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Tight inclusions of C^*-dynamical systems

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Abstract. We study a notion of tight inclusions of C^*- and W^*-dynamical systems, which is meant to capture a tension between topological and measurable rigidity of boundary actions. An important case of such inclusions is $C(X) \subset L^\infty(X, \nu)$ for measurable boundaries with unique stationary compact models. We discuss the implications of this phenomenon in the description of Zimmer amenable intermediate factors. Furthermore, we prove applications in the problem of maximal injectivity of von Neumann algebras.

1. Introduction

One of the key tools in the rigidity theory is the notion of boundary actions in the sense of Furstenberg [14, 15]. These actions are defined in both topological and measurable setups, and exploiting their dynamical and ergodic theoretical properties reveals various rigidity phenomena of the underlying groups.

For example, the fact that the measurable Furstenberg–Poisson boundaries of irreducible lattices in higher rank semisimple Lie groups have few quotients (*the factor theorem*) implies rigidity for normal subgroups (*the normal subgroup theorem*), and a classification of certain spaces related to the Furstenberg–Poisson boundary (*the intermediate factor theorem*) implies rigidity of invariant random subgroups. These rigidity phenomena are "higher rank phenomena" either in the classical sense of semisimple Lie groups or of product groups, and are based on the measure theoretical boundary.

Recently, properties related to the Furstenberg–Poisson boundary (*boundary structures*) have shown to imply strong rigidity results in noncommutative settings [2, 3, 8].

On the other front, dynamical properties of the topological boundaries have been shown to imply certain noncommutative rigidity properties, such as C^*-simplicity and the unique trace property [9, 24].

In many natural examples, measurable boundaries are concretely realized on topological boundaries, and one expects this to be reflected in their dynamical properties. However, the connection between the two notions of boundary actions has barely been systematically investigated.

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A typical instance in which the interaction between the two setups arises is a topological boundary admitting a unique stationary measure turning it into a measurable boundary. A systematic study of such an action in the noncommutative setting was undertaken in the authors' work [20], wherein the framework of this connection, properties of measurable boundaries, were used in C^*-simplicity problems.

An important consequence of having a unique stationary boundary measure is a uniqueness property for equivariant maps from the space of continuous functions into the space of essentially bounded measurable functions on the boundary. This work is devoted to the study of this particular uniqueness phenomenon and several of its applications. As we will see below, this property is not an exclusive feature of certain boundary actions, and it does appear in setups with quite different behavior.

More precisely, this work is about the following notion. Given a locally compact group G, we denote by $\underline{\mathbf{OA}}_G$ the category of all unital G-C^*-algebras and G-W^*-algebras where the morphisms are G-equivariant ucp maps.

By an inclusion $\mathsf{A} \subset \mathsf{B}$ of objects $\mathsf{A}, \mathsf{B} \in \underline{\mathbf{OA}}_G$ we mean a C^*-algebraic inclusion.

Definition (Definition 2.1). We say a C^*-inclusion $\mathsf{A} \subset \mathsf{B}$ of objects $\mathsf{A}, \mathsf{B} \in \underline{\mathbf{OA}}_G$ is *G-tight* if the inclusion map is the unique G-equivariant ucp map from A to B.

This property has already been exploited in some previous work. When A is a commutative C^*-algebra, and B is a commutative von Neumann algebra, this coincides with Furman's notion of alignment systems [13], a key concept in his work on rigidity of homogeneous actions of semisimple groups. Around the same time, in a completely different context, Ozawa [29] proved that for a quasi-invariant and doubly-ergodic measure ν on the Gromov boundary $\partial\mathbb{F}_n$ of the free group, the inclusion $C(\partial\mathbb{F}_n) \subset L^\infty(\partial\mathbb{F}_n, \nu)$ is $\mathbb{F}_n$-tight. He used this property to prove a nuclear embedding result for the reduced C^*-algebra of the free group.

As mentioned earlier, we have the following fact. We use the abbreviation *lcsc* for locally compact and second countable.

Theorem (Theorem 3.4). *Let G be an lcsc group, and let $\mu \in \mathrm{Prob}(G)$ be an admissible probability measure on G. Suppose X is a minimal compact G-space that admits a unique μ-stationary probability measure ν such that (X, ν) is a μ-boundary. Then the canonical embedding $C(X) \subset L^\infty(X, \nu)$ is G-tight.*

The tightness property becomes particularly fruitful in combination with the notion of Zimmer amenability.

Theorem (Theorem 4.8). *Let G be an lcsc group, and let $\mu \in \mathrm{Prob}(G)$ be an admissible measure such that the Furstenberg–Poisson boundary (B, ν) of (G, μ) has a uniquely stationary compact model. Let (Y, η) be a (G, μ)-space, and let $\phi\colon (\tilde{Y}, \tilde{\eta}) \to (Y, \eta)$ be the standard cover in the sense of Furstenberg–Glasner. If $(\tilde{Y}, \tilde{\eta}) \xrightarrow{\varphi} (Z, \omega) \xrightarrow{\psi} (Y, \eta)$ are measurable G-maps such that $\psi \circ \varphi = \phi$ and (Z, ω) is Zimmer amenable, then $(\tilde{Y}, \tilde{\eta}) \overset{\varphi}{\cong} (Z, \omega)$.*

For discrete groups Γ, we prove a noncommutative version of this. Namely, we show that under the same conditions, there are no injective von Neumann algebras M satisfying $\Gamma \ltimes L^\infty(Y,\eta) \subseteq M \subsetneq \Gamma \ltimes L^\infty(\widetilde{Y},\widetilde{\eta})$ (Corollary 4.15).

Examples of tight inclusions involving noncommutative C^*-algebras include the embedding of tight Γ-C^*-algebras in their associated crossed products. This, for instance, yields the following maximal injectivity result.

Theorem (Corollary 4.16). *Let Γ be a discrete group and $\mu \in \mathrm{Prob}(G)$ a generating measure such that the Furstenberg–Poisson boundary (B,ν) of (Γ,μ) has a uniquely stationary compact model. Let $\Gamma \curvearrowright (Z,m)$ be a measure-preserving action. Then the von Neumann algebra $\Gamma \ltimes L^\infty(B \times Z, \nu \times m)$ is maximal injective in $\Gamma \ltimes (\mathcal{B}(L^2(B,\nu)) \bar{\otimes} L^\infty(Z,m))$.*

In [31, Corollary 3.8] Suzuki gave a complete description of the intermediate subalgebras of certain von Neumann crossed product inclusions associated to boundary actions of irreducible higher rank lattices, using the deep theory available for these groups. From this, he concluded a maximal injectivity result, which is a special case of our theorem above. In particular, our result shows that Suzuki's maximal injectivity result is not a higher rank phenomenon but rather follows from the broader framework of tightness. Consequently, this provides a large class of new examples of maximal injective von Neumann algebras (see comments after Corollary 4.16).

However, our notion of tightness is not bound to only certain boundary actions, it is more general even in the commutative setup. Corollary 3.8 and Theorems 3.9 and 3.12 below show that there are examples of tight actions with properties far from boundary actions.

Next, we fix our notation and briefly review some of the definitions and basic facts that will be used in the rest of the paper.

Throughout the paper, unless otherwise stated, G is a locally compact second countable group, and Γ denotes a countable discrete group. We write $G \curvearrowright X$ to mean a continuous action of G on a compact space X by homeomorphisms (all topological spaces in this paper are assumed to be Hausdorff). In this case, we say X is a compact G-space. Given $G \curvearrowright X$ and $G \curvearrowright Y$, we say that Y is a (G-)factor of X, or that X is a (G-)extension of Y, if there is a continuous map φ from X onto Y that is G-equivariant, that is, $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$.

We denote by $\mathrm{Prob}(X)$ the compact convex space of all Borel probability measures on X equipped with the weak$*$ topology. Any continuous action $G \curvearrowright X$ induces a canonical action $G \curvearrowright \mathrm{Prob}(X)$ by affine homeomorphisms. For $\nu \in \mathrm{Prob}(X)$, the Poisson transform (associated to ν) is the map $\mathcal{P}_\nu\colon C(X) \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$ defined by $\mathcal{P}_\nu(f)(g) = \int_X f(gx)\,d\nu(x)$ for $f \in C(X)$ and $g \in G$, where $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ denotes the space of bounded left uniformly continuous functions on G, that is, bounded continuous functions $\phi\colon G \to \mathbb{C}$ satisfying $\|g\phi - \phi\|_{\sup} \to 0$ as $g \to e$.

The map $\mathcal{P}_\nu$ is obviously a continuous unital linear positive map which is also G-equivariant.

We use similar standard terminology for measurable actions of an lcsc group G. All measure spaces considered here are assumed to be standard Borel probability spaces. We write $G \curvearrowright (X, \nu)$ to mean a measurable action of G on a standard Borel probability space (X, ν) by measurable isomorphisms; in particular, in this setup, the measure ν is quasi-invariant, that is, $g\nu$ and ν have the same null sets for any $g \in G$. In this case, we say (X, ν) is a probability G-space. Given $G \curvearrowright (X, \nu)$ and $G \curvearrowright (Y, \eta)$, we say that (Y, η) is a (G-)factor of (X, ν), or that (X, ν) is a (G-)extension of (Y, η), if there is a G-equivariant measurable map φ from X to Y such that $\eta = \varphi_* \nu$. In this case, the map φ yields a canonical G-equivariant von Neumann algebra embedding $\varphi^*\colon L^\infty(Y, \eta) \to L^\infty(X, \nu)$.

For an action $G \curvearrowright (X, \nu)$, similarly to the continuous case, we denote by

$$\mathcal{P}_\nu\colon L^\infty(X) \to L^\infty(G)$$

the Poisson transform $\mathcal{P}_\nu(f)(g) = \int_X f(gx)\, d\nu(x)$, which is a G-equivariant normal unital positive linear map.

Let $G \curvearrowright (X, \nu)$, and $\mu \in \mathrm{Prob}(G)$. We say ν is μ-stationary if $\mu * \nu = \nu$, where $\mu * \nu$ is the convolution of the measures. In this case, we write $(G, \mu) \curvearrowright (X, \nu)$ and say (X, ν) is a (G, μ)-space.

A compact model for a probability G-space (Y, η) is a compact G-space X and a quasi-invariant $\nu \in \mathrm{Prob}(X)$ such that (X, ν) is isomorphic to (Y, η) as probability G-spaces.

We also consider noncommutative actions in this paper, namely, actions of G on C^* and von Neumann algebras. All C^*-algebras considered here are assumed to be unital. By a G-C^*-algebra (resp. G-von Neumann algebra) we mean a unital C^*-algebra (resp. a von Neumann algebra) A, on which G acts continuously by $*$-automorphisms, where continuity is with respect to the point-norm topology (resp. point-weak$*$ topology). For any lcsc group G, the space $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ is a G-C^*-algebra.

Given a discrete group Γ and a Γ-C^* or Γ-von Neumann algebra A, a *covariant representation* of (Γ, A) is a pair (π, ρ), where π is a unitary representation of Γ on a Hilbert space $\mathcal{H}_\pi$, and $\rho\colon \mathsf{A} \to \mathcal{B}(\mathcal{H}_\pi)$ is a Γ-equivariant representation of A, where $\Gamma \curvearrowright \mathcal{B}(\mathcal{H}_\pi)$ by inner automorphisms $\mathrm{Ad}_{\pi(g)}$, $g \in \Gamma$. In this case, we let $\Gamma \ltimes_\pi^\rho \mathsf{A}$ be the C^* or von Neumann algebra (depending on A) generated by the set $\{\rho(a)\pi(g)\colon a \in \mathsf{A}, g \in \Gamma\}$, and we equip it with the action of Γ by inner automorphisms $\mathrm{Ad}_{\pi(g)}$, $g \in \Gamma$. In the case of a regular covariant representation, we use the notation $\Gamma \ltimes \mathsf{A}$ for either the reduced C^*-crossed product or the von Neumann algebra crossed product of the action, again depending on A, and we will explicitly specify the setup if there is any danger of confusion.

We refer the reader to [11] for the definitions and details concerning these constructions and their properties.

In this paper, we often consider equivariant ucp maps from C^*-algebras into von Neumann algebras. In view of the following known fact, these maps should be considered as noncommutative counterparts of *quasi-factor maps* in the sense of Glasner [17, Chapter 8]. We omit the proof of the following, which can be found in [29]. The latter argument was written for a special example, but as noted in [5], the same argument works in general; (see also [3, Proposition 4.10]).

Lemma 1.1. *Let X be a compact metric (G-)space and (Y, η) a standard probability (G-)space. Then there is a one-to-one correspondence between (G-equivariant) η-measurable maps $Y \to \operatorname{Prob}(X)$ and (G-equivariant) ucp maps $C(X) \to L^\infty(Y, \eta)$.*

Boundary actions

The most natural examples in our context are topological and measurable boundary actions in the sense of Furstenberg. We briefly review the definitions and refer the reader to [12, 15, 18, 20] for more details.

An action $G \curvearrowright X$ is said to be a topological boundary action, and X is said to be a topological G-boundary if for every $\nu \in \operatorname{Prob}(X)$ and $x \in X$, there is a net $(g_i)_i$ of elements of G such that $g_i \nu \to \delta_x$ in the weak$*$ topology, where δ_x is the Dirac measure at x. It was shown by Furstenberg [15, Proposition 4.6] that any topological group G admits a unique (up to G-equivariant homeomorphism) maximal G-boundary $\partial_F G$ in the sense that every G-boundary X is a G-factor of $\partial_F G$.

Measurable boundary actions are defined as follows. Let G be an lcsc group, and let $\mu \in \operatorname{Prob}(G)$ be an admissible measure, that is, μ is absolutely continuous with respect to the Haar measure and is not supported in a proper closed subsemigroup, and let ν be a μ-stationary probability measure on a metrizable G-space X. The action $(G, \mu) \curvearrowright (X, \nu)$ is a μ-boundary action if for almost every path $(\omega_k) \in G^{\mathbb{N}}$ of the (G, μ)-random walk, the sequence $\omega_k \nu$ converges to a Dirac measure δ_{x_ω}.

An action $(G, \mu) \curvearrowright (Y, \eta)$ is said to be a μ-boundary action if it has a compact model which is a μ-boundary in the above sense.

Similarly to the topological case, there is a unique (up to G-equivariant measurable isomorphism) maximal μ-boundary (B, ν), called the Furstenberg–Poisson boundary of the pair (G, μ), in the sense that every μ-boundary (X, η) is a G-factor of (B, ν).

2. Tight inclusions

Let G be an lcsc group. We denote by $\underline{\mathbf{OA}}_G$ the category of all unital G-C^*-algebras and G-von Neumann algebras where the morphisms are G-equivariant ucp maps (not assumed normal even between von Neumann algebras).

If Γ is a discrete group, every Γ-von Neumann algebra is also a Γ-C^*-algebra, and therefore $\underline{\mathbf{OA}}_\Gamma$ is just the category of all unital Γ-C^*-algebras.

Given $\mathsf{A} \in \underline{\mathbf{OA}}_\Gamma$, we say A is Γ-injective if it is an injective object in the category $\underline{\mathbf{OA}}_\Gamma$. We refer the reader to [19, 24] for more details on this concept and its connection to boundary actions.

A unital C^*-algebra A is injective if it is injective in the category of unital C^*-algebras with ucp maps as morphisms, equivalently, Γ-injective for the trivial group $\Gamma = \{e\}$.

Note that all positive maps between commutative C^*-algebras are automatically completely positive, but for the sake of consistency, we keep assuming ucp for all our morphisms.

By an inclusion $\mathsf{A} \subset \mathsf{B}$ of objects $\mathsf{A}, \mathsf{B} \in \underline{\mathbf{OA}}_G$ we mean a C^*-algebraic inclusion such that the G-action on B restricts to the G-action on A.

Definition 2.1. An inclusion $\mathsf{A} \subset \mathsf{B}$ of objects $\mathsf{A}, \mathsf{B} \in \underline{\mathbf{OA}}_G$ is called G*-tight* if the inclusion map is the unique G-equivariant ucp map from A to B.

We begin with a few observations on the general properties of G-tight inclusions before getting to some basic examples.

Proposition 2.2. *Suppose* $\mathsf{A} \subset \mathsf{B}$ *is a* G*-tight inclusion of objects in* $\underline{\mathbf{OA}}_G$*. Then for any* $\mathsf{C} \in \underline{\mathbf{OA}}_G$ *with* $\mathsf{A} \subset \mathsf{C} \subset \mathsf{B}$*, the inclusion* $\mathsf{A} \subset \mathsf{C}$ *is* G*-tight; in particular,* $\mathsf{A} \subset \mathsf{A}$ *is* G*-tight.*

Proof. Every G-equivariant ucp map $\mathsf{A} \to \mathsf{C}$ is, in particular, a G-equivariant ucp map from A to B. Thus G-tightness of $\mathsf{A} \subset \mathsf{B}$ implies G-tightness of $\mathsf{A} \subset \mathsf{C}$. ■

We refer to an object A for which $\mathsf{A} \subset \mathsf{A}$ is G-tight as a (G-)self-tight object.

Lemma 2.3. *Suppose* $\mathsf{A} \subset \mathsf{B}$ *is a* G*-tight inclusion of objects in* $\underline{\mathbf{OA}}_G$*. If* A *admits a* G*-invariant state, then* $\mathsf{A} = \mathbb{C}$*.*

Proof. Suppose τ is a G-invariant state on A. Then τ is, in particular, a G-equivariant ucp map from A to $\mathbb{C} \subset \mathsf{B}$. Hence $\tau = \mathrm{id}$ by G-tightness of the $\mathsf{A} \subset \mathsf{B}$, and this implies $\mathsf{A} = \mathbb{C}$. ■

Recall that a topological group G is said to be amenable if any continuous action of G by affine homeomorphisms on a compact convex subset of a topological vector space has a fixed point.

Corollary 2.4. *If there is a* G*-tight inclusion* $\mathsf{A} \subset \mathsf{B}$ *of objects in* $\underline{\mathbf{OA}}_G$ *with* A *non-trivial, then* G *is non-amenable.*

Proof. Suppose $\mathsf{A} \subset \mathsf{B}$ is a G-tight inclusion of objects in $\underline{\mathbf{OA}}_G$. If G is amenable, then A admits a G-invariant state, and therefore we have $\mathsf{A} = \mathbb{C}$ by Lemma 2.3. This implies the claim. ■

Recall that every discrete group Γ admits a minimal normal subgroup N such that Γ/N is an ICC group (every non-trivial element has infinite conjugacy class). This subgroup is called the hyper-FC-center of Γ.

Proposition 2.5. *Let* Γ *be a discrete group and* $\mathsf{A} \subset \mathsf{B}$ *be a* Γ*-tight inclusion of objects in* $\underline{\mathbf{OA}}_\Gamma$*. Then the hyper-FC-center of* Γ *is contained in the kernel of the action of* Γ *on* A*.*

Proof. Suppose $g \in \Gamma$ has a finite conjugacy class C_g. Define $\Phi_g \colon \mathsf{A} \to \mathsf{A}$ by

$$\Phi_g(a) = \frac{1}{\#C_g} \sum_{k \in C_g} ka, \quad a \in \mathsf{A}.$$

Then Φ_g is ucp, and for every $h \in \Gamma$ and $a \in \mathsf{A}$,

$$h\Phi_g(a) = \frac{1}{\#C_g} \sum_{k \in C_g} hka = \frac{1}{\#C_g} \sum_{k \in C_g} kha = \Phi_g(ha),$$

which shows Φ_g is equivariant. Thus, by tightness, $\Phi_g = \mathrm{id}$. Since the identity map on A is an extreme point in the space of all ucp maps on A, it follows that $ka = a$ for every $k \in C_g$ and $a \in \mathsf{A}$. In particular, g acts trivially on A. This implies that the normal subgroup N of Γ consisting of all finite conjugacy elements lies in the kernel of the action $\Gamma \curvearrowright \mathsf{A}$. Repeating the argument for the action $\Gamma/N \curvearrowright \mathsf{A}$ and taking a transfinite induction yield the result. ■

In particular, we conclude the following for discrete groups with faithful actions on tight inclusions.

Corollary 2.6. *Let Γ be a discrete group and $\mathsf{A} \subset \mathsf{B}$ be a Γ-tight inclusion of objects in $\underline{\mathbf{OA}}_\Gamma$. If the action of Γ on A is faithful, then Γ is an ICC group.*

It follows from the definition that for any $\mathsf{B} \in \underline{\mathbf{OA}}_G$, the inclusion $\mathbb{C} \subset \mathsf{B}$ is G-tight.

Proposition 2.7. *Let G be an lcsc group and $\mathsf{B} \in \underline{\mathbf{OA}}_G$. Then B admits a maximal subalgebra A such that $\mathsf{A} \subset \mathsf{B}$ is tight.*

Proof. This follows from a standard Zorn's lemma argument combined with the above observation that the inclusion $\mathbb{C} \subset \mathsf{B}$ is G-tight. ■

We continue with some basic examples of G-tight inclusions. More examples will be studied in later sections.

Example 2.8. For every countable discrete group Γ, $\mathbb{C}$ is a maximal subalgebra of $\ell^\infty(\Gamma)$ which is tight. Indeed, since the right translation by any element of Γ is a ucp equivariant map on $\ell^\infty(\Gamma)$, any function in a tight subalgebra of $\ell^\infty(\Gamma)$ must be invariant under right translation by every $g \in \Gamma$, hence constant.

Example 2.9. If Y is a topological G-boundary, and X is a continuous G-factor of Y, then it follows from [15, Proposition 4.2] that the inclusion $C(X) \subset C(Y)$ is G-tight. In particular, $C(X)$ is self-tight for every topological boundary X.

Example 2.10. Let $\mathcal{H}$ be a separable Hilbert space, and let $G = \mathcal{U}(\mathcal{H})$ be the group of all unitaries on $\mathcal{H}$, considered as an uncountable discrete group. The space $\mathcal{B}(\mathcal{H})$ of all bounded linear maps on $\mathcal{H}$ is canonically a G-C^*-algebra. We show $\mathcal{B}(\mathcal{H})$ is G-self-tight. For this, let ψ be a G-equivariant ucp map on $\mathcal{B}(\mathcal{H})$, and let p be a projection on $\mathcal{H}$. Let $H \leq G$ be the group of unitaries on $\mathcal{H}$ commuting with p. Then by equivariance, $\psi(p)$ also commutes with H, hence $\psi(p) \in \{p\}'' = \mathrm{span}\{p, 1_{\mathcal{B}(\mathcal{H})} - p\}$. Thus, by positivity,

$$\psi(p) = r_p p + s_p p^\perp$$

for some $r_p, s_p \in \mathbb{R}^+$.

Now, given two projections p and q such that both their ranges and orthogonal subspaces to their ranges are infinite-dimensional, there is $u \in G$ such that $upu^* = q$ and $up^{\perp}u^* = q^{\perp}$. Then by equivariance,

$$\psi(q) = \psi(upu^*) = u\psi(p)u^* = u(r_p p + s_p p^{\perp})u^* = r_p q + s_p q^{\perp},$$

therefore $r_p = r_q (=: r)$ and $s_p = s_q (=: s)$. In particular, in this case, since ψ is unital, $1 - s_p = r_{p^{\perp}} = r_p$. Moreover, given such p as above, if we choose subprojections p_1 and p_2 of p with infinite-dimensional ranges such that $p = p_1 + p_2$, then we get

$$\begin{aligned} r p_1 + s p_1{}^{\perp} + r p_2 + s p_2{}^{\perp} &= \psi(p_1 + p_2) = \psi(p) = rp + sp^{\perp} \\ &= r p_1 + r p_2 + s(p_1 + p_2)^{\perp} \\ &= r p_1 + r p_2 + s p_1{}^{\perp} + s p_2{}^{\perp} - s 1_{\mathcal{B}(\mathcal{H})}, \end{aligned}$$

which implies $s = 0$, hence $r = 1$, and $\psi(p) = p$ for all projections p as above. Now, if q is a finite rank projection, then $q \le p$ for some projection p as above. Then, since both $p - q$ and $(p - q)^{\perp}$ have infinite ranks, we get $p = \psi(p) = \psi(p - q) + \psi(q) = p - q + \psi(q)$, which implies $\psi(q) = q$. Hence, ψ restricts to the identity map on projections, and since the set of projections spans a norm dense subspace of $\mathcal{B}(\mathcal{H})$, we conclude $\psi = \mathrm{id}$.

Rigidity properties of locally compact groups are usually passed down to their lattices. This is the case for the tightness condition.

Theorem 2.11. *Let G be an lcsc group, and let Γ be a lattice in G. Then any G-tight inclusion $\mathsf{A} \subset \mathsf{B}$ of objects in $\underline{\mathbf{OA}}_G$ is Γ-tight.*

Proof. Let Γ be a lattice in G. Suppose $\mathsf{A} \subset \mathsf{B}$ is a G-tight inclusion of objects in $\underline{\mathbf{OA}}_G$. Let $\psi\colon \mathsf{A} \to \mathsf{B}$ be a Γ-equivariant ucp map. For each $a \in \mathsf{A}$, the map $g \mapsto g\psi(g^{-1}a)$ is continuous from G to B which is constant on each left Γ-cosets by Γ-equivariance. Thus, it induces a continuous function $\rho_a\colon G/\Gamma \to \mathsf{B}$, $g\Gamma \mapsto g\psi(g^{-1}a)$. Define the map $\phi\colon \mathsf{A} \to \mathsf{B}$ by $\phi(a) := \int_{G/\Gamma} \rho_a(g\Gamma)\, d\mu(g\Gamma)$, where μ is a G-invariant Borel probability measure on G/Γ. Then ϕ is ucp, and for every $h \in G$,

$$\begin{aligned} h\phi(a) &= \int_{G/\Gamma} h\rho_a(g\Gamma)\, d\mu(g\Gamma) = \int_{G/\Gamma} hg\psi(g^{-1}a)\, d\mu(g\Gamma) \\ &= \int_{G/\Gamma} g\psi(g^{-1}ha)\, d\mu(g\Gamma) = \phi(ha), \end{aligned}$$

which shows ϕ is G-equivariant. Hence, by G-tightness, ϕ is the inclusion map. Since the maps $\mathsf{A} \ni a \mapsto g\psi(g^{-1}a) \in \mathsf{B}$ are ucp for every $g \in G$, by extremality of the inclusion map in the set of all ucp maps, it follows $g\psi(g^{-1}a) = a$ for all $a \in \mathsf{A}$ and μ-a.e. $g\Gamma \in G/\Gamma$. By continuity of the action and the fact that μ has full support, we conclude the latter equality for all $g \in G$. Hence, in particular, $\psi = \mathrm{id}_{\mathsf{A}}$. ■

The following lemma, which is essentially [19, Lemma 3.3], provides a useful technical tool for proving the tightness of inclusions.

Lemma 2.12. *Let G be an lcsc group and $\mathsf{A} \subset \mathsf{B} \subset \mathsf{C}$ be inclusions of objects in $\underline{\mathbf{OA}}_G$ such that*

(1) *the inclusion $\mathsf{A} \subset \mathsf{B}$ is G-tight,*

(2) *there is a faithful G-equivariant conditional expectation $\mathbb{E}$ from C onto B,*

then the inclusion $\mathsf{A} \subset \mathsf{C}$ is also G-tight.

Proof. If $\psi: \mathsf{A} \to \mathsf{C}$ is a G-equivariant ucp map, then $\mathbb{E} \circ \psi$ is a G-equivariant ucp map from A to B. Hence, by the tightness of $\mathsf{A} \subset \mathsf{B}$, we get $\mathbb{E} \circ \psi = \mathrm{id}_{\mathsf{A}}$. Then using the fact that $\mathbb{E}$ is a B-bimodule map and applying the Schwarz inequality for the completely positive map ψ, we see that for every $x \in \mathsf{A}$, $\mathbb{E}((x^* - \psi(x^*))(x - \psi(x))) = 0$, which, by faithfulness of $\mathbb{E}$, implies $\psi(x) = x$. Thus, the inclusion $\mathsf{A} \subset \mathsf{C}$ is G-tight. ∎

Recall our notation that $\Gamma \ltimes \mathsf{B}$ denotes either the reduced C^*-crossed product or the von Neumann crossed product for $\mathsf{B} \in \underline{\mathbf{OA}}_\Gamma$, equipped with the canonical Γ-action by inner automorphisms.

Corollary 2.13. *Let Γ be a discrete group and $\mathsf{A} \subset \mathsf{B}$ a Γ-tight inclusion of objects in $\underline{\mathbf{OA}}_\Gamma$. Then the canonical inclusion $\mathsf{A} \subset \Gamma \ltimes \mathsf{B}$ is also Γ-tight.*

Proof. The canonical conditional expectation $\mathbb{E}_0: \Gamma \ltimes \mathsf{B} \to \mathsf{B}$ is Γ-equivariant and faithful. Hence the result follows from Lemma 2.12. ∎

2.1. (Weak) Zimmer amenability

In this section, we consider a property for objects in $\underline{\mathbf{OA}}_G$, which in the case of commutative G-von Neumann algebras gives a weaker property than Zimmer-amenability. This weaker notion of Zimmer amenability is indeed enough for many applications. Furthermore, it has the advantage that it can as well be extended to C^*-algebra setting.

Definition 2.14. An object $\mathsf{B} \in \underline{\mathbf{OA}}_G$ is said to be *weakly Zimmer amenable* if for every $\mathsf{A} \in \underline{\mathbf{OA}}_G$, there is a G-equivariant ucp map $\mathsf{A} \to \mathsf{B}$.

Below are some examples that are easily verified to be weakly Zimmer amenable.

Example 2.15. For any lcsc group G, the G-C^*-algebra $C_b^{lu}(G)$ is weakly Zimmer amenable.

Example 2.16. If G is amenable, then every object $\mathsf{B} \in \underline{\mathbf{OA}}_G$ is weakly Zimmer amenable.

Example 2.17. If $\Gamma \curvearrowright (X, \nu)$ is a Zimmer amenable probability measure-class preserving action of a discrete group Γ, then $L^\infty(X, \nu)$ is a weakly Zimmer amenable Γ-C^*-algebra. In particular, if $\mu \in \mathrm{Prob}(\Gamma)$, and (B, ν) is the Furstenberg–Poisson boundary of (Γ, μ), then $L^\infty(B, \nu)$ is weakly Zimmer amenable.

Example 2.18. More generally than the previous example, since our objects are all unital, every Γ-injective $\mathsf{A} \in \underline{\mathbf{OA}}_\Gamma$ is weakly Zimmer amenable; the converse is not true: if Γ

is non-amenable, then one can see that $\mathcal{B}(\ell^2(\Gamma))$ is not Γ-injective, but it is obviously weakly Zimmer amenable.

The following is a useful characterization of weakly Zimmer amenable G-C^*-algebras. We recall the fact that $C(\partial_F G)$ is G-injective for any locally compact group G (see [24, Theorem 3.11] and [30, Theorem 6]).

Proposition 2.19. *Let G be a locally compact group. A G-C^*-algebra B is weakly Zimmer amenable if and only if $C(\partial_F G) \subseteq \mathsf{B}$ as a G-invariant operator subsystem.*

Proof. Let B be a G-C^*-algebra. Suppose B is weakly Zimmer amenable. Then there is a G-equivariant ucp map $C(\partial_F G) \to \mathsf{B}$. But any such map is isometric (e.g., [24, 30]), hence an embedding of $C(\partial_F G)$ into B as a G-invariant operator subsystem.

Conversely, suppose $C(\partial_F G) \subseteq \mathsf{B}$ as a G-invariant operator subsystem, and let A be a G-C^*-algebra. Then by G-injectivity of $C(\partial_F G)$, there is a G-equivariant ucp map from A to $C(\partial_F G)$, and so in B. Hence, B is weakly Zimmer amenable. ∎

Proposition 2.20. *Let Γ be a discrete group, and let $\mathsf{A} \subset \mathsf{B}$ be a Γ-tight inclusion of objects in $\underline{\mathbf{OA}}_\Gamma$. If B is weakly Zimmer amenable, then $\mathsf{A} = C(X)$ for some topological Γ-boundary X.*

Proof. Since B is weakly Zimmer amenable, we have $C(\partial_F \Gamma) \subseteq \mathsf{B}$ as a Γ-invariant operator subsystem by Proposition 2.19. By Γ-injectivity, there is a Γ-equivariant ucp idempotent $\psi\colon \mathsf{B} \to C(\partial_F \Gamma)$. By tightness of the inclusion $\mathsf{A} \subset \mathsf{B}$, the restriction of ψ to A is the identity map, which implies $\mathsf{A} \subseteq C(\partial_F \Gamma)$ (as a Γ-operator subsystem).

Since $C(\partial_F \Gamma)$ is an injective C^*-algebra, its multiplication coincides with the Choi–Effros product associated to ψ. Since A is a subalgebra of B, it follows that this product agrees with the original product on A. It follows that A is indeed a subalgebra of $C(\partial_F \Gamma)$, and so of the form $\mathsf{A} = C(X)$ for some topological Γ-boundary X. ∎

3. Tight inclusions in commutative setting: tight measure classes

In this section, we focus our attention on a special case of tight inclusions in the commutative setting.

Definition 3.1. Let X be a compact G-space. A non-singular probability measure $\nu \in \mathrm{Prob}(X)$ is said to be *G-tight* (or just tight if the group G is clear from the context) if it has full support, and the canonical embedding $C(X) \subset L^\infty(X, \nu)$ is a G-tight inclusion.

In [13], Furman introduced and studied the notion of the *alignment property*: given a measurable G-space (X, ν) and a compact G-space Z, a Borel measurable G-equivariant map $\pi\colon X \to Z$ is said to have the alignment property if the only Borel measurable G-equivariant map from (X, ν) to $\mathrm{Prob}(Z)$ is the one given by $x \mapsto \delta_{\pi(x)}$.

The correspondence between maps $X \to \mathrm{Prob}(Z)$ and ucp maps $C(Z) \to L^\infty(X, \nu)$ (Lemma 1.1) implies that for X and $\nu \in \mathrm{Prob}(X)$ as in Definition 3.1, ν is G-tight if and only if the identity map id: $(X, \nu) \to X$ has the alignment property in the sense of Furman.

Remark 3.2. Observe that tightness is, in fact, a property of the measure class rather than of a single measure. Furthermore, it can be considered as a property of the algebra $L^\infty(X, \nu)$: the existence of an L^1-dense, C^*-subalgebra A of $L^\infty(X, \nu)$, such that the restriction, $G \curvearrowright \mathsf{A}$, is norm-continuous and there is a unique equivariant ucp map from A into $L^\infty(X, \nu)$.

Let us restate in the case of tight measure classes the facts proven in Section 2 for general tight inclusions.

Proposition 3.3. *Let G be an lcsc group.*

(1) *If a compact G-space X admits both a tight probability measure and an invariant probability measure, then X is a singleton. In particular, for any lcsc group G, the only finite G-space that admits a tight measure class is the trivial one.*

(2) *If the group G admits a non-trivial action with a tight measure class, then G is non-amenable.*

(3) *If Γ is discrete and admits a faithful action on a compact space X supporting a tight measure class, then Γ is ICC.*

(4) *If Γ is a lattice in G, then any G-tight measure class on any compact G-space is Γ-tight.*

A natural source of tight measure classes is the following.

Theorem 3.4. *Let G be an lcsc group, and let $\mu \in \mathrm{Prob}(G)$ be an admissible probability measure on G. Suppose X is a minimal compact metrizable G-space that admits a unique μ-stationary probability measure ν such that (X, ν) is a μ-boundary. Then the measure class of ν is G-tight.*

Proof. By Lemma 1.1, any G-equivariant ucp map from $C(X)$ to $L^\infty(X, \nu)$ corresponds to a measurable equivariant map from (X, ν) to $\mathrm{Prob}(X)$. By [26, Corollary 2.10 (a)], any such map is mapped into delta measures, hence yields a G-equivariant map $(X, \nu) \to (X, \nu)$. But the identity is the unique measurable G-map on (X, ν) (see, e.g., [16, Proposition 3.2 (1)]). We conclude that the identity is the unique G-equivariant ucp map from $C(X)$ to $L^\infty(X, \nu)$. ∎

A space like X in the theorem above is called a μ-USB (stands for uniquely stationary boundary). In [20], we showed that this condition implies that X is a topological boundary.

Recall that topological boundaries yield self-tight algebras of continuous functions (Example 2.9). The same statement fails for general measurable boundaries. Namely,

$L^\infty(B, \nu)$ is not self-tight, and Theorem 3.4 shows that the tightness holds once restricting to a USB (if there is such). Another form of tightness that holds for measurable boundaries is the following well-known fact.

Lemma 3.5. *Let G be an lcsc group, $\mu \in \mathrm{Prob}(G)$ be an admissible measure and (B, ν) be a μ-boundary. Then the only normal equivariant unital positive map on $L^\infty(B, \nu)$ is the identity.*

Proof. Let $\psi \colon L^\infty(B, \nu) \to L^\infty(B, \nu)$ be a normal equivariant unital positive map. We will show that the pre-adjoint map $\psi_* \colon L^1(B, \nu) \to L^1(B, \nu)$ is the identity, which then yields the result.

Since ψ is unital and positive, $\psi_*(\nu)$ is a probability measure, and since ψ is equivariant, $\psi_*(\nu)$ is furthermore μ-stationary and ergodic. Since a measure class can support at most one ergodic stationary measure (e.g., [4, Proposition 2.6]), it follows that $\psi_*(\nu) = \nu$, and therefore $\psi_*(g\nu) = g\nu$ for all $g \in \Gamma$ by equivariance. Since (B, ν) is a μ-boundary, ν is SAT in the sense of [21], which implies that the set $\{g\nu \colon g \in \Gamma\}$ spans a norm-dense subspace of $L^1(B, \nu)$. Hence, we conclude ψ_* is identity on $L^1(B, \nu)$. ∎

Theorem 3.4 already provides a vast class of examples, especially in the case of discrete groups. There is a significant amount of work on realizations of Furstenberg–Poisson boundaries on concrete topological spaces, where the main tool is the strip criterion of Kaimanovich [22]. In many cases, the topological space is compact, and it is proven that the Furstenberg–Poisson measure is the unique stationary measure on the discussed space. These cases include actions of linear groups on flag varieties [10,22,25], hyperbolic groups acting on the Gromov boundary [22], non-elementary subgroups of mapping class groups acting on the Thurston boundary [23], among others.

Tight measure classes vs. topological boundaries

The arguments in [29] imply that for discrete groups Γ, any compact Γ-space that supports a Zimmer amenable tight measure-class must be a topological Γ-boundary. This was noted in [5, Theorem 1.3], although the authors stated the result under topological amenability, which is a stronger assumption. Our Proposition 2.20 is indeed a generalization of this result with a simpler and more conceptual proof.

However, neither arguments directly generalize to non-discrete groups. The existence of the faithful equivariant conditional expectation is a key point in Ozawa's argument, and such conditional expectation does not exist in general non-discrete cases. And the proof of Proposition 2.20 uses the injectivity of $C(\partial_F \Gamma)$ in the case of discrete groups Γ.

Below, we give a simple alternative proof of [5, Theorem 1.3], in the general case of locally compact second countable groups.

Proposition 3.6 (cf. [29] and [5]). *Let G be an lcsc group, and suppose X is a compact G-space that admits a G-tight probability measure ν. If the action $G \curvearrowright (X, \nu)$ is weakly Zimmer-amenable, then $G \curvearrowright X$ is a topological boundary action.*

Proof. For any $\eta \in \mathrm{Prob}(X)$, the Poisson transform $\mathcal{P}_\eta$ is a G-equivariant unital positive map from $C(X)$ into $C_b^{lu}(G)$. Since $G \curvearrowright (X, \nu)$ is weakly Zimmer-amenable, there is a G-equivariant unital positive map ψ_0 from $C_b^{lu}(G)$ to $L^\infty(X, \nu)$. By tightness, $\psi_0 \circ \mathcal{P}_\eta = \mathrm{id}$, which implies $\mathcal{P}_\eta$ is isometric for every $\eta \in \mathrm{Prob}(X)$. By [1, Proposition 1.1], it follows that the weak$*$ closure of the G-orbit of η in $\mathrm{Prob}(X)$ contains X, hence the action $G \curvearrowright X$ is minimal and strongly proximal. ∎

Other examples

All examples of tight measure classes presented so far occur in the setup of boundary actions. However, examples of tight measure classes appear in more general settings. Below, we give examples of actions supporting tight measure classes that are not boundary actions in a topological or measurable sense.

Example 3.7. Let G_1 and G_2 be two lcsc groups, and for $i = 1, 2$, let X_i be a compact G_i-space and $\nu_i \in \mathrm{Prob}(X_i)$ an ergodic non-trivial tight measure class. Consider the action of the product group $G = G_1 \times G_2$ on the disjoint union $X = X_1 \sqcup X_2$, where G_1 and G_2 act trivially on X_2 and X_1, respectively. Then the measure

$$\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$$

is a G-tight measure class on $X = X_1 \sqcup X_2$.

In particular, the above (somewhat superficial) example shows that, unlike the case of boundary actions, neither ergodicity of the measure nor the minimality of the topological action follow from the tightness of the measure class. Furthermore, we conclude another difference of tight measure classes to the case of boundaries.

Corollary 3.8. *In general, tightness of measures does not pass to (measurable or continuous) factors.*

Proof. The G-space (X, ν) from Example 3.7 clearly has a continuous factor, namely, the space with 2 points, equipped with the uniform measure, which is not tight by Lemma 2.3. ∎

We continue with more interesting examples of rigid actions which are not boundaries. In particular, we give examples of purely atomic tight measure classes on non-minimal, non-strongly proximal spaces.

Following [6], we say an open subgroup H of an lcsc group G has *the spectral gap property* if δ_H is the unique H-invariant mean on $\ell^\infty(G/H)$.

Examples of subgroups with the spectral gap property include the following (see [6] for more on these examples):

- $\mathrm{SL}_n(\mathbb{Z}) \leq \mathrm{SL}_{n+1}(\mathbb{Z})$ for $n \geq 2$;
- $H \leq H * K$ for any non-amenable H and any K with $|K| \geq 3$.

Theorem 3.9. *Let H be an open subgroup of G. Then the following are equivalent:*

(1) *H has the spectral gap property.*

(2) *The inclusion $\ell^\infty(G/H) \subset \ell^\infty(G/H)$ is G-tight.*

(3) *The inclusion $\ell^\infty(G/H) \subset \mathcal{B}(\ell^2(G/H))$ is G-tight.*

Proof. (1) $\Rightarrow$ (2): Let $\psi: \ell^\infty(G/H) \to \ell^\infty(G/H)$ be a G-equivariant unital positive map. Then $\psi^*(\delta_H)$ is an H-invariant mean on $\ell^\infty(G/H)$, hence $\psi^*(\delta_H) = \delta_H$ by uniqueness. Since ψ^* is G-equivariant, it follows that $\psi^*(\delta_{gH}) = \delta_{gH}$ for all $g \in G$. Since the set δ_{gH}, $g \in G$, separates points of $\ell^\infty(G/H)$, we conclude that $\psi = \mathrm{id}$.

(2) $\Rightarrow$ (3): The canonical conditional expectation $\mathcal{B}(\ell^2(G/H)) \to \ell^\infty(G/H)$ is G-equivariant and faithful. Hence, it follows from Lemma 2.12 that the inclusion $\ell^\infty(G/H) \subset \mathcal{B}(\ell^2(G/H))$ is G-tight.

(3) $\Rightarrow$ (1): Assume H does not have the spectral gap property, and let ϕ be an H-invariant mean on $\ell^\infty(G/H)$ different from δ_H. Consider the Poisson transform

$$\mathcal{P}_\phi: \ell^\infty(G/H) \to \ell^\infty(G),$$

where G is regarded as a discrete group. Then $\mathcal{P}_\phi$ is indeed mapped into $\ell^\infty(G/H)$, and therefore can be considered as a map from $\ell^\infty(G/H)$ to $\mathcal{B}(\ell^2(G/H))$. We observe that $\mathcal{P}_\phi^*(\delta_H) = \phi$. In particular, $\mathcal{P}_\phi \neq \mathrm{id}$, hence the inclusion $\ell^\infty(G/H) \subset \mathcal{B}(\ell^2(G/H))$ is not G-tight. ∎

In the setting of the above theorem, if, moreover, the action of G on the Stone–Čech compactification $\beta(G/H)$ of the coset space G/H is continuous (e.g., when G is discrete), we get new examples of tight measure classes. Indeed, in this case, any $\nu \in \mathrm{Prob}(G/H)$ with full support considered as a probability measure on the Stone–Čech compactification $\beta(G/H)$ is G-tight.

Denote by $\mathrm{Sub}^{\mathrm{o}}_{\mathrm{sg}}(G)$ the set of open subgroups of G with the spectral gap property. In [6, Theorem A], a representation rigidity result was proved for subgroups $\Lambda \in \mathrm{Sub}^{\mathrm{o}}_{\mathrm{sg}}(\Gamma)$ of a discrete group Γ; namely, it was shown that if Υ is a self-commensurated subgroup of Γ (that is, if $g \in \Gamma$ and $g \notin \Upsilon$, then $\Upsilon \cap g\Upsilon g^{-1}$ has infinite index in either Υ or $g\Upsilon g^{-1}$) such that the quasi-regular representation $\lambda_{\Gamma/\Upsilon}$ is weakly equivalent to $\lambda_{\Gamma/\Lambda}$, then Υ is conjugate to Λ.

Using the self-tightness property of the subgroup $H \in \mathrm{Sub}^{\mathrm{o}}_{\mathrm{sg}}(G)$, we see below that they satisfy a much stronger representation rigidity with themselves.

Definition 3.10. Let π and σ be continuous unitary representations of an lcsc group G. We say π is *barely contained* in σ, denoted $\pi \lll \sigma$, if there is a G-equivariant ucp map from $\mathcal{B}(H_\sigma)$ to $\mathcal{B}(H_\pi)$. We say π and σ are barely equivalent, denoted $\pi \overset{\mathrm{b}}{\sim} \sigma$, if $\pi \lll \sigma$ and $\sigma \lll \pi$.

If π is weakly contained in σ, then π is also barely contained in σ. Indeed, the canonical $*$-homomorphism $C^*_\sigma(G) \to C^*_\pi(G)$ extends to a $*$-homomorphism φ of the multiplier algebras, and any ucp extension $\mathcal{B}(H_\sigma) \to \mathcal{B}(H_\pi)$ of φ is automatically G-equivariant since $\sigma(G)$ is in the multiplicative domain of the ucp extension.

But the converse is very far from being the case. For instance, if G is amenable, then $\pi \overset{\mathrm{b}}{\sim} \sigma$ for any π and σ.

Theorem 3.11. *Let $H, L \in \mathrm{Sub}^{\mathrm{o}}_{\mathrm{sg}}(G)$. Then $\lambda_{G/H} \overset{\mathrm{b}}{\sim} \lambda_{G/L}$ if and only if H and L are conjugate in G.*

Proof. Let $\widetilde{\varphi}\colon \mathcal{B}(\ell^2(G/H)) \to \mathcal{B}(\ell^2(G/L))$ be a G-equivariant ucp map. Restricting $\widetilde{\varphi}$ to $\ell^\infty(G/H)$ and then composing it with the canonical conditional expectation from $\mathcal{B}(\ell^2(G/L)) \to \ell^\infty(G/L)$, we get a G-equivariant unital positive map $\varphi\colon \ell^\infty(G/H) \to \ell^\infty(G/L)$. Analogously, we obtain a G-equivariant unital positive map $\psi\colon \ell^\infty(G/L) \to \ell^\infty(G/H)$.

By Theorem 3.9, both $\ell^\infty(G/H)$ and $\ell^\infty(G/L)$ are G-self-tight, hence $\psi \circ \varphi = \mathrm{id}_{\ell^\infty(G/H)}$ and $\varphi \circ \psi = \mathrm{id}_{\ell^\infty(G/L)}$. It follows that φ and ψ are isometric linear isomorphisms, hence von Neumann algebra isomorphisms. Thus, there is a G-equivariant bijection $G/H \to G/L$. This implies there exists $g \in G$ such that $L = gHg^{-1}$. ∎

Another interesting class of examples of tight measure classes appears as atomic measures on the orbit of "parabolic-type points" as follows.

Theorem 3.12. *Let Γ be a countable discrete group and Λ a subgroup of Γ. Assume there exist a minimal compact Γ-space X and $x_0 \in X$ such that $\Lambda = \Gamma_{x_0}$ and such that δ_{x_0} is the unique Λ-invariant probability measure on X. Then*

(i) *The measure $\nu := \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{g_n x_0}$ is a tight measure on X, where $\{g_n\}_{n\in\mathbb{N}}$ is a complete set of representatives of Γ/Λ.*

(ii) *The inclusion given by the Poisson transform $\mathcal{P}_{\delta_{x_0}}(C(X)) \subset \ell^\infty(\Gamma/\Lambda)$ is a Γ-tight inclusion.*

Proof. Note that $L^\infty(X,\nu)$ is Γ-equivariantly isomorphic to $\ell^\infty(\Gamma/\Lambda)$, and this isomorphism on $C(X)$ is the Poisson transform $\mathcal{P}_{\delta_{x_0}}$. So, we only need to prove (ii). For this, we argue similarly to the proof of Theorem 3.9. Let $\psi\colon C(X) \to \ell^\infty(\Gamma/\Lambda)$ be a Γ-equivariant unital positive map. Then $\psi^*(\delta_\Lambda)$ is a Λ-invariant state on $C(X)$, hence the point evaluation at x_0 by the uniqueness assumption. Since ψ^* is Γ-equivariant, it follows that $\psi^*(\delta_{g\Lambda}) = \delta_{gx_0}$ for all $g \in \Gamma$. This shows $\psi = \mathcal{P}_{x_0}$, hence (ii) follows. ∎

Corollary 3.13. *Let $\Gamma \curvearrowright X$, $\Lambda \le \Gamma$, and let $x_0 \in X$ be as in the statement of Theorem 3.12. Then the inclusion $\mathcal{P}_{\delta_{x_0}}(C(X)) \subset \mathcal{B}(\ell^2(\Gamma/\Lambda))$ is Γ-tight.*

Proof. The canonical conditional expectation $\mathcal{B}(\ell^2(\Gamma/\Lambda)) \to \ell^\infty(\Gamma/\Lambda)$ is Γ-equivariant and faithful. Hence, it follows from Theorem 3.12 and Lemma 2.12 that the inclusion $\mathcal{P}_{\delta_{x_0}}(C(X)) \subset \mathcal{B}(\ell^2(\Gamma/\Lambda))$ is Γ-tight. ∎

We conclude the section with an example of an action satisfying the conditions of Theorem 3.12.

Example 3.14. Consider the action $\mathrm{PSL}_2(\mathbb{Z}) \curvearrowright \mathbb{P}^1(\mathbb{R})$, the element $g = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{Z})$. Then the cyclic subgroup $\Lambda = \langle g \rangle$ generated by g is the stabilizer of the point $x_0 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. For every point $x \in \mathbb{P}^1(\mathbb{R})$, we have $g^n x \to x_0$, which implies that δ_{x_0} is the unique Λ-invariant probability measure on $\mathbb{P}^1(\mathbb{R})$. Moreover, the subgroup generated by g and $h = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{Z})$ is isomorphic to the free group $\mathbb{F}_2$, and the restriction action $\mathbb{F}_2 \curvearrowright \mathbb{P}^1(\mathbb{R})$ also satisfies the conditions of Theorem 3.12.

4. Applications: intermediate objects

In this section, we use properties of tight inclusions and tight measure classes to prove certain rigidity results concerning intermediate operator algebras associated with tight inclusions.

Let $\mathsf{A} \subset \mathsf{B}$ be an inclusion of objects in $\underline{\mathbf{OA}}_G$. By an intermediate object (for the inclusion) we mean a $\mathsf{D} \in \underline{\mathbf{OA}}_G$ such that $\mathsf{A} \subset \mathsf{D} \subset \mathsf{B}$ which is also assumed to be a G-von Neumann algebra in case B is.

Let $\mathsf{B} \in \underline{\mathbf{OA}}_G$, and let $\mathsf{A}, \mathsf{C} \in \underline{\mathbf{OA}}_G$ be G-invariant C^*-subalgebras of B. We write $\mathsf{B} = \mathsf{A} \vee \mathsf{C}$ if B is the object generated by A and C; this means if B is a G-C^*-algebra, then it is the C^*-algebra generated by A and C, and if B is a G-von Neumann algebra, then it is the von Neumann algebra generated by A and C.

Definition 4.1. We say the inclusion $\mathsf{C} \subset \mathsf{B}$ is *co-tight* if there is a G-tight inclusion $\mathsf{A} \subset \mathsf{B}$ such that $\mathsf{B} = \mathsf{A} \vee \mathsf{C}$.

Example 4.2. Let $\mathbb{F}_2 \curvearrowright (Z, m)$ be an ergodic probability measure preserving (pmp) action. Then for any generating $\mu \in \mathrm{Prob}(\mathbb{F}_2)$ and any μ-stationary $\nu \in \mathrm{Prob}(\partial\mathbb{F}_2)$, using Theorem 3.4 and Lemma 2.12, one can show that the inclusion $L^\infty(Z, m) \subset L^\infty(Z \times \partial\mathbb{F}_2, m \times \nu)$ is co-tight.

Example 4.3. If Λ is a subgroup of Γ with spectral gap property, or if it is a subgroup as in the statement of Theorem 3.12, then it follows from Corollary 3.13 that the inclusion $L\Gamma \subset \Gamma \ltimes \ell^\infty(\Gamma/\Lambda)$ is co-tight, where $L\Gamma$ denotes the (left) group von Neumann algebra of Γ, that is, the von Neumann algebra generated by the left regular representation of Γ.

Recall the notion of *G-rigid extension* in the sense of Hamana [19]: if $\mathsf{A} \subset \mathsf{B}$ is an inclusion of G-C^*-algebras, then B is said to be a G-rigid extension of A if the identity map on B is the unique G-equivariant ucp extension of the identity map on A. We should remark that Hamana only considered discrete groups in [19].

Proposition 4.4. *Let G be an lcsc group, and let $\mathsf{C} \subset \mathsf{B}$ be a co-tight inclusion of G-C^*-algebras. Then B is a G-rigid extension of C.*

Proof. By the assumption, there exists a G-tight inclusion $\mathsf{A} \subset \mathsf{B}$ such that $\mathsf{B} = \mathsf{A} \vee \mathsf{C}$. Assume $\psi : \mathsf{B} \to \mathsf{B}$ is a G-equivariant ucp map such that $\psi|_{\mathsf{C}} = \mathrm{id}_{\mathsf{C}}$. By tightness, we also have $\psi|_{\mathsf{A}} = \mathrm{id}_{\mathsf{A}}$. Since ψ is ucp, it follows $\psi|_{\mathsf{A}\vee\mathsf{C}} = \mathrm{id}\,|_{\mathsf{A}\vee\mathsf{C}}$, and this implies the claim. ∎

The notion of (co-)tightness in this section is mainly used in the form of the following two simple observations.

Lemma 4.5. *Let G be an lcsc group, and let $\mathsf{C} \subset \mathsf{B}$ be a co-tight inclusion of objects in $\underline{\mathbf{OA}}_G$, and in the case that B is a von Neumann algebra, we assume that C is a von Neumann subalgebra of B. If there is a G-equivariant ucp map from B to C, then $\mathsf{C} = \mathsf{B}$.*

Proof. Assume $\psi\colon \mathsf{B} \to \mathsf{C}$ is a G-equivariant ucp map. Since $\mathsf{C} \subset \mathsf{B}$ is co-tight, there is a G-tight inclusion $\mathsf{A} \subset \mathsf{B}$ such that B is generated by A and C. By tightness of $\mathsf{A} \subset \mathsf{B}$, the map ψ restricts to identity on A, thus $\mathsf{A} \subset \mathsf{C}$. This implies by our assumptions that $\mathsf{C} = \mathsf{A} \vee \mathsf{C}$, hence $\mathsf{C} = \mathsf{B}$. ■

Lemma 4.6. *Let G be an lcsc group. A co-tight inclusion of objects in $\underline{\mathbf{OA}}_G$ admits no weakly Zimmer-amenable proper intermediate objects.*

Proof. This follows from Lemma 4.5 and the obvious fact that any intermediate object for a co-tight inclusion is also co-tight. ■

Below we use the standard terminology from the ergodic theory of joinings and relatively measure-preserving extensions. In particular, we use the terminology of [16]. We recall that a G-space $(\tilde{Y}, \tilde{\eta})$ is a *joining* of the G-spaces (Y, η) and (X, ν) if both $L^\infty(Y, \eta)$ and $L^\infty(X, \nu)$ are embedded G-equivariantly in $L^\infty(\tilde{Y}, \tilde{\eta})$ and the latter is the von Neumann algebra generated by these two subalgebras.

A factor map $\phi\colon (\tilde{Y}, \tilde{\eta}) \to (X, \nu)$ is said to be *relatively measure-preserving* if the canonical conditional expectation $L^\infty(\tilde{Y}, \tilde{\eta}) \to L^\infty(X, \nu)$ is G-equivariant. A more common definition in the ergodic theory usually uses the adjoint setup and requires equivariance of the disintegration map.

Proposition 4.7. *Let G be an lcsc group, X a compact G-space, and let ν be a G-tight measure on X. Assume that $(\tilde{Y}, \tilde{\eta})$ is a relatively measure-preserving extension of (X, ν), and $\phi\colon (\tilde{Y}, \tilde{\eta}) \to (Y, \eta)$ is a G-factor map such that $(\tilde{Y}, \tilde{\eta})$ is a joining of (Y, η) and (X, ν). If (Z, ω) is a weakly Zimmer amenable G-space, and $(\tilde{Y}, \tilde{\eta}) \xrightarrow{\varphi} (Z, \omega) \xrightarrow{\psi} (Y, \eta)$ are G-factor maps such that $\psi \circ \varphi = \phi$, then $(\tilde{Y}, \tilde{\eta}) \overset{\varphi}{\cong} (Z, \omega)$.*

Proof. First, note that the maps ϕ, φ, and ψ yield canonical embeddings of corresponding L^∞-algebras, which we identify with their images under these embeddings.

Since $(\tilde{Y}, \tilde{\eta})$ is a relatively measure-preserving extension of (X, ν), there is a G-equivariant faithful conditional expectation $L^\infty(\tilde{Y}, \tilde{\eta}) \to L^\infty(X, \nu)$. Hence, by Lemma 2.12, the inclusion $C(X) \subset L^\infty(\tilde{Y}, \tilde{\eta})$ is G-tight.

Since $\tilde{\eta}$ is a joining of ν and η, we have

$$L^\infty(\tilde{Y}, \tilde{\eta}) = L^\infty(X, \nu) \vee L^\infty(Y, \eta) = C(X) \vee L^\infty(Y, \eta),$$

therefore the inclusion $L^\infty(Y, \eta) \subset L^\infty(\tilde{Y}, \tilde{\eta})$ is co-tight. Thus, if $L^\infty(Z, \omega)$ is weakly Zimmer amenable, it follows by Lemma 4.6 that $L^\infty(\tilde{Y}, \tilde{\eta}) = L^\infty(Z, \omega)$, and this completes the proof. ■

In [16], Furstenberg and Glasner introduced and studied the notion of *standard covers*. Let us recall that a (G,μ)-space $(\widetilde{Y},\widetilde{\eta})$ is called *standard* if it is a relatively measure-preserving extension of a μ-boundary. The structure theorem of Furstenberg–Glasner [16, Theorem 4.3] states that for each (G,μ)-space (Y,η), there exists a unique standard space $(\widetilde{Y},\widetilde{\eta})$, called the standard cover of (Y,η), with the property that there exists a μ-boundary (X,θ) which is a relatively measure-preserving factor of $(\widetilde{Y},\widetilde{\eta})$, and such that $(\widetilde{Y},\widetilde{\eta})$ is a joining of (Y,η) and (X,θ). Using this terminology, Nevo–Zimmer's theorem [28] states that for higher rank semisimple Lie groups many stationary actions are standard. Moreover, in this setup, not only that the spaces are standard, but the μ-boundary (X,θ) admits a USB model.

Theorem 4.8. *Let G be an lcsc group and $\mu \in \mathrm{Prob}(G)$ an admissible measure. Let (Y,η) be a (G,μ)-space with the standard cover $(\widetilde{Y},\widetilde{\eta})$ realized by the factor maps*

$$\phi\colon (\widetilde{Y},\widetilde{\eta}) \to (Y,\eta) \quad \textit{and} \quad \phi'\colon (\widetilde{Y},\widetilde{\eta}) \to (X,\theta),$$

where (X,θ) is a μ-boundary and ϕ' is relatively measure-preserving. Let (Z,ω) be a weakly Zimmer amenable (G,μ)-space, and $(\widetilde{Y},\widetilde{\eta}) \xrightarrow{\varphi} (Z,\omega) \xrightarrow{\psi} (Y,\eta)$ are G-factors such that $\psi \circ \varphi = \phi$. If either

(i) *(X,θ) has a metrizable USB model or*

(ii) *the Furstenberg–Poisson boundary of (G,μ) has a metrizable USB model,*

then $(\widetilde{Y},\widetilde{\eta}) \overset{\varphi}{\cong} (Z,\omega)$.

Proof. Case (i): Without loss of generality, we may assume that X is metrizable and (X,θ) is μ-USB. Then θ is a tight measure by Theorem 3.4, and therefore the assertion follows from Proposition 4.7.

Case (ii): Let (B,ν) denote a metrizable USB model of the Furstenberg–Poisson boundary of (G,μ). The space $(\widetilde{Y},\widetilde{\eta})$ is weakly Zimmer amenable as it is a G-extension of a weakly Zimmer amenable space (Z,ω), so there is a G-equivariant ucp map from $C(B)$ to $L^\infty(\widetilde{Y},\widetilde{\eta})$. By Lemma 1.1, the map corresponds to a measurable G-equivariant map $\rho\colon \widetilde{Y} \to \mathrm{Prob}(B)$. Since (B,ν) is USB, it follows from [26, Corollary 2.10 (a)] that ρ is mapped into delta measures, hence a G-equivariant factor map $\rho\colon (\widetilde{Y},\widetilde{\eta}) \to (B,\nu)$ (cf. [27, Theorem 9.2 (1)]). We claim that $\phi' = \mathbf{bnd} \circ \rho$, where $\mathbf{bnd}\colon (B,\nu) \to (X,\theta)$ is the canonical G-factor map. The claim then implies that ρ is relatively measure-preserving by [16, Theorem 4.3] since it is an extension of the Furstenberg–Poisson boundary. Hence, this puts us in the setup of Proposition 4.7, and therefore completes the proof. The claim, indeed, holds in more generality where USB assumption is not needed. We state this in the following lemma. ∎

Lemma 4.9. *Let G be an lcsc group, and let $\mu \in \mathrm{Prob}(G)$ be an admissible measure. Let (X,θ) be a (G,μ)-boundary, let (Y,η) be a (G,μ)-space, and let $\rho\colon (Y,\eta) \to (X,\theta)$ be a relatively measure-preserving factor map. Then ρ^* is the unique normal G-equivariant ucp map from $L^\infty(X,\theta)$ to $L^\infty(Y,\eta)$.*

Proof. Let $\Phi\colon L^\infty(X,\theta) \to L^\infty(Y,\eta)$ be a normal G-equivariant ucp map. Denote by $\mathbb{E}\colon L^\infty(Y,\eta) \to \rho^*(L^\infty(X,\theta))$ the canonical conditional expectation, which is G-equivariant by the assumption. Then $\mathbb{E} \circ \Phi$ is a normal unital positive G-equivariant map from $L^\infty(X,\theta)$ to $\rho^*(L^\infty(X,\theta))$, hence coincides with ρ^* by Lemma 3.5. Since $\mathbb{E}$ is faithful, it follows that $\rho^* = \Phi$. ∎

In the noncommutative setting, co-tight inclusions appear naturally in the setting of covariant representations of actions of tight inclusions.

Proposition 4.10. *Let Γ be a discrete group, and let $\mathsf{A} \in \underline{\mathbf{OA}}_\Gamma$. Suppose that (π,ρ) is a covariant representation of (Γ,A) such that the inclusion $\rho(\mathsf{A}) \subset \Gamma \ltimes_\pi^\rho \mathsf{A}$ is Γ-tight. Let $\mathsf{D} \in \underline{\mathbf{OA}}_\Gamma$ be an intermediate object for the inclusion $C_\pi^*(\Gamma) \subset \Gamma \ltimes_\pi^\rho \mathsf{A}$. If there is a Γ-equivariant ucp map from $\Gamma \ltimes_\pi^\rho \mathsf{A}$ to D, then $\mathsf{D} = \Gamma \ltimes_\pi^\rho \mathsf{A}$.*

In particular, if D is either weakly Zimmer-amenable or injective, then $\mathsf{D} = \Gamma \ltimes_\pi^\rho \mathsf{A}$.

Proof. Note that since the inclusion $\rho(\mathsf{A}) \subset \Gamma \ltimes_\pi^\rho \mathsf{A}$ is Γ-tight and $\Gamma \ltimes_\pi^\rho \mathsf{A}$ is generated by $\rho(\mathsf{A})$ and $C_\pi^*(\Gamma)$, the inclusion $C_\pi^*(\Gamma) \subset \Gamma \ltimes_\pi^\rho \mathsf{A}$ is co-tight. This implies that the inclusion $\mathsf{D} \subset \Gamma \ltimes_\pi^\rho \mathsf{A}$ is also co-tight and hence, the claim follows from Lemma 4.5.

The case of weakly Zimmer-amenable D follows from Lemma 4.6. If D is injective, then there is a conditional expectation

$$\mathbb{E}\colon \Gamma \ltimes_\pi^\rho \mathsf{A} \to \mathsf{D},$$

which is automatically Γ-equivariant since $\pi(\Gamma)$ is in the multiplicative domain of $\mathbb{E}$. The assertion now follows similarly to the weakly Zimmer-amenable case. ∎

We denote by $C_{\mathrm{red}}^*(\Gamma)$ the reduced C^*-algebra of Γ, that is, the C^*-algebra generated by the left regular representation of Γ on $\ell^2(\Gamma)$.

Corollary 4.11. *Let Γ be a discrete group, let $\mathsf{A} \in \underline{\mathbf{OA}}_\Gamma$ be Γ-self-tight, and let D be an intermediate object for the inclusion $C_{\mathrm{red}}^*(\Gamma) \subset \Gamma \ltimes \mathsf{A}$. If D is either weakly Zimmer-amenable or injective, then $\mathsf{D} = \Gamma \ltimes \mathsf{A}$.*

Proof. By Corollary 2.13, the inclusion $\mathsf{A} \subset \Gamma \ltimes \mathsf{A}$ is Γ-tight. Hence, the result follows from Proposition 4.10. ∎

The above corollary applies, for example, in the case $\mathsf{A} = C(X)$, where X is a topological boundary (see Example 2.9).

Corollary 4.12. *Let Γ be a discrete group, B a Γ-von Neumann algebra, and A a weak$*$-dense Γ-C^*-subalgebra of B such that $\mathsf{A} \subset \mathsf{B}$ is Γ-tight. Suppose that D is an intermediate Γ-von Neumann algebra for the crossed product, that is, $L\Gamma \subset \mathsf{D} \subset \Gamma \ltimes \mathsf{B}$. If D is either weakly Zimmer-amenable or injective, then $\mathsf{D} = \Gamma \ltimes \mathsf{B}$.*

Proof. By Corollary 2.13, the inclusion $\mathsf{A} \subset \Gamma \ltimes \mathsf{B}$ is also Γ-tight. Thus, the result follows from Proposition 4.10. ∎

Next, we prove noncommutative generalizations of earlier results regarding intermediate objects, and apply them to prove a maximal injectivity result (Corollary 4.16).

We begin with an extension of Lemma 4.6.

Lemma 4.13. *Let Γ be a countable discrete group, and let $\mathsf{C} \subset \mathsf{B}$ be a co-tight inclusion of objects in $\underline{\mathbf{OA}}_\Gamma$. Then the inclusion $\Gamma \ltimes \mathsf{C} \subset \Gamma \ltimes \mathsf{B}$ is co-tight, hence, in particular, has no injective or weakly Zimmer amenable intermediate proper objects.*

Proof. By co-tightness of $\mathsf{C} \subset \mathsf{B}$, there is a Γ-tight inclusion $\mathsf{A} \subset \mathsf{B}$ such that $\mathsf{B} = \mathsf{A} \vee \mathsf{C}$. Then the inclusion $\mathsf{A} \subset \Gamma \ltimes \mathsf{B}$ is tight by Corollary 2.13, which then implies the inclusion $\Gamma \ltimes \mathsf{C} \subset \Gamma \ltimes \mathsf{B}$ is co-tight. Hence, arguing similarly to the proof of Proposition 4.10, we conclude that the inclusion $\Gamma \ltimes \mathsf{C} \subset \Gamma \ltimes \mathsf{B}$ has no injective or weakly Zimmer amenable intermediate proper objects. ∎

The following is the noncommutative extension of Proposition 4.7.

Proposition 4.14. *Let Γ be a discrete group, X a compact Γ-space, and let ν be a Γ-tight measure on X. Assume that $(\widetilde{Y}, \widetilde{\eta})$ is a relatively measure-preserving extension of (X, ν), and (Y, η) a Γ-factor of $(\widetilde{Y}, \widetilde{\eta})$ such that $(\widetilde{Y}, \widetilde{\eta})$ is a joining of (Y, η) and (X, ν). Suppose $\Gamma \ltimes L^\infty(Y, \eta) \subseteq M \subseteq \Gamma \ltimes L^\infty(\widetilde{Y}, \widetilde{\eta})$ is an inclusion of von Neumann algebras. If M is injective, then $M = \Gamma \ltimes L^\infty(\widetilde{Y}, \widetilde{\eta})$.*

Proof. As seen in the proof of Proposition 4.7, the assumptions imply that the inclusion $L^\infty(Y, \eta) \subseteq L^\infty(\widetilde{Y}, \widetilde{\eta})$ is co-tight. Thus, the result follows from Lemma 4.13. ∎

This result provides large classes of examples. We single out two interesting cases in the following.

Corollary 4.15. *Let Γ be a discrete group and $\mu \in \mathrm{Prob}(\Gamma)$ a generating measure such that the Furstenberg–Poisson boundary (B, ν) of (Γ, μ) has a metrizable μ-USB model. Then the conclusion of Proposition 4.14 holds in the following two cases, for any Γ-factor map $(\widetilde{Y}, \widetilde{\eta}) \to (Y, \eta)$:*

(i) *$(\widetilde{Y}, \widetilde{\eta})$ is Zimmer amenable and is the standard cover of (Y, η); or*

(ii) *$(\widetilde{Y}, \widetilde{\eta}) = (B \times Z, \nu \times m)$ and $(Y, \eta) = (Z, m)$, with the canonical factor map, for some ergodic pmp action $\Gamma \curvearrowright (Z, m)$.*

Proof. Under the assumptions of part (i), it was shown in the proof of Theorem 4.8 (ii) that $(\widetilde{Y}, \widetilde{\eta})$ is a relatively measure-preserving extension of (B, ν), and also is a joining of (Y, η) and (B, ν). Since ν is Γ-tight by Theorem 3.4, the assertion (i) follows from Proposition 4.14.

Now, assume (ii). Then $(\widetilde{Y}, \widetilde{\eta})$ is a relatively measure-preserving extension of (B, ν), indeed the integration of the Z-component with respect to m is the canonical conditional expectation $L^\infty(B \times Z, \nu \times m) \to L^\infty(B, \nu)$, which is obviously Γ-equivariant. Also, $(\widetilde{Y}, \widetilde{\eta})$ is clearly a joining of (Y, η) and (B, ν). Thus, assertion (ii) also follows from Proposition 4.14. ∎

The above statements can be considered as "minimal ambient injectivity" results. Since the commutant of an injective von Neumann algebra is also injective (regardless of the representation), by taking commutants in the above inclusions, we obtain (new) examples of maximal injective von Neumann subalgebras.

In particular, in the product space case as in part (ii) of Corollary 4.15, we get the following.

Given a non-singular action $\Gamma \curvearrowright (X, \nu)$, we have an action $\Gamma \curvearrowright \mathcal{B}(L^2(X, \nu))$ by inner automorphisms associated with the corresponding Koopman representation.

Corollary 4.16. *Let Γ be a discrete group and $\mu \in \mathrm{Prob}(G)$ a generating measure such that the Furstenberg–Poisson boundary (B, ν) of (Γ, μ) has a metrizable USB model. Let $\Gamma \curvearrowright (Z, m)$ be a measure-preserving action. Then the von Neumann algebra $\Gamma \ltimes L^\infty(B \times Z, \nu \times m)$ is maximal injective in $\Gamma \ltimes (\mathcal{B}(L^2(B, \nu)) \bar{\otimes} L^\infty(Z, m))$.*

Proof. Assume that N is an injective von Neumann algebra, and that we have

$$\Gamma \ltimes L^\infty(B \times Z, \nu \times m) \subseteq N \subseteq \Gamma \ltimes (\mathcal{B}(L^2(B, \nu)) \bar{\otimes} L^\infty(Z, m)).$$

Taking commutants in $B\big(\ell^2(\Gamma) \otimes L^2(B \times Z, \nu \times m)\big)$, we get (see, e.g., [32, Proposition V.7.14]) the inclusion of von Neumann algebras

$$\Gamma \ltimes L^\infty(Z, m) \subseteq N' \subseteq \Gamma \ltimes L^\infty(B \times Z, \nu \times m),$$

and N' is injective. Thus, $N' = \Gamma \ltimes L^\infty(B \times Z, \nu \times m)$ by Proposition 4.14, and hence $N = \Gamma \ltimes L^\infty(B \times Z, \nu \times m)$. ∎

We note that the above corollary can be, of course, stated in the more general setup of Proposition 4.14.

The special case of Corollary 4.16, where Γ is an irreducible lattice in a higher rank lattice G, and Z is the trivial space, yields a stronger version of Suzuki's maximal injectivity result in [31], where he proves that $\Gamma \ltimes L^\infty(B, \nu)$ is maximal injective in $\Gamma \ltimes L^\infty(Y, \eta)'$, where (Y, η) is any essentially-free measurable Γ-factor of (B, ν), and the commutant of $L^\infty(Y, \eta)$ is taken in $\mathcal{B}(L^2(B, \nu))$ (see the last assertion in [31, Corollary 3.8]). In Suzuki's proof, the essential freeness assumption was needed to apply the splitting result for the intermediate von Neumann algebras that he proved in the same paper. Another key step of his proof is the use of Margulis' factor theorem, which is a deep result concerning higher rank lattices. In particular, Corollary 4.16 shows that the source of such rigidity of intermediate von Neumann algebras is not a higher rank, but a much broader phenomenon of tightness. In fact, we obtain a large class of new examples of maximal injective von Neumann algebras. As remarked before, examples of Furstenberg–Poisson boundary actions with a uniquely stationary compact metrizable model include actions of hyperbolic groups on their Gromov boundaries, linear groups on flag varieties, mapping class groups on the Thurston boundary, and $\mathrm{Out}(\mathbb{F}_n)$ on the boundary of the outer space, all for suitable μ's.

Remark 4.17. Methods similar to those described above may be applied to the case of tight inclusions, as in Theorem 3.12, to prove maximal injectivity of $L\Lambda$ in $L\Gamma$. More precisely, assume that $\Gamma \curvearrowright X$, and $x_0 \in X$ and $\Lambda \leq \Gamma$ are as in the statement of Theorem 3.12. In addition, assume that $\Lambda = \Gamma_{x_0}$ is a maximal amenable subgroup of Γ. We will show that $L\Lambda$ is maximal injective in $L\Gamma$. Obviously, it is equivalent to show this for the right group von Neumann algebras $R\Lambda \subseteq R\Gamma$. If $R\Lambda \subseteq N \subseteq R\Gamma$ is an inclusion of von Neumann algebras with N injective, then $L\Gamma \subseteq N' \subseteq R\Lambda' = L\Gamma \vee \ell^\infty(\Gamma/\Lambda)$, and N' is injective. The canonical conditional expectation $B(\ell^2(\Gamma)) \to \ell^\infty(\Gamma)$, which is Γ-equivariant and faithful, restricts to a conditional expectation $R\Lambda' \to \ell^\infty(\Gamma/\Lambda)$. By Theorem 3.12, $\mathcal{P}_{\delta_{x_0}}(C(X)) \subset \ell^\infty(\Gamma/\Lambda)$ is Γ-tight. Furthermore, since $\Lambda = \Gamma_{x_0}$, it follows that $\mathcal{P}_{\delta_{x_0}}(C(X))$ is weak$*$-dense in $\ell^\infty(\Gamma/\Lambda)$, thus the inclusion $L\Gamma \subset L\Gamma \vee \ell^\infty(\Gamma/\Lambda)$ is co-tight. Invoking Proposition 4.10, we conclude $N' = L\Gamma \vee \ell^\infty(\Gamma/\Lambda)$, hence $N \subseteq R\Gamma$.

This applies to the case of maximal abelian subgroups of free groups as in Example 3.14. However, the result of Boutonnet–Carderi [7, Theorem A] covers these examples, but our methods offer a new approach, which may result in new examples in this setup.

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