_\Gamma)/\mathrm{IA}_\Gamma$ with a group $\mathcal{H}_\Lambda$ as follows. Take $m_1 = 1$ and m_2 to be the number of vertices in Γ. Order the vertices of Γ as per Convention 6.1. Let $V_1, \dots, V_r$ be the equivalence classes so that if $i < j$, then given $v_a \in V_i$ and $v_b \in V_j$, we have $a < b$. To construct Λ, take the directed graph with r vertices labelled by V_i and add an edge from V_i to V_j whenever $V_i \leq V_j$. Note that this includes edges from each V_i to itself.

In the following, we explain how to realise $Q \ltimes A_C^X$ as a matrix group of this form, starting with $Q \cong \mathcal{H}_\Lambda$. You may think of the rows/columns of a matrix in $\mathcal{H}_\Lambda$ as corresponding to vertices of Γ. To obtain the corresponding matrix representation of $Q \ltimes A_C^X$, we add a new row/column above/in front of the existing entries. The new row/column can be thought of, roughly, as representing C.

Lemma 7.3. *Let $X = V_i$ be an equivalence class in Γ, and let C be an X-component. Construct $\widehat{\Lambda}$ from Λ by adding a vertex labelled by $V_0 = \{0\}$, and adding an edge from V_0 to itself, and from V_0 to any V_j where an edge from V_i terminates.*

Then $Q \ltimes A_C^X \cong \mathcal{H}_{\widehat{\Lambda}}$.

Proof. The quotient map $\mathcal{H}_{\widehat{\Lambda}} \to \mathcal{H}_\Lambda \cong Q$ that kills the first coordinate, corresponding to the integer 0, has kernel K isomorphic to A_C^X, seen as follows. The kernel is generated by the matrices E_0^b for any $b \in V_j$ where there is an edge from V_i to V_j in Λ, or equivalently so that $V_i \leq V_j$. The isomorphism $K \cong A_C^X$ comes from identifying E_0^b with $\pi_{C'}^{v_b}$ for each $\pi_{C'}^{v_b} \in P_C^X$. Both groups K and A_C^X are free abelian of the same rank, namely $|P_C^X|$, and the above describes an identification of bases.

By Lemma 2.6, the action of Q on A_C^X agrees with the action of $\mathcal{H}_\Lambda$ on K, giving $Q \ltimes A_C^X \cong \mathcal{H}_{\widehat{\Lambda}}$ as required. ■

We are now ready to apply the result of [1] that gives sufficient conditions for groups $\mathcal{H}_\Lambda$ to have property (T) in order to complete the proof of Theorem 4. These conditions are as follows. Recall that n_i is the size of V_i.

Proposition 7.4 ([1, Proposition 4.2]). *Let Λ be constructed as above. Suppose the following conditions hold:*

- ($\mathcal{H}1$) *for each i, there is an edge from V_i to itself,*
- ($\mathcal{H}2$) *$n_i \neq 2$ for each i,*
- ($\mathcal{H}3$) *whenever $n_i = n_j = 1$ and there is an edge from V_i to V_j, with $i \neq j$, there are a third vertex V_k and edges from V_i to V_k and from V_k to V_j.*

Then $\mathcal{H}_\Lambda$ has property (T).

Proof of Theorem 4. Assume that Γ has no SIL and that conditions (A1) and (A2′) hold. Let $V_1, \ldots, V_r$ be the equivalence classes of Γ. Note that property (A1) implies each equivalence class of Γ has size not equal to 2.

Construct the graph Λ from Γ as above. Condition ($\mathcal{H}1$) holds in Λ by construction, while condition (A1) implies that ($\mathcal{H}2$) and ($\mathcal{H}3$) also hold.

Let $X = V_i$ and C be an X-component, chosen so that (X, C) is principal. Now construct $\widehat{\Lambda}$ as described in Lemma 7.3. By construction, $\widehat{\Lambda}$ inherits both ($\mathcal{H}1$) and ($\mathcal{H}2$) from Λ. For ($\mathcal{H}3$), if there is an edge from V_0 to V_j, for $j \neq i$, in $\widehat{\Lambda}$, and $n_j = 1$, then there are also edges from V_0 to V_i and from V_i to V_j. This is sufficient since condition (A2′) prevents us from having $n_i = 1$. Proposition 7.4 therefore implies that $\mathcal{H}_{\widehat{\Lambda}}$, and hence $Q \ltimes A_C^X$ by Lemma 7.3, has property (T).

To complete the proof, we apply Lemmas 7.1 and 4.5. Denote by A the subgroup of IA_Γ generated by the sets A_C^X when (X, C) is principal. By Lemma 7.1, we get that $Q \ltimes A$ has property (T). Since A has finite index in IA_Γ by Lemma 4.5, so does $Q \ltimes A$ in $\mathrm{Out}(A_\Gamma)$, and the result follows. ■

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