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Inertia of Kwong matrices

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Abstract. Let r be any real number and for any n let $p_1, \ldots, p_n$ be distinct positive numbers. A Kwong matrix is the $n \times n$ matrix whose (i, j) entry is $(p_i^r + p_j^r)/(p_i + p_j)$. We determine the signatures of eigenvalues of all such matrices. The corresponding problem for the family of Loewner matrices $[(p_i^r - p_j^r)/(p_i - p_j)]$ has been solved earlier.

1. Introduction

Let f be a nonnegative C^1 function on $(0, \infty)$. Let n be a positive integer and $p_1 < p_2 < \cdots < p_n$ distinct positive real numbers. The $n \times n$ matrix

$$L_f(p_1, \ldots, p_n) = \Big[\frac{f(p_i) - f(p_j)}{p_i - p_j}\Big]$$

is called a *Loewner matrix* associated with f. These matrices play an important role in several areas of analysis, one of them being Loewner's theory of operator monotone functions. A central theorem in this theory asserts that f is operator monotone if and only if all Loewner matrices associated with f are positive semidefinite. See [2,3,14].

Closely related to Loewner matrices are the matrices

$$K_f(p_1, \ldots, p_n) = \Big[\frac{f(p_i) + f(p_j)}{p_i + p_j}\Big].$$

These too have been studied in several papers. In [11] Kwong showed that all matrices K_f are positive semidefinite if (but not only if) f is operator monotone. Because of this, the matrices K_f are sometimes called *Kwong matrices*. Audenaert [1] has characterised all functions f for which all K_f are positive semidefinite.

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Of particular interest are the functions $f(t) = t^r$, where r is any real number. For these functions, we denote L_f and K_f by L_r and K_r, respectively. Thus,

$$L_r(p_1, \ldots, p_n) = \left[\frac{p_i^r - p_j^r}{p_i - p_j}\right]$$

and

$$K_r(p_1, \ldots, p_n) = \left[\frac{p_i^r + p_j^r}{p_i + p_j}\right].$$

Another fundamental theorem by Loewner says that the function $f(t) = t^r$ is operator monotone if and only if $0 \le r \le 1$. Thus, all matrices L_r are positive semidefinite if and only if $0 \le r \le 1$. From the work of Kwong and Audenaert cited above, it follows that all matrices K_r are positive semidefinite if and only if $-1 \le r \le 1$, and they are positive definite if and only if $-1 < r < 1$.

In their work [5] Bhatia and Holbrook studied the matrices L_r for values of r outside the interval [0, 1]. Among other things, they showed that, when $1 < r < 2$, every matrix L_r has exactly one positive eigenvalue. This is in striking contrast to the case $0 < r < 1$, in which all eigenvalues of L_r are positive. This led them to make a conjecture about the signature of eigenvalues of L_r as r varies over real numbers.

Let A be any $n \times n$ Hermitian matrix. The *inertia* of A is the triplet

$$\operatorname{In}(A) = (\pi(A), \zeta(A), \nu(A)),$$

where $\pi(A)$, $\zeta(A)$, and $\nu(A)$ are respectively the numbers of positive, zero and negative eigenvalues of A. By the results of Loewner cited above, $\operatorname{In} L_r = (n, 0, 0)$ when $0 < r < 1$, and the result of Bhatia and Holbrook says that $\operatorname{In} L_r = (1, 0, n-1)$ when $1 < r < 2$. The conjecture in [5] described the inertia of L_r for other values of r.

In [7] Bhatia and Sano made two essential contributions to this problem. They provided a better understanding of the problem for the range $1 < r < 2$, and they also obtained a solution for the range $2 < r < 3$. Let $\mathcal{H}_1$ be the $(n-1)$-dimensional subspace of $\mathbb{C}^n$ defined as

$$\mathcal{H}_1 = \left\{x \in \mathbb{C}^n : \sum_{i=1}^{n} x_i = 0\right\}.$$

An $n \times n$ Hermitian matrix is said to be *conditionally positive definite* (cpd) if one has $\langle x, Ax \rangle \ge 0$ for all $x \in \mathcal{H}_1$. It is said to be *conditionally negative definite* (cnd) if $-A$ is cpd. If A is nonsingular and cnd with all entries nonnegative, then $\operatorname{In} A = (1, 0, n-1)$. Bhatia and Sano [7] showed that the matrix L_r is cnd when $1 < r < 2$, thus explaining the result in [5]. They also showed that L_r is cpd when $2 < r < 3$.

In the same paper [7], the authors found an interesting difference between the inertial properties of L_r and K_r in the range $2 < r < 3$. They showed that K_r is nonsingular and cnd in the interval $1 < r < 3$, and hence $\text{In}\, K_r = (1, 0, n-1)$ for such r. Thus, there arises the problem of studying $\text{In}\, K_r$ parallel to that of $\text{In}\, L_r$.

The inertia of L_r was completely determined by Bhatia, Friedland, and Jain in [4]. The corresponding theorem on K_r was proved by us shortly afterwards. This was announced in [6]. The aim of the present paper is to publish our proof. Our main result is the following.

Theorem 1. *Let $p_1 < p_2 < \cdots < p_n$ and r be any positive real numbers and let K_r be the matrix defined in (1).*

(i) *K_r is singular if and only if r is an odd integer smaller than n.*

(ii) *When r is an odd integer smaller than or equal to n, the inertia of K_r is given as follows:*

$$\text{In}\, K_r = \begin{cases} \left(\left\lceil \frac{r}{2} \right\rceil, n-r, \left\lfloor \frac{r}{2} \right\rfloor\right) & r = 1 \pmod 4, \\ \left(\left\lfloor \frac{r}{2} \right\rfloor, n-r, \left\lceil \frac{r}{2} \right\rceil\right) & r = 3 \pmod 4. \end{cases}$$

(iii) *If $0 \leq r < 1$, then K_r is positive definite, and hence* $\text{In}\, K_r = (n, 0, 0)$.

(iv) *Suppose $k < r < k+2 < n$, where k is an odd integer. Then*

$$\text{In}\, K_r = \begin{cases} \left(\left\lceil \frac{k}{2} \right\rceil, 0, n - \left\lceil \frac{k}{2} \right\rceil\right) & k = 1 \pmod 4, \\ \left(n - \left\lceil \frac{k}{2} \right\rceil, 0, \left\lceil \frac{k}{2} \right\rceil\right) & k = 3 \pmod 4. \end{cases}$$

(v) *If n is odd, then $\text{In}\, K_r = \text{In}\, K_n$ for $r > n-2$; and if n is even, then $\text{In}\, K_r = \text{In}\, K_n = \left(\frac{n}{2}, 0, \frac{n}{2}\right)$ for $r > n-1$.*

At the beginning of Section 1 we observe that for every real number r, $\text{In}\, K_{-r} = \text{In}\, K_r$. So, Theorem 1 describes the inertia of K_r for every real number r.

There is a striking similarity and a striking difference between the behaviour of the signs of eigenvalues of L_r and K_r. As r moves over $(0, \infty)$, the eigenvalues of both flip signs at certain integral values of r. For L_r these flips take place at all integers $r \leq n-1$, and each time all but one eigenvalue change signs. For K_r the flips take place at all odd integers $r \leq n-1$. At $r = 1$ all but one eigenvalue change signs, and after that all but two eigenvalues change signs.

Figure 1 is a schematic representation of the eigenvalues of K_r for $n = 6$ and $r \geq 0$.

In an earlier paper [6], we studied the inertia of another family $P_r = [(p_i + p_j)^r]$. The structure of the proof there has been the template of subsequent works on this kind of problem. See, e.g., [4] and the recent work [13] on the Kraus matrix. (Warning: The authors of [13] use the symbol K_r for something different from our Kwong matrix.)

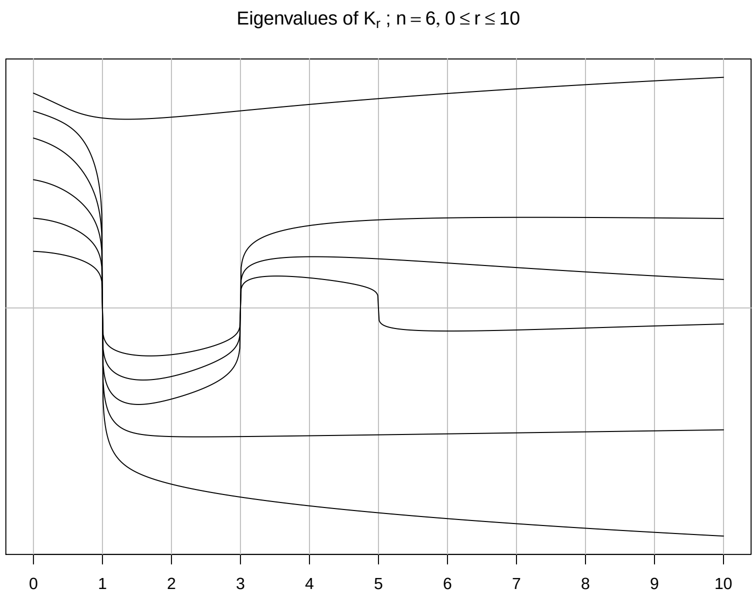

Figure 1

Our proof here follows the same steps as in these papers; the details are different at some crucial points.

2. Proof of Theorem 1

Two Hermitian matrices A and B are said to be *congruent* if there exists an invertible matrix X such that $B = X^*AX$. The Sylvester law of inertia says that A and B are congruent if and only if $\operatorname{In} A = \operatorname{In} B$.

Let D be the diagonal matrix $D = \operatorname{diag}(p_1, \ldots, p_n)$. Then, for every $r > 0$,

$$K_{-r} = D^{-r} K_r D^{-r}, \tag{2.1}$$

and hence,

$$\operatorname{In} K_{-r} = \operatorname{In} K_r.$$

The substitution $p_i = \mathrm{e}^{2x_i}$, $x_i \in \mathbb{R}$, gives

$$K_r = \Delta \widetilde{K}_r \Delta,$$

where $\Delta = \mathrm{diag}(\mathrm{e}^{(r-1)x_1}, \ldots, \mathrm{e}^{(r-1)x_n})$, and

$$\widetilde{K}_r = \left[\frac{\cosh r(x_i - x_j)}{\cosh(x_i - x_j)}\right].$$

By Sylvester's law, $\mathrm{In}\, \widetilde{K}_r = \mathrm{In}\, K_r$. When $n = 2$, we have

$$\widetilde{K}_r = \begin{bmatrix} 1 & \dfrac{\cosh r(x_1 - x_2)}{\cosh(x_1 - x_2)} \\ \dfrac{\cosh r(x_1 - x_2)}{\cosh(x_1 - x_2)} & 1 \end{bmatrix}.$$

So, $\det \widetilde{K}_r = 1 - \frac{\cosh^2 r(x_1 - x_2)}{\cosh^2(x_1 - x_2)}$. This is positive if $0 < r < 1$, zero if $r = 1$, and negative if $r > 1$. The inertia of K_r is $(2, 0, 0)$ in the first case, $(1, 1, 0)$ in the second case, and $(1, 0, 1)$ in the third. All assertions of Theorem 1 are thus valid in the case $n = 2$.

We will use the following extension of Sylvester's law. A proof is given in [6].

Proposition 2. *Let $n \geq r$, and let A be an $r \times r$ Hermitian matrix and X an $r \times n$ matrix of rank r. Then*

$$\mathrm{In}\, X^* A X = \mathrm{In}\, A + (0, n - r, 0). \tag{2.2}$$

We now prove part (ii) of the theorem. Let r be an odd integer, $r \leq n$. Then

$$\frac{p_i^r + p_j^r}{p_i + p_j} = p_i^{r-1} - p_i^{r-2} p_j + p_i^{r-3} p_j^2 - \cdots + p_j^{r-1}.$$

So, the matrix K_r can be factored as

$$K_r = W^* V W,$$

where W is the $r \times n$ Vandermonde matrix given by

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{r-1} & p_2^{r-1} & \cdots & p_n^{r-1} \end{bmatrix}$$

and V is the $r \times r$ antidiagonal matrix with entries $(1, -1, 1, -1, \ldots, -1, 1)$ down its sinister diagonal. So, by the generalised Sylvester's law (2.2), we have for every odd integer $r \leq n$,

$$\mathrm{In}\, K_r = \mathrm{In}\, V + (0, n - r, 0).$$

The matrix V is nonsingular and its eigenvalues are ± 1. In the case $r = 1 \pmod 4$, $\operatorname{tr} V = 1$ and the multiplicity of 1 as an eigenvalue of V exceeds by one the multiplicity of -1. In the case $r = 3 \pmod 4$, $\operatorname{tr} V = -1$ and the multiplicity of -1 as an eigenvalue of V exceeds by one the multiplicity of 1. This establishes part (ii) of Theorem 1.

Next, let $c_1, c_2, \ldots, c_n$ be real numbers, not all of which are zero, and let f be the function on $(0, \infty)$ defined as

$$f(x) = \sum_{j=1}^{n} c_j \frac{x^r + p_j^r}{x + p_j}. \tag{2.3}$$

Theorem 3. *Let n be an odd number. Then for every positive real number $r > n - 1$, the function f in equation (2.3) has at most $n - 1$ zeros in $(0, \infty)$.*

Proof. Consider the function g defined as

$$g(x) = f(x) \prod_{j=1}^{n} (x + p_j).$$

Expanding the product, we can write

$$\begin{aligned} g(x) = {} & \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1} \\ & + \beta_0 x^r + \beta_1 x^{r+1} + \cdots + \beta_{n-1} x^{r+n-1}. \end{aligned}$$

The function g can be written as

$$g(x) = x^r h_1(x) + h_2(x),$$

where

$$h_1(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i} (x + p_j) \quad \text{and} \quad h_2(x) = \sum_{i=1}^{n} c_i p_i^r \prod_{j \neq i} (x + p_j).$$

Since both h_1 and h_2 are Lagrange interpolation polynomials of degree at most $n - 1$ and not all c_i are zero, neither of the polynomials h_1 and h_2 is identically zero. Hence, g is not identically zero. Now, consider the function g_0 defined as

$$g_0(x) = \sum_{i=1}^{n} c_i \frac{x^r - p_i^r}{x - p_i} \prod_{j=1}^{n} (x - p_j).$$

Then a calculation shows that

$$\begin{aligned} g_0(x) = {} & -\alpha_0 + \alpha_1 x + \cdots + \alpha_{n-2} x^{n-1} - \alpha_{n-1} x^{n-1} \\ & + \beta_0 x^r - \beta_1 x^{r+1} + \cdots + \beta_{n-1} x^{r+n-1}. \end{aligned} \tag{2.4}$$

By the Descartes rule of signs [12, p. 46], the number of positive zeros of g is no more than the number of sign changes in the sequence of coefficients

$$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \beta_0, \beta_1, \ldots, \beta_{n-1}).$$

Let this number of sign changes be s, and let $s_0$be the number of sign changes in the coefficients in (2.4). Since n is odd, we have $s + s_0 \leq 2n - 1$. We know that g_0 has at least n positive zeros $p_1, \ldots, p_n$. So, $s_0 \geq n$, and hence $s \leq n - 1$. Hence, g has at most $n - 1$ positive zeros, and therefore so does f. ■

We can deduce the following.

Corollary 4. *Let n be an odd number, and let $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ be two n-tuples of distinct positive numbers. Then for every $r > n - 1$, the $n \times n$ matrix*

$$\left[\frac{p_i^r + q_j^r}{p_i + q_j}\right] \tag{2.5}$$

is nonsingular. So, in particular if n is odd, then for every $r > n - 1$, the matrix K_r is nonsingular.

Proof. If the matrix (2.5) is singular, then there exists a nonzero tuple $(c_1, \ldots, c_n)$ such that

$$f(x) = \sum_{j=1}^{n} c_j \frac{x^r + q_j^r}{x + q_j}$$

has at least n zeros $p_1, \ldots, p_n$. But this is not possible by Theorem 3. So, the matrix (2.5) and hence, the matrix K_r is nonsingular for all odd n and $r > n - 1$. ■

We complete the proof of Theorem 1 using "snaking" process: the validity of the theorem is extended by alternatively increasing n and r.

For any positive numbers p and q and any real r, we have

$$\frac{p^r + q^r}{p + q} = p^{r-1} - p\frac{p^{r-2} + q^{r-2}}{p + q}q + q^{r-1}.$$

This gives us the identity

$$K_r = D^{r-1}E - DK_{r-2}D + ED^{r-1}, \tag{2.6}$$

where D is the diagonal matrix $\operatorname{diag}(p_1, \ldots, p_n)$ and E the matrix all whose entries are one. For $1 \leq j \leq n$, let $\mathcal{H}_j$ be the subspace of $\mathbb{C}^n$ defined as

$$\begin{aligned}\mathcal{H}_j &= \left\{x : \sum x_i = 0, \sum p_i x_i = 0, \ldots, \sum p_i^{j-1} x_i = 0\right\}\\ &= \left\{x : Ex = 0, EDx = 0, \ldots, ED^{j-1}x = 0\right\}.\end{aligned}$$

Evidently, $\dim \mathcal{H}_j = n - j$ and $\mathcal{H}_{j+1} \subset \mathcal{H}_j$.

It will be convenient to use the notation $K_r^{(n)}$ to indicate an $n \times n$ matrix of the type K_r. When the superscript n is not used, it will be understood that a statement about K_r is true for all n.

Recall that part (iii) of the theorem is known, i.e., K_r is positive definite for $0 \leq r < 1$. Relation (2.1) then shows that it has the same property for $-1 < r < 0$. K_0 is the Cauchy matrix $\left[\frac{2}{p_i + p_j}\right]$ and is positive definite. Thus, K_r is positive definite for $-1 < r < 1$.

Now, let $1 < r < 3$. Then $-1 < r - 2 < 1$. Using identity (2.6), we see that if x is a nonzero vector in $\mathcal{H}_1$, then

$$\langle x, K_r x \rangle = -\langle Dx, K_{r-2} Dx \rangle < 0.$$

So, the matrix K_r is conditionally negative definite and has at least $n - 1$ negative eigenvalues. Since all entries of K_r are positive, it has at least one positive eigenvalue. Thus,

$$\operatorname{In} K_r = (1, 0, n-1),$$

for $1 < r < 3$. By Corollary 4, $K_r^{(3)}$ is nonsingular for $r > 2$. So, $\operatorname{In} K_r^{(3)}$ does not change for $r > 2$. This shows that $\operatorname{In} K_r^{(3)} = (1, 0, 2)$ for all $r > 1$. So, the theorem is established when $n = 3$.

Now, let $n > 3$ and $3 < r < 5$. Using the identity (2.6) and the case $1 < r < 3$ of the theorem that has been established we see that if x is a nonzero vector in $\mathcal{H}_2$, then $\langle x, K_r x \rangle > 0$. So, the matrix K_r has at least $n - 2$ positive eigenvalues. By the $n = 3$ case already proved, we know that K_r has a 3×3 principal submatrix with two negative eigenvalues. So, by Cauchy's interlacing principle, K_r must have at least two negative eigenvalues. We conclude that

$$\operatorname{In} K_r = (n-2, 0, 2), \quad \text{if } 3 < r < 5.$$

In particular, this shows that $\operatorname{In} K_r^{(5)} = (3, 0, 2)$ if $3 < r < 5$, and since $K_r^{(5)}$ is nonsingular for $r > 4$, it has the same inertia for all $r > 3$. So, Theorem 1 is established when $n = 5$. Next consider the case $n = 4$. Let $r > 3$. The matrix $K_r^{(4)}$ has a principal submatrix $K_r^{(3)}$ whose inertia is $(1, 0, 2)$. So, by the interlacing principle $K_r^{(4)}$ has at least two negative eigenvalues. On the other hand, $K_r^{(4)}$ is a principal submatrix of $K_r^{(5)}$ whose inertia is $(3, 0, 2)$. So, again by the interlacing principle $K_r^{(4)}$ has at least two positive eigenvalues. Thus, $\operatorname{In} K_r^{(4)} = (2, 0, 2)$ for all $r > 3$, and Theorem 1 is established for $n = 4$.

This line of reasoning can be continued. Use the space $\mathcal{H}_3$ at the next stage to go to the interval $5 < r < 7$. Then use the established case $n \leq 5$ to extend the validity of the theorem to first the case $n = 7$, and then $n = 6$.

3. Remarks

Remark 1. In [4, Theorem 1.1 (v)] it was shown that all nonzero eigenvalues of L_r are simple. We have not been able to prove a corresponding statement for K_r.

Remark 2. An $n \times n$ real matrix A is said to be *strictly sign-regular* (SSR) if for every $1 \leq k \leq n$, all $k \times k$ sub-determinants of A are nonzero and have the same sign. If this is true for all $1 \leq k \leq m$ for some $m < n$, then we say A *is in the class* $\mathrm{SSR_m}$. See [9] for a detailed study of such matrices. In [4] it was shown that the matrix L_r is in the class $\mathrm{SSR_r}$ if $r = 1, 2, \ldots, n-1$, and in the class SSR for all other $r > 0$. This fact was then used to prove the simplicity of nonzero eigenvalues of L_r.

Let $n = 4$ and consider the matrix $K_3(1, 2, 5, 10)$. It can be seen that the leading 2×2 principal subdeterminant of this matrix is -5, while the determinant of the top right 2×2 submatrix is 35. So, this matrix is not in the class $\mathrm{SSR_2}$.

Remark 3. Let $p_1 < p_2$ and $q_1 < q_2$ be two ordered pairs of distinct positive numbers such that $\{p_1, p_2\} \cap \{q_1, q_2\}$ is nonempty. With a little work, it can be shown that the determinant of the 2×2 matrix $\left[\frac{p_i^r + q_j^r}{p_i + q_j}\right]$ is positive if $0 < r < 1$ and negative if $r > 1$. Using this, one sees that for $n = 3$ and $r \neq 1$, the matrix K_r is SSR.

Remark 4. There is a curious and intriguing connection between the inertia of K_r and that of another family. For $r \geq 0$, let B_r be the $n \times n$ matrix

$$B_r = [|p_i - p_j|^r].$$

This family has been studied widely in connection with interpolation of scattered data and splines. The inertias of these matrices were studied by Dyn, Goodman, and Micchelli in [8]. In [8, Theorems 4], they prove results akin to our Theorem 1 for the matrices B_r. Together, these results imply that

$$\operatorname{In} B_r = \operatorname{In} K_{r+1} \tag{3.1}$$

for all $r \geq 0$. It will be good to have an understanding of what leads to this remarkable coincidence. By Sylvester's law (3.1) is equivalent to saying that B_r and K_{r+1} are congruent. In a recent work [10] the authors construct an explicit congruence between the matrices B_r and K_{r+1}. This provides an alternative proof of (3.1), and thus also an alternative (but indirect) proof of our theorem.

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