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Groups with minimal harmonic functions as small as you like

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(with an appendix by Nicolás Matte Bon)

Abstract. For any order of growth $f(n) = o(\log n)$, we construct a finitely-generated group G and a set of generators S such that the Cayley graph of G with respect to S supports a harmonic function with growth f but does not support any harmonic function with slower growth. The construction uses permutational wreath products $\mathbb{Z}/2 \wr_X \Gamma$ in which the base group Γ is defined via its properly chosen action on X.

1. Introduction

A harmonic function on a graph is a function f from its vertices into $\mathbb{R}$ such that for every vertex v, $f(v)$ is equal to the average of f over all the neighbours of v. Finite connected graphs have no nonconstant harmonic functions, and infinite ones have, in many interesting examples, surprisingly small families of harmonic functions. For example, on the graph $\mathbb{Z}^d$, the only harmonic functions with polynomial growth are polynomials [12]. When the graph is the Cayley graph of some finitely generated group, it is natural to try to relate properties of harmonic functions to properties of the group. Graphs for which there are no nonconstant bounded harmonic functions are called Liouville graphs, and they are deeply connected with random walk entropy and amenability [1, 3, 9, 11, 14]. In a different regime, harmonic functions with linear growth were used by Kleiner to give a new proof of Gromov's famous polynomial growth theorem [15, 24].

There is an interesting quantitative version of the Liouville question which goes as follows: for a given Cayley graph, what is the largest $f:\mathbb{N} \to [0,\infty)$ such that any harmonic function h with $h(x) = o(f(\mathrm{dist}(1,x)))$ is constant? (1 is of course the identity element of the group, and dist is the graph distance in the Cayley graph.) This question was addressed in [4] where a number of examples were analysed. In particular, for the two-dimensional lamplighter group $\mathbb{Z}/2 \wr \mathbb{Z}^2$, it was shown that it supports a harmonic function with logarithmic growth, but that any h with $h(x) = o(\log(\mathrm{dist}(1,x)))$ is constant. Surprisingly, though, it turns out that this cannot be changed by using more complicated lamps. Indeed, it was shown in [4] that for $G = (\cdots(\mathbb{Z}^2 \wr \mathbb{Z}^2) \wr \mathbb{Z}^2)\cdots \wr \mathbb{Z}^2$, the same behaviour holds,

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namely, any sublogarithmic harmonic function must be constant (in sharp contrast to the behaviour of random walk return probabilities and entropy on G).

In this paper, we construct examples of groups with any f between log and constant. Here is the precise statement.

Theorem. *Let f be a positive C^1 function on $[1,\infty)$ such that $f(x)\to\infty$ and such that $xf'(x)$ is decreasing. Then there exist a finitely generated group G and a finite, symmetric set of generators S such that the Cayley graph $\mathrm{Cay}(G;S)$ has the following properties:*

(1) *There exists a nonconstant harmonic function h on $\mathrm{Cay}(G;S)$ such that $|h(x)|\le Cf(\mathrm{dist}(1,x))$.*

(2) *Any harmonic function h on $\mathrm{Cay}(G;S)$ with $h(x)=o(f(\mathrm{dist}(1,x))$ is constant.*

It is a famous open problem whether the Liouville property is a group property, i.e., whether it is possible for the same group to have two sets of generators with respect to which one Cayley graph is Liouville and the other is not. (It is known that the Liouville property is not *quasi-isometrically invariant* for general graphs, see [2, 19].) In our case too, we do not know whether the minimal growth rate of a harmonic function is a group property or it might depend on the generators. For the *specific* groups we are constructing, though, it is possible to show that any set of generators has the same minimal growth of harmonic functions. We will not prove it, though, as it only adds technical complications to the proof.

Let us warn the reader against confusing a harmonic function of minimal growth with a minimal harmonic function. A positive harmonic function h is called minimal if any other positive harmonic function g with $g(x)\le h(x)$ for all x is a multiple of h by a constant. Such functions play a role in the construction of the Martin boundary of a group. Surprisingly, perhaps, minimal harmonic functions in fact grow very fast. For example, in $\mathbb{Z}/2\wr\mathbb{Z}$ they actually grow exponentially fast. A minimal harmonic function h on $\mathbb{Z}/2\wr\mathbb{Z}$ will have a very specific "direction" in which it decays exponentially fast and this prevents any other harmonic function from being smaller than h *everywhere*, but in a typical direction h will increase exponentially. We will not prove these claims as they are somewhat off topic, but they follow in a more or less straightforward manner from the description of minimal harmonic functions using the Martin kernel.

Our construction uses *permutational wreath products* – an approach with a very successful track record in constructing groups with interesting behaviour – but with the following twist. A permutational wreath product (exact definitions will be given below) starts from a group acting on a set X. In most constructions so far the groups were automaton groups, and these act naturally on certain sets such that the result is a "graphical fractal". These groups and their actions have been studied extensively, and the construction requires deep knowledge of this theory. Here, instead of starting with a well-studied group, we start with a graph (which we also denote by X, the group will act on its vertices). We colour the edges of the graph such that any vertex is incident to all colours, and then use the colouring to construct a group acting on X. Each colour (say azure) will correspond to the permutation of X given by taking every vertex x to the vertex on the other side of the edge e incident to x and coloured azure. The group will simply be the group of permutations generated by all the colours. We shall see that if the colouring is chosen to have enough repetitions, then the group may be analysed directly and quite simply, with no need for the heavy combinatorial analysis typically associated with automaton groups. While not a truly different technique, more of a different way to think about existing techniques, we believe it is useful, and in fact it has already been used in [16].

2. Preliminaries

2.1. Graph and random walk preliminaries

For a graph G and two vertices x and y, we denote by $x \sim y$ the case where (x, y) is an edge of G, and say that x and y are neighbours. By $d(x, y)$ or $\operatorname{dist}(x, y)$ we denote the graph distance between them, i.e., the length of the shortest path between them in the graph (if one exists; ∞ otherwise). We denote by $B(x, r)$ the closed ball $B(x, r) = \{y : d(x, y) \le r\}$. If we need to stress that this is taken in some graph G, we shall denote the ball by $B_G(x, r)$. The sphere will be denoted by ∂B, i.e., $\partial B(x, r) = \{y : d(x, y) = r\}$. We shall also use G to denote the set of vertices of G, so $x \in G$ means that x is a vertex of G. If we need the set of edges of G, we shall denote it by $E(G)$. For two graphs G and H, we denote by $G \times H$ the standard graph product, i.e., the graph with vertex set $\{(g, h) : g \in G, h \in H\}$ and with $(g, h) \sim (g', h')$ if and only if $g = g'$ and $h \sim h'$ or $g \sim g'$ and $h = h'$.

A weighted graph is a pair (G, m), where $m : E(G) \to (0, \infty)$ is called the weight function. We consider every graph also as a weighted graph with $m \equiv 1$. Similarly, a multigraph (i.e., a graph where multiple edges are allowed between vertices) is considered as a weighted graph with $m(x, y)$ being the number of edges between x and y. The Laplacian Δ of a weighted graph G is the operator on $\ell^2(G)$ defined by

$$(\Delta f)(x) = \sum_{y \sim x} m(x, y)(f(x) - f(y)).$$

For a weighted graph G, the simple random walk on G is the stochastic process on the vertices of G which, whenever it is in some vertex x, moves to any neighbour y of x with probability proportional to $m(x, y)$. In particular, if the graph is simple, it chooses among the neighbours with equal probability. For an $x \in G$ and an $A \subset G$, the hitting time of A (from x) is the random time

$$\min\{t \in \{0, 1, 2, \dots\} : R(t) \in A\},$$

where R is the simple random walk on G with $R(0) = x$. If R never hits A, we consider the minimum to be ∞.

For a weighted graph G and two sets A, $B \subset G$, we define $\operatorname{Res}(A, B)$ to be the electrical resistance between them, i.e., construct an electrical network where each edge has resistance 1, and where the sets A and B are fused to be one point each, and then measure the resulting effective resistance between these two points. Formally, the definition is as follows. Assume first that G is finite and connected, and that each of A and B is a point. Find the function $h\colon G \to \mathbb{R}$ satisfying $\Delta h = \delta_A - \delta_B$ (it is unique up to addition of constants) and define

$$\operatorname{Res}(A, B) = h(B) - h(A).$$

For general graphs, take the resistance inside finite concentric balls, take the radius to infinity and define $\operatorname{Res}(A, B)$ to be the limit of the finite resistances (by Rayleigh monotonicity [10, §1.4], this is a decreasing series so the limit exists). If A and B are sets, define $\operatorname{Res}(A, B)$ by identifying each one to a point (adding up weights if multiple edges are generated) and measuring the resistance on the identified graph. For more details see, say, the book [10].

We shall consider the effective resistance between a point and the boundary of a ball around it with radius R, and the main property of effective resistance that we shall use is that this resistance is inversely proportional to the probability that a random walk starting from the point will hit the boundary of the ball before returning. See [10, §1.3.4] for more details. Other properties of electrical resistance will only be used to estimate the resistance in the Schreier graphs (Lemmas 4.7 and 3.5).

We denote by C and c constants, which might change from place to place or even within the same formula. Throughout C will denote constants which are sufficiently large, and c will denote constants which are sufficiently small. The constants would depend only on the graph at hand (usually denoted by T) unless otherwise specified. The notation $X \approx Y$ is short for $cX \le Y \le CX$.

2.2. Group preliminaries: Cayley and Schreier graphs, permutational wreath products

Let X be a set, and let G be a group acting on X from the right (denoted by $X \curvearrowleft G$). We denote the action of a $g \in G$ on an $x \in X$ by $x.g$ (so, of course, $x.gh = (x.g).h$). For a (finite, symmetric) subset $S \subset G$, the right Schreier graph $\operatorname{Sch}(X; S)$ is the graph whose vertices are X and whose edges are all $(x, x.s)$ for all $x \in X$ and $s \in S$. The (right) Cayley graph $\operatorname{Cay}(G; S)$ is the Schreier graph of the action of G on itself by right multiplication.

The wreath product $\mathbb{Z}/2 \wr_X G$ is the group

$$(\mathbb{Z}/2)^X \rtimes G,$$

where the action of G on $(\mathbb{Z}/2)^X$ implicit in the notation $\rtimes$ is $g(\omega)(x) = \omega(x.g)$. In other words, the product on $\mathbb{Z}/2 \wr_X G$ is

$$(\omega, g) \cdot (\omega', g') = (\omega + \omega'(\cdot.g), gg').$$

The action $X \curvearrowleft G$ induces an action $X \curvearrowleft \mathbb{Z}/2 \wr_X G$ by having the G component act on X, and the ω component doing nothing at all.

Lemma 2.1. *Let $X \curvearrowleft G$ and let $S \subset G$ be finite. Let $o \in X$ and let $a\colon X \to \mathbb{R}$ satisfy $\Delta a = \delta_o$, where Δ is the Laplacian of* $\operatorname{Sch}(X;S)$ *and δ_o is the Kronecker delta at o. Assume $a(o) = -1/2$. Then the function*

$$f(\omega, g) = (-1)^{\omega(o)} \cdot a(o.g)$$

is (right) harmonic on $\mathbb{Z}/2 \wr_X G$ with respect to the switch-or-move generators, that is, $\{(\delta_o, 1)\} \cup \{(0,s)\colon s \in S\}$.

Here and below, when we write $(-1)^{\omega(o)} a(o.g)$, we consider the group $\mathbb{Z}/2$ to be embedded in $\mathbb{R}$ as $\{0,1\}$, and $(-1)^{\omega(o)} a(o.g)$ is then just powering and multiplication in $\mathbb{R}$. Some readers might benefit from thinking about a as the *harmonic potential* (see, e.g., [23]) at o, though we are not making any requirement that a be minimal in any sense.

Proof. This is a straightforward, if confusing, calculation. Denote by $\Delta_\wr$ the Laplacian of $\mathbb{Z}/2 \wr_X G$ with respect to the switch-or-move generators. We first examine g for which $o.g \neq o$. For such g, we have

$$\begin{aligned}
(\Delta_\wr f)(\omega, g) &\overset{(*)}{=} f(\omega,g) - f(\omega + \delta_{o.g^{-1}}, g) + \sum_{s\in S}(f(\omega,g) - f(\omega, gs)) \\
&\overset{(**)}{=} (-1)^{\omega(o)} a(o.g) - (-1)^{\omega(o)} a(o.g) \\
&\qquad + \sum_{s\in S}((-1)^{\omega(o)} a(o.g) - (-1)^{\omega(o)} a(o.gs)) \\
&\overset{(***)}{=} (-1)^{\omega(o)} (\Delta a)(o.g) = 0,
\end{aligned}$$

where $(*)$ follows from the definitions of $\mathbb{Z}/2 \wr_X G$ and $\Delta_\wr$; $(**)$ follows from the definition f and from the fact that if $o.g \neq o$, then $o.g^{-1} \neq o$ and then $(\omega + \delta_{o.g^{-1}})(o) = \omega(o)$; and in $(***)$, Δ stands for the Laplacian on $\operatorname{Sch}(X;S)$.

In the case that $o.g = o$, we get a similar formula, except that for the generator $(\delta_o, 1)$, we have $(\omega, g)(\delta_o, 1) = (\omega + \delta_o, g)$ which inverts the sign at o. Hence

$$\begin{aligned}
(\Delta_\wr f)(\omega, g) &= (-1)^{\omega(o)} a(o) + (-1)^{\omega(o)} a(o) + \sum_{s\in S}((-1)^{\omega(o)} a(o) - (-1)^{\omega(o)} a(o.s)) \\
&= (-1)^{\omega(o)} (2a(o) + (\Delta a)(o)) = 0
\end{aligned}$$

since we assumed $\Delta a(o) = 1$ and $a(o) = -1/2$. ∎

3. Spherically symmetric trees

As explained in the introduction, we shall construct a group from a coloured graph, and properties of the random walk on that graph will allow us to infer properties of the group. We now reveal the nature of the graph in question: it will be a product of a spherically symmetric tree with $\mathbb{Z}$. We therefore start with some properties of spherically symmetric trees and random walks on them.

A tree is a graph with no cycles. For a tree T and a marked vertex o (also called the root of T), we say that T is spherically symmetric (with respect to o) if the degree of a vertex depends only on its graph distance from o. For a spherically symmetric tree T with all degrees of all vertices 2 or 3 (except the root which can have degree 1 or 2), we call the points of degree 3, branch points. The root of T is considered a branch point if its degree is 2.

Definition 3.1. Let $\mathcal{T}$ be the family of spherically symmetric trees with degrees as in the previous paragraph such that the distances b_i of the branch points from o satisfy $\inf_i b_{i+1}/b_i > 2$.

Lemma 3.2. *Let $T \in \mathcal{T}$. Then for any $r \in \mathbb{N}$ and any $x \in B(o, r)$, the expected hitting time of o from x in the graph $B(o, r)$ is $\le Cr^2$.*

Proof. Let H be the graph given by taking $B(o, r)$ and "projecting it on $\mathbb{Z}$", i.e., for every $h \le r$, identifying all the vertices with distance h from o (so H would have multiple edges). By spherical symmetry, the hitting time of o in H is exactly as in $B(o, r)$. To estimate the hitting time in H, we use the commute-time identity, which states that in any finite graph G and for any two vertices x and y, we have $C(x, y) = \mathrm{Res}(x, y) \cdot |E(G)|$, where $C(x, y)$, the commute time, is the time a random walk starting from x takes to hit y and then return to x. See [6] for the proof of this identity. Let

$$\ell = \max\{2^i : b_i < r\},$$

i.e., the number of branches of the tree at r (in particular, we define $\ell = 1$ in the case that $b_1 \ge r$). Hence $|E(H)| \le \ell r$. The resistance between x and o can be bounded above by the resistance to the leaves, for which we have the formula

$$\mathrm{Res}(o, \text{leaves}) = b_1 + \sum \frac{\min(b_{i+1}, r) - b_i}{2^i} \le C\frac{r}{\ell},$$

where C depends only on $\min b_{i+1}/b_i$. We get that the commute-time is $\le Cr^2$ which of course bounds the hitting time in H and hence also in $B(o, r)$. ∎

Lemma 3.3. *Let $T \in \mathcal{T}$. Then $T \times \mathbb{Z}$ satisfies a "spherically symmetric Harnack inequality", i.e., there exists C such that for any $x \in T \times \mathbb{Z}$, any $r > 0$ and any h which is positive harmonic in $B(x, 2r)$ and spherically symmetric, $\max_{B(x,r)} h \le C \min_{B(x,r)} h$.*

Before starting the proof proper, let us give the main idea: we will show the Harnack inequality by showing a *Poincaré inequality*, which, under volume doubling, implies the Harnack inequality. The Poincaré inequality is essentially equivalent to an inequality for the second eigenvalue of the Laplacian. This, in turn, follows from the hitting time estimate of Lemma 3.2. This chain of conclusions allows to prove a Harnack inequality without calculating pointwise exit probabilities, which would have probably been longer and more technical. The Poincaré inequality will be reused later in the proof of Lemma 3.6.

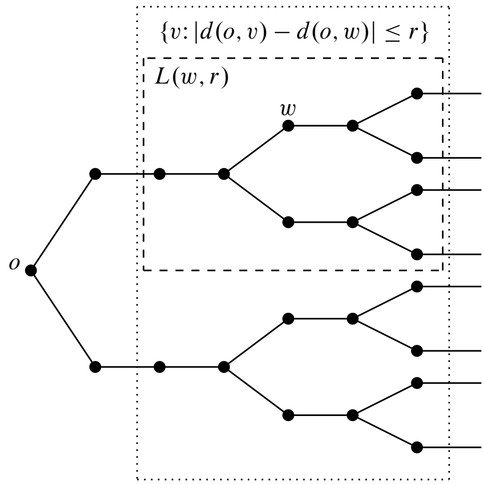

Figure 1. The subtree $L(w,r)$.

Proof. For any $w \in T$ and any r, let $L(w,r)$ be the following subgraph of T: we take all $v \in T$ such that $|d(o,v) - d(o,w)| \le r$, examine the induced subgraph of T and take the connected component of w; see Figure 1.

We first want to bound the second eigenvalue λ_L of the Laplacian on L. We note that L is itself a tree and satisfies the conditions of Lemma 3.2. Hence the hitting time of its root is bounded by Cr^2 (strictly speaking, we apply Lemma 3.2 for an extension of L to an infinite tree in $\mathcal{T}$). This bound for the hitting time is well known to bound the mixing time of L, say by the equivalence of the mixing time and the forget time (see [17]) or by [20, Corollary 1.2]. Since the mixing time bounds the inverse of the second eigenvalue, we get that $\lambda_L \ge cr^{-2}$.

Next we examine $Q = L \times \{-r, \dots, r\}$. Since the eigenvalues of a graph product are simply sums of all couples of eigenvalues of the two factors, we get a similar estimate for the second eigenvalue of Q, i.e., $\lambda_Q \ge cr^{-2}$.

We now rewrite this inequality in a functional form that resembles the Poincaré inequality. For this, we first note that the first eigenvalue of the Laplacian is 0, and corresponds to the eigenvector which is constant 1. Hence by the minimax representation of eigenvalues (note that the Laplacian is self-adjoint), we have

$$\|f\|^2 \le Cr^2 \langle \Delta f, f\rangle \quad \forall f \text{ such that } \langle f, 1\rangle = 0.$$

A simple resummation shows that $\langle \Delta f, f\rangle = \sum |f(x) - f(y)|^2$, where the sum is over the edges of Q. The condition $\langle f, 1\rangle = 0$ can be removed by subtracting the average of f. Hence we arrive at (in weighted graph notation)

$$\sum_{x\in Q} m(x)|f(x) - f_Q|^2 \le Cr^2 \sum_{(x,y)\in E(Q)} m(x,y)|f(x) - f(y)|^2, \tag{3.1}$$

where $m(x, y)$ is 1 when $x \sim y$ and 0 otherwise, $m(x) = \sum_{y \in Q} m(x, y)$, $E(Q)$ is the set of edges of Q and $f_Q = (1/|Q|) \sum_{x \in Q} f(x)$.

We now "flatten" $T \times \mathbb{Z}$. By this we mean that we define a new weighted graph F. Set $F = \mathbb{Z}^+ \times \mathbb{Z}$ and define $\pi : T \times \mathbb{Z} \to F$ by $\pi(x, n) = (d(x, o), n)$. We consider π also as a map from $E(T \times \mathbb{Z})$ to $E(F)$. We then define the weight of each edge e in F to be $|\pi^{-1}(e)|$ (we denote the weights in F by $m(x, y)$ and $m(x)$ too). We remark that F is not a product graph itself. Inequality (3.1) translates to a Poincaré inequality for squares in F. Indeed, a square of F is lifted to a disjoint collection of copies of Q. Hence to show the Poincaré inequality, we take a function f on the square, lift it (i.e., consider $f \circ \pi$) to said disjoint collection, use (3.1) for each copy and sum.

Next we show that F satisfies the volume doubling condition, i.e.,

$$m(B(x, 2r)) \le C m(B(x, r)) \quad \forall x, r,$$

where $B(x, r)$ is a ball in F (in the usual graph metric which ignores the weights), and where for any set of vertices S, $m(S) = \sum_{x \in S} m(x)$. The easiest way to see this is to start with T and note that

$$m(\{v \in T : |d(o, v) - d(o, w)| \le 2r\}) \le C m(\{v \in T : |d(o, v) - d(o, w)| \le r\})$$

(this is a straightforward calculation, though we note that it uses the condition that the branching points b_i satisfy $b_{i+1}/b_i \ge 2$), conclude the same after multiplying the left-hand side by $\{-2r, \dots, 2r\}$ and the right-hand side by $\{-r, \dots, r\}$, and then by flattening for squares in F. Moving from squares to balls follows by inscribing $B(x, 2r)$ in a square of side length $2r + 1$ and inscribing in $B(x, r)$ a square of side length $r/2$.

Similarly, we need to move the Poincaré inequality from squares to balls, since the usual weak Poincaré inequality is stated for balls, namely

$$\sum_{x \in B(z,r)} m(x) |f - f_{B(z,r)}|^2 \le C r^2 \sum_{(x,y) \in E(B(z,2r))} m(x, y) |f(x) - f(y)|^2 \quad \forall z, r$$

(it is called "weak" because the ball on the left has radius r and on the right has radius $2r$). Nevertheless, it is well known that the squares and balls versions are equivalent under volume doubling (see, e.g., [13, §5], the setting there is a little different but the proof is the same).

The purpose of all these manoeuvres was to be able to apply Delmotte's theorem [8] which states that the weak Poincaré inequality and volume doubling imply Harnack's inequality. Lifting to $T \times \mathbb{Z}$ gives a Harnack inequality for spherically symmetric functions. ■

Lemma 3.4. *Let $T \in \mathcal{T}$, and let $T \times \mathbb{Z} \curvearrowleft G$. Then there exists a harmonic function u on $\mathbb{Z}/2 \wr_{T \times \mathbb{Z}} G$ such that*

$$u(x) \le C \operatorname{Res}_{T \times \mathbb{Z}}(d(o, x)),$$

where $\operatorname{Res}_{T \times \mathbb{Z}}(n)$ is the electrical resistance in $T \times \mathbb{Z}$ from o to $\partial B(o, n)$.

Proof. Let r be arbitrary, and let $a = a_r \colon B_{T\times\mathbb{Z}}(o,r) \to \mathbb{R}$ satisfy that Δa restricted to $B(o, r-1)$ is δ_o, $a(o) = 0$ and is constant on $\partial B(o,r)$. Here and below, $o = o_{T\times\mathbb{Z}} = (o_T, 0)$. Since these conditions define a uniquely, it is spherically symmetric.

Let $s < r/2$ and examine the random walk X_t on $B(o,s)$ starting from o. Let τ_s be the hitting time of $\partial B(o,s)$, τ_o^+ the first return time to o and $\tau = \min(\tau_o^+, \tau_s)$. Since $a(X_t)$ is a (bounded) martingale for $1 \le t \le \tau$, we have

$$\mathbb{E}(a(X_1)) = \mathbb{E}(X_\tau) = \mathbb{P}(\tau = \tau_s)\mathbb{E}(a(X(\tau_s)) \mid \tau = \tau_s) + \mathbb{P}(\tau = \tau_o^+)a(o).$$

Now $a(o) = 0$, $\mathbb{E}(a(X_1)) = 1$ since $\Delta a(o) = 1$ and by the Markov property,

$$\mathbb{E}(a(X(\tau_s)) \mid \tau = \tau_s) = \mathbb{E}(a(X(\tau_s))).$$

Thus

$$\operatorname{Res}(s) = \frac{1}{\mathbb{P}(X \text{ hits } \partial B(o,s) \text{ before returning to } o)} = \mathbb{E}(a(X(\tau_s))). \tag{3.2}$$

By Harnack's inequality (Lemma 3.3), all values of a on $\partial B(o,s)$ are equal up to constants, and hence are also equal to the expectation in (3.2) up to constants. We get

$$a_r(x) \le C \operatorname{Res}(d(o,x)) \quad \forall d(o,x) \le \frac{1}{2}r.$$

This means that we may take a pointwise converging subsequence a_{r_k}. The limit a_∞ satisfies $\Delta a_\infty = \delta_o$ and $a_\infty(x) \le C \operatorname{Res}(d(o,x))$. An appeal to Lemma 2.1 finishes the proof (we use Lemma 2.1 with $a_{\text{Lemma 2.1}} = a_\infty - 1/2$). ■

Lemma 3.5. *For any $T \in \mathcal{T}$,*

$$\operatorname{Res}_{T\times\mathbb{Z}}(n) \approx \sum_{k=1}^{n} \frac{1}{k\ell(k)},$$

where $\ell(k)$ is the number of branches of T at level k.

Proof. For the lower bound, we use the fact that identifying vertices only decreases the resistance (see, e.g., [10, §2.2]). Let S_n be the square

$$S_n = \{(t,k) \in T \times \mathbb{Z} \colon d(o,t) = n,\ |k| \le n \text{ or } k = \pm n,\ d(o,t) \le n\},$$

and note that

$$S_{N/2} \subset B_{T\times\mathbb{Z}}(o,N).$$

Let E_n be the number of edges between S_{n-1} and S_n. We identify all vertices in S_n, for all n, and get

$$\operatorname{Res}_{T\times\mathbb{Z}}(N) \ge \operatorname{Res}_{T\times\mathbb{Z}}(o, S_{N/2}) \ge \sum_{n=1}^{N/2} \frac{1}{E_n}.$$

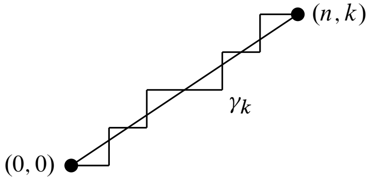

Figure 2. The path γ_k is a discretisation of the line.

To estimate E_n, examine edges between the two different pieces of S_n. The number of edges between the part $\{d(o,t) = n\}$ in S_{n-1} and S_n is $(2n+1)\ell(n)$. The number of edges between the other part of S_{n-1} is (twice) the size of the tree up to level n, i.e.,

$$2\sum_{k=1}^{n} \ell(k) \le 2n\ell(n).$$

The difference between summing $1/E_n$ up to N and up to $N/2$ is no more than a constant. This proves the lower bound.

For the upper bound, we construct a unit flow from $(o,0)$ to S_n (in fact, only to the "vertical" part of S_n, but this is not important) and use the fact that the energy of any unit flow from $(o,0)$ to S_n is an upper bound on $\mathrm{Res}_{T\times\mathbb{Z}}(o, S_N)$ (see, e.g., [10, §1.3.5]). Let $t \in T$ satisfy $d(o,t) = n$, and let $|k| \le n$. Examine the line in $\mathbb{R}^2$ from $(0,0)$ to (n,k). Discretise it to a path γ_k in $\mathbb{Z}^2$ which does not go left, and such that any edge of γ_k is at distance ≤ 1 from the line (if there is more than one way to do it, choose one arbitrarily). See Figure 2. Translate γ_k to a collection of $\ell(n)$ paths in $T \times \mathbb{Z}$ as follows: assume the vertex $(a,b) \in \mathbb{Z}^2$ translates to a $(t,b) \in T \times \mathbb{Z}$. If the next edge in γ_k is vertical, translate $(a, b \pm 1)$ (as the case might be) to $(t, b \pm 1)$. If it is horizontal, translate $(a+1, b)$ to (t', b), where t' is one of the children of t in T. This gives $\ell(n)$ different paths in $T \times \mathbb{Z}$. Send $1/((2n+1)\ell(n))$ mass through each such path. This is the desired flow.

Let us now calculate the energy of this flow. Since $T \times \mathbb{Z}$ has bounded degree, we may examine flows through vertices instead of through edges, losing only a constant. Examine therefore a vertex (t,b), and assume $|b| \le d(o,t) + 2$, since otherwise the flow is zero. To check how many paths go through it, note that the line in $\mathbb{R}^2$ must pass no further than distance 2 from it. This restricts to an angle of opening $\le C/d(o,t)$ and hence, there are no more than $C/d(o,t)$ of the flow in such paths. Further, the flow divides evenly between the different $\ell(d(o,t))$ branches at this level, so the flow through each particular branch is $1/\ell(d(o,t))$ of the total flow. We get that the flow is bounded by $C/d(o,t)\ell(d(o,t))$. Summing the squares gives

$$\mathrm{Res}_{T\times\mathbb{Z}}(o, S_N) \le \sum_{\substack{(t,b):\\ |b|-2\le d(o,t)\le N}} \frac{C}{d(o,t)^2\ell(d(o,t))^2} \le \sum_{n=1}^{N} \frac{C}{n\ell(n)}.$$

This finishes the proof of the lemma. ∎

Lemma 3.6. *Let $T \in \mathcal{T}$, let $x, y \in T \times \mathbb{Z}$, and let $r > d(x, y)$. Let τ be the hitting time of y by a random walk on $T \times \mathbb{Z}$ started from x. Let τ' be an independent exponential variable with expectation r^2. Then*

$$\mathbb{P}(\tau > \tau') \approx \frac{1}{\operatorname{Res}_{T\times\mathbb{Z}}(r)},$$

where the constants implicit in the $\approx$ might depend on x and y.

Proof. Since the claim is trivial in the case that $T \times \mathbb{Z}$ is transient, we shall assume it is recurrent, i.e., $\operatorname{Res}(r) = \operatorname{Res}_{T\times\mathbb{Z}}(r) \to \infty$ as $r \to \infty$. We may assume that r is sufficiently large. Recall from the proof of Lemma 3.3 that the flattening F of $T \times \mathbb{Z}$ satisfies volume doubling and Poincaré's inequality. Hence, by Delmotte's theorem [8], the random walk on F satisfies Gaussian bounds

$$\mathbb{P}^x(R_t = y) \le \frac{Cm(y)}{m(B(x, \sqrt{t}))} \exp\Big(-\frac{cd(x, y)^2}{t}\Big), \tag{3.3}$$

where m is as in the proof of Lemma 3.3 as well. In particular, we may conclude that $\mathbb{P}^x(d(x, R_t) > \lambda\sqrt{t}) \le C \exp(-c\lambda^2)$. From this we get (using a simple dyadic decomposition of $[0, t]$) the same bound for $\max_{s\le t} d(x, R_s)$. This last bound extends from F back to $T \times \mathbb{Z}$ since if $R = R[0, t]$ is a random walk on $T \times \mathbb{Z}$ and if the image of R in F stayed in a ball of radius $\lambda\sqrt{t}$ throughout the time interval $[0, t]$, then R must have stayed in a ball of radius $C\lambda\sqrt{t}$.

Another fact we ask the reader to recall, this time from Lemma 3.4, is that the probability that a random walk starting from x hits y before hitting $\partial B(y, s)$ is $\approx 1/\operatorname{Res}(s)$ (recall that all $\approx$ signs and all c and C may depend on the points x and y).

With these two facts, we first conclude the lower bound on the probability. With probability $\approx 1/\operatorname{Res}(r)$, the random walk reaches $\partial B(y, r)$ before hitting y. It then has probability $> c$ to walk cr^2 time without getting to distance bigger than $r/2$, which of course prevents it from returning to y during this time interval. During this period of length cr^2, there is a positive probability that τ' will occur. Hence

$$\mathbb{P}(\tau > \tau') > \frac{c}{\operatorname{Res}(r)}.$$

For the other direction, we first note that

$$\mathbb{P}\Big(\tau' < \frac{r^2}{\operatorname{Res}(r)}\Big) \le \frac{1}{\operatorname{Res}(r)}.$$

Denote this event by $\mathcal{B}_1$. Next we sum (3.3) over $B(y, s)$ for some $s \le r/\sqrt{\operatorname{Res}(r)}$ to claim that, for the random walk on our flattened graph F,

$$\mathbb{P}^x(d(R_{r^2/\operatorname{Res}(r)}, y) < s) \le \frac{Cm(B(y, s))}{m(B(x, r/\sqrt{\operatorname{Res}(r)}))}.$$

Examining only the $\mathbb{Z}$ direction gives $m(B(y,s)) \le m(B(y,s')) \cdot (s/s')$, so

$$\mathbb{P}^x\Big(d(R_{r^2/\operatorname{Res}(r)}, y) < \frac{r}{\operatorname{Res}(r)^{3/2}}\Big) \le \frac{C}{\operatorname{Res}(r)}.$$

This estimate extends from F to $T \times \mathbb{Z}$ as distances in $T \times \mathbb{Z}$ are no less than the distances in the projection to F. Denote this event by $\mathcal{B}_2$ and put $s = r/\operatorname{Res}(r)^{3/2}$.

Thus if neither of $\mathcal{B}_1$, $\mathcal{B}_2$ occurred, then the random walk exited a ball of radius s before time τ'. Adding the estimate for the probability for exiting a ball of radius s before τ mentioned above gives

$$\mathbb{P}(\tau > \tau') \le \frac{C}{\operatorname{Res}(r)} + \frac{C}{\operatorname{Res}(s)}. \tag{3.4}$$

Applying Lemma 3.5 twice gives

$$\operatorname{Res}(s) \ge c\sum_{n=1}^{s} \frac{1}{n\ell(n)} \ge c\sum_{n=1}^{r} \frac{1}{n\ell(n)} - C\log\frac{r}{s} \ge c\operatorname{Res}(r) - C\log\frac{r}{s}.$$

We insert this into (3.4) to get

$$\mathbb{P}(\tau > \tau') \le \frac{C}{\operatorname{Res}(r)} + \frac{C}{\operatorname{Res}(r) - C\log\operatorname{Res}(r)}.$$

The lemma is proved. ■

4. Proof of the theorem

At this point, we set out to construct an action of some group G on a tree T, and we need G to be "small" in a sense to be prescribed later.

Definition 4.1. Let $T \in \mathcal{T}$. Colour the edges of T with three colours, azure, bordeaux and chartreuse such that no two neighbouring edges have the same colour. Edges between $b_i + 1$ and b_{i+1} will be coloured azure and bordeaux alternatingly, with all edges of a fixed distance from o having the same colour, and chartreuse will be used in the branching points for one of the children. See Figure 3.

Let G be the subgroup of the group of permutations of T (acting from the right) generated by the following three involutions termed a, b and c. The generator a, for example, maps every vertex x to the vertex on the other end of the edge coloured azure containing x, or, if no such edge exists, maps x to itself (if you prefer, add loops to all vertices of T of degree 2 and colour them in the missing colour). We call G the coloured involution group of T.

Definition 4.2. We define a subpolynomially growing tree to be a $T \in \mathcal{T}$ such that

$$|B(o,r)| = r^{1+o(1)}.$$

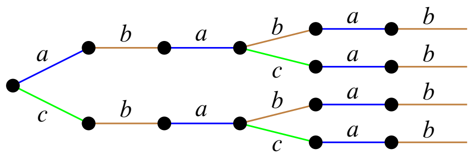

Figure 3. Colouring by azure, bordeaux and chartreuse, marked by a, b and c, respectively, for readers with a monochrome copy.

Lemma 4.3. *Let T be a subpolynomially growing tree. Then for every r, the number of different coloured graphs that can arise as balls of radius r in T is $r^{1+o(1)}$.*

Proof. Let $x \in T$ and examine the number of branch points in $B(x, r)$. If there are none, then the ball is a line and there are exactly 2 possibilities for the colouring. If there is one, then there are $r + 1$ possibilities for the distance of x from the branch point, 3 possibilities for the "side" of x (father's side, chartreuse child's side or non-chartreuse child's side), and two more possibilities which depend on whether the non-chartreuse child of the branch point is azure or bordeaux. All in all, we get no more than $12r + 12$ possibilities.

Finally, if there is more than one branch point, then we have two branch points with distance less than $2r$ between them. Since $b_{i+1}/b_i > 2$, this means that both branch points are no further than $4r$ from the root, and x is no further than $5r$ from the root. Since T is subpolynomially growing, the number of possibilities for this is no more than $r^{1+o(1)}$. ■

Lemma 4.4. *Let T be a subpolynomially growing tree, and let G be its coloured involution group. Then G satisfies that $H_n \le n^{1/2+o(1)}$, where H_n is the entropy of simple random walk on the Cayley graph of G with respect to the generators a, b and c, i.e., $H_n := H(X_n)$, where X_n is the random walk and H is the Shannon entropy.*

Proof. Examine the random walk X_n on G. For every $x \in T$, $x.X_n$ is a simple random walk on T. By the Carne–Varopoulos theorem (see [5, 21, 25] or [18, §13.2]),

$$\mathbb{P}(d(x.X_n, x) > C\sqrt{n \log n}) \le n^{-3}$$

for some C sufficiently large. By Lemma 4.3, there are only $n^{1+o(1)}$ different balls we need to consider, hence

$$\mathbb{P}(\exists x \in T \text{ such that } d(x.X_n, x) > C\sqrt{n \log n}) \le n^{-2+o(1)}.$$

Denote the event above by $\mathcal{B}$. We now divide H_n as follows:

$$H_n \le \mathbb{P}(\mathcal{B})H(X_n|\mathcal{B}) + \mathbb{P}(\mathcal{B}^c)H(X_n|\mathcal{B}^c) + 1.$$

For both terms, we now bound the entropy by the logarithm of the state space. For $H(X_n|\mathcal{B})$, we bound it simply by Cn while for $H(X_n|\mathcal{B}^c)$ we have a smaller state space:

since there are only $(C\sqrt{n\log n})^{1+o(1)} = n^{1/2+o(1)}$ different relevant balls, and since each such ball has volume less than $n^{1/2+o(1)}$, the state space is smaller than

$$(n^{1/2+o(1)})^{n^{1/2+o(1)}}$$

and its logarithm is less than $n^{1/2+o(1)}$. Combining this with $\mathbb{P}(\mathcal{B}) \le n^{-2+o(1)}$, the lemma is proved. ∎

Theorem 4.5. *Let T be a subpolynomially growing tree, and let G be its coloured involution group. Let $H = \mathbb{Z}/2 \wr_{(T\times\mathbb{Z})} (G \times \mathbb{Z})$, where $G \times \mathbb{Z}$ acts on $T \times \mathbb{Z}$ by*

$$(t,n).(g,m) = (t.g, n+m) \quad \forall t \in T, g \in G,\ n,m \in \mathbb{Z}.$$

Let $S = \{(a,0),(b,0),(c,0),(1,1),(1,-1)\}$ so that S generates $G \times \mathbb{Z}$. Consider H as a Cayley graph with respect to the switch-or-move generators $\{(\delta_o,1)\} \cup \{(0,s)\colon s \in S\}$, where o is the root of T. Then H supports a harmonic function u with

$$u(x) \le \mathrm{Res}_{T\times\mathbb{Z}}(d(o,x)),$$

and any harmonic function h with $h(x) = o(\mathrm{Res}_{T\times\mathbb{Z}}(d(o,x)))$ is constant.

Proof. The claim that u exists follows from Lemma 3.4. Let therefore h be a harmonic function with $h(x) = o(\mathrm{Res}_{T\times\mathbb{Z}}(d(o,x)))$. Recall that $x = (\omega, g, n)$, where $\omega \in (\mathbb{Z}/2)^{T\times\mathbb{Z}}$ is the lamp configuration, $g \in G$ and $n \in \mathbb{Z}$. Our first step is the following lemma.

Lemma 4.6. *The function h does not depend on the lamp configuration, i.e., for every ω, ω', g and n, $h(\omega,g,n) = h(\omega',g,n)$.*

Proof. We follow an argument from [4]. It is enough to prove the claim when ω differs from ω' in exactly one point of $T \times \mathbb{Z}$, call it v. Examine lazy random walks R and R' on H started from $x = (\omega,g,n)$ and $x' = (\omega',g,n)$. Couple them as follows: they walk exactly the same unless $v.R_t = o$ at some time t (recall that $T \times \mathbb{Z} \curvearrowleft H$ via its $G \times \mathbb{Z}$ coordinate). In this case, if R does a $(\delta_0, 1)$, let R' do a lazy step and vice versa (other kinds of steps are still done together). In the former case, we get $R_{t+1} = R'_{t+1}$ at this time, and we then change the coupling rule so that they walk together for all time.

Fix some $r > 0$ and examine our two coupled walks. Let τ' be the stopping time when the two walks coupled as in the previous paragraph. Let τ'' be an exponential variable with expectation r^2, independent of everything else. Let $\tau = \min\{\tau', \tau''\}$. We now claim that

$$\mathbb{P}(\tau = \tau'') \le \frac{C}{\mathrm{Res}(r)}, \tag{4.1}$$

where C might depend on x and v but not on r. To see (4.1), note that at every visit of $v.R_t = v.R'_t$ to o, there is a positive probability that they will couple. Let B_k be the event that $v.R_t$ reached o exactly k times before time τ'', but no coupling occurred. On the one hand, there is probability e^{-ck} that in k attempts, coupling always failed. On the other hand, after the k^{th} visit, τ'' is independent of the past, so B_k implies that τ'' happened before $v.R_t$ returned to o. By Lemma 3.6, because $v.R_t$ is a simple random walk on $T \times \mathbb{Z}$, this probability is $\le C/\operatorname{Res}(r)$. We get

$$\mathbb{P}(\tau = \tau'') \le \sum_{k=0}^{\infty} e^{-ck} \frac{C}{\operatorname{Res}(r)},$$

the constant C here comes from Lemma 3.6, so it depends on the starting point of the walk, but there are only two starting points to consider, $v.(g,n)$ (for $k = 0$) and o (for $k \ge 1$). This shows (4.1).

Returning to our two coupled walks, since $h(R_t)$ is a martingale,

$$h(x) = \mathbb{E}(h(R_\tau)) = \mathbb{E}(h(R_\tau) \cdot \mathbf{1}\{\tau = \tau''\}) + \mathbb{E}(h(R_\tau) \cdot \mathbf{1}\{\tau \ne \tau''\})$$

(justifying integrability is easy because $h(x) \le Cd(o,x)$ and τ is bounded by τ'' which is an exponential variable). The same holds for $h(x')$ with R'_t instead of R_t. However, if $\tau \ne \tau''$, then $R_\tau = R'_\tau$, so these terms give equal contribution to $h(x)$ and $h(x')$. We get

$$|h(x) - h(x')| \le |\mathbb{E}(h(R_\tau) \cdot \mathbf{1}\{\tau = \tau''\})| + |\mathbb{E}(h(R'_\tau) \cdot \mathbf{1}\{\tau = \tau''\})|.$$

A crude bound over τ'' will give

$$|h(x) - h(x')| \le 2\mathbb{P}(\tau = \tau'') \cdot \max\{|h(x)| : d(o,x) \le r^3\} + O(e^{-cr}).$$

By (4.1) and our assumption on h, we get

$$|h(x) - h(x')| \le \frac{o(\operatorname{Res}(r^3))}{\operatorname{Res}(r)} + O(e^{-cr}).$$

Lemma 4.6 now follows from the next simple result. ∎

Lemma 4.7. *There exists an infinite sequence $r_k \to \infty$ such that*

$$\frac{\operatorname{Res}(r_k^3)}{\operatorname{Res}(r_k)} \le C.$$

Proof. Examine $\operatorname{Res}(2^{3^k})$. Because our graph contains $\mathbb{Z}^2$, $\operatorname{Res}(2^{3^k}) \le C3^k$ (according to Rayleigh's monotonicity, see, e.g., [10, §2.2]). Further, this is an increasing sequence. Hence for infinitely many choices of k, $\operatorname{Res}(2^{3^k}) \le 4\operatorname{Res}(2^{3^{k-1}})$. ∎

We return to the proof of Theorem 4.5. We have just established Lemma 4.6, i.e., that h does not depend on the lamps. We get that $h(\omega, g, n) = h'(g,n)$, and h' is harmonic on $G \times \mathbb{Z}$. To show that it is constant, we use our entropy estimate. Indeed, by [11, Theorem B], if for some f and a sequence n_k, we have

$$\lim_{k\to\infty} f(n_k)\sqrt{H_{n_k+1} - H_{n_k}} = 0,$$

then a harmonic function with growth at most f is constant. By Lemma 4.4, the entropy of simple random walk on G (with respect to the generators a, b and c) is at most $n^{1/2+o(1)}$. Thus the same holds for $G \times \mathbb{Z}$ (with the added generators of $\mathbb{Z}$). Denote the entropy of $G \times \mathbb{Z}$ by H_n. Thus one may find a sequence n_k such that $H_{n_k+1} - H_{n_k} \le n_k^{-1/2+o(1)}$. We take the function f to be, say, $f(n) = n^{1/8}$ and get that any harmonic function h' on $G \times \mathbb{Z}$ with $h'(x) \le Cd(o,x)^{1/8}$ is constant. Since our function h' satisfies

$$h'(x) = o(\mathrm{Res}_{T\times\mathbb{Z}}(d(o,x))) = o(\mathrm{Res}_{\mathbb{Z}\times\mathbb{Z}}(d(o,x))) = o(\log d(o,x)),$$

it must be constant, and Theorem 4.5 is proved. ■

4.1. Eligible growth rates

Theorem 4.5 is almost our stated result. We merely need to demonstrate that for any function f with $f(x) \to \infty$ and $xf'(x)$ decreasing, one may construct a subpolynomially growing tree T such that $\mathrm{Res}_{T\times\mathbb{Z}}(n) \approx f(n)$. This is no more than a calculus exercise, but let us do it in details nonetheless.

Lemma 4.8. *Let f be a positive C^1 function on $[1,\infty)$ such that $f(x) \to \infty$ and that $xf'(x)$ is decreasing. Then there exist $\ell(n)$ such that for all N,*

$$\sum_{n=1}^{N} \frac{1}{n\ell(n)} \approx f(N),$$

and such that $\ell(n)$ can be taken to be the branching values of a subpolynomially growing tree. The constant implicit in the $\approx$ may depend on f, but not on N.

Proof. We would have liked to define $\ell(n) = 1/nf'(n)$ but this might not satisfy the requirement that $\ell(2n) \le 2\ell(n)$, needed for a tree in $\mathcal{T}$. Define therefore

$$w(n) = \min_{k=0}^{\infty} \frac{4^k}{nf'(n2^{-k})}, \qquad w_2(n) = \min\{w(n), \log^2 8n\},$$

where in the definition of w, we extend f below 1 to be $f(x) = f(1) + f'(1)\log x$, an extension which preserves the condition that $xf'(x)$ be decreasing, and ensures that the minimum is achieved. We note that $w(2n) \le 2w(n)$. Indeed, $w(n) = 4^k/nf'(n2^{-k})$ for some k, and using $k+1$ in the definition of $w(2n)$ gives $w(2n) \le 4^{k+1}/2nf'(2n\cdot 2^{-k-1}) = 2w(n)$. The same holds for $\log^2 8n$ and hence, also $w_2(2n) \le 2w_2(n)$. Let us now estimate $\sum 1/nw_2(n)$. On the one hand, we have

$$\sum_{n=1}^{N} \frac{1}{nw_2(n)} \ge \sum_{n=1}^{N} \frac{1}{n\cdot\frac{1}{nf'(n)}} \ge f'(N) + \sum_{n=1}^{N-1} f(n+1) - f(n)$$
$$= f(N) - f(1) + f'(N) \approx f(N),$$

where in the first inequality, we used the definition of w with $k = 0$ and in the second inequality, the fact that f' is also decreasing, so $f'(n) \geq f'(\xi) = f(n+1) - f(n)$. For the other direction, we write

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n w_2(n)} &= \sum_{n=1}^{N} \max\left(\frac{1}{n \log^2 8n}, \max_{k=0}^{\infty} \frac{f'(n2^{-k})}{4^k}\right) \\
&\overset{(*)}{\leq} \sum_{n=1}^{N} \frac{1}{n \log^2 8n} + \sum_{k=0}^{\infty} \frac{1}{4^k} \sum_{n=1}^{N} f'(n2^{-k}) \\
&\leq C + \sum_{k=0}^{\infty} \frac{1}{4^k}\left(f'(2^{-k}) + 2^k \sum_{n=2}^{N} f(n2^{-k}) - f((n-1)2^{-k})\right) \\
&\overset{(**)}{\leq} C + \sum_{k=0}^{\infty} \frac{1}{2^k} f(N2^{-k}) \overset{(***)}{\leq} C + 2f(N),
\end{aligned}$$

where in $(*)$ we estimated both maxima by a sum, and rearranged the summands; $(**)$ follows from the fact that $f(x) = C + C\log x$ for $x < 1$; and $(***)$ is due to the fact that f is increasing (f cannot decrease, since if at some x we have $f'(x) < 0$, then f' must be negative forever, contradicting the requirement $f(x) \to \infty$). Defining

$$\ell(n) = 2^{\lfloor \log_2 w_2(n) \rfloor},$$

we see that $\sum 1/(n\ell(n)) \approx f$ while at the same time, $\ell(2n) \leq 2\ell(n)$ which is the only condition necessary for $\ell(n)$ to be the branching numbers of a tree in $\mathcal{T}$. Since $\ell(n) \leq \log^2 8n$, T is also subpolynomially growing. ∎

Combining Lemmas 3.5 and 4.8 and Theorem 4.5 proves our main result.

A. A remark on groups defined by slowly growing trees (by Nicolás Matte Bon)

To prove their main result, the authors introduce a new idea to construct groups, defined by an explicit Schreier graph obtained by labelling a slowly growing tree, which is of independent interest. We remark in this appendix that this construction yields elementary amenable groups (in fact, (locally finite)-by-dihedral, see Proposition A.1 below). In particular, the main result of this paper, which is pertinent to the realm of amenable groups, holds already in the more restricted realm of elementary amenable groups. (Recall that the class of elementary amenable groups is the smallest class of groups that contain finite and abelian groups and is stable under direct limits and extensions [7]; certain types of asymptotic and algebraic behaviours are possible for amenable groups but not for elementary amenable groups.)

A variant of this group construction, named *bubble groups*, were introduced and used in [16] and further studied in [22], where their amenability was shown with an analytic method, and the isoperimetric profile of (permutational wreath products over) the bubble groups was studied and shown to realise a wide family of behaviors. Our remark also applies to the bubble groups (showing that they are locally finite-by-metabelian). It would be interesting to know if these group constructions can be used to study the behaviour of other asymptotic invariants among elementary amenable groups.

It is interesting to notice that these group constructions were somewhat inspired by previous ones based on automata groups, which are instead often non-elementary amenable.

Definition of the groups

We work in parallel with two different groups belonging to two different families, denoted by G and Γ, that we shall define precisely below. The group G is essentially the same group denoted G in Section 4, with the difference that we also allow the degree of branching points of the tree T to vary (and to be possibly unbounded). The group Γ is one of the *bubble groups* from [16], in a similar more general setting considered in [22].

Choose and fix two sequences of positive integers: the *scaling sequence* (b_i), assumed to be strictly increasing, and the *degree sequence* (d_i). We also assume that $d_i \geq 3$ for every $i \geq 1$. We set $d_* = \sup d_i$ (possibly, $d_* = \infty$).

To define the group G, let T be a spherically homogeneous rooted tree as in Figure 1, where the root has degree $d_0 \geq 1$, and every other vertex has degree 2, except if its distance from the root is equal to b_i for some $i > 0$, in which case it has degree d_i. A vertex with degree greater than 2 will be called a *branching point*. We label the edges of T using the letters a, b and c_n, $n \in [1, d_* - 2]$ as follows. Edges that are strictly between two branching points will be labelled by a and b alternatively, as in Figure 2. Edges adjacent to a branching point of degree d_i will be labelled by $a, b, c_1, \ldots, c_{d_i - 2}$. Each letter a, b or c_n corresponds to a permutation of the vertex set of T, denoted by the same letter, that exchanges the endpoints of every edge with the corresponding label. As in Section 4, we let G be the group of permutations of T generated by a, b and c_n, $1 \geq n \geq d_* - 2$. The group G is finitely generated if and only if the degree sequence is bounded. Note that the groups appearing in the proof of the main theorem of the paper are of the form $\mathbb{Z}/2 \wr_{T \times \mathbb{Z}} (G \times \mathbb{Z})$ (see Theorem 4.5), for particular choices of the scaling sequence, and for a degree sequence constant and equal to 3. Therefore, they are elementary amenable if and only if the group G is so.

To define the bubble group Γ, we let Θ be the (oriented) graph obtained from the graph T as follows. Every branching point is replaced by an oriented cycle of length d_i, called a *branching cycle*, and every path between a branching point at generation i and a branching point at generation $i + 1$ is replaced by an oriented cycle of length $2(b_{i+1} - b_i)$, called a *bubble*; see Figure 4. Edges on a bubble will be labelled by the letter α, and edges on a branching cycle will be labelled by β. We still denote by α, β the corresponding

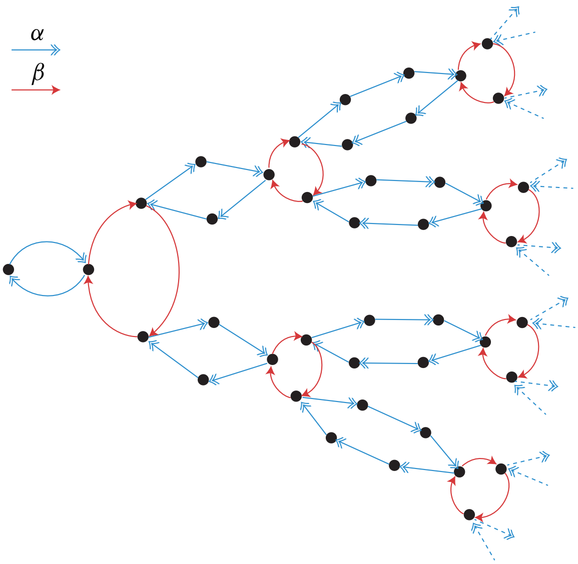

Figure 4. The graph Θ.

permutations that permute cyclically every bubble or branching cycles, respectively. The *bubble group* is the group Γ generated by α and β. Note that the group Γ, unlike the group G, is always 2-generated, even if the degree sequence (d_i) is unbounded.

Structure of Γ and G

In the statements below, we make the assumption that $b_{i+1} - b_i$ tends to infinity, while no assumption is made on (d_i). Under the same assumptions, the fact that the group Γ is amenable was first established in [22, Proposition 5.13].

Proposition A.1. *Let (b_i), (d_i) be a scaling sequence and a degree sequence. Assume that $b_i - b_{i-1}$ tends to infinity, and let (d_i) be arbitrary. Then*

(i) *The group G splits as a semi-direct product of the form*

$$G \simeq N \rtimes D_\infty,$$

where N is locally finite, and D_∞ is the infinite dihedral group. The surjection $G \to D_\infty$ maps a and b to two generating involutions of D_∞, and all the generators c_j, $j \geq 1$ to the identity.

(ii) *The group Γ is an extension of the form*

$$1 \to K \to \Gamma \to C \wr \mathbb{Z} \to 1,$$

where K is locally finite, and C is a non-trivial cyclic group, which is finite if the degree sequence (d_i) is bounded, and infinite cyclic if (d_i) is unbounded. The *surjection* $\Gamma \to C \wr \mathbb{Z}$ *maps* α *to a standard generator of* $\mathbb{Z}$*, and* β *to a standard generator of the lamp group over* 0.

In particular, the groups G *and* Γ *are elementary amenable.*

Before the proof, let us fix some terminology on spaces of Schreier graphs. If H is a finitely generated group endowed with a finite generating set S, we denote $\mathrm{Sch}(H, S)$ the space of rooted, connected, oriented graphs with edges labelled by S, that arise as Schreier graphs of the pair (H, S), up to isomorphisms of rooted connected labelled graphs. Endowed with the topology induced by the space of marked graphs, $\mathrm{Sch}(H, S)$ is a compact space, on which H acts continuously by moving the root in the natural way. As it is well known and easy to see, this action is conjugate to the conjugation action of H on its space of subgroups $\mathrm{Sub}(H)$ endowed with the Chabauty topology (however, we will not need this point of view here). In our conventions, edges of Schreier graphs are always oriented, with the exception that we represent edges corresponding to a generator of order two by a single unoriented one rather than two oriented ones (this is consistent with the fact that T is not oriented, while Θ is). We systematically omit the loops when representing Schreier graphs. All Schreier graphs will be intended as *rooted* and *labelled* graphs, and we will specify *unrooted* when we forget the root.

Proposition A.1 follows by analysing the closure of the orbits of T and Θ in the spaces of Schreier graphs of G and Γ. In one sentence, each graph in the closure provides a quotient of the groups, and these can be readily described.

Proof of Proposition A.1. Throughout the proof, we say that an integer $n \geq 3$ is *admissible* if $d_i = n$ for infinitely many n's. For simplicity, let us first consider the case where the degree sequence d is bounded. We begin by proving (i) and then explain the modifications needed for (ii). Let $S = \{a, b, c_1, \ldots\}$ be the standard generating set of G. Let $o \in T$ be the root, and view (T, o) as an element of $\mathrm{Sch}(G, S)$. We look at the closure of the G-orbit of (T, o) in $\mathrm{Sch}(G, S)$. Each Schreier graph in the closure has underlying unrooted graph of one of the following three types: the graph T, the graph $\overline{T}_n$ as in Figure 5 (a), where the degree n of its unique branching point runs over admissible integers, and the graph $\widehat{T}$, shown in Figure 5 (b). Each of these labelled graphs naturally defines a group of permutations of its vertices, generated by the permutations of its vertices that correspond to each letter (as in Section 2). Since all these graphs are Schreier graphs of (G, S), the group defined by each of them is a quotient of G. The group defined by $\widehat{T}$ is simply the dihedral group D_∞. Therefore, we have a surjection $p\colon G \to D_\infty$ mapping a, b to two generating involutions of D_∞. Since a, b already generate a subgroup isomorphic to D_∞ in G, the group G splits as a semi-direct product of the form $G \simeq \ker(p) \rtimes D_\infty$. We shall now check that $\ker(p)$ is locally finite. For each admissible n, we denote $\overline{G}_n$ the group of permutations of vertices of $\overline{T}_n$ defined by the labelling of edges of $\overline{T}_n$, and $\pi_n\colon G \to \overline{G}_n$ the associated natural surjection. Observe that $\widehat{T}$ is also in the closure of the orbit of the rooted graphs whose underlying graph is $\overline{T}_n$, therefore it is also a Schreier graph of $\overline{G}_n$ for every n. Thereby, we also have a surjection $p_n\colon \overline{G}_n \to D_\infty$. Clearly,

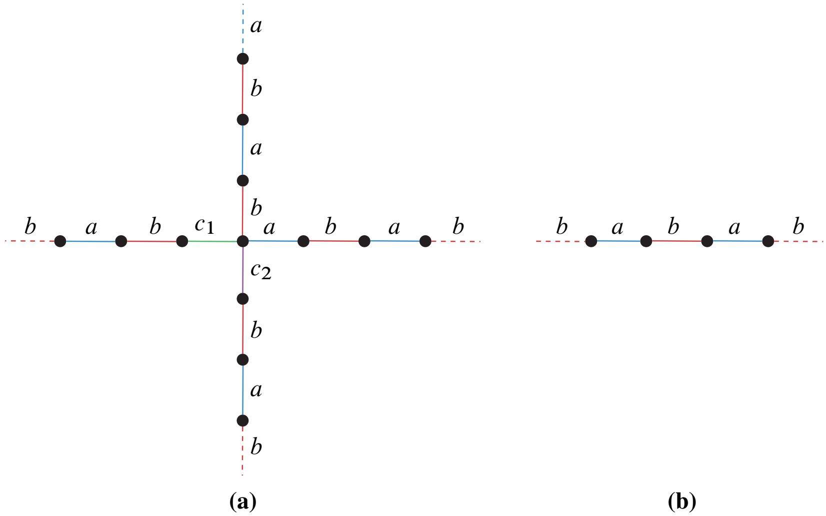

Figure 5. (a) The graph $\overline{T}_n$ for $n = 4$. (b) The graph $\widehat{T}$.

$\ker(p_n)$ is a locally finite subgroup of $\overline{G}_n$: in fact every element of $\ker(p_n)$ acts trivially sufficiently far from the branching point of $\overline{T}_n$, hence $\ker(p_n)$ consists of permutations of $\overline{T}_n$ with finite support. Moreover, it is clear that $p = p_n \circ \pi_n$, as it is seen by looking at the images of generators. It follows that π_n maps $\ker(p)$ inside $\ker(p_n)$ for every admissible n. Consider the diagonal map $\pi\colon \prod_n \pi_n\colon G \to \prod_n \overline{G}_n$, where the product is taken over all admissible n's. It follows from the discussion above that π maps $\ker(p)$ to the locally finite group $\prod_n \ker p_n$. Hence to check that $\ker(p)$ is locally finite, it is enough to check that $\ker \pi = \bigcap_n \ker \pi_n$ is locally finite (since an extension of two locally finite groups is still locally finite). To see this, let $g \in \ker \pi$, and let $|g|$ be its word length with respect to the standard generating set. Since $b_i - b_{i-1}$ tends to infinity, the ball of radius $|g|$ around any vertex $v \in T$ sufficiently far from the root contains at most one branching point. Hence for every sufficiently far vertex $v \in T$, the ball of radius $|g|$ around v coincides with a ball in some $\overline{T}_n$, for n admissible. Since g has trivial projection in $\overline{G}_n$ for every admissible n, we deduce that g fixes v, and thus it fixes all vertices sufficiently far from the root. It follows that $\ker(\pi)$ consists of permutations with finite support, hence it is locally finite. As noted, this shows that $\ker p$ is locally finite and concludes the proof of (i) under the assumption that (d_i) is bounded.

To remove this assumption, write G as the ascending union of its finitely generated subgroups $G|_j$ generated by $S_j = \{a, b, c_1, \dots, c_j\}$, and let $T|_j$ be the graph obtained from T by removing all edges labelled by c_i, $i > j$. Essentially, the same argument as in the case of a bounded degree sequence shows that we have surjections $p|_j\colon G|_j \to D_\infty$ with locally finite kernel (a minor modification is needed since the graph $T|_j$ is no longer connected, hence strictly speaking, it is not an element of $\mathrm{Sch}(G|_j, S_j)$: consider instead the closure of the orbits of all its connected components and argue in a similar way). This

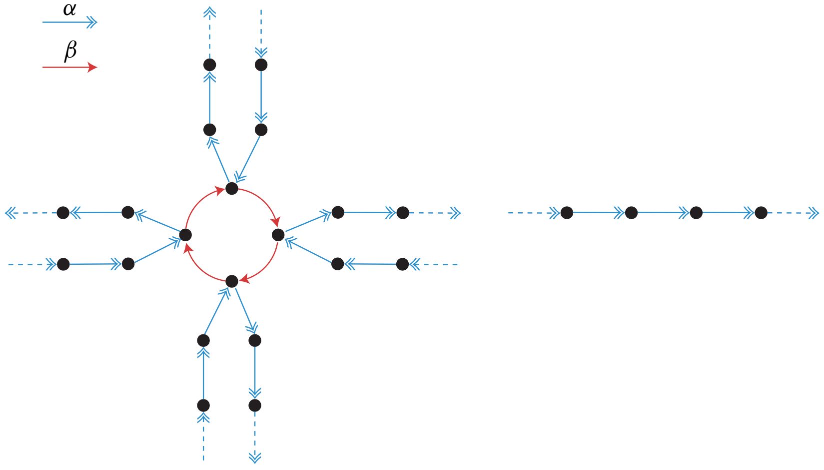

Figure 6. (a) The graph $\overline{\Theta}_n$ for $n = 4$. (b) The graph $\widehat{\Theta}$.

shows that we have surjections $p|_j : G|_j \to D_\infty$ with locally finite kernel, mapping a and b to the standard generators of D_∞ and each c_i to 1. Thus they globally define a surjection $p: G \to D_\infty$ with locally finite kernel.

We now prove (ii). Assume first that the sequence (d_i) is bounded. We consider the closure of (θ, o) in $\mathrm{Sch}(\Gamma, S)$. Similarly, to the previous case, the graphs in the closure have underlying unrooted graph of three types: the graph Θ, the graphs of the form $\overline{\theta}_n$ for n admissible (see Figure 6 (a)) and the graph $\widehat{\Theta}$ (see Figure 6 (b)).

Denote by $\overline{\Gamma}_n$ the group defined by the graph $\overline{\theta}_n$, by $\overline{\pi}_n : \Gamma \to \widehat{\Gamma}_n$ the corresponding projection, and by $\overline{\Gamma}$ the image of Γ into $\prod_n \overline{\Gamma}_n$ under the diagonal map $\pi = \prod_n \pi_n$. As in the case of G, $\ker \pi$ is locally finite. Closer inspection shows that the group $\overline{\Gamma}_n$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \wr \mathbb{Z}$, and the projection π_n sends the generator α to a standard generator of $\mathbb{Z}$, and the generator β to a standard generator of the cyclic lamp group over 0. It follows that $\overline{\Gamma}$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \wr \mathbb{Z}$, where ℓ is the least common multiple of all the admissible integers $n \in \mathbb{N}$. This concludes the proof for Γ under the assumption that (d_i) is bounded.

Let us assume now that (d_i) is unbounded. In this case, the argument for Γ is slightly different from the one used for G, since Γ is still finitely generated, and we may still work in the space $\mathrm{Sch}(\Gamma, S)$. The closure of the orbit of (Θ, o) now contains the same graphs as in the bounded degree case, plus the graph $\overline{\Theta}_\infty$ obtained by taking a limit of graphs of the form $(\overline{\Theta}_n, x)$ when n goes to ∞ and the root x belongs to a branching cycle (in plain words, $\overline{\Theta}_\infty$ consists of a bi-infinite oriented line labelled by β, to each vertex of which is glued a bi-infinite line labelled by α). The permutation group $\overline{\Gamma}_\infty$ defined by its labelling is easily seen to be isomorphic to $\mathbb{Z} \wr \mathbb{Z}$. To conclude the proof, it is enough to show that the kernel of the natural surjection $\pi_\infty : \Gamma \to \overline{\Gamma}_\infty$ consists of finitely supported permutations of Θ. To this extent, let $g \in \ker \pi_\infty$, and write $g = s_1 \dots s_n$ as a product of generators. Observe that $\overline{\Theta}_\infty$ covers all graphs of the form $\overline{\Theta}_n$, for all $n \in \mathbb{N}$ (the covering map is given by folding the line labelled by β to an n-cycle, and identifying two α-lines if the vertices of the β-line to which they are glued are identified). If a vertex $v \in \Theta$ lies sufficiently far from the root, then the ball of a fixed radius r around v is isomorphic to a ball of radius r in a graph of the form $\overline{\Theta}_n$, for some $n \in \mathbb{N}$. Since $g \in \ker \pi_\infty$, the path labelled by $s_1, \dots, s_n$ in this ball lifts to a closed path in $\overline{\Theta}_\infty$, hence it was already closed. This shows that $gv = v$. It follows that g is a permutation with finite support, concluding the proof. ∎

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