Interfaces Free Bound. 26 (2024), 1–30
DOI 10.4171/IFB/504

Functional inequalities and strong Lyapunov functionals for free surface flows in fluid dynamics

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Abstract. This paper is motivated by the study of Lyapunov functionals for the Hele-Shaw and Mullins-Sekerka equations describing free surface flows in fluid dynamics. We prove that the L^2-norm of the free surface elevation and the area of the free surface are Lyapunov functionals. The proofs combine exact identities for the dissipation rates with functional inequalities. We introduce a functional which controls the L^2-norm of three-half spatial derivative. Under a mild smallness assumption on the initial data, we show that the latter quantity is also a Lyapunov functional for the Hele-Shaw equation, implying that the area functional is a strong Lyapunov functional. Precise lower bounds for the dissipation rates are established, showing that these Lyapunov functionals are in fact entropies.

1. Introduction

The equations

Consider a time-dependent surface Σ given as the graph of some function h so that, at time $t \geq 0$,

$$\Sigma(t) = \{(x, y) \in \mathbf{T}^d \times \mathbf{R};\ y = h(t, x)\},$$

where $\mathbf{T}^d$ denotes a d-dimensional torus. We are interested in several free boundary problems described by nonlinear parabolic equations. A free boundary problem is described by an evolution equation which expresses the velocity of Σ at each point in terms of some nonlinear expressions depending on h. The most popular example is the *mean-curvature* equation, which stipulates that the normal component of the velocity of Σ is equal to the mean curvature at each point. It follows that

$$\partial_t h + \sqrt{1 + |\nabla h|^2}\kappa = 0, \quad \text{where } \kappa = -\operatorname{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right). \tag{1.1}$$

The previous equation plays a fundamental role in differential geometry. Many other free boundary problems appear in fluid dynamics. Among these, we are chiefly concerned with the equations modeling the dynamics of a free surface transported by the flow of

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2020 Mathematics Subject Classification. Primary 35Q35; Secondary 76B07, 76N06.
Keywords. Free surface, Hele-Shaw, Mullins-Sekerka, Lyapunov, entropy.

an incompressible fluid evolving according to Darcy's law[1]. We begin with the Hele-Shaw equations with or without surface tension. One formulation of this problem reads

$$\partial_t h + G(h)(gh + \mu\kappa) = 0, \tag{1.2}$$

where κ is as in (1.1), g and μ are real numbers in $[0, 1]$, and $G(h)$ is the (normalized) Dirichlet-to-Neumann operator, defined as follows: for any functions $h = h(x)$ and $\psi = \psi(x)$,

$$G(h)\psi(x) = \sqrt{1 + |\nabla h|^2}\partial_n \mathcal{H}(\psi)\big|_{y=h(x)},$$

where $\nabla = \nabla_x$, $\partial_n = n \cdot \nabla$, and n is the outward unit normal to Σ given by

$$n = \frac{1}{\sqrt{1 + |\nabla h|^2}} \begin{pmatrix} -\nabla h \\ 1 \end{pmatrix},$$

and $\mathcal{H}(\psi)$ is the harmonic extension of ψ in the fluid domain, solution to

$$\begin{cases} \Delta_{x,y}\mathcal{H}(\psi) = 0 \quad \text{in } \Omega := \{(x, y) \in \mathbf{T}^d \times \mathbf{R} : y < h(x)\}, \\ \mathcal{H}(\psi)|_{y=h} = \psi. \end{cases} \tag{1.3}$$

Given $f = f(x, y)$, we use $f|_{y=h}$ as a short notation for $x \mapsto f(x, h(x))$.

When $g = 1$ and $\mu = 0$, equation (1.2) is called the Hele-Shaw equation without surface tension. Hereafter, we will refer to this equation simply as the *Hele-Shaw* equation. If $g = 0$ and $\mu = 1$, the equation is known as the Hele-Shaw equation with surface tension, also known as the *Mullins-Sekerka* equation. Let us record the terminologies

$$\partial_t h + G(h)h = 0 \quad \text{(Hele-Shaw)},$$
$$\partial_t h + G(h)\kappa = 0 \quad \text{(Mullins-Sekerka)}.$$

1.1. Lyapunov functionals and entropies

Our main goal is to find some time monotonicity properties for the previous free boundary flows in a unified way. Before going any further, let us fix the terminology used in this paper.

Definition 1.1. (a) Consider one of the evolution equations stated above and a function

$$I : C^\infty(\mathbf{T}^d) \to [0, +\infty).$$

We say that I is a *Lyapunov functional* if the following property holds: for any smooth solution h in $C^\infty([0, T] \times \mathbf{T}^d)$ for some $T > 0$, we have

$$\forall t \in [0, T], \quad \frac{d}{dt} I(h(t)) \leq 0.$$

The quantity $-\frac{d}{dt} I(h)$ is called the *dissipation rate* of the functional $I(h)$.

[1] A long version of our paper exists on arXiv (see [4]) which makes the link with functional inequalities (some classical, some not) for other problems with free boundaries in fluid dynamics.

(b) We say that a Lyapunov functional I is an *entropy* if the dissipation rate satisfies, for some $C > 0$,

$$-\frac{d}{dt}I(h(t)) \geq CI(h(t)).$$

(c) Eventually, we say that I is a *strong Lyapunov functional* if

$$\frac{d}{dt}I(h(t)) \leq 0 \quad \text{and} \quad \frac{d^2}{dt^2}I(h(t)) \geq 0.$$

This means that $t \mapsto I(h(t))$ decays in a convex manner.

Remark 1.2. (i) The Cauchy problems for various free boundary equations have been studied by different techniques, for weak solutions, viscosity solutions, or also classical solutions. We refer the reader to [5, 6, 16–20, 23, 25, 28, 29, 31, 32, 34, 35, 37, 39]. Thanks to the parabolic smoothing effect, classical solutions are smooth for positive times (the elevation h belongs to $C^\infty((0,T] \times \mathbf{T}^d)$). This is why we consider functionals I defined only on smooth functions $C^\infty(\mathbf{T}^d)$.

(ii) Assume that I is an entropy for an evolution equation and consider a global in time solution of the latter problem. Then, $t \mapsto I(h(t))$ decays exponentially fast. In the literature, there are more general definitions of entropies for various evolution equations. The common idea is that entropy dissipation methods allow to study the large-time behavior or to prove functional inequalities (see [9, 11–15, 24, 26, 33, 41, 42]).

(iii) To say that $I(h)$ is a strong Lyapunov functional is equivalent to saying that the dissipation rate $-\frac{d}{dt}I(h)$ is also a Lyapunov functional. This notion was introduced in [3] as a tool to find Lyapunov functionals which control higher order Sobolev norms (see also Pazy [38]). Indeed, in general, the dissipation rate is expected to be a higher-order energy because of the smoothing effect of a parabolic equation. Notice that the idea to compute the second-order derivative in time is related to the celebrated work of Bakry and Émery [10].

1.2. Examples

For the reader's convenience, we begin by discussing some examples which are well known in certain communities.

Example 1.3. Consider the heat equation $\partial_t h - \Delta h = 0$. The energy identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^d} h^2 dx + \int_{\mathbf{T}^d} |\nabla h|^2 dx = 0$$

implies that the square of the L^2-norm is a Lyapunov functional. It is in addition a strong Lyapunov functional since, by differentiating the equation, the quantity $\int_{\mathbf{T}^d} |\nabla h|^2 dx$ is also a Lyapunov functional. Furthermore, if one assumes that the mean value of $h(0,\cdot)$ vanishes, then the Poincaré inequality implies that the square of the L^2-norm is an entropy. Now, let us discuss another important property, which holds for positive solutions. Assume that $h(t,x) \geq 1$ and introduce the Boltzmann entropy, defined by

$$H(h) = \int_{\mathbf{T}^d} h \log h \, dx.$$

Then, $H(h)$ is a strong Lyapunov functional. This classical result (see Evans [27]) follows directly from the pointwise identities

$$\begin{aligned}(\partial_t - \Delta)(h \log h) &= -\frac{|\nabla h|^2}{h},\\ (\partial_t - \Delta)\frac{|\nabla h|^2}{h} &= -2h\left|\frac{\nabla^2 h}{h} - \frac{\nabla h \otimes \nabla h}{h^2}\right|^2.\end{aligned}$$

Recall that the L^1-norm of $|\nabla h|^2/h$, called the Fisher information, plays a key role in entropy methods and information theory (see Villani's lecture notes [40] and his book [41, Chapters 20, 21, and 22]).

Example 1.4 (Mean-curvature equation). Consider the mean curvature equation

$$\partial_t h + \sqrt{1+|\nabla h|^2}\kappa = 0.$$

If h is a smooth solution, then

$$\frac{d}{dt}\mathcal{H}^d(\Sigma) \leq 0, \quad \text{where } \mathcal{H}^d(\Sigma) = \int_{\mathbf{T}^d} \sqrt{1+|\nabla h|^2}\, dx. \tag{1.4}$$

This is proved by an integration by parts argument

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^d(\Sigma) &= \int_{\mathbf{T}^d} \nabla_x(\partial_t h)\cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\, dx = \int_{\mathbf{T}^d} (\partial_t h)\kappa \, dx\\ &= -\int_{\mathbf{T}^d} \sqrt{1+|\nabla h|^2}\kappa^2 \, dx \leq 0.\end{aligned}$$

In fact, the mean-curvature equation is a gradient flow for $\mathcal{H}^d(\Sigma)$; see [22].

Example 1.5 (Hele-Shaw equation). Consider the equation $\partial_t h + G(h)h = 0$. Recall that $G(h)$ is a non-negative operator. Indeed, denoting by $\varphi = \mathcal{H}(\psi)$ the harmonic extension of ψ given by (1.3), it follows from Stokes theorem that

$$\int_{\mathbf{T}^d} \psi G(h)\psi \, dx = \int_{\partial\Omega} \varphi \partial_n \varphi \, d\mathcal{H}^d = \iint_{\Omega} |\nabla_{x,y}\varphi|^2 \, dy\, dx \geq 0. \tag{1.5}$$

Consequently, if h is a smooth solution to $\partial_t h + G(h)h = 0$, then

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^d} h^2 \, dx = -\int_{\mathbf{T}^d} hG(h)h \, dx \leq 0.$$

This shows that $\int_{\mathbf{T}^d} h^2 dx$ is a Lyapunov functional. In [6], it is proved that in fact $\int_{\mathbf{T}^d} h^2 dx$ is a strong Lyapunov functional and also an entropy. This result is generalized in [3] to functionals of the form $\int_{\mathbf{T}^d} \Phi(h)dx$, where Φ is a convex function whose derivative is also convex.

Example 1.6 (Mullins-Sekerka). Assume that h solves $\partial_t h + G(h)\kappa = 0$, and denote by $\mathcal{H}^d(\Sigma)$ the area functional (see (1.4)). Then, (1.5) implies that

$$\frac{d}{dt}\mathcal{H}^d(\Sigma) = \int_{\mathbf{T}^d} (\partial_t h)\kappa\, dx = -\int_{\mathbf{T}^d} \kappa G(h)\kappa\, dx \le 0,$$

so $\mathcal{H}^d(\Sigma)$ is a Lyapunov functional. In fact, the Mullins-Sekerka equation is a gradient flow for $\mathcal{H}^d(\Sigma)$; see [7, 30].

2. Statements of the main results

Our main goal is to study the decay properties of several natural coercive quantities for the Hele-Shaw and the Mullins-Sekerka equations.

2.1. Entropies for the Hele-Shaw and Mullins-Sekerka equations

The first two coercive quantities which we want to study are the L^2-norm and the area functional (that is, the d-dimensional surface measure)

$$\left(\int_{\mathbf{T}^d} h(t,x)^2\, dx\right)^{\frac{1}{2}}, \quad \mathcal{H}^d(\Sigma) = \int_{\mathbf{T}^d} \sqrt{1 + |\nabla h|^2}\, dx.$$

Our first main result states that these are Lyapunov functionals for the Hele-Shaw and Mullins-Sekerka equations in any dimension.

Theorem 2.1. *Let $d \ge 1$, $(g,\mu) \in [0,+\infty)^2$, and assume that h is a smooth solution to*

$$\partial_t h + G(h)(gh + \mu\kappa) = 0. \tag{2.1}$$

Then,

$$\frac{d}{dt}\int_{\mathbf{T}^d} h^2\, dx \le 0 \quad \text{and} \quad \frac{d}{dt}\mathcal{H}^d(\Sigma) \le 0.$$

Remark 2.2. The main point is that this result holds uniformly with respect to g and μ. For comparison, let us recall some results which hold for the special cases, where either $g = 0$ or $\mu = 0$.

(i) When $g = 0$, the fact that the area functional $\mathcal{H}^d(\Sigma)$ decays in time follows from a well-known gradient flow structure for the Mullins-Sekerka equation. However, the decay of the L^2-norm in this case is new.

(ii) When $\mu = 0$, the decay of the L^2-norm follows from an elementary energy estimate. However, the proof of the decay of the area functional $t \mapsto \mathcal{H}^d(\Sigma(t))$ requires a more subtle argument. It is implied (but only implicitly) by some computations by Antontsev, Meirmanov, and Yurinsky in [8]. The main point is that we will give a different approach which holds uniformly with respect to g and μ. In addition, we will obtain a precise lower bound for the dissipation rate showing that $\mathcal{H}^d(\Sigma)$ is an entropy when $\mu = 0$ and not only a Lyapunov functional.

To prove these two uniform decay results, the key ingredient will be to study the following functional:

$$J(h) := \int_{\mathbf{T}^d} \kappa\, G(h)h\, dx.$$

It appears naturally when performing energy estimates. Indeed, by multiplying equation (2.1) with h or κ and integrating by parts, one obtains

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^d} h^2 dx + g\int_{\mathbf{T}^d} hG(h)h\, dx + \mu J(h) = 0, \\ \frac{d}{dt}\mathcal{H}^d(\Sigma) + gJ(h) + \mu\int_{\mathbf{T}^d}\kappa G(h)\kappa\, dx = 0. \end{aligned} \tag{2.2}$$

We will prove that $J(h)$ is non-negative. Since the Dirichlet-to-Neumann operator is a non-negative operator (see (1.5)), this will be sufficient to conclude that the L^2-norm and the area functional $\mathcal{H}^d(\Sigma)$ are non-increasing along the flow.

An important fact is that $J(h)$ is a nonlinear analog of the homogeneous $H^{3/2}$-norm. A first way to give this statement a rigorous meaning consists in noticing that $G(0)h = |D_x|h = \sqrt{-\Delta_x}h$ and the linearized version of κ is $-\Delta_x h$. Therefore, if $h = \varepsilon\zeta$, then

$$J(\varepsilon\zeta) = \varepsilon^2\int_{\mathbf{T}^d}\left(|D_x|^{3/2}\zeta\right)^2 dx + O(\varepsilon^3).$$

We will prove a functional inequality (see Proposition 3.4 below) which shows that $J(h)$ controls the $L^2(\Omega)$-norm of the Hessian of the harmonic extension $\mathcal{H}(h)$ of h, given by (1.3) with $\psi = h$. Consequently, $J(h)$ controls three-half spatial derivative of h in L^2 by means of a trace theorem.

2.2. The area functional is a strong Lyapunov functional

As seen in Example 1.4, for the mean-curvature equation in space dimension $d = 1$, there exist Lyapunov functionals which control all the spatial derivatives of order less than 2. On the other hand, for the Hele-Shaw and Mullins-Sekerka equations, it is more difficult to find higher-order energies which control some derivatives of the solution. This is because it is harder to differentiate these equations. For the Mullins-Sekerka problem, one can quote two recent papers by Chugreeva–Otto–Westdickenberg [21] and Acerbi–Fusco–Julin–Morini [1]. In both papers, the authors compute the second derivative in time of some coercive quantities to study the long-time behavior of the solutions in perturbative regimes. Here, we will prove a similar result for the Hele-Shaw equation. However, the analysis will be entirely different. On the one hand, it is easier in some sense to differentiate the Hele-Shaw equation. On the other hand, we will be able to exploit some additional identities and inequalities which allow us to obtain a result under a very mild smallness assumption.

Here, we consider the Hele-Shaw equation

$$\partial_t h + G(h)h = 0.$$

It is known that the Cauchy problem for the latter equation is well posed on the Sobolev spaces $H^s(\mathbf{T}^d)$ provided that $s > 1 + d/2$, and moreover, the critical Sobolev exponent is $1 + d/2$ (see [6, 19, 36, 37]). On the other hand, the natural energy estimate only controls the L^2-norm. It is thus natural to seek higher-order energies, which are bounded in time and which control Sobolev norms $H^\mu(\mathbf{T}^d)$ of order $\mu > 0$. It was proved in [3, 6] that one can control one-half derivative of h by exploiting some convexity argument. More precisely, it is proved in the previous references that

$$\frac{d}{dt}\int_{\mathbf{T}^d} hG(h)h\,dx \le 0.$$

This inequality gives a control of a higher-order Lyapunov functional of order 1/2. Indeed,

$$\int_{\mathbf{T}^d} hG(h)h\,dx = \iint_{\Omega} |\nabla_{x,y}\mathcal{H}(h)|^2\,dy\,dx,$$

where $\mathcal{H}(h)$ is the harmonic extension of h (solution to (1.3), where ψ is replaced by h). Hence, by using a trace theorem, it follows that $\int_{\mathbf{T}^d} hG(h)h\,dx$ controls the $H^{1/2}$-norm of h.

The search for higher-order functionals leads to interesting new difficulties. Our strategy here is to try to prove that the area functional is a strong Lyapunov functional. This means that the function $t \mapsto \mathcal{H}^d(\Sigma(t))$ decays in a convex manner. This is equivalent to $d^2\mathcal{H}^d(\Sigma)/dt^2 \ge 0$. Now, remembering (cf. (2.2)) that

$$\frac{d}{dt}\mathcal{H}^d(\Sigma) + J(h) = 0, \quad \text{where } J(h) = \int_{\mathbf{T}^d} \kappa G(h)h\,dx,$$

the previous convexity argument suggests that $dJ(h)/dt \le 0$, which implies that $J(h)$ is a Lyapunov function. This gives us a very interesting higher-order energy since the functional $J(h)$ controls three-half spatial derivative of h (as seen above, and as will be made precise in Proposition 3.4). The next result states that the previous strategy applies under a very mild smallness assumption on the first-order derivatives of the elevation h at time 0.

Theorem 2.3. *Consider a smooth solution to $\partial_t h + G(h)h = 0$. There exists a universal constant c_d depending only on the dimension d such that if initially*

$$\sup_{\mathbf{T}^d} |\nabla h_0|^2 \le c_d, \quad \sup_{\mathbf{T}^d} |G(h_0)h_0|^2 \le c_d,$$

then

$$\frac{d}{dt}J(h) + \frac{1}{2}\int_{\mathbf{T}^d} \frac{(|\nabla^2 h|^2 + |\nabla\partial_t h|^2)}{(1 + |\nabla h|^2)^{3/2}}\,dx \le 0.$$

Remark 2.4. (i) The constant c_d is the unique solution in $[0, 1)$ to

$$2c_d(d + (d + \sqrt{d})c_d) + 4\left(c_d(d + (d + 1)c_d)\left(\frac{36(d + 1)}{1 - c_d} + 1\right)\right)^{\frac{1}{2}} = \frac{1}{2}.$$

(ii) Since

$$\frac{d}{dt}J(h) = -\frac{d^2}{dt^2}\mathcal{H}^d(\Sigma),$$

it is equivalent to say that the area functional $\mathcal{H}^d(\Sigma)$ is a strong Lyapunov functional.

3. Uniform Lyapunov functionals for the Hele-Shaw and Mullins-Sekerka equations

In this section, we prove Theorem 2.1.

3.1. Maximum principles for the pressure

In this paragraph, the time variable does not play any role, and we ignore it to simplify notations.

We will need the following elementary result.

Lemma 3.1. *Consider a smooth function h in $C^\infty(\mathbf{T}^d)$, and set*

$$\Omega = \{(x,y) \in \mathbf{T}^d \times \mathbf{R} : y < h(x)\}.$$

For any ζ in $C^\infty(\mathbf{T}^d)$, there is a unique function $\phi \in C^\infty(\overline{\Omega})$ such that $\nabla_{x,y}\phi \in L^2(\Omega)$, solution to the Dirichlet problem

$$\begin{cases} \Delta_{x,y}\phi = 0 & \text{in } \Omega, \\ \phi(x,h(x)) = \zeta(x) & \text{for all } x \in \mathbf{T}^d. \end{cases} \tag{3.1}$$

Moreover, for any multi-index $\alpha \in \mathbf{N}^d$ and any $\beta \in \mathbf{N}$ with $|\alpha| + \beta > 0$, one has

$$\partial_x^\alpha \partial_y^\beta \phi \in L^2(\Omega) \quad \text{and} \quad \lim_{y\to-\infty} \sup_{x\in\mathbf{T}^d} |\partial_x^\alpha \partial_y^\beta \phi(x,y)| = 0. \tag{3.2}$$

Proof. The existence and smoothness of the solution ϕ is a classical elementary result. We prove only property (3.2).

Let y_0 be an arbitrary real number such that $\mathbf{T}^d \times \{y_0\}$ is located underneath the boundary $\partial\Omega = \{y = h\}$, and then set $\psi(x) = \phi(x, y_0)$. This function belongs to $C^\infty(\mathbf{T}^d)$ since ϕ belongs to $C^\infty(\overline{\Omega})$. Now, in the domain $\Pi := \{(x,y);\ y < y_0\}$, ϕ coincides with the harmonic extension of ψ by the uniqueness of the harmonic extension. Since Π is invariant by translation in x, we can compute the latter function by using the Fourier transform in x. It results that

$$\forall x \in \mathbf{T}^d, \quad \forall y < y_0, \quad \phi(x,y) = (e^{(y-y_0)|D_x|}\psi)(x). \tag{3.3}$$

(Here, for $\tau < 0$, $e^{\tau|D_x|}$ denotes the Fourier multiplier with symbol $e^{\tau|\xi|}$.) Indeed, the function $(e^{(y-y_0)|D_x|}\psi)(x)$ is clearly harmonic and is equal to ψ on $\{y = y_0\}$. Then, for $|\alpha| + \beta > 0$, it easily follows from (3.3) and the Plancherel theorem that $\partial_x^\alpha \partial_y^\beta \phi$ belongs to $L^2(\Pi)$. On the other hand, on the strip $\{(x, y);\ y_0 < y < h(x)\}$, the function $\partial_x^\alpha \partial_y^\beta \phi$ is bounded and hence square integrable. By combining the two previous results, we obtain that $\partial_x^\alpha \partial_y^\beta \phi$ belongs to $L^2(\Omega)$. To prove the second half of (3.2), we use again formula (3.3) and the Plancherel theorem to infer that $\partial_x^\alpha \partial_y^\beta \phi(\cdot, y)$ converges to 0 in any Sobolev space $H^\mu(\mathbf{T}^d)$ ($\mu \geq 0$) when y goes to $-\infty$. The desired decay result now follows from the Sobolev embedding $H^\mu(\mathbf{T}^d) \subset L^\infty(\mathbf{T}^d)$ for $\mu > d/2$. ∎

Let us fix some notations used in the rest of this section. Now, we consider a smooth function $h = h(x)$ in $C^\infty(\mathbf{T}^d)$ and set

$$\Omega = \{(x, y) \in \mathbf{T}^d \times \mathbf{R} : y < h(x)\}.$$

We denote by φ the harmonic extension of h in $\overline{\Omega}$. This is the solution to (3.1) in the special case, where $\zeta = h$. Namely, φ solves

$$\begin{cases} \Delta_{x,y}\varphi = 0 & \text{in } \Omega, \\ \varphi(x, h(x)) = h(x) & \text{for all } x \in \mathbf{T}^d. \end{cases} \tag{3.4}$$

Introduce $Q : \overline{\Omega} \to \mathbf{R}$ defined by

$$Q(x, y) = \varphi(x, y) - y.$$

We call Q the pressure. In this paragraph, we gather some results for the pressure which are all consequences of the maximum principle. For further references, the main result states that $\partial_y Q < 0$ everywhere in the fluid.

Proposition 3.2. (i) *On the free surface $\Sigma = \{y = h(x)\}$, the function Q satisfies the following properties:*

$$\partial_n Q = -|\nabla_{x,y} Q| \quad \textit{and} \quad n = -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|},$$

where n denotes the normal to Σ, given by

$$n = \frac{1}{\sqrt{1 + |\nabla h|^2}} \begin{pmatrix} -\nabla h \\ 1 \end{pmatrix}. \tag{3.5}$$

Moreover, the Taylor coefficient a defined by

$$a(x) = -\partial_y Q(x, h(x))$$

satisfies $a(x) > 0$ for all $x \in \mathbf{T}^d$.

(ii) *For all (x, y) in $\overline{\Omega}$, there holds*

$$\partial_y Q(x, y) < 0. \tag{3.6}$$

Furthermore,

$$\inf_{\overline{\Omega}}(-\partial_y Q) \geq \min\Big\{ \inf_{x\in\mathbf{T}^d} a(x), 1\Big\}. \tag{3.7}$$

(iii) *The function* $|\nabla_{x,y} Q|$ *belongs to* $C^\infty(\overline{\Omega})$.

(iv) *We have the following bound:*

$$\sup_{(x,y)\in\overline{\Omega}} |\nabla_{x,y} Q(x,y)|^2 \leq \max_{\mathbf{T}^d} \frac{(1-G(h)h)^2}{1+|\nabla h|^2}. \tag{3.8}$$

Remark 3.3. Consider the evolution problem for the Hele-Shaw equation

$$\partial_t h + G(h)h = 0.$$

Then, in [6], it is proved that

$$\inf_{x\in\mathbf{T}^d} a(t,x) \geq \inf_{x\in\mathbf{T}^d} a(0,x), \qquad \sup_{x\in\mathbf{T}^d} |G(h)h(t,x)| \leq \sup_{x\in\mathbf{T}^d} |G(h)h(0,x)|.$$

Therefore, (3.7) and (3.8) give two different control of the derivatives of the pressure, which are uniform in time.

Proof. In this proof, it is convenient to truncate the domain Ω to work with a compact domain. Consider $\beta > 0$ such that the line $\mathbf{T}^d \times \{-\beta\}$ is located underneath the free surface $\Sigma = \{y = h(x)\}$, and set

$$\Omega_\beta = \{(x,y) \in \mathbf{T}^d \times \mathbf{R};\, -\beta < y < h(x)\}.$$

We will apply the maximum principle in Ω_β and then let β goes to $+\infty$.

(i) This point is well known in certain communities, but we recall the proof for the reader's convenience. We begin by observing that, since $Q|_{y=h} = 0$, on the free surface we have $|\nabla_{x,y} Q| = |\partial_n Q|$. So, to prove that $\partial_n Q = -|\nabla_{x,y} Q|$, it remains only to prove that $\partial_n Q \leq 0$. To do so, we begin by noticing that Q is solution to the following elliptic problem:

$$\Delta_{x,y} Q = 0, \quad Q|_{y=h} = 0.$$

We will apply the maximum principle in Ω_β with β large enough. In view of (3.2), there is $\beta > 0$ such that

$$\forall y \leq -\frac{\beta}{2}, \quad \|\partial_y \varphi(\cdot, y)\|_{L^\infty(\mathbf{T}^d)} \leq \frac{1}{2}.$$

In particular, on $\{y = -\beta\}$, there holds

$$\forall x \in \mathbf{T}^d, \quad \partial_y Q(x,-\beta) = \partial_y \varphi(x,-\beta) - 1 \leq -\frac{1}{2}. \tag{3.9}$$

On the other hand, by using the classical maximum principle for harmonic functions in Ω_β, we see that Q reaches its minimum on the boundary $\partial\Omega_\beta$. In light of (3.9), the minimum is not attained on $\{y = -\beta\}$, so it is attained on Σ. Since Q vanishes there, this means that $Q \geq 0$ in Ω_β. This immediately implies the wanted result $\partial_n Q \leq 0$. In addition, since the boundary is smooth, we can apply the classical Hopf–Zaremba principle to infer that $\partial_n Q < 0$ on Σ.

Let us now prove that $a > 0$. Recall that, by notation, ∇ denotes the gradient with respect to the horizontal variable only, $\nabla = (\partial_{x_1}, \dots, \partial_{x_d})^t$. Since Q vanishes on Σ, we have

$$0 = \nabla(Q|_{y=h}) = (\nabla Q)|_{y=h} + (\partial_y Q)|_{y=h}\nabla h, \tag{3.10}$$

which implies that, on $y = h$, we have

$$\begin{aligned} a = -(\partial_y Q)|_{y=h} &= -\frac{1}{1+|\nabla h|^2}\big(\partial_y Q|_{y=h} - \nabla h \cdot (\nabla Q)|_{y=h}\big) \\ &= -\frac{1}{\sqrt{1+|\nabla h|^2}}\partial_n Q\Big|_{y=h}. \end{aligned} \tag{3.11}$$

Since $\partial_n Q < 0$ on Σ, this implies that a is a positive function. Eventually, remembering that $n = \frac{1}{\sqrt{1+|\nabla h|^2}}\binom{-\nabla h}{1}$ and using (3.10), we verify that

$$n = -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}\Big|_{y=h}.$$

This completes the proof of statement (i).

(ii) Since the function $-\partial_y Q$ is harmonic in Ω, the maximum principle applied in Ω_β implies that $-\partial_y Q$ reaches its minimum on the boundary $\partial\Omega_\beta$, so

$$-\partial_y Q \geq \min\big\{\inf_{\Sigma}(-\partial_y Q), \inf_{\{y=-\beta\}}(-\partial_y Q)\big\}.$$

By letting β go to $+\infty$, we obtain (3.7) since $-\partial_y Q$ converges to 1 (see (3.2) applied with $\alpha = 0$ and $\beta = 1$). This in turn implies (3.6) in view of the fact that $a > 0$, as proved in the previous point.

(iii) Since we assume that h is smooth, the function Q belongs to $C^\infty(\overline{\Omega})$. As a consequence, to prove that $|\nabla_{x,y} Q|$ is smooth, it is sufficient to prove that $|\nabla_{x,y} Q|^2$ is bounded from below by a positive constant, which is an immediate consequence of (3.6).

(iv) Since Q is an harmonic function, we have

$$\Delta_{x,y}|\nabla_{x,y} Q|^2 = 2|\nabla_{x,y}^2 Q| \geq 0.$$

Consequently, the maximum principle for sub-harmonic functions implies that

$$\sup_{\overline{\Omega}_\beta}|\nabla_{x,y} Q|^2 = \sup_{\partial\Omega_\beta}|\nabla_{x,y} Q|^2,$$

where Ω_β is as above. By letting β go to $+\infty$, we obtain that

$$\sup_{\overline{\Omega}}|\nabla_{x,y} Q|^2 = \max\big\{\sup_{\Sigma}|\nabla_{x,y} Q|^2, 1\big\}, \tag{3.12}$$

where we used as above the fact that $|\nabla_{x,y}Q|$ tends to 1 when y goes to $-\infty$. We are thus reduced to estimating $|\nabla_{x,y}Q|^2$ on Σ. To do so, observe that the identity (3.10) implies that, on Σ, we have

$$|\nabla_{x,y}Q|^2 = (1+|\nabla h|^2)(\partial_y Q)^2 = (1+|\nabla h|^2)a^2. \tag{3.13}$$

Using the computations already performed in (3.11) and remembering that $Q = \varphi - y$, we obtain

$$a = -\frac{1}{1+|\nabla h|^2}\big(-1+\partial_y\varphi|_{y=h} - \nabla h\cdot(\nabla\varphi)|_{y=h}\big).$$

On the other hand, since φ is the harmonic extension of h, by definition of the Dirichlet-to-Neumann operator $G(h)$, one has

$$G(h)h = \partial_y\varphi|_{y=h} - \nabla h\cdot(\nabla\varphi)|_{y=h}.$$

We conclude that

$$a = \frac{1-G(h)h}{1+|\nabla h|^2},$$

which in turn implies that

$$(1+|\nabla h|^2)a^2 = \frac{(1-G(h)h)^2}{1+|\nabla h|^2}.$$

By combining this with (3.12) and (3.13), we conclude the proof of statement (iv). ∎

3.2. The key functional identity

Let us recall some notations: we denote by κ the mean curvature

$$\kappa = -\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right). \tag{3.14}$$

Also, we denote by $\varphi = \varphi(x,y)$ the harmonic extension of h in Ω given by (3.4), and we use the notation

$$Q(x,y) = \varphi(x,y) - y.$$

Proposition 3.4. *Let $d \geq 1$, assume that $h\colon \mathbf{T}^d \to \mathbf{R}$ is a smooth function, and set*

$$J(h) := \int_{\mathbf{T}^d} \kappa\, G(h)h\,dx.$$

Then,

$$J(h) = \iint_\Omega \frac{|\nabla_{x,y}Q|^2|\nabla_{x,y}^2 Q|^2 - |\nabla_{x,y}Q\cdot\nabla_{x,y}\nabla_{x,y}Q|^2}{|\nabla_{x,y}Q|^3}\,dy\,dx \geq 0. \tag{3.15}$$

Remark 3.5. (i) Since $|\nabla_{x,y}Q| \geq |\partial_y Q|$, it follows from (3.6) and the positivity of the Taylor coefficient a (see statement (i) in Proposition 3.2) that $|\nabla_{x,y}Q|$ is bounded by a positive constant on $\overline{\Omega}$. On the other hand, directly from (3.2), the function

$$|\nabla_{x,y} Q|^2 |\nabla_{x,y}^2 Q|^2 - |\nabla_{x,y} Q \cdot \nabla_{x,y} \nabla_{x,y} Q|^2$$

belongs to $L^2(\Omega)$. It follows that the right-hand side of (3.15) is a well-defined integral.

(ii) To clarify notations, set $\partial_i = \partial_{x_i}$ for $1 \leq i \leq d$ and $\partial_{d+1} = \partial_y$. Then,

$$\begin{cases} |\nabla_{x,y}^2 Q|^2 = \displaystyle\sum_{1 \leq i,j \leq d+1} (\partial_i \partial_j Q)^2, \\ |\nabla_{x,y} Q \cdot \nabla_{x,y} \nabla_{x,y} Q|^2 = \displaystyle\sum_{1 \leq i \leq d+1} \Big(\sum_{1 \leq j \leq d+1} (\partial_j Q) \partial_i \partial_j Q \Big)^2. \end{cases}$$

So, it follows from the Cauchy–Schwarz inequality that

$$|\nabla_{x,y} Q \cdot \nabla_{x,y} \nabla_{x,y} Q|^2 \leq |\nabla_{x,y} Q|^2 |\nabla_{x,y}^2 Q|^2.$$

This shows that $J(h) \geq 0$.

(iii) If $d = 1$, then one can simplify the previous expression. Remembering that

$$\Delta_{x,y} Q = 0,$$

one can verify that

$$J(h) = \frac{1}{2} \iint_{\Omega} \frac{|\nabla_{x,y}^2 Q|^2}{|\nabla_{x,y} Q|} dy dx.$$

Notice that, for the Hele-Shaw equation, one has a uniform in time estimate for $|\nabla_{x,y} Q|$ as explained in Remark 3.3. Consequently, $J(h)$ controls the L^2-norm of the second-order derivative of Q.

Proof. To prove Proposition 3.4, the main identity is given by the following result.

Lemma 3.6. *There holds*

$$J(h) = \int_{\Sigma} \partial_n |\nabla_{x,y} Q| \, d\mathcal{H}^d, \tag{3.16}$$

where $\Sigma = \{y = h(x)\}$.

Proof. By definition of the Dirichlet-to-Neumann operator, one has

$$G(h)h = \sqrt{1 + |\nabla h|^2} \partial_n \varphi|_{y=h},$$

so

$$\int_{\mathbf{T}^d} \kappa \, G(h) h \, dx = \int_{\mathbf{T}^d} \kappa \, \partial_n \varphi \sqrt{1 + |\nabla h|^2} dx.$$

Using expression (3.5) for the normal n, we observe that

$$\partial_n Q = \partial_n \varphi - \frac{1}{\sqrt{1 + |\nabla h|^2}}.$$

Directly from definition (3.14) of κ, we get that

$$\int_{\mathbf{T}^d} \kappa \, dx = 0.$$

So, by combining the previous identities, we deduce that

$$J(h) = \int_{\mathbf{T}^d} \kappa \, (\partial_n Q)|_{y=h} \sqrt{1+|\nabla h|^2} dx,$$

which can be written under the form

$$J(h) = \int_{\Sigma} \kappa \, \partial_n Q \, d\mathcal{H}^d. \tag{3.17}$$

Now, we recall from Proposition 3.2 that, on the free surface Σ, we have

$$\partial_n Q\big|_{\Sigma} = -|\nabla_{x,y} Q|\big|_{\Sigma} \quad \text{and} \quad n = -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}\bigg|_{\Sigma}.$$

Notice that $m = -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}$ is defined not only on the graph but also in some neighborhood of points $(x_0, h(x_0))$ on Σ. Since

$$\kappa = \mathrm{div}_{\Sigma} \, n = \mathrm{trace}((\mathrm{Id} - n(x_0,h(x_0)) \otimes n(x_0,h(x_0)))(\nabla_{x,y} m)(x_0,h(x_0)))$$

at a point $(x_0, h(x_0))$, it follows that

$$\kappa = \mathrm{trace}(\nabla_{x,y} m(x_0,h(x_0))) - \mathrm{trace}(n(x_0,h(x_0)) \otimes n(x_0,h(x_0))(\nabla_{x,y} m)(x_0,h(x_0)))$$

and therefore the formula

$$\kappa = -\,\mathrm{div}_{x,y} \left(\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|} \right)\bigg|_{\Sigma}$$

because m is a unit vector field near x_0 which implies

$$\mathrm{trace}(n(x_0,h(x_0)) \otimes n(x_0,h(x_0))(\nabla_{x,y} m)(x_0)) = 0.$$

Indeed,

$$\begin{aligned}
&\mathrm{trace}(n(x_0,h(x_0)) \otimes n(x_0,h(x_0))(\nabla m)(x_0,h(x_0))) \\
&\quad = \sum_{i,j} n_i(x_0,h(x_0)) n_j(x_0,h(x_0)) (\partial_i m_j)(x_0,h(x_0)) \\
&\quad = \frac{1}{2} \sum_i n_i(x_0,h(x_0)) \partial_i \Big(\sum_j |m_j|^2 \Big)(x_0,h(x_0)) = 0.
\end{aligned}$$

In conclusion, we get

$$\int_{\Sigma} \kappa \partial_n Q \, d\mathcal{H}^d = \int_{\Sigma} \mathrm{div} \left(\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|} \right) |\nabla_{x,y} Q| \, d\mathcal{H}^d. \tag{3.18}$$

Remembering that $\operatorname{div}_{x,y} \nabla_{x,y} Q = 0$, one can further simplify

$$\begin{aligned}\operatorname{div}_{x,y}\left(\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}\right)|\nabla_{x,y} Q| &= \operatorname{div}_{x,y}\left(\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}|\nabla_{x,y} Q|\right) - \frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|} \cdot \nabla_{x,y}|\nabla_{x,y} Q| \\ &= \operatorname{div} \nabla_{x,y} Q - \frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|} \cdot \nabla_{x,y}|\nabla_{x,y} Q| \\ &= -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|} \cdot \nabla_{x,y}|\nabla_{x,y} Q|.\end{aligned}$$

Now, we use again the identity $n = -\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}$ to infer that, on Σ, we have

$$\operatorname{div}_{x,y}\left(\frac{\nabla_{x,y} Q}{|\nabla_{x,y} Q|}\right)|\nabla_{x,y} Q| = n \cdot \nabla_{x,y}|\nabla_{x,y} Q| = \partial_n |\nabla_{x,y} Q|.$$

Consequently, it follows from (3.17) and (3.18) that

$$J(h) = \int_\Sigma \partial_n |\nabla_{x,y} Q| \, d\mathcal{H}^d.$$

This completes the proof of the lemma. ∎

We have proved that $J(h)$ is equal to the integral over Σ of $\partial_n |\nabla_{x,y} Q|$. This suggests applying the Stokes theorem. To do so, as in the proof of Proposition 3.2, it is convenient to truncate the domain Ω to work with a compact domain. Again, we consider $\beta > 0$ such that the hyperplane $\{y = -\beta\}$ is located underneath the free surface Σ and set

$$\Omega_\beta = \{(x, y) \in \mathbf{T}^d \times \mathbf{T};\ -\beta < y < h(x)\}.$$

Let us check that the contribution from the fictitious bottom disappears when β goes to $+\infty$.

Lemma 3.7. *Denote by Γ_β the bottom $\Gamma_\beta = \{(x, y) \in \mathbf{T}^d \times \mathbf{R};\ y = -\beta\}$. Then,*

$$\lim_{\beta \to +\infty} \int_{\Gamma_\beta} \partial_n |\nabla_{x,y} Q| \, d\mathcal{H}^d = 0. \tag{3.19}$$

Proof. We have

$$\int_{\Gamma_\beta} \partial_n |\nabla_{x,y} Q| \, d\mathcal{H}^d = -\int_{\mathbf{T}^d} \partial_y |\nabla_{x,y} Q| \, dx = -\int_{\mathbf{T}^d} \frac{\nabla_x Q \cdot \nabla_x \partial_y Q + \partial_y Q \partial_y^2 Q}{|\nabla_{x,y} Q|} dx.$$

As we have seen in Remark 3.5, the function $|\nabla_{x,y} Q|$ is bounded from below by a positive constant in Ω. Consequently, it is bounded from below on Γ_β uniformly with respect to β. On the other hand, it follows from (3.2) that

$$\lim_{\beta \to +\infty} \|(\nabla_x Q \cdot \nabla_x \partial_y Q + \partial_y Q \partial_y^2 Q)(\cdot, -\beta)\|_{L^\infty(\mathbf{T}^d)} = 0.$$

This immediately gives the wanted result. ∎

Now, we are in a position to conclude the proof. It follows from (3.16) that

$$J(h)=\int_{\partial\Omega_\beta}\partial_n|\nabla_{x,y}Q|\,d\mathcal{H}^d-\int_{\Gamma_\beta}\partial_n|\nabla_{x,y}Q|\,d\mathcal{H}^d.$$

Now, remembering that $|\nabla_{x,y}Q|$ belongs to $C^\infty(\overline{\Omega})$ (see statement (iii) in Proposition 3.2), one may apply the Stokes theorem to infer that

$$J(h)=\int_{\Omega_\beta}\Delta_{x,y}|\nabla_{x,y}Q|\,dy\,dx-\int_{\Gamma_\beta}\partial_n|\nabla_{x,y}Q|\,d\mathcal{H}^d.$$

Since $|\nabla_{x,y}Q|>0$ belongs to $C^\infty(\overline{\Omega})$, one can compute $\Delta_{x,y}|\nabla_{x,y}Q|$. To do so, we apply the general identity

$$\Delta_{x,y}u^2=2u\Delta_{x,y}u+2|\nabla_{x,y}u|^2,$$

with $u=|\nabla_{x,y}Q|$. This gives that

$$\begin{aligned}\Delta|\nabla_{x,y}Q|&=\frac{1}{2|\nabla_{x,y}Q|}\left(\Delta_{x,y}|\nabla_{x,y}Q|^2-2|\nabla_{x,y}|\nabla_{x,y}Q||^2\right)\\&=\frac{1}{2|\nabla_{x,y}Q|}\left(\Delta_{x,y}|\nabla_{x,y}Q|^2-2\frac{|\nabla_{x,y}Q\cdot\nabla_{x,y}\nabla_{x,y}Q|^2}{|\nabla_{x,y}Q|^2}\right).\end{aligned}$$

On the other hand, since $\Delta_{x,y}Q=0$, one has

$$\Delta_{x,y}|\nabla_{x,y}Q|^2=\sum_{1\le j,k\le d+1}\partial_j^2(\partial_kQ)^2=2\sum_{1\le j,k\le d+1}(\partial_j\partial_kQ)^2=2|\nabla_{x,y}^2Q|^2.$$

By combining the two previous identities, we conclude that

$$\Delta|\nabla_{x,y}Q|=\frac{1}{|\nabla_{x,y}Q|^3}\left(|\nabla_{x,y}Q|^2|\nabla_{x,y}^2Q|^2-|\nabla_{x,y}Q\cdot\nabla_{x,y}\nabla_{x,y}Q|^2\right).$$

As we have seen in Remark 3.5, the previous term is integrable on Ω. So, we can use the dominated convergence theorem and let β go to $+\infty$. Then, (3.19) implies that the contribution from the bottom disappears from the limit, and we obtain the wanted result (3.15). This completes the proof. ∎

3.3. Proof of Theorem 2.1

We are now ready to prove Theorem 2.1. Let $(g,\mu)\in[0,+\infty)^2$, and assume that h is a smooth solution to

$$\partial_t h+G(h)(gh+\mu\kappa)=0.$$

We want to prove that

$$\frac{d}{dt}\int_{\mathbf{T}^d}h^2\,dx\le 0\quad\text{and}\quad\frac{d}{dt}\mathcal{H}^d(\Sigma)\le 0.$$

Multiplying the equation $\partial_t h + G(h)(gh + \mu\kappa) = 0$ by h and integrating over $\mathbf{T}^d$, one obtains that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{T}^d} h^2 dx = -g\int_{\mathbf{T}^d} hG(h)h\,dx - \mu\int_{\mathbf{T}^d} hG(h)\kappa\,dx. \tag{3.20}$$

The first term in the right-hand side is non-positive since $G(h)$ is a non-negative operator. Indeed, as we recalled in the introduction, considering an arbitrary function ψ and denoting by φ its harmonic extension, it follows from Stokes theorem that

$$\int_{\mathbf{T}^d} \psi G(h)\psi\,dx = \int_{\partial\Omega} \varphi\partial_n\varphi\,d\mathcal{H}^d = \iint_{\Omega} |\nabla_{x,y}\varphi|^2 dy\,dx \geq 0. \tag{3.21}$$

This proves that

$$-g\int_{\mathbf{T}^d} hG(h)h\,dx \leq 0.$$

We now prove that the second term in the right-hand side of (3.20) is also non-positive. To see this, we use (3.15) and the fact that $G(h)$ is self-adjoint to obtain

$$\int_{\mathbf{T}^d} hG(h)\kappa\,dx = \int_{\mathbf{T}^d} \kappa G(h)h\,dx = J(h) \geq 0.$$

This proves that

$$\frac{d}{dt}\int_{\mathbf{T}^d} h^2 dx \leq 0.$$

It remains to prove that $\frac{d}{dt}\mathcal{H}^d(\Sigma) \leq 0$. Write

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^d(\Sigma) &= \frac{d}{dt}\int_{\mathbf{T}^d} \sqrt{1+|\nabla h|^2}\,dx\\ &= \int_{\mathbf{T}^d} \nabla_x(\partial_t h)\cdot\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\,dx\\ &= \int_{\mathbf{T}^d} (\partial_t h)\kappa\,dx\end{aligned}$$

to obtain

$$\frac{d}{dt}\mathcal{H}^d(\Sigma) = -\mu\int_{\mathbf{T}^d} \kappa G(h)\kappa\,dx - gJ(h) \leq 0,$$

where we used again (3.15) and the property (3.21) applied with $\psi = \kappa$.

This completes the proof.

4. Strong decay for the Hele-Shaw equation

In this section, we prove Theorem 2.3 about the monotonicity of $J(h)$ for solutions of the Hele-Shaw equation. Recall that, by notation,

$$J(h) = \int_{\mathbf{T}^d} \kappa G(h)h\,dx, \quad \text{where } \kappa = -\operatorname{div}\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right).$$

We want to prove that $J(h)$ is non-increasing under a mild smallness assumption on $\nabla_{x,t}h$ at initial time.

Proposition 4.1. *Assume that h is a smooth solution to the Hele-Shaw equation $\partial_t h + G(h)h = 0$. Then,*

$$\frac{d}{dt}J(h) + \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2 + |\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}}\,dx - \int_{\mathbf{T}} \kappa\theta\,dx \leq 0, \tag{4.1}$$

where

$$\theta = G(h)\left(\frac{|\nabla_{t,x}h|^2}{1+|\nabla h|^2}\right) - \operatorname{div}\left(\frac{|\nabla_{t,x}h|^2}{1+|\nabla h|^2}\nabla h\right), \tag{4.2}$$

with $|\nabla_{t,x}h|^2 = (\partial_t h)^2 + |\nabla h|^2$. In addition, if $d = 1$, then (4.1) is in fact an equality.

Proof. If h solves the Hele-Shaw equation $\partial_t h + G(h)h = 0$, one can rewrite $J(h)$ under the form

$$J(h) = -\int_{\mathbf{T}^d} \kappa\partial_t h\,dx.$$

Consequently,

$$\frac{d}{dt}J(h) + \int_{\mathbf{T}^d} \kappa_t\partial_t h\,dx + \int_{\mathbf{T}^d} \kappa h_{tt}\,dx = 0. \tag{4.3}$$

Let us compute the first integral. To do so, we use the Leibniz rule and then integrate by parts to obtain

$$\begin{aligned}\int_{\mathbf{T}^d} \kappa_t\partial_t h\,dx &= -\int_{\mathbf{T}^d} \operatorname{div}\left(\frac{\nabla\partial_t h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\cdot\nabla\partial_t h}{(1+|\nabla h|^2)^{3/2}}\nabla h\right)\partial_t h\,dx\\ &= \int_{\mathbf{T}^d} \frac{(1+|\nabla h|^2)|\nabla\partial_t h|^2 - (\nabla h\cdot\nabla\partial_t h)^2}{(1+|\nabla h|^2)^{3/2}}\,dx.\end{aligned}$$

Now, the Cauchy–Schwarz inequality implies that

$$(1+|\nabla h|^2)|\nabla\partial_t h|^2 - (\nabla h\cdot\nabla\partial_t h)^2 \geq |\nabla\partial_t h|^2.$$

(Notice that this is an equality in dimension $d = 1$.) It follows from (4.3) that

$$\frac{d}{dt}J(h) + \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2}{(1+|\nabla h|^2)^{3/2}}\,dx + \int_{\mathbf{T}^d} \kappa h_{tt}\,dx \leq 0. \tag{4.4}$$

We now move to the most interesting part of the proof, which is the study of the second term $\int \kappa h_{tt}$. The main idea is to use the fact that the Hele-Shaw equation can be written under the form of a modified Laplace equation. Let us pause to recall the argument introduced in [3]. For the reader's convenience, we begin by considering the linearized equation, which reads $\partial_t h + G(0)h = 0$. Since the Dirichlet-to-Neumann operator $G(0)$ associated to a flat half-space is given by $G(0) = |D|$, that is, the Fourier multiplier defined by $|D|e^{ix\cdot\xi} = |\xi|e^{ix\cdot\xi}$, the linearized Hele-Shaw equation reads

$$\partial_t h + |D|h = 0.$$

Since $-|D|^2 = \Delta$, we find that

$$\Delta_{t,x} h = \partial_t^2 h + \Delta h = 0.$$

The next result generalizes this observation to the Hele-Shaw equation.

Theorem 4.2 (From [3]). *Consider a smooth solution h to $\partial_t h + G(h)h = 0$. Then,*

$$\Delta_{t,x} h + B(h)^* \big(|\nabla_{t,x} h|^2\big) = 0, \tag{4.5}$$

where $B(h)^*$ is the adjoint (for the $L^2(\mathbf{T}^d)$-scalar product) of the operator defined by

$$B(h)\psi = \partial_y \mathcal{H}(\psi)|_{y=h},$$

where $\mathcal{H}(\psi)$ is the harmonic extension of ψ, solution to

$$\Delta_{x,y} \mathcal{H}(\psi) = 0 \quad \text{in } \Omega, \quad \mathcal{H}(\psi)|_{y=h} = \psi.$$

We next replace the operator $B(h)^*$ by an explicit expression which is easier to handle. Directly from the definition of $B(h)$ and the chain rule, one can check that (see, for instance, [6, Proposition 5.1])

$$B(h)\psi = \frac{G(h)\psi + \nabla h \cdot \nabla \psi}{1 + |\nabla h|^2}.$$

Consequently,

$$B(h)^* \psi = G(h)\left(\frac{\psi}{1 + |\nabla h|^2}\right) - \operatorname{div}\left(\frac{\psi}{1 + |\nabla h|^2} \nabla h\right).$$

It follows that

$$B(h)^* \big(|\nabla_{t,x} h|^2\big) = \theta, \tag{4.6}$$

where θ is as defined in the statement of Proposition 4.1.

We now go back to the second term in the right-hand side of (4.4) and write that

$$\int_{\mathbf{T}^d} \kappa h_{tt} \, dx = \int_{\mathbf{T}^d} \kappa \Delta_{t,x} h \, dx - \int_{\mathbf{T}^d} \kappa \Delta h \, dx.$$

(To clarify notations, recall that Δ denotes the Laplacian with respect to the variable x only.) By plugging this in (4.4) and using (4.5)–(4.6), we get

$$\frac{d}{dt} J(h) + \int_{\mathbf{T}^d} \frac{|\nabla \partial_t h|^2}{(1 + |\nabla h|^2)^{3/2}} \, dx - \int_{\mathbf{T}} \kappa \theta \, dx - \int_{\mathbf{T}^d} \kappa \Delta h \, dx \le 0.$$

As a result, to complete the proof of Proposition 4.1, it remains only to show that

$$-\int_{\mathbf{T}^d} \kappa \Delta h \, dx \ge \int_{\mathbf{T}^d} \frac{|\nabla^2 h|^2}{(1 + |\nabla h|^2)^{3/2}} \, dx. \tag{4.7}$$

Notice that, in dimension $d = 1$, we have

$$\kappa = -\frac{\partial_x^2 h}{(1 + (\partial_x h)^2)^{3/2}},$$

so (4.7) is in fact an equality. To prove (4.7) in arbitrary dimension, we begin by applying the Leibniz rule to write

$$-\kappa = \frac{\Delta h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \otimes \nabla h : \nabla^2 h}{(1 + |\nabla h|^2)^{3/2}}, \tag{4.8}$$

where we use the standard notations

$$\nabla h \otimes \nabla h = ((\partial_i h)(\partial_j h))_{1 \le i,j \le d}, \quad \nabla^2 h = (\partial_i \partial_j h)_{1 \le i,j \le d}$$

together with $A : B = \sum_{i,j} a_{ij} b_{ij}$. So,

$$-\int_{\mathbf{T}^d} \kappa \Delta h \, dx = \int_{\mathbf{T}^d} \frac{(\Delta h)^2}{\sqrt{1 + |\nabla h|^2}} dx - \int_{\mathbf{T}^d} \frac{(\Delta h) \nabla h \otimes \nabla h : \nabla^2 h}{(1 + |\nabla h|^2)^{3/2}} dx. \tag{4.9}$$

On the other hand, by integrating by parts twice, we get

$$\begin{aligned}
\int_{\mathbf{T}^d} \frac{(\Delta h)^2}{\sqrt{1 + |\nabla h|^2}} dx &= \sum_{i,j} \int_{\mathbf{T}^d} \frac{(\partial_i^2 h)(\partial_j^2 h)}{\sqrt{1 + |\nabla h|^2}} dx \\
&= \sum_{i,j} \int_{\mathbf{T}^d} \frac{(\partial_i \partial_j h)^2}{\sqrt{1 + |\nabla h|^2}} dx \\
&\quad + \sum_{i,j,k} \frac{(\partial_i h)(\partial_k h)(\partial_j^2 h)(\partial_i \partial_k h) - (\partial_i h)(\partial_k h)(\partial_i \partial_j h)(\partial_j \partial_k h)}{(1 + |\nabla h|^2)^{3/2}} dx \\
&= \int_{\mathbf{T}^d} \frac{(1 + |\nabla h|^2)|\nabla^2 h|^2 + (\Delta h) \nabla h \otimes \nabla h : \nabla^2 h - (\nabla h \cdot \nabla^2 h)^2}{(1 + |\nabla h|^2)^{3/2}} dx.
\end{aligned}$$

By combining this with (4.9) and simplifying, we obtain

$$-\int_{\mathbf{T}^d} \kappa \Delta h \, dx = \int_{\mathbf{T}^d} \frac{(1 + |\nabla h|^2)|\nabla^2 h|^2 - (\nabla h \cdot \nabla^2 h)^2}{(1 + |\nabla h|^2)^{3/2}} dx.$$

Now, by using the Cauchy–Schwarz inequality in $\mathbf{R}^d$, we obtain the wanted inequality (4.7), and the proposition follows. ■

In view of the previous proposition, to prove that $J(h)$ is non-increasing, it remains to show that the last term in the left-hand side of (4.1) can be absorbed by the second one. It does not seem feasible to get such a result by exploiting some special identity for the solutions, but, as we will see, we do have an inequality which holds under a very mild smallness assumption. We begin by making a smallness assumption on the space and time derivatives of the unknown h. We will next apply a maximum principle to bound these derivatives in terms of the initial data only.

Lemma 4.3. *Let $c_d < 1$, and assume that*

$$\sup_{t,x} |\nabla h(t,x)|^2 \le c_d, \qquad \sup_{t,x} (\partial_t h(t,x))^2 \le c_d. \tag{4.10}$$

Then,

$$\int_{\mathbf{T}^d} \kappa\theta dx \le \gamma_d \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2 + |\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}} dx \tag{4.11}$$

with

$$\gamma_d = 2c_d(d+\sqrt{d}) + 4\Big(c_d(d+(d+1)c_d)\Big(\frac{36(d+1)}{1-c_d}+1\Big)\Big)^{\frac{1}{2}}.$$

Proof. To shorten notations, let us set

$$H := \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2 + |\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}} dx,$$

and

$$\zeta := \frac{|\nabla_{t,x} h|^2}{1+|\nabla h|^2} = \frac{(\partial_t h)^2 + |\nabla h|^2}{1+|\nabla h|^2}.$$

Then, by definition of θ (see (4.2)), we have

$$\theta = G(h)\zeta - \operatorname{div}(\zeta\nabla h) = I_1 + I_2$$

with

$$I_1 = -\zeta\Delta h$$

and

$$I_2 = G(h)\zeta - \nabla\zeta\cdot\nabla h.$$

We will study the contributions of I_1 and I_2 to $\int \kappa\theta dx$ separately.

(1) *Contribution of I_1.* We claim that

$$-\int_{\mathbf{T}^d} \kappa\zeta\Delta h dx \le \int_{\mathbf{T}^d} \zeta\big(d + (d+\sqrt{d})|\nabla h|^2\big) \frac{|\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}} dx. \tag{4.12}$$

To see this, we use again (4.8) to write

$$-\kappa\zeta\Delta h = \zeta\frac{(\Delta h)^2}{\sqrt{1+|\nabla h|^2}} - \zeta\frac{(\Delta h)\nabla h\otimes\nabla h : \nabla^2 h}{(1+|\nabla h|^2)^{3/2}}.$$

Then, we recall that for all $v\colon \mathbf{R}^d \mapsto \mathbf{R}^d$

$$(\operatorname{div} v)^2 = \sum_i\sum_j \partial_i v_i \partial_j v_j \le \sum_i\sum_j \frac{1}{2}\big((\partial_i v_i)^2 + (\partial_j v_j)^2\big) \le d|\nabla v|^2,$$

and therefore,

$$(\Delta h)^2 \le d\,|\nabla^2 h|^2. \tag{4.13}$$

Then, by using the Cauchy–Schwarz inequality, we prove claim (4.12).

Now, observe that, by definition of ζ, we have $\zeta \leq |\nabla_{t,x}h|^2$. So, by assumption (4.10), we deduce that

$$\begin{aligned}\zeta\big(d+(d+\sqrt{d})|\nabla h|^2\big) &\leq |\nabla_{t,x}h|^2 \frac{d+(d+\sqrt{d})|\nabla h|^2}{1+|\nabla h|^2}\\ &\leq 2c_d(d+\sqrt{d}).\end{aligned}$$

Therefore, it follows from (4.12) that

$$-\int_{\mathbf{T}^d} \kappa\zeta\Delta h\,dx \leq 2c_d(d+\sqrt{d})H. \tag{4.14}$$

(2) *Contribution of I_2.* We now estimate the quantity

$$\int_{\mathbf{T}^d} \kappa\big(G(h)\zeta-\nabla\zeta\cdot\nabla h\big)dx. \tag{4.15}$$

We will prove that the absolute value of this term is bounded by

$$4\bigg(c_d(d+(d+1)c_d)\bigg(\frac{36(d+1)}{1-c_d}+1\bigg)\bigg)^{\frac{1}{2}}H. \tag{4.16}$$

By combining this estimate with (4.14), this will imply the wanted inequality (4.11).

To begin, we apply the Cauchy–Schwarz inequality to bound the absolute value of (4.15) by

$$\bigg(\int_{\mathbf{T}^d}(1+|\nabla h|^2)^{3/2}\kappa^2\,dx\bigg)^{\frac{1}{2}}\bigg(\int_{\mathbf{T}^d}\frac{(G(h)\zeta-\nabla\zeta\cdot\nabla h)^2}{(1+|\nabla h|^2)^{3/2}}dx\bigg)^{\frac{1}{2}}.$$

We claim that

$$\int_{\mathbf{T}^d}(1+|\nabla h|^2)^{3/2}\kappa^2\,dx \leq 2(d+(d+1)c_d)H, \tag{4.17}$$

and

$$\int_{\mathbf{T}^d}\frac{(G(h)\zeta-\nabla\zeta\cdot\nabla h)^2}{(1+|\nabla h|^2)^{3/2}}dx \leq 8c_d\bigg(\frac{36(d+1)}{1-c_d}+1\bigg)H. \tag{4.18}$$

It will follow from these claims that the absolute value of (4.15) is bounded by (4.16), which in turn will complete the proof of the lemma.

We begin by proving (4.17). Recall from (4.8) that

$$-\kappa = \frac{\Delta h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h\otimes\nabla h : \nabla^2 h}{(1+|\nabla h|^2)^{3/2}},$$

and therefore, using the inequality $(\Delta h)^2 \leq d\,|\nabla^2 h|^2$ (see (4.13)),

$$\kappa^2 \leq 2\big(d+(d+1)|\nabla h|^2\big)\frac{|\nabla^2 h|^2}{(1+|\nabla h|^2)^3},$$

which implies (4.17), remembering that $|\nabla h|^2 \leq c_d$, by assumption (4.10).

We now move to the proof of (4.18). Since

$$\frac{(G(h)\zeta - \nabla\zeta\cdot\nabla h)^2}{(1+|\nabla h|^2)^{3/2}} \le 2(G(h)\zeta)^2 + 2|\nabla\zeta|^2,$$

it is sufficient to prove that

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 dx + \int_{\mathbf{T}^d} |\nabla\zeta|^2 dx \le 4c_d\left(\frac{36(d+1)}{1-c_d}+1\right)H. \tag{4.19}$$

To establish (4.19), the crucial point will be to bound the L^2-norm of $G(h)\zeta$ in terms of the L^2-norm of $\nabla\zeta$. Namely, we now want to prove the following estimate: if

$$|\nabla h|^2 \le c_d$$

with $c_d < 1$, then

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 dx \le \frac{12(d+1)}{1-c_d}\int_{\mathbf{T}^d} |\nabla\zeta|^2 dx. \tag{4.20}$$

To do so, we will exploit the following Rellich-type inequality (proved in Appendix A):

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 dx \le \int_{\mathbf{T}^d} (1+|\nabla h|^2)|\nabla\zeta - \mathcal{B}\nabla h|^2 dx, \tag{4.21}$$

where

$$\mathcal{B} = \frac{G(h)\zeta + \nabla\zeta\cdot\nabla h}{1+|\nabla h|^2}. \tag{4.22}$$

The previous identity is a Rellich-type identity which gives a control on the boundary of the normal derivative in terms of the tangential one. Then, by replacing $\mathcal{B}$ in (4.21) by its expression (4.22), we obtain that

$$\begin{aligned}&\int_{\mathbf{T}^d} (G(h)\zeta)^2 dx\\ &\quad\le \int_{\mathbf{T}^d} (1+|\nabla h|^2)\left|\frac{(1+|\nabla h|^2)\mathrm{Id} - \nabla h\otimes\nabla h}{1+|\nabla h|^2}\nabla\zeta - \frac{\nabla h}{1+|\nabla h|^2}G(h)\zeta\right|^2 dx.\end{aligned}$$

So, expanding the right-hand side and simplifying, we get

$$\begin{aligned}&\int_{\mathbf{T}^d} \frac{1}{1+|\nabla h|^2}(G(h)\zeta)^2 dx\\ &\quad= \int_{\mathbf{T}^d} \frac{|((1+|\nabla h|^2)\mathrm{Id} - \nabla h\otimes\nabla h)\nabla\zeta|^2}{1+|\nabla h|^2} dx\\ &\qquad - 2\int_{\mathbf{T}^d} \nabla h\cdot\frac{(((1+|\nabla h|^2)\mathrm{Id} - \nabla h\otimes\nabla h)\nabla\zeta)}{1+|\nabla h|^2} G(h)\zeta dx.\end{aligned}$$

Hence, by using the Young inequality,

$$\int_{\mathbf{T}^d} \frac{1}{1+|\nabla h|^2}(G(h)\zeta)^2 dx \le \int_{\mathbf{T}^d} \frac{\big|(1+|\nabla h|^2)\mathrm{Id}-\nabla h\otimes\nabla h\big|^2}{1+|\nabla h|^2}|\nabla\zeta|^2 dx + \int_{\mathbf{T}^d} \frac{|\nabla h|^2}{1+|\nabla h|^2}(G(h)\zeta)^2 dx + \int_{\mathbf{T}^d} \frac{|(1+|\nabla h|^2)\mathrm{Id}-\nabla h\otimes\nabla h|^2}{1+|\nabla h|^2}|\nabla\zeta|^2 dx.$$

Now, we write

$$\frac{|((1+|\nabla h|^2)\mathrm{Id}-\nabla h\otimes\nabla h)|^2}{1+|\nabla h|^2} \le (d+1)\frac{(1+2|\nabla h|^2)^2}{1+|\nabla h|^2}$$

to obtain

$$\int_{\mathbf{T}} \frac{1-|\nabla h|^2}{1+|\nabla h|^2}(G(h)\zeta)^2 dx \le 2(d+1)\int_{\mathbf{T}} \frac{(1+2|\nabla h|^2)^2}{1+|\nabla h|^2}|\nabla\zeta|^2 dx.$$

Now, recalling that $|\nabla h|^2 \le c_d < 1$, we get

$$\int_{\mathbf{T}} (G(h)\zeta)^2 dx \le 2(d+1)\frac{(1+c_d)(1+2c_d)^2}{1-c_d}\int_{\mathbf{T}}|\nabla\zeta|^2 dx \le \frac{36(d+1)}{1-c_d}\int_{\mathbf{T}}|\nabla\zeta|^2 dx.$$

In view of (4.20), to prove the wanted inequality (4.19), we are reduced to establishing

$$\int_{\mathbf{T}}|\nabla\zeta|^2 dx \le 4c_d\int_{\mathbf{T}} \frac{|\nabla\partial_t h|^2+|\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}} dx.$$

Since

$$\nabla\zeta = 2\frac{\partial_t h}{(1+|\nabla h|^2)^{1/4}}\frac{\nabla\partial_t h}{(1+|\nabla h|^2)^{3/4}} + 2\frac{(1-(\partial_t h)^2)\nabla h}{(1+|\nabla h|^2)^{5/4}}\cdot\frac{\nabla^2 h}{(1+|\nabla h|^2)^{3/4}},$$

the latter inequality will be satisfied provided that

$$\frac{((1-(\partial_t h)^2)|\nabla h|)^2}{(1+|\nabla h|^2)^{5/2}} \le c_d, \qquad \frac{(\partial_t h)^2}{(1+|\nabla h|^2)^{1/2}} \le c_d.$$

The latter couple of conditions are obviously satisfied when

$$|\nabla h|^2 \le c_d, \quad |\partial_t h|^2 \le c_d \quad \text{with } c_d < 1. \tag{4.23}$$

This completes the proof of Lemma 4.3. ∎

We are now in a position to complete the proof. Recall that Proposition 4.1 implies that

$$\frac{d}{dt}J(h) + \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2+|\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}} dx \le \int_{\mathbf{T}}\kappa\theta dx.$$

On the other hand, Lemma 4.3 implies that

$$\int_{\mathbf{T}} \kappa\theta\,dx \leq \gamma_d \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2 + |\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}}\,dx$$

with

$$\gamma_d = 2c_d(d+\sqrt{d}) + 4\Big(c_d(d+(d+1)c_d)\Big(\frac{36(d+1)}{1-c_d}+1\Big)\Big)^{\frac{1}{2}}$$

provided that

$$\sup_{t,x} |\nabla h(t,x)|^2 \leq c_d, \quad \sup_{t,x} (\partial_t h(t,x))^2 \leq c_d. \tag{4.24}$$

We now fix $c_d \in [0,1)$ by solving the equation $\gamma_d = 1/2$ (the latter equation has a unique solution since $c_d \mapsto \gamma_d$ is strictly increasing). It follows that

$$\frac{d}{dt} J(h) + \frac{1}{2} \int_{\mathbf{T}^d} \frac{|\nabla\partial_t h|^2 + |\nabla^2 h|^2}{(1+|\nabla h|^2)^{3/2}}\,dx \leq 0.$$

The expected decay of $J(h)$ is thus seen to hold as long as the solution $h = h(t,x)$ satisfies assumption (4.24). Consequently, to conclude the proof of Theorem 2.3, it remains only to show that assumption (4.24) on the solution will hold provided that it holds initially. To see this, we use the fact that there is a maximum principle for the Hele-Shaw equation for space *and* time derivatives (the maximum principle for spatial derivatives is well known (see [6,16,34]); the one for time derivative is given by [6, Theorem 2.11]). This means that assumption (4.23) holds for all time $t \geq 0$ provided that it holds at time 0. This concludes the proof of Theorem 2.3.

A. A Rellich-type estimate

This appendix gives a proof of inequality (4.21).

Lemma A.1. *For any smooth functions h and ζ in $C^\infty(\mathbf{T}^d)$, there holds*

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2\,dx \leq \int_{\mathbf{T}^d} (1+|\nabla h|^2)|\nabla\zeta - \mathcal{B}\nabla h|^2\,dx, \tag{A.1}$$

where

$$\mathcal{B} = \frac{G(h)\zeta + \nabla\zeta\cdot\nabla h}{1+|\nabla h|^2}. \tag{A.2}$$

Remark A.2. (i) It is elementary to extend this inequality to functions which are not smooth.

(ii) This extends an estimate proved in [2] when $d = 1$ for the Dirichlet-to-Neumann operator associated to a domain with finite depth. When $d = 1$, the main difference is that this is an identity (and not only an inequality). This comes from the fact that, in the proof below, to derive (A.4) we use the inequality $(\nabla h \cdot \mathcal{V})^2 \leq |\nabla h|^2 \cdot |\mathcal{V}|^2$, which is clearly an identity when $d = 1$.

Proof. We follow the analysis in [2]. Set

$$\Omega = \{(x, y) \in \mathbf{T}^d \times \mathbf{R};\, y < h(x)\},$$

and denote by ϕ the harmonic function defined by

$$\begin{cases} \Delta_{x,y}\phi = 0 & \text{in } \Omega = \{(x, y) \in \mathbf{T} \times \mathbf{R};\; y < h(x)\}, \\ \phi(x, h(x)) = \zeta(x). \end{cases}$$

As recalled in Lemma 3.1, this is a classical elliptic boundary problem, which admits a unique smooth solution. Moreover, it satisfies

$$\lim_{y \to -\infty} \sup_{x \in \mathbf{T}^d} |\nabla_{x,y}\phi(x, y)| = 0. \tag{A.3}$$

Introduce the notations

$$\mathcal{V} = (\nabla\phi)_{|y=h}, \quad \mathcal{B} = (\partial_y\phi)_{|y=h}.$$

(We parenthetically recall that ∇ denotes the gradient with respect to the horizontal variables $x = (x_1, \dots, x_d)$ only.) It follows from the chain rule that

$$\mathcal{V} = \nabla\zeta - \mathcal{B}\nabla h,$$

while $\mathcal{B}$ is given by (A.2). On the other hand, by definition of the Dirichlet-to-Neumann operator, one has the identity

$$G(h)\zeta = (\partial_y\phi - \nabla h \cdot \nabla\phi)_{|y=h},$$

so

$$G(h)\zeta = \mathcal{B} - \nabla h \cdot \mathcal{V}.$$

Squaring this identity yields

$$(G(h)\zeta)^2 = \mathcal{B}^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (\nabla h \cdot \mathcal{V})^2.$$

Since $(\nabla h \cdot \mathcal{V})^2 \le |\nabla h|^2 \cdot |\mathcal{V}|^2$, this implies the inequality

$$(G(h)\zeta)^2 \le \mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V} + (1 + |\nabla h|^2)\mathcal{V}^2. \tag{A.4}$$

Integrating this gives

$$\int_{\mathbf{T}^d} (G(h)\zeta)^2 dx \le \int_{\mathbf{T}^d} (1 + |\nabla h|^2)|\mathcal{V}|^2 dx + R,$$

where

$$R = \int_{\mathbf{T}^d} \big(\mathcal{B}^2 - |\mathcal{V}|^2 - 2\mathcal{B}\nabla h \cdot \mathcal{V}\big) dx.$$

Since $|\mathcal{V}| = |\nabla\zeta - \mathcal{B}\nabla h|$, we immediately see that, to obtain the wanted estimate (A.1), it is sufficient to prove that $R = 0$. To do so, we begin by noticing that R is the flux associated to a vector field. Indeed,

$$R = \int_{\partial\Omega} X \cdot n \, d\sigma,$$

where $X : \Omega \to \mathbf{R}^{d+1}$ is given by

$$X = \big(-(\partial_y\phi)\nabla\phi ; |\nabla\phi|^2 - (\partial_y\phi)^2 \big).$$

Then, the key observation is that this vector field satisfies $\operatorname{div}_{x,y} X = 0$ since

$$\partial_y\big((\partial_y\phi)^2 - |\nabla\phi|^2\big) + 2\operatorname{div}((\partial_y\phi)\nabla\phi) = 2(\partial_y\phi)\Delta_{x,y}\phi = 0,$$

as can be verified by an elementary computation. Now, we see that the cancellation of $R = 0$ comes from the Stokes theorem. To rigorously justify this point, we truncate Ω in order to work in a smooth bounded domain. Given a parameter $\beta > 0$, set

$$\Omega_\beta = \{(x, y) \in \mathbf{T}^d \times \mathbf{T}; -\beta < y < h(x)\}.$$

An application of the divergence theorem in Ω_β gives that

$$0 = \iint_{\Omega_\beta} \operatorname{div}_{x,y} X \, dy \, dx = R + \int_{\{y=-\beta\}} X \cdot n \, d\sigma.$$

Sending β to $+\infty$ and remembering that X converges to 0 uniformly when y goes to $-\infty$ (see (A.3)), we obtain the expected result $R = 0$. This completes the proof. ■

Acknowledgments. We would like to thank the referees for their numerous comments.

Funding. This research was supported by the SingFlows project, grant ANR-18-CE40-0027 of the French National Research Agency (ANR).

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Received 1 October 2021; revised 24 August 2023.

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