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Symbolic group varieties and dual surjunctivity

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Abstract. Let G be a group. Let X be an algebraic group over an algebraically closed field K. Denote by $A = X(K)$ the set of rational points of X. We study algebraic group cellular automata $\tau: A^G \to A^G$ whose local defining map is induced by a homomorphism of algebraic groups $X^M \to X$, where M is a finite memory. When G is sofic and K is uncountable, we show that if τ is post-surjective, then it is weakly pre-injective. Our result extends the dual version of Gottschalk's conjecture for finite alphabets proposed by Capobianco, Kari, and Taati. When G is amenable, we prove that if τ is surjective, then it is weakly pre-injective, and conversely, if τ is pre-injective, then it is surjective. Hence, we obtain a complete answer to a question of Gromov on the Garden of Eden theorem in the case of algebraic group cellular automata.

1. Introduction

We recall basic notations in symbolic dynamics. Fix a set A called the *alphabet*, and a group G, the *universe*. A *configuration* $c \in A^G$ is a map $c: G \to A$. The Bernoulli shift action $G \times A^G \to A^G$ is defined by $(g, c) \mapsto gc$, where $(gc)(h) = c(g^{-1}h)$ for $g, h \in G$ and $c \in A^G$. For $\Omega \subset G$ and $c \in A^G$, the *restriction* $c|_\Omega \in A^\Omega$ is given by $c|_\Omega(g) = c(g)$ for all $g \in \Omega$.

Following von Neumann [26], a *cellular automaton* over the group G and the alphabet A is a map $\tau: A^G \to A^G$ admitting a finite *memory set* $M \subset G$ and a *local defining map* $\mu: A^M \to A$ such that

$$(\tau(c))(g) = \mu((g^{-1}c)|_M)$$

for all $c \in A^G$ and $g \in G$.

Two configurations $c, d \in A^G$ are *asymptotic* if $c|_{G\setminus E} = d|_{G\setminus E}$ for some finite subset $E \subset G$. Let $\tau: A^G \to A^G$ be a cellular automaton. Then τ is *pre-injective* if $\tau(c) = \tau(d)$ implies $c = d$ whenever $c, d \in A^G$ are asymptotic. We say that τ is *post-surjective* if for every $x, y \in A^G$ with y asymptotic to $\tau(x)$, we can find $z \in A^G$ asymptotic to x such that $\tau(z) = y$.

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The cellular automaton $\tau\colon A^G \to A^G$ is said to be *linear* if A is a finite-dimensional vector space and τ is a linear map.

The important Gottschalk's conjecture [15] asserts that over any universe, an injective cellular automaton with finite alphabet must be surjective.

The conjecture was shown to hold over sofic groups (cf. [16, 27], see also [6, 9, 21]) while no examples of non-sofic groups are known in the literature. The dual version of Gottschalk's conjecture was introduced recently by Capobianco, Kari, and Taati in [3] and states the following.

Conjecture 1.1. *Let G be a group, and let A be a finite set. Suppose that $\tau\colon A^G \to A^G$ is a post-surjective cellular automaton. Then τ is pre-injective.*

As for Gottschalk's conjecture, the above dual surjunctivity conjecture is also known when the universe is a sofic group (cf. [3, Theorem 2]).

Theorem 1.2 (Capobianco–Kari–Taati). *Let G be a sofic group, and let A be a finite set. Suppose that $\tau\colon A^G \to A^G$ is a post-surjective cellular automaton. Then τ is pre-injective.*

Moreover, as Bartholdi pointed out in [1, Theorem 1.6], Conjecture 1.1 also holds for linear cellular automata over sofic groups.

Theorem 1.3. *Let G be a sofic group, and let V be a finite-dimensional vector space over a field. Suppose that $\tau\colon V^G \to V^G$ is a post-surjective linear cellular automaton. Then τ is pre-injective.*

Several related applications of groups satisfying Conjecture 1.1 are investigated in the papers [14, 22].

Fix a group G and an algebraic group X over an algebraically closed field K. Denote by $A = X(K)$ the set of K-points of X. We regard $A \subset X$ as a subset which consists of closed points of X (see, e.g., [9, Remark A.21]).

We denote by $CA_{\text{algr}}(G, X, K)$ the set of *algebraic group cellular automata* over (G, X, K), which consists of cellular automata $\tau\colon A^G \to A^G$ which admit a memory set M with local defining map $\mu\colon A^M \to A$ induced by some homomorphism of algebraic groups $f\colon X^M \to X$, i.e., $\mu = f|_{A^M}$, where X^M is the fibered product of copies of X indexed by M.

In [21, Definition 8.1], two notions of weak pre-injectivity, namely, (•)-pre-injectivity and (••)-pre-injectivity, are introduced for the class CA_{algr} (cf. Section 4). We prove in Corollary 4.3 that in CA_{algr}, we have

$$(\bullet)\text{-pre-injectivity} \Rightarrow (\bullet\bullet)\text{-pre-injectivity}. \tag{1.1}$$

Note that for linear cellular automata, pre-injectivity, (•)-pre-injectivity, and (••)-pre-injectivity are equivalent (cf. [21, Proposition 8.8]).

Generalizing Theorem 1.3, we establish Conjecture 1.1 for the class CA_{algr} where the universe is a sofic group and the alphabet is an arbitrary algebraic group not necessarily connected (cf. Theorem 6.1).

Theorem A. *Let G be a sofic group, and let X be an algebraic group over an uncountable algebraically closed field K. Suppose that $\tau \in CA_{\mathrm{algr}}(G, X, K)$ is post-surjective. Then τ is $(\bullet)$-pre-injective.*

We observe in Example 6.2 that for every group G, there exist a complex algebraic group X and $\tau \in CA_{\mathrm{algr}}(G, X, \mathbb{C})$ such that τ is post-surjective but not pre-injective. Moreover, in characteristic zero, we prove (Theorem 9.2) that every post-surjective, pre-injective $\tau \in CA_{\mathrm{algr}}$ is reversible with $\tau^{-1} \in CA_{\mathrm{algr}}$. Such property was known in the literature for cellular automata with finite alphabet [3, Theorem 1] and for linear cellular automata [1].

The classical Myhill–Moore Garden of Eden theorem for finite alphabets (cf. [19,20]) asserts that a cellular automaton over the group universe $\mathbb{Z}^d$ is pre-injective if and only if it is surjective. Over amenable groups, the theorem was extended to cellular automata with finite alphabet in [13] and to linear cellular automata in [5]. The theorem fails over non-amenable groups (cf. [1,2], see also [8]). In [16, 8.J. Question], Gromov asked

> *Does the Garden of Eden theorem generalize to the proalgebraic category? First, one asks if pre-injective $\Rightarrow$ surjective, while the reverse implication needs further modification of definitions.*

Let G be an amenable group, and let K be an algebraically closed field. The papers [10] and [21], respectively, give a positive answer to Gromov's question for the class $CA_{\mathrm{alg}}(G, X, K)$ (cf. Section 2.6) when X is a complete irreducible algebraic variety over K, and for the class $CA_{\mathrm{algr}}(G, X, K)$ when X is a connected algebraic group over K.

In this paper, we obtain the following complete answer to Gromov's question for the class $CA_{\mathrm{algr}}(G, X, K)$, where X is an arbitrary algebraic group (cf. Theorems 7.2 and 8.1).

Theorem B. *Let G be an amenable group, and let X be an algebraic group over an algebraically closed field K. Suppose that $\tau \in CA_{\mathrm{algr}}(G, X, K)$. Then the following hold:*

(i) *If τ is pre-injective, then it is surjective.*

(ii) *If τ is surjective, then it is both $(\bullet)$- pre-injective and $(\bullet\bullet)$-pre-injective.*

In Proposition 7.3, we show that one cannot replace the pre-injectivity hypothesis in Theorem B (i) by the weaker $(\bullet\bullet)$-pre-injectivity. Moreover, we obtain a very general result (cf. Theorem 5.2) saying that post-surjectivity implies surjectivity in CA_{algr} and CA_{alg}. Consequently, when the universe G is an amenable group, Theorem B (ii) implies Theorem A.

The paper is organized as follows. In Section 2, we present briefly important properties of sofic groups as well as amenable groups. Section 2.6 recalls basic definitions about the classes CA_{alg} and CA_{algr}. In Section 3, we introduce the useful tool of induced maps on the set of connected components of algebraic varieties and give some applications to the class CA_{algr}. Then in Section 4, we investigate at length $(\bullet)$-pre-injectivity and $(\bullet\bullet)$-pre-injectivity in the class CA_{algr} and prove (1.1). In Section 5, we establish a certain uniform post-surjectivity property (Lemma 5.3) and show, in particular, that post-surjectivity implies surjectivity in CA_{alg} and CA_{algr} (Theorem 5.2). We present the proof of Theorem A in Section 6. Then Theorem 7.2 establishes the Myhill property for CA_{algr} as stated in Theorem B (i). Finally, the Moore property for CA_{algr}, i.e., Theorem B (ii), is proved in Section 8 (Theorem 8.1).

2. Preliminaries

2.1. Convention and notation

To simplify the presentation, we suppose throughout the paper that the universe G is always a finitely generated group. The symbol e denotes the neutral element of algebraic groups and group alphabets while the neutral element of a universe G is denoted by 1_G. We denote cardinality by $|\cdot|$. Given a group alphabet A and a subset $E \subset G$ of a universe G, we identify naturally every subset D of A^E with the corresponding subset $D_e = D \times \{e\}^{G \setminus E}$ of $A^E \times \{e\}^{G \setminus E}$.

2.2. Algebraic varieties and algebraic groups

Following [17, Corollaire 6.4.2], an algebraic variety X over an algebraically closed field K is a reduced K-scheme of finite type and is identified with the set of K-points $X(K)$. Algebraic subvarieties are Zariski closed subsets, and algebraic subgroups are subgroups which are also algebraic subvarieties. An algebraic group is a group that is an algebraic variety with group operations given by algebraic morphisms (cf. [18]). See also [9, Appendix A], [10, Section 2] for standard definitions and basic properties of algebraic varieties.

2.3. Sofic groups

The important class of sofic groups was introduced by Gromov [16] and Weiss [27] as a common generalization of residually finite groups and amenable groups. Many conjectures for groups have been established for the sofic ones such as Gottschalk's surjunctivity conjecture and its dual surjunctivity conjecture (cf. [3]). See also [4, 7] for some more details.

Let S be a finite set. Then an S-*label graph* is a pair $\mathcal{G} = (V, E)$, where V is the set of *vertices*, and $E \subset V \times S \times V$ is the set of *edges*.

Denote by $l(\rho)$ the *length* of a path ρ in $\mathcal{G}$. If $v, v' \in V$ are not connected by a path in $\mathcal{G}$, we set $d_{\mathcal{G}}(v, v') = \infty$. Otherwise, we define

$$d_{\mathcal{G}}(v, v') = \min\{l(\rho) : \rho \text{ is a path from } v \text{ to } v'\}.$$

For $v \in V$ and $r \geq 0$, we define

$$B_{\mathcal{G}}(v, r) = \{v' \in V : d_{\mathcal{G}}(v, v') \leq r\}.$$

Observe that $B_{\mathcal{G}}(v, r)$ is naturally a finite S-labeled subgraph of $\mathcal{G}$.

Let (V_1, E_1) and (V_2, E_2) be two S-label graphs. A map $\phi\colon V_1 \to V_2$ is called an *S-labeled graph homomorphism* from (V_1, E_1) to (V_2, E_2) if $(\phi(v), s, \phi(v')) \in E_2$ for all $(v, s, v') \in E_1$. A bijective S-labeled graph homomorphism $\phi\colon V_1 \to V_2$ is an *S-labeled graph isomorphism* if its inverse $\phi^{-1}\colon V_2 \to V_1$ is an S-labeled graph homomorphism.

Let G be a finitely generated group, and let $S \subset G$ be a finite *symmetric* generating subset, i.e., $S = S^{-1}$. The *Cayley graph of G with respect to S* is the connected S-labeled graph $C_S(G) = (V, E)$, where $V = G$ and $E = \{(g, s, gs)\colon g \in G \text{ and } s \in S\}$.

For $g \in G$ and $r \geq 0$, we denote $B_S(r) = B_{C_S(G)}(1_G, r)$.

We can characterize sofic groups as follows [7, Theorem 7.7.1].

Theorem 2.1. *Let G be a finitely generated group. Let $S \subset G$ be a finite symmetric generating subset. Then the following are equivalent:*

(a) *the group G is sofic;*

(b) *for all $r, \varepsilon > 0$, there exists a finite S-labeled graph $\mathcal{G} = (V, E)$ satisfying*

$$|V(r)| \geq (1 - \varepsilon)|V|,$$

where $V(r) \subset V$ consists of $v \in V$ such that there exists a (unique) S-labeled graph isomorphism $\psi_{v,r}\colon B_S(r) \to B_{\mathcal{G}}(v, r)$ with $\psi_{v,r}(1_G) = v$.

Let $0 \leq r \leq r'$. Then $V(r') \subset V(r)$ since every S-labeled graph isomorphism

$$\psi_{v,r'}\colon B_S(r') \to B_{\mathcal{G}}(v, r')$$

induces by restriction an S-labeled graph isomorphism $B_S(r) \to B_{\mathcal{G}}(v, r)$. We shall need the following well-known packing lemma (cf. [27], [7, Lemma 7.7.2], see also [21] for (ii)).

Lemma 2.2. *With the notation as in Theorem 2.1, the following hold:*

(i) *$B_{\mathcal{G}}(v, r) \subset V(kr)$ for all $v \in V((k + 1)r)$ and $k \geq 0$.*

(ii) *There exists a finite subset $V' \subset V(3r)$ such that the balls $B_{\mathcal{G}}(v, r)$ are pairwise disjoint for all $v \in V'$ and that $V(3r) \subset \bigcup_{v \in V'} B_{\mathcal{G}}(v, 2r)$.*

2.4. Tilings of groups

Let G be a group, and let $E, E' \subset G$. A subset $T \subset G$ is called an *(E, E')-tiling* if the following hold:

(T-1) the subsets gE, $g \in T$, are pairwise disjoint,

(T-2) $G = \bigcup_{g \in T} gE'$.

We shall need the following existence result which is an immediate consequence of Zorn's lemma (see [7, Proposition 5.6.3]).

Proposition 2.3. *Let E be a nonempty finite subset of a group G and let $E' = \{gh^{-1}\colon g, h \in E\}$. Then there exists an (E, E')-tiling $T \subset G$.*

2.5. Amenable group and algebraic mean dimension

Amenable groups were introduced by von Neumann in [25]. A group G is *amenable* if it admits a *Følner net*, i.e., a family $(F_i)_{i \in I}$ over a directed set I consisting of nonempty finite subsets of G such that

$$\lim_{i \in I} \frac{|F_i \setminus F_i g|}{|F_i|} = 0 \quad \text{for all } g \in G.$$

In [10], algebraic mean dimension is introduced as an analog of topological and measure-theoretic entropy, as well as various notions of mean dimension studied by Gromov in [16].

Definition 2.4. Let G be an amenable group, and let $\mathcal{F} = (F_i)_{i \in I}$ be a Følner net for G. Let X be an algebraic variety over an algebraically closed field K, and let $A = X(K)$. The *algebraic mean dimension* of a subset $\Gamma \subset A^G$ with respect to $\mathcal{F}$ is the quantity $\operatorname{mdim}_{\mathcal{F}}(\Gamma)$ defined by

$$\operatorname{mdim}_{\mathcal{F}}(\Gamma) = \limsup_{i \in I} \frac{\dim(\Gamma_{F_i})}{|F_i|},$$

where $\dim(\Gamma_{F_i})$ denotes the Krull dimension of

$$\Gamma_{F_i} = \{x|_{F_i} : x \in \Gamma\} \subset A^{F_i}$$

with respect to the Zariski topology.

We shall need the following technical lemma in Sections 7 and 8.

Lemma 2.5. *Let G be an amenable group, and let $\mathcal{F} = (F_i)_{i \in I}$ be a Følner net for G. Let X be an algebraic variety over an algebraically closed field K, and let $A = X(K)$. Suppose that $\Gamma \subset A^G$ satisfies the following condition:*

(C) *there exist finite subsets $E, E' \subset G$ and an (E, E')-tiling $T \subset G$ such that $\dim \Gamma_{gE} < \dim A^{gE}$ for all $g \in T$.*

Then one has $\operatorname{mdim}_{\mathcal{F}}(\Gamma) < \dim(X)$.

Proof. See [10, Lemma 5.2]. ∎

2.6. Strongly irreducible subshifts

Let G be a group, and let A be a set. A *subshift* of A^G is a G-invariant subset. A subshift of A^G is called a *closed subshift* if it is closed in A^G with respect to the prodiscrete topology.

We say that a subshift $\Sigma \subset A^G$ is *strongly irreducible* if there exists a finite subset $\Delta \subset G$ such that for all finite subsets $E, F \subset G$ with $E\Delta \cap F = \varnothing$ and $x, y \in \Sigma$, there exists $z \in \Sigma$ such that

$$z|_E = x|_E \quad \text{and} \quad z|_F = y|_F.$$

2.7. Algebraic subshifts and algebraic cellular automata

Let G be a group. Let X be an algebraic variety over an algebraically closed field K, and let $A = X(K)$. Then *algebraic subshifts of finite type* of A^G are closed subshifts of the form

$$\Sigma(A^G; W, D) = \{x \in A^G : (gx)|_D \in W \text{ for all } g \in G\},$$

where $D \subset G$ is a finite subset and $W \subset A^D$ is an algebraic subvariety.

Following [9, Definition 1.1], the set $CA_{\text{alg}}(G, X, K)$ of *algebraic cellular automata* consists of cellular automata $\tau: A^G \to A^G$ which admit a local defining map $\mu: A^M \to A$ induced by any K-morphism of algebraic varieties $f: X^M \to X$, i.e., $\mu = f|_{A^M}$ (note that we always have $f(A^M) \subset A$).

Let $\Lambda \subset A^G$ be a subshift. If $\Lambda = \tau(\Sigma)$ for some $\tau \in CA_{\text{alg}}(G, X, K)$ and an algebraic subshift of finite type $\Sigma \subset A^G$, then we call Λ an *algebraic sofic subshift* (cf. [11]). See also [12, 24] for the simpler linear case.

3. Induced maps on the set of connected components

Fix an algebraically closed field K. For every K-algebraic variety U, we denote by U_0 the finite set of connected components of U, and let $i_U: U \to U_0$ be the map sending every point $u \in U$ to the connected component of U which contains u. It is clear that $(U^n)_0 = (U_0)^n$ for every $n \in \mathbb{N}$.

For every morphism of K-algebraic varieties $\pi: R \to T$, we denote by $\pi_0: R_0 \to T_0$ the map which sends every $p \in R_0$ to $q_0 \in T$, where q_0 is the connected component of T containing $\pi(u)$ for any point $u \in R$ that belongs to p_0. Note that π_0 is well defined since the image of every connected component is connected. Moreover, it follows from the definition that

$$i_T \circ \pi = \pi_0 \circ i_R. \tag{3.1}$$

If in addition the map π is surjective, then clearly $|T_0| \leq |R_0|$.

Let X be a K-algebraic variety, and let G be a group. Suppose $\tau: X(K)^G \to X(K)^G$ is an algebraic cellular automaton with an algebraic local defining map $f: X^M \to X$ for some finite symmetric subset $M \subset G$, i.e., $M = M^{-1}$. Then we obtain a well-defined cellular automaton $\tau_0: X_0^G \to X_0^G$ admitting $f_0: X_0^M \to X_0$ as a local defining map,

$$\tau_0(c)(g) = f_0((g^{-1}c)|_M)$$

for all $c \in X_0^G$ and $g \in G$. Let $i_{X^G}: X^G \to X_0^G$ be the induced map $i_{X^G} = \prod_G i_X$. Then it is clear that

$$i_{X^G} \circ \tau = \tau_0 \circ i_{X^G}.$$

We also infer from relation (3.1) the functoriality of our construction of induced cellular automata: for all $\tau, \sigma \in CA_{\text{alg}}(G, X, K)$, we have

$$(\sigma \circ \tau)_0 = \sigma_0 \circ \tau_0.$$

Indeed, let $f: X^M \to X$ and $h: X^M \to X$ be the algebraic local defining maps of τ and σ, respectively, for some finite memory set $M \in G$. Let $f_M^+: X^{M^2} \to X^M$ be the induced map given by $f_M^+(c)(g) = f((g^{-1}c)|_M)$ for every $c \in A^{M^2}$ and $g \in M$. Then $h \circ f_M^+: X^{M^2} \to X$ is an algebraic local defining map of $\sigma \circ \tau$ associating with the memory set M^2. Since

$$(h \circ f_M^+)_0 = h_0 \circ (f_M^+)_0,$$

we deduce without difficulty that $(\sigma \circ \tau)_0 = \sigma_0 \circ \tau_0$.

Suppose that $\pi: R \to T$ is a homomorphism of algebraic groups over K. Then R_0 and T_0 inherit naturally a group structure from R and T, respectively. For example, the multiplication map $R_0 \times R_0 \to R_0$ is defined by $pq = r$ for $(p, q) \in R_0 \times R_0$, where $r \in R_0$ is the connected component of R containing xy for any $x, y \in R$ such that $x \in p$ and $y \in q$. Therefore, it follows immediately from (3.1) that $\pi_0: R_0 \to T_0$ is a group homomorphism. With this observation, we obtain the following lemma.

Lemma 3.1. *Let G be a group, and let X be an algebraic group over K. Suppose that $\tau \in CA_{\text{algr}}(G, X, K)$. Then the induced cellular automaton $\tau_0: X_0^G \to X_0^G$ is a group cellular automaton.*

Proof. By definition, τ admits an algebraic local defining map $f: X^M \to X$ for some finite subset $M \subset G$. Then the induced map $f_0: X_0^M \to X_0$ is a local defining map of the cellular automaton $\tau_0: X_0^G \to X_0^G$. Since f_0 is a homomorphism of groups, τ_0 is a group cellular automaton. ∎

4. Weak pre-injectivity

We recall the following two notions of weak pre-injectivity introduced in [21, Definition 8.1].

Definition 4.1. Let G be a group. Let X be a K-algebraic group, and let $A = X(K)$. For every $\tau \in CA_{\text{algr}}(G, X, K)$, we say that

(a) τ is $(\bullet)$*-pre-injective* if there is no finite subset $\Omega \subset G$ and no Zariski closed subset $H \subsetneq A^\Omega$ such that

$$\tau((A^\Omega)_e) = \tau(H_e).$$

(b) τ is $(\bullet\bullet)$*-pre-injective* if for every finite subset $\Omega \subset G$, we have

$$\dim(\tau((A^\Omega)_e)) = \dim(A^\Omega).$$

We establish first the following lemma.

Lemma 4.2. *Let $f: X \to Y$ be a homomorphism of algebraic groups over a field K. Suppose that* $\dim X > \dim f(X)$. *Then there exists a closed subset $Z \subsetneq X$ such that* $\dim Z < \dim X$ *and* $f(Z) = f(X)$.

Proof. Let us write $X = \bigcup_{i \in I} X_i$ as a disjoint union of connected components of X, where I is a finite set. For each $i \in I$, consider the restriction algebraic morphism $f_i = f|_{X_i} \colon X_i \to f(X_i)$.

By [18, Theorems 5.80 and 5.81], we know that the image $f(X)$ is an algebraic group. Since connected components of an algebraic group are precisely irreducible components and have the same dimension as the dimension of the algebraic group, it follows that $f_i(X_i)$ is a connected component of $f(X)$ for every $i \in I$. The morphisms f_i are surjective morphisms of irreducible algebraic varieties with $\dim X_i > \dim f_i(X_i)$. Hence, [21, Lemma 8.2] implies that for every $i \in I$, there exists a proper closed subset $Z_i \subsetneq X_i$ such that $f_i(Z_i) = f_i(X_i) = f(X_i)$. In particular, since X_i is irreducible, it follows that $\dim Z_i < \dim X_i$ for every $i \in I$.

Let $Z = \bigcup_{i \in I} Z_i \subset X$, then we find by construction that

$$f(Z) = \bigcup_{i \in I} f(Z_i) = \bigcup_{i \in I} f(X_i) = f(X)$$

and clearly,

$$\dim Z = \max_{i \in I} \dim Z_i < \max_{i \in I} X_i = \dim X.$$

Therefore, Z verifies the desired properties and the proof is complete. ∎

Lemma 4.2 allows us to show the following general logical implication in the class CA_{algr}:

$$(\bullet)\text{-pre-injectivity} \Rightarrow (\bullet\bullet)\text{-pre-injectivity}.$$

Corollary 4.3. *Let G be a group, and let X be an algebraic group over K. Let $\tau \in CA_{\text{algr}}(G, X, K)$. Suppose that τ is $(\bullet)$-pre-injective. Then τ is also $(\bullet\bullet)$-pre-injective.*

Proof. Let $A = X(K)$, and suppose on the contrary that τ is not $(\bullet\bullet)$-pre-injective. Then we can find a finite subset $E \subset G$ such that $\dim \tau((A^E)_e) < \dim A^E$. Hence, we infer from Lemma 4.2 that there exists a closed subset $Z \subset A^E$ such that $\dim Z < \dim A^E$ and that $\tau((A^E)_e) = \tau(Z_e)$. Since $\dim Z < \dim A^E$, we have $Z \subsetneq A^E$ and we can conclude that τ is not $(\bullet)$-pre-injective, which is a contradiction. The proof is thus complete. ∎

Let X be a connected algebraic group over an algebraically closed field K, and let G be a group. Let $\tau \in CA_{\text{algr}}(G, X, K)$. Then it was shown in [21, Proposition 8.3] that τ is $(\bullet)$-pre-injective if and only if it is $(\bullet\bullet)$-pre-injective. However, the following result shows that the converse of Corollary 4.3 fails as soon as the alphabet is not a connected algebraic group.

Proposition 4.4. *Let G be a group, and let K be an algebraically closed field. Then there exist a finite algebraic group X over K and $\tau \in CA_{\text{algr}}(G, X, K)$ such that τ is $(\bullet\bullet)$-pre-injective but is not $(\bullet)$-pre-injective.*

Proof. Let $X = \mathbb{Z}/4\mathbb{Z}$ and consider the homomorphism $\varphi \colon X \to X$ given by $x \mapsto 2x$. Let $Y = \operatorname{Ker} \varphi \simeq \mathbb{Z}/2\mathbb{Z}$, then we also have $\varphi(X) = Y$. Let us denote $H = X \setminus \{e\} \subsetneq X$.

We define $\tau: X^G \to X^G$ by $\tau(c)(g) = \varphi(c(g))$ for all $g \in G$ and $c \in X^G$.

Now let $E \subset G$ be a finite subset. Then it is clear by the construction that we have an equality of Krull dimensions $\dim \tau((X^E)_e) = \dim X^E = 0$. It follows that τ is (••)-pre-injective. However, we have

$$\tau((X^E)_e) = \tau((H^E)_e) = (Y^E)_e,$$

and $H^E \subsetneq X^E$ is a proper closed subset. Consequently, τ is not (•)-pre-injective, and the proof is complete. ∎

5. Uniform post-surjectivity

We will show in this section that the class CA_{algr} admits a uniform post-surjectivity property (cf. Lemma 5.3). We also prove in Theorem 5.2 that in the class CA_{alg}, we have the implication

$$\text{post-surjectivity} \Rightarrow \text{surjectivity}.$$

5.1. Post-surjectivity implies surjectivity

This subsection is independent of the rest of the paper. We begin with the following uniform property of strong irreducibility which is a generalization of [3, Proposition 1].

Lemma 5.1. *Let G be a countable group. Let X be an algebraic variety over an uncountable algebraically closed field K, and let $A = X(K)$. Let $\Sigma \subset A^G$ be a strongly irreducible closed algebraic subshift. Then there exists a finite subset $\Delta \subset G$ such that for every $x, y \in \Sigma$ and for every finite subset $E \subset G$, we can find $z \in \Sigma$ which coincides with x outside of $E\Delta$, and $z|_E = y|_E$.*

Proof. Since Σ is strongly irreducible, there exists a finite subset $\Delta \subset G$ with $1_G \in \Delta$ such that for all finite subsets $E_1, E_2 \subset G$ with $E_1\Delta \cap E_2 = \varnothing$ and all $z_1, z_2 \in \Sigma$, there exists $z \in \Sigma$ such that $z|_{E_1} = z_1|_{E_1}$ and $z|_{E_2} = z_2|_{E_2}$.

Fix a finite subset $E \subset G$, and let $x, y \in \Sigma$. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} F_n$ and $E\Delta \subset F_n$ for every $n \in \mathbb{N}$. We set $H_n = F_n \setminus E\Delta$. Then for every $n \in \mathbb{N}$, there exists by the strong irreducibility of Σ a configuration $z_n \in \Sigma$ such that $z_n|_E = y|_E$ and such that $z_n|_{H_n} = x|_{H_n}$. Let us define for $n \in \mathbb{N}$

$$\Lambda_n = \{c \in \Sigma_{F_n} : c|_E = y|_E,\ c|_{H_n} = x|_{H_n}\}.$$

Then by the above paragraph, we deduce that $(\Lambda_n)_{n\in\mathbb{N}}$ forms an inverse system of nonempty algebraic varieties over K. The transition maps are simply induced by the restriction maps $A^{F_m} \to A^{F_n}$ for $0 \leq n \leq m$. Hence, by [9, Lemma B.2], $\varprojlim_{n\in\mathbb{N}} \Lambda_n$ is nonempty, and thus we can find

$$z \in \varprojlim_{n\in\mathbb{N}} \Lambda_n \subset \varprojlim_{n\in\mathbb{N}} \Sigma_{F_n} = \Sigma.$$

The latter equality follows from the closedness of Σ in A^G with respect to the prodiscrete topology.

Note that $z \in \Sigma$ is asymptotic to x and satisfies $z|_E = y|_E$. In fact, z and x coincide outside of $E\Delta$. The proof is thus complete. ∎

We obtain the following generalization of [3, Proposition 2].

Theorem 5.2. *Let G be a countable group, and let K be an uncountable algebraically closed field. Let X be an algebraic variety over K, and let $A = X(K)$. Let $\Sigma \subset A^G$ be a strongly irreducible algebraic sofic subshift. Suppose that $\tau\colon \Sigma \to \Sigma$ is the restriction of some $\sigma \in CA_{\text{alg}}(G, X, K)$. Then if τ is post-surjective, it is also surjective.*

Proof. Let us fix $x_0, y \in \Sigma$ and a memory set $M \subset G$ of τ. Let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} E_n$. Then for every $n \in \mathbb{N}$, we infer from Lemma 5.1 that there exists $z_n \in \Sigma$ asymptotic to $\tau(x_0)$ and such that $z_n|_{E_n} = y|_{E_n}$.

Since τ is post-surjective and $\tau(x_0) \in \mathrm{Im}(\tau)$, it follows that $z_n \in \mathrm{Im}(\tau)$ for every $n \in \mathbb{N}$. As $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of G such that $G = \bigcup_{n\in\mathbb{N}} E_n$, we deduce that y belongs to the closure of $\mathrm{Im}(\tau)$ with respect to the prodiscrete topology.

Since $\mathrm{Im}(\tau)$ is closed by [11, Theorem 8.1], it follows that $y \in \mathrm{Im}(\tau)$. Therefore, τ is surjective and the proof is complete. ∎

We note here that by the exactly same proof, Theorem 5.2 still holds if X is an algebraic group over an arbitrary algebraically closed field K, $\Sigma \subset A^G$ is a strongly irreducible algebraic group subshift (cf. [23, Definition 1.2]), and $\tau\colon \Sigma \to \Sigma$ is the restriction of some $\sigma \in CA_{\text{algr}}(G, X, K)$. It suffices to observe that in this situation, $\mathrm{Im}(\tau)$ is still closed in A^G thanks to [23, Theorem 4.4].

5.2. Uniform post-surjectivity

We have the following key uniform property for the post-surjectivity in the class CA_{algr}.

Lemma 5.3 (Uniform post-surjectivity). *Let G be a countable group. Let X be an algebraic group over an uncountable algebraically closed field K, and suppose that $\tau \in CA_{\text{algr}}(G, X, K)$ is post-surjective. Let $A = X(K)$. Then there exists a finite subset $E \subset G$ with the following property. For all $x, y \in A^G$ such that $y|_{G\setminus\{1_G\}} = \tau(x)|_{G\setminus\{1_G\}}$, there exists $x' \in A^G$ such that $\tau(x') = y$ and $x'|_{G\setminus E} = x|_{G\setminus E}$.*

Proof. Let $M \subset G$ be a finite memory set of τ such that $1_G \in M$ and $M = M^{-1}$. Let $\mu\colon A^M \to A$ be the corresponding local defining map.

Since $\tau \in CA_{\text{algr}}(G, X, K)$, it follows that μ is induced by a homomorphism of algebraic groups $f\colon X^M \to X$. Let $(E_n)_{n\in\mathbb{N}}$ be an exhaustion of G consisting of finite subsets such that $1_G \in E_0$. For each $n \in \mathbb{N}$, we define

$$V_n = \left\{x \in A^G \colon \tau(x)|_{G\setminus\{1_G\}} = \{e\}^{G\setminus\{1_G\}},\ x|_{G\setminus E_n} = \{e\}^{G\setminus E_n}\right\}. \tag{5.1}$$

Consider the following homomorphism of algebraic groups:

$$\varphi_n\colon A^{E_n} \times \{e\}^{E_n M^2 \setminus E_n} \to A^{E_n M}$$

defined by $\varphi_n(x)(g) = \mu((g^{-1}x)|_M)$ for all $x \in A^{E_n} \times \{e\}^{E_n M^2 \setminus E_n}$ and $g \in E_n M$. Note that $1_G \in E_n M$ for every $n \in \mathbb{N}$. We denote respectively by $p_n\colon A^{E_n M} \to A^{\{1_G\}}$ and $q_n\colon A^{E_n M} \to A^{E_n M \setminus \{1_G\}}$, the canonical projections.

It is clear that for all $n \in \mathbb{N}$, we can identify $V_n = \operatorname{Ker} q_n \circ \varphi_n$ which is an algebraic subgroup of $A^{E_n} \times \{e\}^{E_n M^2 \setminus E_n} = A^{E_n}$. Let us consider

$$Z_n = p_n(\varphi_n(V_n)) = \tau(V_n)_{\{1_G\}}, \quad T_n = A \setminus Z_n.$$

Then Z_n is a closed algebraic subgroup of A, and thus T_n is a Zariski open subset of A for every $n \in \mathbb{N}$. Since $E_n \subset E_{n+1}$, we deduce from (5.1) that $V_n \subset V_{n+1}$. Consequently, we find that $Z_n \subset Z_{n+1}$ for all $n \in \mathbb{N}$.

We claim that $\bigcup_{n\in\mathbb{N}} Z_n = A$. Indeed, let $y \in A$ and consider $c \in A^G$ defined by $c(g) = e$ for all $g \in G \setminus \{1_G\}$ and $c(1_G) = y$. Since τ is post-surjective and $\tau(e^G) = e^G$, it follows that there exist $x \in A^G$ and $n \in \mathbb{N}$ such that $x|_{G\setminus E_n} = e^{G\setminus E_n}$ and such that $\tau(x) = c$. We deduce that $\tau(x)|_{G\setminus\{1_G\}} = e^{G\setminus\{1_G\}}$ and thus $x \in V_n$. Moreover, because $\tau(x)(1_G) = y$, it follows that $y \in Z_n$. Hence, we have proven the claim that $\bigcup_{n\in\mathbb{N}} Z_n = A$.

Therefore, $(T_n)_{n\in\mathbb{N}}$ is a decreasing sequence of Zariski open (thus constructible, see [9, Section A.1]) subsets of A and satisfies

$$\bigcap_{n\in\mathbb{N}} T_n = A \setminus \Big(\bigcup_{n\in\mathbb{N}} Z_n\Big) = \varnothing.$$

Since the field K is uncountable and algebraically closed, we infer from [9, Lemma B.3] that there exists $N \in \mathbb{N}$ such that $T_N = \varnothing$. It follows that $Z_N = A$. We claim that $E = E_N$ satisfies the desired property in the conclusion of the lemma.

Indeed, suppose that $x, y \in A^G$ satisfy $y|_{G\setminus\{1_G\}} = \tau(x)|_{G\setminus\{1_G\}}$. Let us define $c \in A^G$, where $c(g) = y(g)(\tau(x)(g))^{-1}$ for all $g \in G$. Then $c|_{G\setminus\{1_G\}} = e^{G\setminus\{1_G\}}$. Since $Z_N = \tau(V_N)_{\{1_G\}} = A$, we can find $d \in V_N$ such that $\tau(d)(1_G) = c(1_G)$. Therefore, $d|_{G\setminus E_N} = e^{G\setminus E_N}$ and $\tau(d)|_{G\setminus\{1_G\}} = e^{G\setminus\{1_G\}}$.

Consequently, since τ is a homomorphism, we find for $x' = dx \in A^G$ and for every $g \in G$ that

$$\begin{aligned}
\tau(x')(g) &= \tau(d)(g)\tau(x)(g) \\
&= \begin{cases} \tau(x)(g) & \text{if } g \in G \setminus \{1_G\}, \\ y(1_G)(\tau(x)(1_G))^{-1}\tau(x)(1_G) & \text{if } g = 1_G \end{cases} \\
&= \begin{cases} y(g) & \text{if } g \in G \setminus \{1_G\}, \\ y(1_G) & \text{if } g = 1_G \end{cases} \\
&= y(g).
\end{aligned}$$

Therefore, $\tau(x') = y$. On the other hand, since $d|_{G\setminus E} = e^{G\setminus E}$ and $x' = dx$, we have $x'|_{G\setminus E} = x|_{G\setminus E}$. The conclusion thus follows. ∎

6. Dual surjunctivity for CA_{algr}

In this section, we will present the proof of Theorem A and the construction of Example 6.2 showing that in a certain sense, Theorem A is optimal.

Theorem 6.1. *Let G be a sofic group, and let X be an algebraic group over an uncountable algebraically closed field K. Suppose that $\tau \in CA_{\mathrm{algr}}(G, X, K)$ is post-surjective. Then τ is both $(\bullet)$-pre-injective and $(\bullet\bullet)$-pre-injective.*

Proof. Let $A = X(K)$, and let $S \subset G$ be a memory set of τ such that $1_G \in S$, $S = S^{-1}$ and that S generates G. Let $f\colon A^S \to S$ be the corresponding local defining map which is a homomorphism of algebraic groups.

As $(\bullet)$-pre-injectivity implies $(\bullet\bullet)$-pre-injectivity in the class CA_{algr} (cf. Lemma 4.3), it suffices to show that τ is $(\bullet)$-pre-injective.

Suppose on the contrary that τ is not $(\bullet)$-pre-injective. Then there exist a finite subset $\Omega \subset G$ and a proper closed subset $H \subsetneq A^\Omega$ such that

$$\tau((A^\Omega)_e) = \tau(H_e). \tag{6.1}$$

Since τ is post-surjective, there exists a finite subset $E \subset G$ with the property described in Lemma 5.3, i.e., for $x, y \in A^G$ with $y|_{G\setminus\{1_G\}} = \tau(x)|_{G\setminus\{1_G\}}$, there exists $x' \in A^G$ such that $\tau(x') = y$ and $x'|_{G\setminus E} = x|_{G\setminus E}$.

In the Cayley graph $C_S(G)$, note that $B_S(1) = S$ (see Section 2.3). Up to enlarging E, we can suppose without loss of generality that

$$\Omega \subset B_S(r-1) \subset E = B_S(r) \quad \text{for some } r \geq 2.$$

If $\dim A = 0$, then A is a finite group, so Theorem 1.2 implies that τ is pre-injective. By [10, Proposition 6.4.1, Example 8.1] and [21, Proposition 8.3.(ii)], pre-injectivity is equivalent to $(\bullet)$-pre-injectivity for finite group alphabets. Thus we deduce that τ is $(\bullet)$-pre-injective.

Suppose from now on that $\dim A > 0$. Let X_0 be the set of connected components of X. Let us fix $0 < \varepsilon < 1/2$ small enough so that

$$|X_0|^{\varepsilon}(1 - |X_0|^{-|B_S(r)|})^{\frac{1}{2|B_S(2r)|}} < 1, \tag{6.2}$$

and that

$$0 < (1-\varepsilon)^{-1} < 1 + \frac{1}{|B_S(2r)| \dim A}. \tag{6.3}$$

Since the group G is sofic, it follows from Theorem 2.1 that there exists a finite S-labeled graph $\mathcal{G} = (V, E)$ associated to the pair $(3r, \varepsilon)$ such that

$$|V(3r)| \geq (1-\varepsilon)|V| > \frac{1}{2}|V|, \tag{6.4}$$

where for each $s \geq 0$, the subset $V(s) \subset V$ consists of $v \in V$ such that there exists a unique S-labeled graph isomorphism $\psi_{v,s}\colon B_{\mathcal{G}}(v,s) \to B_S(s)$ sending v to 1_G (cf. Theorem 2.1). Note that $V(s) \subset V(s')$ for all $0 \leq s' \leq s$.

We denote $B(v,s) = B_{\mathcal{G}}(v,s)$ for $v \in V$ and $s \geq 0$. Define $V' \subset V(3r)$ as in Lemma 2.2 (ii), so that $B(v,r)$ are pairwise disjoint for all $v \in V'$ and $V(3r) \subset \bigcup_{v \in V'} B(v,2r)$. In particular,

$$|V(3r)| \leq |V'||B_S(2r)|. \tag{6.5}$$

Note that the local map f induces a homomorphism of algebraic groups $\Phi\colon A^V \to A^{V(3r)}$ given by $\Phi(x)(v) = f(\psi_{v,1}(x|_{B(v,1)}))$ for all $x \in A^V$ and $v \in V(3r)$. As $E = B_S(r) \subset B_S(r)S \subset B_S(3r)$, we deduce applying repeatedly Lemma 5.3 that Φ is surjective (cf. the [3, proof of Lemma 2]),

$$\Phi(A^V) = A^{V(3r)}. \tag{6.6}$$

We claim that $\dim \operatorname{Ker} \tau|_{(A^\Omega)_e} = 0$. Indeed, suppose on the contrary that

$$\dim \operatorname{Ker} \tau|_{(A^\Omega)_e} \geq 1. \tag{6.7}$$

For $s \geq r-1$ and $v \in V(s)$, we denote by $\varphi_{v,s}\colon A^{B_S(s)} \to A^{B(v,s)}$ and $\varphi_{v,s,\Omega}\colon A^\Omega \to A^{\psi_{v,s}(\Omega)}$ the isomorphisms induced by the bijections $\psi_{v,s}$ and $\psi_{v,s}|_\Omega$, respectively.

Since $\Omega \subset B_S(r-1)$, we can regard $\operatorname{Ker} \tau|_{(A^\Omega)_e}$ as a closed subgroup of $A^{B_S(r-1)} \times \{e\}^{B_S(r)\setminus B_S(r-1)}$. Let us denote $\overline{V'} = \coprod_{v \in V'} B(v,r)$.

As Φ is naturally induced by the local defining map $f\colon A^S \to A$ of τ and as the balls $B(v,r)$, $v \in V'$, are disjoint, we deduce that

$$\{e\}^{V \setminus \overline{V'}} \times \prod_{v \in V'} \varphi_{v,r}(\operatorname{Ker} \tau|_{(A^\Omega)_e}) \subset \operatorname{Ker} \Phi.$$

Consequently, relation (6.7) implies that

$$\dim \operatorname{Ker} \Phi \geq \sum_{v \in V'} \dim \varphi_{v,r}(\operatorname{Ker} \tau|_{(A^\Omega)_e}) = \sum_{v \in V'} \dim(\operatorname{Ker} \tau|_{(A^\Omega)_e}) \geq |V'|. \tag{6.8}$$

The Fiber dimension theorem (see, e.g., [18, Proposition 5.23]) implies that

$$\begin{aligned}
\dim \Phi(A^V) &\overset{(6.8)}{=} \dim A^V - \dim \operatorname{Ker} \Phi \overset{(6.4)}{\leq} |V| \dim A - |V'| \\
&\overset{(6.5)}{\leq} (1-\varepsilon)^{-1}|V(3r)| \dim A - \frac{|V(3r)|}{|B_S(2r)|} \\
&\overset{(6.3)}{\leq} |V(3r)| \dim A \Big((1-\varepsilon)^{-1} - \frac{1}{|B_S(2r)| \dim A} \Big) \\
&< |V(3r)| \dim A = \dim A^{V(3r)}.
\end{aligned}$$

However, since $\Phi(A^V) = A^{V(3r)}$ by (6.6), we arrive at a contradiction. Thus, we have proven the claim that $\dim \operatorname{Ker} \tau|_{(A^\Omega)_e} = 0$.

In what follows, we shall distinguish two cases according to whether

$$\dim H < \dim A^{B_S(r)} \quad \text{or} \quad \dim H = \dim A^{B_S(r)}.$$

Case 1: $\dim H < \dim A^{\Omega}$. Then we infer from (6.1) that

$$\dim \tau((A^{\Omega})_e) = \dim \tau(H_e) < \dim A^{\Omega}.$$

Therefore, the Fiber dimension theorem (cf. [18, Proposition 5.23]) implies

$$\dim \operatorname{Ker} \tau|_{(A^{\Omega})_e} = \dim A^{\Omega} - \dim \tau((A^{\Omega})_e) \geq 1,$$

which is a contradiction since $\dim \operatorname{Ker} \tau|_{(A^{\Omega})_e} = 0$.

Case 2: $\dim H = \dim A^{\Omega}$. Since $\dim \operatorname{Ker} \tau|_{(A^{\Omega})_e} = 0$, it follows from the Fiber dimension theorem (cf. [18, Proposition 5.23]) that

$$\dim \tau(H_e) = \dim \tau((A^{\Omega})_e) = \dim A^{\Omega}.$$

From the decomposition of H into irreducible components (see, for example, [10, Section 2.1]), we can write $H = H' \cup H''$, where H' is the union of irreducible components of H of dimension equal to $\dim H$, and H'' is the union of other irreducible components. Consequently, $\tau(H_e) = \tau(H'_e) \cup \tau(H''_e)$ and $\dim H'' < \dim H$. Moreover, H' contains precisely components which are irreducible components of A^G as H is closed in A^{Ω} and $\dim H = \dim A^{\Omega}$. Since irreducible components of an algebraic group are also connected components, we deduce that H' is a union of some connected components of the algebraic group A^{Ω}.

Note that since $\tau(H_e) = \tau((A^{\Omega})_e)$ is an algebraic group, all of its connected components have the same dimension $\dim \tau((A^{\Omega})_e)$.

On the other hand, since $\dim \tau(H''_e) \leq \dim H'' < \dim A^{\Omega} = \dim \tau((A^{\Omega})_e)$, we deduce that $\dim \tau(H'_e) = \dim \tau((A^{\Omega})_e)$ and also

$$\tau((A^{\Omega})_e) = \tau(H'_e) \cup \tau(H''_e) = \tau(H'_e). \tag{6.9}$$

Since $\Phi\colon A^V \to A^{V(3r)}$ is surjective, the induced map $\Phi_0\colon X_0^V \to X_0^{V(3r)}$ is also a surjective homomorphism (cf. Section 3). Let $Y \subset X$ be the neutral connected component of X and $B = Y(K)$. We deduce from (6.9) that $\tau(H' \times B^{B_S(r) \setminus \Omega})$ has nonempty intersection with every connected component of $\tau(A^{\Omega} \times B^{B_S(r) \setminus \Omega})$. In particular, for every $v \in V'$, we find that

$$\Phi_0((A^{\psi_{v,r}(\Omega)} \times B^{V \setminus \psi_{v,r}(\Omega)})_0) = \Phi_0((\varphi_{v,r,\Omega}(H') \times B^{V \setminus \psi_{v,r}(\Omega)})_0). \tag{6.10}$$

Note that since H' is a union of some connected components of A^{Ω}, we have

$$(\varphi_{v,r,\Omega}(H') \times B^{V \setminus \psi_{v,r}(\Omega)})_0 \in X_0^V.$$

Therefore, in (6.10), the expression $\Phi_0((\varphi_{v,r,\Omega}(H') \times B^{V \setminus \psi_{v,r}(\Omega)})_0)$ is well defined.

For each $v \in V'$, we consider the following subset of $X_0^{B_S(r)}$:

$$I_v = (X_0^{B_S(r)} \setminus (A^{\psi_{v,r}(\Omega)} \times B^{B_S(r) \setminus \psi_{v,r}(\Omega)})_0) \cup (\varphi_{v,r,\Omega}(H') \times B^{B_S(r) \setminus \psi_{v,r}(\Omega)})_0.$$

Then since $(H')_0 \subsetneq X_0^\Omega$ is a proper subset, we deduce that

$$|I_v| \le |X_0^{B_S(r)}| - 1. \tag{6.11}$$

Moreover, since $\overline{V'} = \coprod_{v \in V'} B(v,r)$ is a disjoint union of the balls $B(v,r)$ and since $\psi_{v,r}(\Omega) \subset B(v, r-1)$ for all $v \in V'$, we infer from (6.10) that

$$\Phi_0((A^V)_0) = \Phi_0\Big(X_0^{V \setminus \overline{V'}} \times \prod_{v \in V'} I_v\Big).$$

Taking the cardinality of both sides, we deduce from relations (6.11), (6.5), (6.4), and (6.2) that

$$\begin{aligned}
|\Phi_0(X_0^V)| &\le \Big|X_0^{V \setminus \overline{V'}} \times \prod_{v \in V'} I_v\Big| \overset{(6.11)}{\le} |X_0|^{|V| - |V'||B_S(r)|}(|X_0|^{|B_S(r)|} - 1)^{|V'|} \\
&= |X_0|^{|V|}(1 - |X_0|^{-|B_S(r)|})^{|V'|} \overset{(6.5)}{\le} |X_0|^{|V|}(1 - |X_0|^{-|B_S(r)|})^{\frac{|V(3r)|}{|B_S(2r)|}} \\
&\overset{(6.4)}{<} |X_0|^{|V|}(1 - |X_0|^{-|B_S(r)|})^{\frac{|V|}{2|B_S(2r)|}} \overset{(6.2)}{<} |X_0|^{|V|}|X_0|^{-\varepsilon|V|} \\
&= |X_0|^{(1-\varepsilon)|V|} \overset{(6.4)}{<} |X_0|^{|V(3r)|},
\end{aligned}$$

which is again a contradiction since $\Phi_0(X_0^V) = X_0^{V(3r)}$. Therefore, we can conclude that τ must be $(\bullet)$-pre-injective. The proof of the theorem is thus complete. ■

6.1. A counterexample

Using nontrivial covering maps, we present a simple example which shows that in the class CA_{algr}, the implication

$$\text{post-surjectivity} \Rightarrow \text{pre-injectivity}$$

fails over any universe.

Example 6.2. Let G be a group, and let E be a complex elliptic curve with origin $O \in E$. Consider the algebraic group cellular automaton $\tau \colon E^G \to E^G$ defined by $\tau(c)(g) = 2c(g)$ for every $c \in E^G$ and $g \in G$. We claim that τ is post-surjective but it is not pre-injective.

Indeed, consider the multiplication-by-2 map $\varphi \colon E \to E$, $P \mapsto 2P$. Then φ is a covering map of E of degree 4. Hence, there exists $P \in I \setminus \{O\}$ such that $2P = O$. Consider $c \in E^G$ given by $c(1_G) = P$, and $c(g) = O$ if $g \in G \setminus \{1_G\}$. It is immediate that c and O^G are asymptotic and distinct but $\tau(c) = \tau(O^G) = O^G$. This proves that τ is not pre-injective.

Now let $x, y \in E^G$ such that $y|_{G \setminus \Omega} = \tau(x)|_{G \setminus \Omega}$ for some finite subset $\Omega \subset G$. Since φ is surjective, we can find $p \in E^\Omega$ such that $2p(g) = y(g)$ for all $g \in \Omega$. Consider $z \in E^G$ given by $z|_{G \setminus \Omega} = x|_{G \setminus \Omega}$ and $z|_\Omega = p$; then it is clear that $\tau(z) = y$. This shows that τ is post-surjective.

More generally, we obtain with a similar argument the following example.

Example 6.3. Let G be a group, and let A be a set equipped with a surjective non-injective self map $f: A \to A$. Then the induced cellular automaton $\tau: A^G \to A^G$ given by $\tau(x)(g) = f(x(g))$ for all $x \in A^G$, and $g \in G$ is post-surjective but not pre-injective.

7. Myhill property of CA_{algr}

We shall need the following technical result in the proof of Theorem 7.2.

Proposition 7.1. *Let G be an amenable group, and let $\mathcal{F} = (F_i)_{i \in I}$ be a Følner net for G. Let X be an algebraic group over an algebraically closed field K, and let $A = X(K)$. Suppose that $\tau \in CA_{\text{alg}}(G, X, K)$ is (••)-pre-injective. Then one has*

$$\operatorname{mdim}_{\mathcal{F}}(\tau(A^G)) = \dim(X).$$

Proof. It is a direct consequence of [10, Proposition 6.5]. It suffices to observe there that $CA_{\text{algr}} \subset CA_{\text{alg}}$ and in the class CA_{algr}, the two notions ($**$)-pre-injectivity and (••)-pre-injectivity are in fact equivalent by [21, Proposition 8.3]. ■

We can now state and prove the Myhill property for the class CA_{algr}, which is the content of Theorem B (i).

Theorem 7.2. *Let G be an amenable group, and let X be an algebraic group over K. Suppose that $\tau \in CA_{\text{algr}}(G, X, K)$ is pre-injective. Then τ is surjective.*

Proof. Let $A = X(K)$, and let $\Gamma = \tau(A^G)$. Then it follows from [21, Theorem 5.1] that Γ is closed in A^G with respect to the prodiscrete topology.

Since τ is pre-injective, it is (••)-pre-injective (cf. [21, Proposition 8.3]). We can thus deduce from Proposition 7.1 that

$$\operatorname{mdim}_{\mathcal{F}}(\Gamma) = \dim(X),$$

where $\mathcal{F} = (F_i)_{i \in I}$ is an arbitrary fixed Følner net for G.

Therefore, it follows immediately from Lemma 2.5 and Proposition 2.3 that we have an equality of Krull dimensions $\dim \Gamma_E = \dim X^E$ for every finite subset $E \subset G$.

On the other hand, [11, Theorem 7.1] implies that Γ_E is an algebraic subgroup of A^E for every finite subset $E \subset G$.

Now consider the induced group cellular automaton $\tau_0: X_0^G \to X_0^G$, where the alphabet X_0 is the set of connected components of X (cf. Lemma 3.1). We are going to show that τ_0 is also pre-injective.

Let $f: A^M \to A$, where $M \subset G$ is a finite symmetric subset, be a homomorphism of algebraic groups which is also a local defining map of τ.

Suppose on the contrary that τ_0 is not pre-injective. Consequently, we can find a finite subset $E \subset G$, and subvarieties $V_1, V_2 \subset A^E$, and a subvariety $U \subset A^{ME \setminus E}$ with the following properties:

(a) U is a connected component of $A^{EM \setminus E}$, and V_1, V_2 are distinct connected components of A^E;

(b) the images $\tau_E^+(U \times V_1)$ and $\tau_E^+(U \times V_2)$ belong to the same connected component Z of the algebraic group A^E, where the induced homomorphism $\tau_E^+\colon A^{EM} \to A^E$ of algebraic groups is given by $\tau_E^+(c)(g) = f((g^{-1}c)|_M)$ for every $c \in A^{EM}$ and $g \in E$.

Let us choose an arbitrary point $u \in U$. Then as τ is pre-injective and as $\dim V_i = \dim Z = \dim A^E$, we must have $\tau_E^+(\{u\} \times V_i) = Z$ for $i = 1, 2$.

Indeed, since otherwise we would have $\dim \tau_E^+(\{u\} \times V_i) < \dim Z = \dim V_i$. Note that $\{u\} \times V_i$ is an irreducible variety. Therefore, applying [10, Proposition 2.11], we can find distinct points $s, t \in V_i$ such that $\tau_E^+(u, s) = \tau_E^+(u, t)$. Hence, the map $\tau_E^+|_{\{u\} \times V_i}$ cannot be injective. It follows that τ is not pre-injective, which is a contradiction.

Therefore, for any $z \in Z$, we can find $v_i \in V_i$ for $i = 1, 2$ such that $\tau_E^+(u, v_i) = z$. Since V_1 and V_2 are disjoint, $v_1 \neq v_2$ and it follows that τ is not pre-injective, which is a contradiction. We conclude that τ_0 is indeed pre-injective.

Hence, since the alphabet X_0 is finite and G is an amenable group, we can deduce from the classical Garden of Eden theorem for finite alphabets that τ_0 is surjective.

Let $E \subset G$ be any finite subset. As τ_0 is surjective, we deduce from the definition of τ_0 that Γ_E contains points in every connected component of A^E. On the other hand, we have seen that Γ_E is an algebraic subgroup of A^E such that $\dim \Gamma_E = \dim A^E$. It follows that $\Gamma_E = X^E$ for every finite subset $E \subset G$.

Since the image $\Gamma = \tau(A^G)$ is closed in A^G with respect to the prodiscrete topology, we find that

$$\Gamma = \varprojlim_{E \subset G} \Gamma_E = \varprojlim_{E \subset G} A^E = A^G.$$

It follows that τ is surjective and the proof is complete. ■

Our next result shows that in the class CA_{algr}, the implication

$$(\bullet\bullet)\text{-pre-injectivity} \Rightarrow \text{surjectivity}$$

does not hold in any universe G.

Proposition 7.3. *Let G be a group. Then there exist a finite algebraic group X over K and $\tau \in CA_{\text{algr}}(G, X, K)$ such that τ is $(\bullet\bullet)$-pre-injective but is not surjective.*

Proof. Let X and $\tau \in CA_{\text{algr}}(G, X, K)$ be given by Proposition 4.4. Keep the notations as in the proof of Proposition 4.4. Then we know that τ is $(\bullet\bullet)$-pre-injective but it is not surjective since $\tau(X^G) = Y^G \subsetneq X^G$. The proof is thus complete. ■

8. Moore property of CA_{algr}

To complete the proof of Theorem B, we will prove the following Moore property of the class CA_{algr}.

Theorem 8.1. *Let G be an amenable group, and let X be an algebraic group over an algebraically closed field K. Suppose that $\tau \in CA_{\mathrm{algr}}(G, X, K)$ surjective. Then τ is both $(\bullet)$-pre-injective and $(\bullet\bullet)$-pre-injective.*

Proof. Let $A = X(K)$, and let $\mathcal{F}$ be a Følner net for G. Thanks to Corollary 4.3, it suffices to show that τ is $(\bullet)$-pre-injective. For this, we shall proceed by contradiction.

Suppose that τ is not $(\bullet)$-pre-injective. Thus, there exist a finite subset $E \subset G$ and a proper closed subset $H \subsetneq A^E$ such that

$$\tau((A^E)_e) = \tau(H_e). \tag{8.1}$$

We will distinguish two cases according to whether $\dim H = \dim A^E$.

Case 1: $\dim H < \dim A^E$. By Proposition 2.3, we can find a finite subset $E' \subset G$ such that G contains an (E, E')-tiling T. For every $t \in T$, we define $H(t) \subset A^{tE}$ to be the image of H under the canonical bijective map $A^E \to A^{tE}$ that is induced by the left-multiplication by t^{-1}. Since τ is a G-equivariant homomorphism, we deduce from (8.1) that for each $t \in T$, we have that

$$\tau(A^{tE} \times \{p\}) = \tau(H(t) \times \{p\}) \quad \text{for all } p \in A^{G \setminus tE}.$$

Consider the subset $\Gamma \subset A^G$ defined by

$$\Gamma = A^{G \setminus TE} \times \prod_{t \in T} H(t).$$

We can check that $\tau(A^G) = \tau(\Gamma)$ (cf. [10, proof of Proposition 6.6]). Therefore, we find that

$$\mathrm{mdim}_{\mathcal{F}}(\tau(A^G)) = \mathrm{mdim}_{\mathcal{F}}(\tau(\Gamma)) \overset{[10,\text{ Proposition 5.1}]}{\leq} \mathrm{mdim}_{\mathcal{F}}(\Gamma) \overset{\text{Lemma 2.5}}{<} \dim(X),$$

which contradicts the surjectivity of τ. Observe that the hypothesis of Lemma 2.5 is satisfied since we have $\dim H(t) < \dim A^E$ for all $t \in T$.

Case 2: $\dim H = \dim A^E$. According to whether $\dim \tau((A^E)_e) = \dim A^E$, we distinguish two subcases as follows.

Case 2a: $\dim \tau((A^E)_e) < \dim A^E$. Then Lemma 4.2 tells us that there exists a proper closed subset $Z \subset A^E$ such that $\dim Z < \dim A^E$ and that

$$\tau((A^E)_e) = \tau(Z_e).$$

We are thus in the situation of case 1 and obtain a contradiction.

Case 2b: $\dim \tau((A^E)_e) = \dim A^E$. Hence, we deduce that

$$\dim \tau((A^E)_e) = \dim \tau(H_e) = \dim H = \dim A^E.$$

Let V_i, $i \in I$, be the connected components of the algebraic group A^E, where I is a finite set. As $H \subset A^E$ and $\dim H = \dim A^E$, we can write $H = Z \cup V$, where $V = \bigcup_{j\in J} V_j$ for some $J \subsetneq I$, and Z is a closed subset of A^E such that $\dim Z < \dim A^E$. We find that

$$\tau(H_e) = \tau(Z_e) \cup \tau(V_e).$$

Note that $\tau(H_e) = \tau((A^E)_e)$ is an algebraic group, all of its connected components are therefore irreducible and have the same dimension. But since $\dim \tau(Z_e) \leq \dim Z < \dim A^E = \dim \tau(H_e)$, we deduce immediately that $\tau(H_e) = \tau(V_e)$.

Let us consider the induced cellular automaton $\tau_0 \colon X_0^G \to X_0^G$ where the alphabet X_0 is the set of connected components of X. Let $\varepsilon \in X_0$ denote the connected component of X containing e. We claim that τ_0 is not pre-injective. Indeed, since $J \subsetneq I$ and

$$\tau((A^E)_e) = \tau(H_e) = \tau(V_e) = \tau\Big(\Big(\bigcup_{j\in J} V_j\Big)_e\Big),$$

we find that $\tau_0((X_0^E)_\varepsilon) = \tau_0(Q_\varepsilon)$, where $Q \subset X_0^E$ is the set of connected components of $\bigcup_{j\in J} V_j$. Hence $|Q| = |J|$. Since $|J| < |I| = |X_0^E|$, it follows immediately that the map τ_0 is not pre-injective.

As the alphabet X_0 is finite and the group G is amenable, we deduce from the classical Garden of Eden theorem that τ_0 is not surjective. In particular, we deduce that τ is not surjective. Hence, we also arrive at a contradiction in this case.

Therefore, we can conclude that τ must be $(\bullet)$-pre-injective and the proof of the theorem is complete. ∎

9. Reversibility in CA_{algr}

We have seen in Theorem 5.2 that post-surjectivity implies surjectivity in the classes CA_{alg} and CA_{algr}. On the other hand, pre-injectivity is weaker than injectivity. As shown by Capobianco, Kari, and Taati in [3, Theorem 1], such trade-off between injectivity and surjectivity preserves bijectivity for cellular automata with finite alphabet.

Theorem 9.1 (Capobianco–Kari–Taati). *Let G be a group, and let A be a finite set. Then every pre-injective, post-surjective cellular automaton $\tau \colon A^G \to A^G$ is reversible.*

It turns out that the same property holds for the class CA_{algr} at least in characteristic zero. Moreover, we can show that the inverse is also an algebraic group cellular automaton.

Theorem 9.2. *Let G be a group, and let X be an algebraic group over an algebraically closed field K of characteristic zero. Let $\tau \in CA_{\text{algr}}(G, X, K)$ be a post-surjective, pre-injective group cellular automaton. Then τ is reversible and $\tau^{-1} \in CA_{\text{algr}}(G, X, K)$.*

Proof. Suppose $\tau \in CA_{\text{algr}}(G, X, K)$ is post-surjective and pre-injective. Let $A = X(K)$. Using Lemma 5.3 instead of [3, Lemma 1], we have a similar result as stated in [3, Corollary 2] for the class CA_{algr}. Thus, the exact same construction given in [3, Theorem 1] shows that τ is reversible, i.e., there exists a cellular automaton $\sigma: A^G \to A^G$ such that $\sigma \circ \tau = \tau \circ \sigma = \text{Id}$. In particular, τ is bijective.

Therefore, we can apply directly [21, Proposition 6.2] to see that for some memory set $M \subset G$, the cellular automaton σ admits a local defining map $A^M \to A$ which is a homomorphism of algebraic groups. It follows that $\sigma \in CA_{\text{algr}}(G, X, K)$, and the proof is complete. ■

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