Interfaces Free Bound. 26 (2024), 31–44
DOI 10.4171/IFB/501
© 2023 European Mathematical Society
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Quantitative convergence of the "bulk" free boundary in an oscillatory obstacle problem

Farhan Abedin and William M. Feldman

Abstract. We consider an oscillatory obstacle problem where the coincidence set and free boundary are also highly oscillatory. We establish a rate of convergence for a regularized notion of free boundary to the free boundary of a corresponding classical obstacle problem, assuming the latter is regular. The convergence rate is linear in the minimal length scale determined by the fine properties of a corrector function.

1. Introduction

Let $U \subset \mathbb{R}^n$ be a smooth, bounded domain, and let $\varphi_0 \in C^2(U) \cap C(\overline{U})$ be an obstacle that is positive somewhere in U, negative on ∂U, and satisfies the ellipticity condition

$$\lambda \leq -\Delta\varphi_0 \leq \lambda^{-1} \tag{1}$$

for some $1 \geq \lambda > 0$. Consider the following obstacle minimal supersolution above φ_0:

$$u_0(x) := \min\big\{v : \Delta v \leq 0 \text{ in } U,\ v \geq \varphi_0(x) \text{ in } U,\ v \geq 0 \text{ on } \partial U\big\}. \tag{2}$$

Then, u_0 satisfies

$$\min\big\{\Delta u_0, u_0 - \varphi_0\big\} = 0.$$

The non-contact set of u_0 is $\Omega_0 := \{u_0 > \varphi_0\} \cap U$, the contact (or coincidence) set of u_0 is $\Lambda_0 := \{u_0 = \varphi_0\} \cap U$, and the free boundary is the set $\Gamma_0 := \partial\{u_0 = \varphi_0\} \cap U$.

In this work we study a natural toy model for the behavior of an elastic membrane resting on a rough surface. Let ψ be $\mathbb{Z}^n$-periodic, with the normalization

$$\min_{\mathbb{R}^n} \psi = -1 \quad \text{and} \quad \max_{\mathbb{R}^n} \psi = 0, \tag{3}$$

and let $p \in \mathbb{R}$ be a given exponent. For each $\varepsilon > 0$, define the rough obstacle

$$\varphi_\varepsilon(x) := \varphi_0(x) + \varepsilon^p \psi(x/\varepsilon).$$

2020 Mathematics Subject Classification. Primary 35B27; Secondary 35J86, 35R35.
Keywords. Obstacle problem, homogenization, free boundaries, large scale regularity.

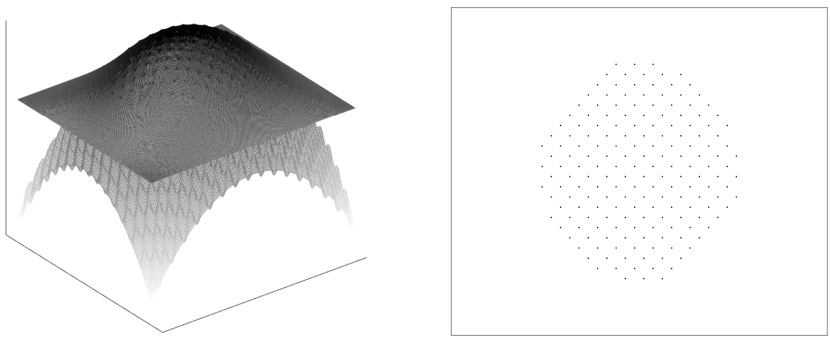

Figure 1. Left: simulation of the obstacle problem solution above a parabolic obstacle perturbed by an oscillating sinusoid. Right: contact set of the solution with the obstacle.

Consider the obstacle minimal supersolution

$$u_\varepsilon(x) = \min\bigl\{v : \Delta v \le 0 \text{ in } U,\ v \ge \varphi_0(x) + \varepsilon^p \psi(x/\varepsilon) \text{ in } U,\ v \ge 0 \text{ on } \partial U\bigr\} \tag{4}$$

which satisfies

$$\min\{\Delta u_\varepsilon, u_\varepsilon - \varphi_\varepsilon\} = 0.$$

The non-contact set of u_ε is $\Omega_\varepsilon := \{u_\varepsilon > \varphi_\varepsilon\} \cap U$, the contact (or coincidence) set of u_ε is $\Lambda_\varepsilon := \{u_\varepsilon = \varphi_\varepsilon\} \cap U$, and the free boundary is the set $\Gamma_\varepsilon := \partial\{u_\varepsilon = \varphi_\varepsilon\} \cap U$.

Our goal is to quantitatively compare the functions u_ε and u_0, as well as the contact sets Λ_ε and Λ_0. Note that the obstacle φ_ε and contact set Λ_ε may be highly oscillatory. Generally speaking, when $p < 2$ the obstacle solution u_ε rests on the peaks of ψ and the contact set is effectively "discretized"; see Figure 1. Furthermore, as illustrated by the example in Figure 1, one cannot expect the free boundary Γ_ε of the oscillatory obstacle problem to converge in Hausdorff distance to the free boundary Γ_0 of the unperturbed obstacle problem.

The purpose of this note is to show that certain analogues of the basic regularity theory for the classical obstacle problem can be developed for the oscillatory obstacle problem and used to define the notion of the *bulk contact set* $\widetilde{\Lambda}_\varepsilon$ and *bulk free boundary* $\widetilde{\Gamma}_\varepsilon$, which can be compared directly with the effective contact set Λ_0 and Γ_0, respectively. The rate of convergence of $\widetilde{\Gamma}_\varepsilon$ to Γ_0 is determined by fine properties of a corrector-type function arising from an appropriate cell problem, which is studied in Section 2.

Let us state our main result more precisely. We first introduce the corrector function. For each $\mu > 0$, let χ_μ be the $\mathbb{Z}^n$-periodic minimal supersolution of the problem

$$\chi_\mu := \min\bigl\{v : \Delta v \le \mu \text{ in } \mathbb{R}^n,\ v \ge \psi(x) \text{ in } \mathbb{R}^n,\ v \text{ is } \mathbb{Z}^n\text{-periodic}\bigr\}. \tag{5}$$

Since the zero function is a supersolution for each $\mu > 0$, we have $\psi \le \chi_\mu \le 0$ in $\mathbb{R}^n$. Define

$$\mathcal{E}(\mu) = -\inf \chi_\mu = \|\chi_\mu\|_\infty. \tag{6}$$

We show in Section 2 that $\mathcal{E}(\mu) \to 0$ as $\mu \to 0$, but the rate depends sensitively on the behavior of ψ near its maxima. In some cases this rate is linear in μ, but it also may be Hölder or worse.

Next we define the minimal length scale coming from the corrector function; this is the quantity that will determine the rate of convergence in our main result.

Definition 1.1. The *minimal length scale* of the ε-oscillatory obstacle problem is

$$\mathrm{r}(\varepsilon) := (\varepsilon^p \mathcal{E}(\lambda^{-1}\varepsilon^{2-p}))^{1/2}. \tag{7}$$

We will assume henceforth that $\lim_{\varepsilon\to 0} \mathrm{r}(\varepsilon) = 0$; this is a requirement on the exponent p. The condition $p \geq 0$ is always sufficient, but we may consider $p < 0$ as well. For instance, when $\mathcal{E}(\mu) = O(\mu)$, this holds for all $p \in \mathbb{R}$, while when $\mathcal{E}(\mu) = O(\mu^\alpha)$, this holds when $(1-\alpha)p + 2\alpha > 0$.

Finally, we define the notion of the "bulk" free boundary for the ε-oscillatory problem, which we consider to be an appropriate proxy for the free boundary Γ_ε when establishing quantitative convergence results.

Definition 1.2. The *bulk contact set* of the ε obstacle problem, denoted $\widetilde{\Lambda}_\varepsilon$, is the union of cubes in the $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice that intersect Λ_ε. The *bulk free boundary* of the ε obstacle problem is the set $\widetilde{\Gamma}_\varepsilon := \partial\widetilde{\Lambda}_\varepsilon \cap U$.

We can now state our main result.

Theorem 1.1. *Assume Γ_0 consists only of regular points, in the sense of Caffarelli* [8]*. Then, there exists $C \geq 1$ depending on the solution of* (2) *such that for all $\varepsilon \leq 1$,*

$$d_H(\Lambda_0, \widetilde{\Lambda}_\varepsilon) \leq C\,\mathrm{r}(\varepsilon) \quad \textit{and} \quad d_H(\Gamma_0, \widetilde{\Gamma}_\varepsilon) \leq C\,\mathrm{r}(\varepsilon).$$

We refer to Section 4 for the precise regularity properties of Γ_0 that are assumed. We also prove a rate of convergence of the gradients which can be found in Section 5.

Remark 1.3. The estimate $d_H(\Lambda_0, \Lambda_\varepsilon) \leq C\,\mathrm{r}(\varepsilon)$ also holds, and the notion of bulk coincidence set / bulk free boundary are really needed for comparing the free boundaries.

1.1. Literature

The obstacle problem is a classical and much studied example of a PDE problem featuring a free boundary. It was realized some time ago, maybe first by De Giorgi, Dal Maso, and Longo [14], that the Γ-limit of an obstacle-type minimization problem may be of a different type depending critically on the capacity of the peaks of the obstacle. This phenomenon has been studied significantly in [2, 3, 6, 9–11, 13], although those results are not quantitative.

There is also work on the stability of the obstacle problem under perturbations of the obstacle [4, 5, 7, 19]. The primary difference between these works and ours is that the stability is measured with respect to strong norms on the Laplacian of the obstacle (L^∞

or L^1). In our case the perturbation of the Laplacian of the obstacle is only small in a weak- / negative-order sense.

One motivation for studying oscillatory obstacle problem (4) is its connection with the Hele-Shaw flow in periodic media [15, 17, 18]. The obstacle problem studied in [18] resembles (4) when $p = 2$, but the nature of the transformation from the Hele-Shaw problem precludes the kind of oscillatory contact set that we are interested in here.

The works that are closest to ours are [12] and [1]. Our results quantify the qualitative convergence results obtained in [12], and we go further by establishing the convergence of the bulk free boundary of the oscillatory obstacle problem to the free boundary of the unperturbed problem. The recent work [1] establishes a large scale regularity theory for the obstacle problem with an oscillatory divergence-form elliptic operator. They make an assumption on "compatibility" of the obstacle with the operator, which avoids the kind of oscillatory contact set that we study here. However, removing this compatibility assumption in the context of oscillatory divergence-form PDE operator would result in a cell problem that is more singular and apparently much more difficult than the one we study in Section 2, so it is not clear if the notion of bulk contact set would be useful in that context.

We also mention the paper of the second author and Kim [16] which considers a capillary problem on a rough surface. This is quite a different problem, but there are loose analogies. In [16] the surface roughness also results in a singular oscillatory contact set and contact line, and a notion of "bulk" can be used in a similar way to recover free boundary regularity at large scales.

2. Correctors and a cell problem

This section is devoted to the study of a cell problem that is meant to describe the local behavior of the ε-obstacle solution u_ε above its contact set Λ_ε. Recall the corrector function χ_μ defined in (5) and its height $\mathcal{E}(\mu)$ defined in (6). Our goal is to show (with a quantitative estimate) that $\mathcal{E}(\mu) \to 0$ as $\mu \to 0$. We point out that a simple integration-by-parts argument yields the following bound on the Dirichlet energy in a unit cell:

$$\int_{\mathbb{T}^n} |\nabla \chi_\mu|^2 dx = \int_{\mathbb{T}^n} (-\chi_\mu) \Delta \chi_\mu dx \le \mu \mathcal{E}(\mu).$$

The rate of convergence $\mathcal{E}(\mu) \to 0$ is sensitive to the structure of ψ near its zero level set, particularly the co-dimension of the zero level set and the regularity of ψ near the zero level set.

We will assume that ψ satisfies the following growth bound near its maximum (recall the normalization given in (3) on ψ):

$$\psi(x) \ge -B \operatorname{dist}(x, \{\psi = 0\})^s \quad \text{for some } \ B,\ s > 0. \tag{8}$$

Note that such a bound holds naturally with $s = 2$ when $\psi \in C^{1,1}$ near its maximum set, or with $s \in (0, 2]$ when $\psi \in C^{[s], s-[s]}$.

Theorem 2.1. *Assume assumption* (8) *holds. The corrector* $\chi_\mu \to 0$ *as* $\mu \to 0$ *with the following estimates for* $\mu < 1$*:*

(1) *For* $n = 2$ *there is a constant* C *depending on* s *and* B *so that*

$$\mathcal{E}(\mu) \le C\mu(1 + |\log \mu|).$$

(2) *For* $n \ge 3$ *there is a constant* C *depending on* s *and* B *so that*

$$\mathcal{E}(\mu) \le C\mu^{\frac{s}{s+n-2}}.$$

In particular, when $s = 2$,

$$\mathcal{E}(\mu) \le C\mu^{\frac{2}{n}}.$$

(3) *If* $\partial\{\psi < 0\}$ *contains a regular submanifold* Σ *of codimension* $k \in \{1, \dots, n\}$, *then*

$$\mathcal{E}(\mu) \le \begin{cases} C\mu & k = 1, \\ C\mu(1 + |\log \mu|) & k = 2, \\ C\mu^{\frac{s}{s+k-2}} & 2 < k \le n, \end{cases}$$

where the constants C *depend on* B, s, *and the regularity of the parametrization of* Σ.

Example 2.2. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is laminar, that is, it only depends on $m < n$ variables, then part (3) applies with $k = m$.

Proof. Let $G(x)$ be the Green's function for the Laplace operator on the torus $\mathbb{T}^n = \mathbb{R}^n$ mod $\mathbb{Z}^n$ solving

$$\Delta G = 1 - \sum_{k \in \mathbb{Z}^n} \delta_k$$

and normalized so that $\min G = 0$. Standard Green's function estimates give

$$A = \sup_{B_{1/2}(0)} |G(x) + \frac{1}{2\pi} \log |x|| < +\infty \ \text{ in } \ n = 2$$

and

$$A = \sup_{B_{1/2}(0)} |G(x) - \alpha_n |x|^{2-n}| < +\infty \quad \text{in } \ n \ge 3.$$

Recall the normalization of ψ given in(3). Suppose, without loss of generality, that $0 \in \{\psi = 0\}$. By (8),

$$\psi(x) \ge -B|x|^s.$$

Then define, for some $r < 1/2$ to be chosen,

$$h(x) = \mu\Big(G(x) + \frac{1}{2\pi}\log(r) - A\Big) - Br^s \quad \text{when } \ n = 2$$

or

$$h(x) = \mu(G(x) - \alpha_n r^{2-n} - A) - Br^s \quad \text{when } n \geq 3,$$

so that

$$h(x) \leq \psi(x) \quad \text{on } \partial B_r(0).$$

By the comparison principle,

$$h(x) \leq \chi_\mu(x) \quad \text{in } \mathbb{T}^n \setminus B_r(0).$$

Note that, when $n \geq 3$,

$$-\min h \leq \alpha_n \mu r^{2-n} + A\mu + Br^s,$$

the right-hand side is minimized when $r^{s+n-2} = \mu$, and

$$-\min h \lesssim \mu^{\frac{s}{s+n-2}}.$$

When $n = 2$,

$$-\min h \leq \frac{\mu}{2\pi}|\log r| + A\mu + Br^s,$$

so we choose $r^s = \mu$ to get

$$-\min h \lesssim \mu(1 + |\log \mu|).$$

Note that the power s only appears in the constants in this case.

The lower bound for χ_μ obtained above only holds in $\mathbb{T}^n \setminus B_r(0)$, but in $B_r(0)$ we still have $\chi_\mu \geq \psi \geq -Br^s$ which is a lower bound of the same order given the choice of r.

Next we consider the case when $\partial\{\psi < 0\}$ contains a regular submanifold Σ of co-dimension k. Then, define

$$g(x) = \int_\Sigma G(x-y)dS(y).$$

Then, $\Delta g = 1$ in $\mathbb{T}^n \setminus \Sigma$ and standard integral estimates show that

$$g(x) \leq \begin{cases} C & k = 1, \\ C(1 + |\log d(x, \Sigma)|) & k = 2, \\ Cd(x, \Sigma)^{2-k} & 3 \leq k \leq n. \end{cases}$$

Then, we do the same barrier argument as in the previous argument, replacing G with g. ■

Remark 2.3. If $\{\psi = 0\}$ consists of a finite number of points, and the lower bound given in (8) comes with an upper bound of matching order, a similar barrier from the proof of Theorem 2.1 can act as a supersolution to establish a matching asymptotic lower bound of $\mathcal{E}(\mu)$. In general, it seems tricky to establish an exact asymptotic for $\mathcal{E}(\mu)$, as that would depend, in a complicated manner, on the set $\{\psi = 0\}$ and the growth of ψ near that set.

3. L^∞ estimates for the obstacle solutions

Our goal in this section is to obtain L^∞ estimates for the difference of u_0 and u_ε; an important result we will obtain along the way is the non-degeneracy property given by Lemma 3.3. It will be convenient, at this stage, to work with appropriate height functions for each of the obstacle problems given by (2) and (4).

Let $w_0 := u_0 - \varphi_0$ be the height function for obstacle problem (2). Then, w_0 solves the obstacle problem

$$w_0(x) = \min\big\{v : \Delta v \le -\Delta\varphi_0 \text{ in } U,\ v \ge 0 \text{ in } U,\ v \ge -\varphi_0 \text{ on } \partial U\big\}.$$

We note that $\Omega_0 = \{w_0 > 0\} \cap U$ and $\Lambda_0 = \{w_0 = 0\} \cap U$. From the theory for the classical obstacle problem [8], we know that w_0 is $C^{1,1}$ and satisfies

$$\Delta w_0 = -\Delta\varphi_0 \quad \text{in } \Omega_0 \qquad \text{and} \qquad w_0 = |Dw_0| = 0 \quad \text{on } \Lambda_0.$$

Next consider the function $w_\varepsilon := u_\varepsilon - \varphi_0$ which solves the obstacle problem

$$w_\varepsilon(x) = \min\big\{v : \Delta v \le -\Delta\varphi_0 \text{ in } U,\ v \ge \varepsilon^p\psi(x/\varepsilon) \text{ in } U,\ v \ge -\varphi_0 \text{ on } \partial U\big\}. \tag{9}$$

Although w_ε can be negative, we refer to it as a "height function" for the oscillatory problem. Note that

$$\Omega_\varepsilon = \big\{w_\varepsilon > \varepsilon^p\psi(x/\varepsilon)\big\} \cap U \quad \text{and} \quad \Lambda_\varepsilon = \big\{w_\varepsilon = \varepsilon^p\psi(x/\varepsilon)\big\} \cap U.$$

Also, w_ε satisfies

$$\Delta w_\varepsilon = -\Delta\varphi_0 \quad \text{in } \Omega_\varepsilon.$$

Certainly, $w_\varepsilon \ge \varepsilon^p\psi(x/\varepsilon)$, but there is actually a much stronger lower bound in terms of the corrector, as presented in the next lemma.

Lemma 3.1. *Let $w_\varepsilon = u_\varepsilon - \varphi_0$ as above. Then,*

$$w_\varepsilon(x) \ge \varepsilon^p \chi_{\lambda^{-1}\varepsilon^{2-p}}(x/\varepsilon) \quad \textit{for all } x \in U.$$

Proof. Consider the function $v_\varepsilon(x) = \varepsilon^p\chi_{\lambda^{-1}\varepsilon^{2-p}}(x/\varepsilon) - w_\varepsilon(x)$ and the set $V = \{v_\varepsilon > 0\}$. Since $w_\varepsilon > 0$ on ∂U, we have $V \subset\subset U$. Thus, v_ε vanishes on $\partial V \cap U$.

For any $x \in V$, we have $\varepsilon^p\chi_{\lambda^{-1}\varepsilon^{2-p}}(x/\varepsilon) > w_\varepsilon(x) \ge \varepsilon^p\psi(x/\varepsilon)$. So, by (5),

$$(\Delta\varepsilon^p\chi_{\lambda^{-1}\varepsilon^{2-p}})\Big(\frac{x}{\varepsilon}\Big) = \lambda^{-1}.$$

Since w_ε satisfies $\Delta w_\varepsilon \le -\Delta\varphi_0(x) \le \lambda^{-1}$ in U, it follows that v_ε is subharmonic in V and vanishes on ∂V, from which it follows that $v_\varepsilon \equiv 0$ in V, implying that V is empty. ■

As a consequence, we have the next lemma, which is an estimate for the difference of the height functions w_0 and w_ε. Note that, in many cases, this is a significant improvement on the trivial L^∞ estimate between w_ε and w_0 of order ε^p; this is because, recalling the definition of $\mathcal{E}(\mu)$ from (6), $\min_U \varepsilon^p\chi_{\lambda^{-1}\varepsilon^{2-p}}(x/\varepsilon) = -\mathrm{r}(\varepsilon)^2$ as long as U contains a single $\varepsilon\mathbb{Z}^n$-periodic cell.

Proposition 3.2. *Let* $\mathrm{r}(\varepsilon)$ *be as defined in* (7). *Then,*

$$w_0 - \mathrm{r}(\varepsilon)^2 \le w_\varepsilon \le w_0 \quad \text{in } U.$$

Proof. To prove $w_\varepsilon \le w_0$, we observe that $w_0 \ge 0 \ge \varepsilon^p \psi(x/\varepsilon)$ in U and $w_0 = \varphi_0$ on ∂U. Therefore, w_0 is admissible for the minimization given by (9), and so $w_0 \ge w_\varepsilon$.

Next, to show $w_0 - \mathrm{r}(\varepsilon)^2 \le w_\varepsilon$, we observe that, by translation invariance, the solution of the obstacle problem on U with boundary condition $-\varphi_0 - \mathrm{r}(\varepsilon)^2$ and obstacle $-\mathrm{r}(\varepsilon)^2$ is $w_0 - \mathrm{r}(\varepsilon)^2$. Since $w_\varepsilon \ge \varepsilon^p \chi_{\lambda^{-1}\varepsilon^{2-p}}(x/\varepsilon) \ge -\mathrm{r}(\varepsilon)^2$ in U and $w_\varepsilon = -\varphi_0 \ge \varphi_0 - \mathrm{r}(\varepsilon)^2$ on ∂U, we conclude that $w_0 - \mathrm{r}(\varepsilon)^2 \le w_\varepsilon$ in U. ∎

3.1. Non-degeneracy

It is well known that the height function w_0 satisfies the following non-degeneracy property: for all $z \in \Gamma_0$ and $r > 0$ such that $B_r(z) \Subset U$, we have

$$\sup_{B_r(z)} w_0 \ge \frac{\lambda}{2n} r^2.$$

For the height function w_ε of the oscillatory problem, we will establish an analogous non-degeneracy statement at scales larger than $\mathrm{r}(\varepsilon)$.

Lemma 3.3. *For all* $z \in U$ *with* $\operatorname{dist}(z, \Lambda_\varepsilon) > (\lambda^{-1} 2n)^{\frac{1}{2}} \mathrm{r}(\varepsilon)$ *and* $r > 0$ *such that* $B_r(z) \Subset U$, *we have*

$$\sup_{B_r(z)} w_\varepsilon \ge \frac{\lambda}{2n} r^2 - \mathrm{r}(\varepsilon)^2.$$

Proof. On the set $D := B_r(z) \cap \Omega_\varepsilon$, consider the function

$$\zeta_\varepsilon(x) := w_\varepsilon(x) - \frac{\lambda}{2n}|x - z|^2.$$

Then, by (1),

$$\Delta \zeta_\varepsilon = \Delta w_\varepsilon - \lambda = -\Delta \varphi_0 - \lambda \ge 0 \quad \text{in } D.$$

The maximum principle implies ζ_ε attains its maximum on ∂D. Furthermore, since $\zeta_\varepsilon(z) = w_\varepsilon(z) \ge -\mathrm{r}(\varepsilon)^2$, we have $\max_D \zeta_\varepsilon \ge -\mathrm{r}(\varepsilon)^2$.

Now let $x_{\max} \in \partial D$ be such that $\zeta_\varepsilon(x_{\max}) = \max_D \zeta_\varepsilon$. We decompose ∂D as the disjoint union $\partial D = (\partial B_r(z) \cap \Omega_\varepsilon) \cup (\overline{B_r(z)} \cap \Lambda_\varepsilon)$. If $x_{\max} \in \partial B_r(z) \cap \Omega_\varepsilon$, we have $|x_{\max} - z| = r$, and so

$$-\mathrm{r}(\varepsilon)^2 \le \zeta_\varepsilon(x_{\max}) = w_\varepsilon(x_{\max}) - \frac{\lambda}{2n} r^2.$$

Consequently, $w_\varepsilon(x_{\max}) \ge \frac{\lambda}{2n} r^2 - \mathrm{r}(\varepsilon)^2$, from which it follows that $\sup_{B_r(z)} w_\varepsilon \ge \frac{\lambda}{2n} r^2 - \mathrm{r}(\varepsilon)^2$, as claimed.

If, on the other hand, we have $x_{\max} \in \overline{B_r(z)} \cap \Lambda_\varepsilon$, then

$$-\mathrm{r}(\varepsilon)^2 \le \zeta_\varepsilon(x_{\max}) = w_\varepsilon(x_{\max}) - \frac{\lambda}{2n}|x_{\max} - z|^2 \le -\frac{\lambda}{2n}|x_{\max} - z|^2.$$

This implies $\frac{\lambda}{2n}|x_{\max} - z|^2 \le \mathrm{r}(\varepsilon)^2$, which contradicts the assumption that $\operatorname{dist}(z, \Lambda_\varepsilon) > (\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)$. ■

Remark 3.4. The scale $\mathrm{r}(\varepsilon)$ is highly dependent on ψ; see Theorem 2.1 above. Of particular interest is when $\mathrm{r}(\varepsilon) = O(\varepsilon)$, which holds when $\mathcal{E}(\mu) = O(\mu)$ and for any value of $p \in \mathbb{R}$. We believe that $\mathrm{r}(\varepsilon)$ is the "correct" length scale to measure the contact set. This is essentially because we postulate that the dominant term in the asymptotic expansion of the height function w_ε above the "bulk" contact set is the corrector $\varepsilon^p \chi_{\varepsilon^{2-p}}(x/\varepsilon)$, which has height scaling $\mathrm{r}(\varepsilon)^2$. In order to grow away from this corrector via quadratic non-degeneracy at this same height scaling, one needs to move distance $\mathrm{r}(\varepsilon)$ away from the "bulk" contact set.

4. Distance estimates for the free boundaries

In this final section, we combine the results from the previous sections to prove Theorem 1.1. Before we can do this, it will be necessary to make precise the regularity assumptions we make on the free boundary Γ_0.

A well-known consequence of the classical regularity theory for the obstacle problem [8] is the following $C^{1,1}$ estimate for the height function w_0:

(i) $C^{1,1}$ bound: $\sup_U |D^2 w_0| \le M$ with M depending on the lower bound of $-\varphi_0$ on ∂U, on λ, and on $\|\Delta\varphi_0\|_{C^\gamma}$.

We will also assume that Γ_0 consists only of regular points in the sense of Caffarelli [8]. This leads to the following regularity properties:

(ii) Strong non-degeneracy: there exists $c_1 > 0$ such that if $x \in \Omega_0$, then $w_0(x) \ge c_1 d(x, \Gamma_0)^2$.

(iii) Uniform positive density of contact region: there exists a constant $c_2 \in (0, \frac{1}{2})$ such that for any $r > 0$ and $x \in \Lambda_0$, there exists $y \in \Lambda_0 \cap B_r(x)$ such that $B_{c_2 r}(y) \Subset \Lambda_0 \cap B_r(x)$.

Both properties follow from well-known regularity results for the classical obstacle problem. We make some convenient citations: for property (ii), apply [1, Lemma 5.5]; and for property (iii), apply [8, Theorem 7] and a compactness argument.

We recall the definition of the bulk free boundary from Definition 1.2.

Definition. The *bulk contact set* of the ε obstacle problem, denoted $\widetilde{\Lambda}_\varepsilon$, is the union of cubes in the $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice that intersect Λ_ε. The *bulk free boundary* of the ε obstacle problem is the set $\widetilde{\Gamma}_\varepsilon := \partial\widetilde{\Lambda}_\varepsilon \cap U$.

Note that if $x \in \widetilde{\Gamma}_\varepsilon$, then x belongs to some $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice cube Q_x and there is a neighboring $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice cube $\widehat{Q}_x$ such that $x \in \partial Q_x \cap \partial \widehat{Q}_x$ and $\widehat{Q}_x \subset \Omega_\varepsilon$. The center z of $\widehat{Q}_x$ then satisfies $\mathrm{dist}(z, \Lambda_\varepsilon) \geq 2(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon) > (\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)$. In particular, Lemma 3.3 can be applied at z.

The following non-degeneracy statement at the bulk free boundary is an immediate consequence of Lemma 3.3; such a non-degeneracy property can be viewed as an essential attribute of a "good" notion of bulk free boundary:

Corollary 4.1. *There is $c(n,\lambda) > 0$ so that for all $r > 0$, if $x \in \widetilde{\Gamma}_\varepsilon$, then*

$$\sup_{B_r(x)} w_\varepsilon \geq c(n,\lambda)r^2 - 2\mathrm{r}(\varepsilon)^2.$$

Proof. Let z be as defined in the preceding paragraph. For $r \leq 2|x-z|$, use $w_\varepsilon \geq -\mathrm{r}(\varepsilon)^2$; and for $r \geq 2|x-z|$, use Lemma 3.3 centered at z. ∎

Properties (i), (ii), and (iii) of Γ_0 are sufficient to derive an $\mathrm{r}(\varepsilon)$ rate of convergence of $\widetilde{\Gamma}_\varepsilon$ to Γ_0 in the Hausdorff distance, thus proving Theorem 1.1.

Proposition 4.2. *There exists $C > 0$ depending on n, λ and the quantity c_1 from property (ii) such that for all $\varepsilon \leq 1$ and $x \in \widetilde{\Gamma}_\varepsilon$, we have*

$$d(x, \Gamma_0) \leq C\,\mathrm{r}(\varepsilon).$$

Proof. Let $r = d(x, \Gamma_0)$. There are two possibilities:

Case 1: $B_r(x) \subset \Omega_0$.

Let Q_x be the $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice cube containing x. By definition of $\widetilde{\Gamma}_\varepsilon$, there is a $4(\lambda^{-1}2n)^{\frac{1}{2}}\mathrm{r}(\varepsilon)\mathbb{Z}^n$ lattice cube Q_x such that $x \in \partial Q_x$ and $Q_x \cap \Lambda_\varepsilon \neq \emptyset$; that is, there is $z \in Q_x$ such that $w_\varepsilon(z) = \varepsilon^p \psi(z/\varepsilon) \leq 0$.

Applying the strong non-degeneracy of w_0 at z, we have

$$c_1 d(z, \Gamma_0)^2 \leq w_0(z) \leq \mathrm{r}(\varepsilon)^2 + w_\varepsilon(z) \leq \mathrm{r}(\varepsilon)^2.$$

Thus, $d(z,\Gamma_0) \leq c_1^{-1/2}\mathrm{r}(\varepsilon)$. Since $d(x,z) \leq c(n,\lambda)\mathrm{r}(\varepsilon)$, it follows that $d(x,\Gamma_0) \leq C\,\mathrm{r}(\varepsilon)$, as claimed.

Case 2: $B_r(x) \subset \Lambda_0$.

Applying Corollary 4.1, we get

$$cr^2 - 2\mathrm{r}(\varepsilon)^2 \leq \sup_{B_r(x)} w_\varepsilon = \sup_{B_r(x)} (w_\varepsilon - w_0) \leq \mathrm{r}(\varepsilon)^2.$$

It follows that $d(x, \Gamma_0) \leq C\,\mathrm{r}(\varepsilon)$. ∎

Proposition 4.3. *There is $C \geq 1$ depending on n, λ, and the parameters from properties (i) and (iii) above so that for all $x \in \Gamma_0$,*

$$d(x, \widetilde{\Gamma}_\varepsilon) \leq C\,\mathrm{r}(\varepsilon).$$

Proof. Fix $\varepsilon \leq \varepsilon_0$ and let $r = d(x, \widetilde{\Gamma}_\varepsilon)$. There are two possibilities:

Case 1: $B_r(x) \subset U \setminus \widetilde{\Lambda}_\varepsilon$.

By the uniform positive density of Λ_0, we know there exists $y \in \Lambda_0 \cap B_r(x)$ such that $B_{c_2 r}(y) \Subset \Lambda_0 \cap B_r(x)$. We may also assume that $\operatorname{dist}(y, \Lambda_\varepsilon) > (\lambda^{-1} 2n)^{\frac{1}{2}} \mathrm{r}(\varepsilon)$, for otherwise we would already have $r \leq C \mathrm{r}(\varepsilon)$. Applying Corollary 4.1 in $B_{c_2 r}(y)$, we find

$$c(c_2 r)^2 - 2\mathrm{r}(\varepsilon)^2 \leq \sup_{B_{c_2 r}(y)} w_\varepsilon = \sup_{B_{c_2 r}(y)} (w_\varepsilon - w_0) \leq \mathrm{r}(\varepsilon)^2.$$

It follows that $r \leq C \mathrm{r}(\varepsilon)$.

Case 2: $B_r(x) \subset \widetilde{\Lambda}_\varepsilon$.

Suppose w_0 attains its maximum on $B_r(x)$ at $x_{\max} \in B_r(x)$. Assume without loss of generality that $r \geq \mathrm{r}(\varepsilon)$. By definition of $\widetilde{\Lambda}_\varepsilon$, we can find a point y such that $|x_{\max} - y| \leq C \mathrm{r}(\varepsilon)$ and $w_\varepsilon(y) \leq 0$. Therefore, by the non-degeneracy of w_0 and the Lipschitz estimate for w_0 (from property (i) above), we have

$$\begin{aligned} C r^2 \leq \sup_{B_r(x)} w_0 &= w_0(x_{\max}) \\ &= (w_0(x_{\max}) - w_0(y)) + (w_0(y) - w_\varepsilon(y)) + w_\varepsilon(y) \\ &\leq 2M r \mathrm{r}(\varepsilon) + \mathrm{r}(\varepsilon)^2 \leq (2M + 1) r \mathrm{r}(\varepsilon). \end{aligned}$$

Consequently, $r \leq C \mathrm{r}(\varepsilon)$, as claimed. ■

We remark that the Hausdorff distance estimate

$$d_H(\Lambda_0, \widetilde{\Lambda}_\varepsilon) \leq C \mathrm{r}(\varepsilon)$$

holds by the same arguments as in Case 1 of the previous two propositions.

5. Gradient convergence

In this section we show another notion of convergence at the level of the gradient. Specifically, we show that

$$\left(\fint_{B_{\mathrm{r}(\varepsilon)}(x)} |\nabla w_0 - \nabla w_\varepsilon|^2 dy \right)^{1/2} \leq C \mathrm{r}(\varepsilon) \quad \text{in } \{x \in U : d(x, \partial U) \geq \mathrm{r}(\varepsilon)\}.$$

This can be considered as an L^∞ estimate at scales above $\mathrm{r}(\varepsilon)$.

First of all, note that $w_\varepsilon - w_0$ is harmonic in the complement of $\Lambda_\varepsilon \cup \Lambda_0$, so

$$|\nabla w_\varepsilon(x) - \nabla w_0(x)| \leq \frac{C}{r} \sup_{B_r(x)} |w_\varepsilon(x) - w_0(x)| \leq \frac{C}{r} \mathrm{r}(\varepsilon)^2 \quad \text{for } B_r(x) \subset U \setminus (\Lambda_\varepsilon \cup \Lambda_0).$$

We apply that estimate with $r = \mathrm{r}(\varepsilon)$ to obtain

$$|\nabla w_\varepsilon(x) - \nabla w_0(x)| \leq C \mathrm{r}(\varepsilon) \quad \text{for } d(x, \Lambda_\varepsilon \cup \Lambda_0) \geq \mathrm{r}(\varepsilon).$$

If $d(x, \Lambda_0 \cup \Lambda_\varepsilon) \leq \mathrm{r}(\varepsilon)$, then the Hausdorff distance estimate established in the previous section implies

$$d(x, \Lambda_0) \leq C\,\mathrm{r}(\varepsilon).$$

The $C^{1,1}$ estimate of w_0 shows that

$$w_0(x) \leq C\,\mathrm{r}(\varepsilon)^2, \quad |\nabla w_0(x)| \leq C\,\mathrm{r}(\varepsilon)$$

and the L^∞ estimate of $w_\varepsilon - w_0$ also shows

$$w_\varepsilon(x) \leq C\,\mathrm{r}(\varepsilon)^2.$$

We just need to establish an analogous supremum estimate of $\nabla w_\varepsilon(x)$.

First of all, notice that if $w_\varepsilon(x) > 0$ then $\Delta w_\varepsilon(x) = -\Delta\varphi_0(x)$ so $-w_\varepsilon(x)\Delta w_\varepsilon(x) < 0$, while if $w_\varepsilon(x) \leq 0$, then

$$-w_\varepsilon(x)\Delta w_\varepsilon(x) \leq -w_\varepsilon(x)(-\Delta\varphi_0(x)) \leq C\,\mathrm{r}(\varepsilon)^2,$$

since $\Delta w_\varepsilon \leq -\Delta\varphi_0$ everywhere. Thus, $-w_\varepsilon(x)\Delta w_\varepsilon(x) \leq C r(\varepsilon)^2$ in either case.

Now we apply a Caccioppoli-type estimate. Take a standard cut-off function ζ which is 1 in $B_{\mathrm{r}(\varepsilon)}(x)$ and zero outside of $B_{2\mathrm{r}(\varepsilon)}(x)$ with $|\nabla\zeta| \leq \frac{C}{\mathrm{r}(\varepsilon)}$. The argument of the previous paragraph shows that

$$\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} -w_\varepsilon \Delta w_\varepsilon \zeta^2 dy \leq C\,\mathrm{r}(\varepsilon)^2.$$

On the other hand, we can compute

$$\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} -w_\varepsilon \Delta w_\varepsilon \zeta^2 dy = \fint_{B_{2\mathrm{r}(\varepsilon)}(x)} |\nabla w_\varepsilon|^2 \zeta^2 dy + \fint_{B_{2\mathrm{r}(\varepsilon)}(x)} \zeta \nabla w_\varepsilon \cdot 2 w_\varepsilon \nabla\zeta dy.$$

By Young's inequality

$$\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} \zeta \nabla u_\varepsilon \cdot 2 w_\varepsilon \nabla\zeta dy \geq -\frac{1}{2}\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} |\nabla w_\varepsilon|^2 \zeta^2 dy - 2\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} w_\varepsilon^2 |\nabla\zeta|^2 dy$$

and

$$\fint_{B_{2\mathrm{r}(\varepsilon)}(x)} w_\varepsilon^2 |\nabla\zeta|^2 dy \leq C\,\mathrm{r}(\varepsilon)^4 \frac{1}{\mathrm{r}(\varepsilon)^2}.$$

Combining the previous inequalities leads to

$$\left(\fint_{B_{\mathrm{r}(\varepsilon)}(x)} |\nabla w_\varepsilon|^2 dy\right)^{1/2} \leq C\,\mathrm{r}(\varepsilon).$$

Note that we have not proved a $\mathrm{r}(\varepsilon)$ rate for the gradient in L^∞ even for the corrector problem, so this kind of averaged estimate is basically the best we can do without knowing something more about corrector problem.

Acknowledgements. A significant portion of this work was carried out while F.A. held a postdoctoral position in the Department of Mathematics at the University of Utah, whose support is gratefully acknowledged.

Funding. F.A. acknowledges support from the AMS and Simons Foundation through an AMS-Simons Travel Grant. W.F. acknowledges the support of NSF grant DMS-2009286.

References

[1] G. Aleksanyan and T. Kuusi, Quantitative homogenization for the obstacle problem and its free boundary. 2021, arXiv:2112.10879

[2] H. Attouch and C. Picard, *Asymptotic analysis of variational problems with constraints of obstacle type*. Publ. Math. Orsay 82 [Mathematical Publications of Orsay 82] 7, Université de Paris-Sud, Département de Mathématiques, Orsay, 1982 Zbl 0498.49007 MR 683876

[3] H. Attouch and C. Picard, Variational inequalities with varying obstacles: the general form of the limit problem. *J. Functional Analysis* **50** (1983), no. 3, 329–386 MR 695419

[4] I. Blank, Sharp results for the regularity and stability of the free boundary in the obstacle problem. *Indiana Univ. Math. J.* **50** (2001), no. 3, 1077–1112 Zbl 1032.35170 MR 1871348

[5] I. Blank and J. LeCrone, Perturbed obstacle problems in Lipschitz domains: linear stability and nondegeneracy in measure. *Rocky Mountain J. Math.* **49** (2019), no. 5, 1407–1418 Zbl 1423.35460 MR 4010568

[6] L. Caffarelli and K.-a. Lee, Viscosity method for homogenization of highly oscillating obstacles. *Indiana Univ. Math. J.* **57** (2008), no. 4, 1715–1741 Zbl 1158.35109 MR 2440878

[7] L. A. Caffarelli, A remark on the Hausdorff measure of a free boundary, and the convergence of coincidence sets. *Boll. Un. Mat. Ital. A (5)* **18** (1981), no. 1, 109–113 Zbl 0453.35085 MR 607212

[8] L. A. Caffarelli, The obstacle problem revisited. *J. Fourier Anal. Appl.* **4** (1998), no. 4–5, 383–402 MR 1658612

[9] L. A. Caffarelli and A. Mellet, Random homogenization of an obstacle problem. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **26** (2009), no. 2, 375–395 Zbl 1180.35069 MR 2504035

[10] L. Carbone and F. Colombini, On convergence of functionals with unilateral constraints. *J. Math. Pures Appl. (9)* **59** (1980), no. 4, 465–500 Zbl 0415.49010 MR 607050

[11] D. Cioranescu and F. Murat, A strange term coming from nowhere. In *Topics in the mathematical modelling of composite materials*, pp. 45–93, Progr. Nonlinear Differential Equations Appl. 31, Birkhäuser, Boston, MA, 1997 MR 1493040

[12] M. Codegone and J.-F. Rodrigues, Convergence of the coincidence set in the homogenization of the obstacle problem. *Ann. Fac. Sci. Toulouse Math. (5)* **3** (1981), no. 3–4, 275–285 (1982) Zbl 0493.49008 MR 658736

[13] G. Dal Maso and P. Longo, Γ-limits of obstacles. *Ann. Mat. Pura Appl. (4)* **128** (1981), 1–50 Zbl 0467.49004 MR 640775

[14] E. De Giorgi, G. Dal Maso, and P. Longo, Γ-limits of obstacles. (Italian) *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8)* **68** (1980), no. 6, 481–487 Zbl 0512.49011 MR 639976

[15] C. M. Elliott and V. Janovský, A variational inequality approach to Hele-Shaw flow with a moving boundary. *Proc. Roy. Soc. Edinburgh Sect. A* **88** (1981), no. 1–2, 93–107 Zbl 0455.76043 MR 611303

[16] W. M. Feldman and I. C. Kim, Liquid drops on a rough surface. *Comm. Pure Appl. Math.* **71** (2018), no. 12, 2429–2499 Zbl 1418.76027 MR 3869035

[17] I. C. Kim, Error estimates on homogenization of free boundary velocities in periodic media. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **26** (2009), no. 3, 999–1019 Zbl 1177.35260 MR 2526413

[18] I. C. Kim and A. Mellet, Homogenization of a Hele-Shaw problem in periodic and random media. *Arch. Ration. Mech. Anal.* **194** (2009), no. 2, 507–530 Zbl 1251.76015 MR 2563637

[19] S. Serfaty and J. Serra, Quantitative stability of the free boundary in the obstacle problem. *Anal. PDE* **11** (2018), no. 7, 1803–1839 Zbl 1391.35432 MR 3810473

Received 11 August 2022; revised 2 May 2023.

Farhan Abedin
Department of Mathematics, Lafayette College, 224 Pardee Hall, PA 18042 Easton, US; abedinf@lafayette.edu

William M. Feldman
Department of Mathematics, University of Utah, 155 SOUTH 1400 EAST, UT 84112 Salt Lake City, US; feldman@math.utah.edu