

Asymptotic growth patterns for class field towers

Arindam Bhattacharyya, Vishnu Kadiri, and Anwesh Ray

Abstract. Let p be an odd prime number. We study growth patterns associated with finitely ramified Galois groups considered over the various number fields varying in a \mathbb{Z}_p -tower. These Galois groups can be considered as non-commutative analogues of ray class groups. For certain \mathbb{Z}_p -extensions in which a given prime above p is completely split, we prove precise asymptotic lower bounds. Our investigations are motivated by the classical results of Iwasawa, who showed that there are growth patterns for p -primary class numbers of the number fields in a \mathbb{Z}_p -tower.

1. Introduction

1.1. Motivation from Iwasawa theory

Let \mathbb{L} be a number field and p be an odd prime number. Choose an algebraic closure $\bar{\mathbb{L}}$ of \mathbb{L} . Let $H_p(\mathbb{L})$ be the maximal abelian unramified p -extension of \mathbb{L} , and let $\mathcal{A}_p(\mathbb{L})$ denote the Galois group $\text{Gal}(H_p(\mathbb{L})/\mathbb{L})$. By class field theory, $\mathcal{A}_p(\mathbb{L})$ is naturally isomorphic to the p -primary part of the class group of \mathbb{L} . Setting $h_p(\mathbb{L}) := \#\mathcal{A}_p(\mathbb{L})$, we refer to $h_p(\mathbb{L})$ as the p -class number of \mathbb{L} . We set \mathbb{Z}_p to denote the ring of p -adic integers. A \mathbb{Z}_p -extension of a number field \mathbb{L} is an infinite Galois extension $\mathbb{L}_\infty/\mathbb{L}$, such that $\text{Gal}(\mathbb{L}_\infty/\mathbb{L})$ is isomorphic to \mathbb{Z}_p (as a topological group). Given a \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$, and $n \in \mathbb{Z}_{\geq 0}$, we set \mathbb{L}_n/\mathbb{L} to be the extension such that $\mathbb{L}_n \subseteq \mathbb{L}_\infty$ and $[\mathbb{L}_n : \mathbb{L}] = p^n$. The field \mathbb{L}_n is the n -th layer, and we obtain a tower of number fields

$$\mathbb{L} = \mathbb{L}_0 \subseteq \mathbb{L}_1 \subseteq \cdots \subseteq \mathbb{L}_n \subseteq \mathbb{L}_{n+1} \subseteq \cdots .$$

Let $e_n \in \mathbb{Z}_{\geq 0}$ be such that $p^{e_n} = h_p(\mathbb{L}_n)$. In his seminal work, Iwasawa [6] showed that there are constants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ and $v \in \mathbb{Z}$, such that for all large enough values of n , one has that $e_n = p^n \mu + n\lambda + v$.

Thus, Iwasawa's results show that there are interesting growth patterns for p -class numbers in certain infinite towers of number fields. These results motivate the study of asymptotic growth properties of p Hilbert class field towers along \mathbb{Z}_p -extensions. Throughout, p will be a fixed odd prime number, and let $\mathbb{L}_\infty/\mathbb{L}$ be a \mathbb{Z}_p -extension. Set $\mathcal{F}(\mathbb{L}_n)$ to denote the maximal unramified pro- p extension of \mathbb{L}_n , and set $G_n := \text{Gal}(\mathcal{F}(\mathbb{L}_n)/\mathbb{L}_n)$. We identify the abelianization of G_n with

$$\mathcal{A}_p(\mathbb{L}_n) = \text{Gal}(H_p(\mathbb{L}_n)/\mathbb{L}_n).$$

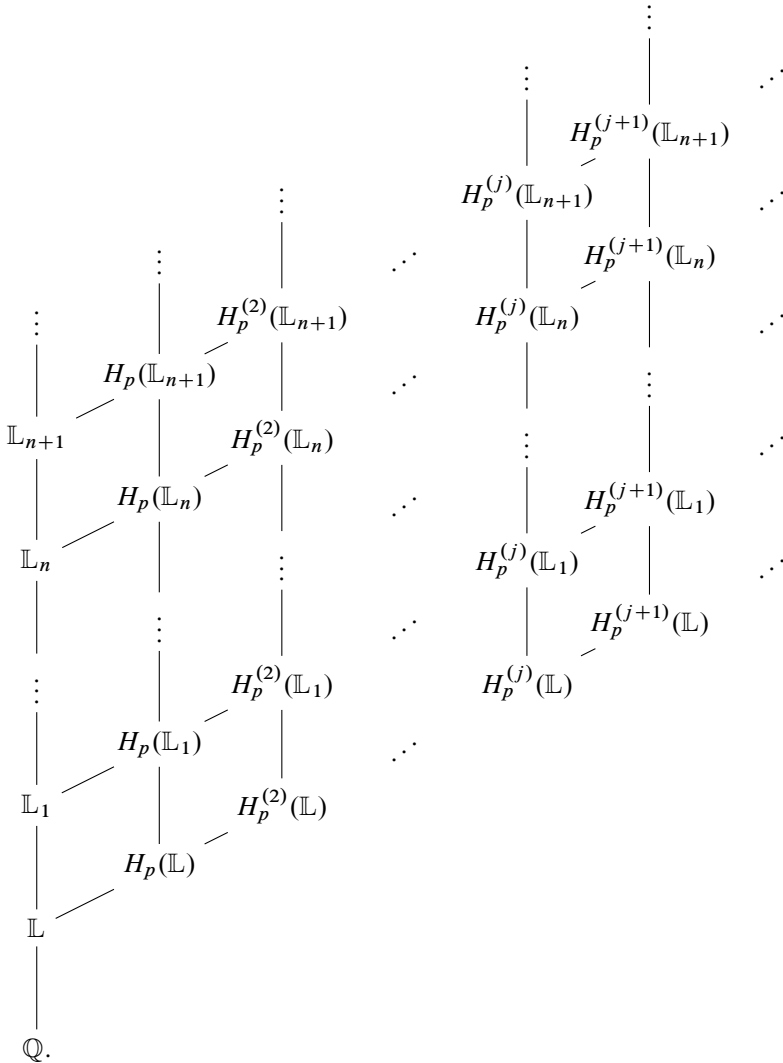
Mathematics Subject Classification 2020: 11R23.

Keywords: Hilbert class field towers, Iwasawa theory, growth patterns.

The field $\mathcal{F}(\mathbb{L}_n)$ contains a p Hilbert class field tower over \mathbb{L}_n . In greater detail, for $j \in \mathbb{Z}_{\geq 0}$, define $H_p^{(j)}(\mathbb{L}_n)$ inductively as follows:

$$\begin{aligned} H_p^{(0)}(\mathbb{L}_n) &:= \mathbb{L}_n, \\ H_p^{(1)}(\mathbb{L}_n) &:= H_p(\mathbb{L}_n), \\ H_p^{(j)}(\mathbb{L}_n) &:= H_p(H_p^{(j-1)}(\mathbb{L}_n)) \quad \text{for } j \geq 2. \end{aligned}$$

In this way, we obtain a tower of p -extensions of \mathbb{L}_n whose union is equal to $\mathcal{F}(\mathbb{L}_n)$. The field $\mathcal{F}(\mathbb{L}_n)$ is infinite if and only if $H_p^{(j)}(\mathbb{L}_n) \subsetneq H_p^{(j+1)}(\mathbb{L}_n)$ for all $j \geq 1$. These Hilbert class field towers, when viewed along the \mathbb{Z}_p -extension, are represented by the following diagram:



Iwasawa studied asymptotic growth patterns for the degrees $[H_p(\mathbb{L}_n) : \mathbb{L}_n]$ considered along the tower that is the second column (from the left margin) in the above diagram. In the spirit of Iwasawa theory, we consider natural growth questions for the pro- p groups G_n as $n \rightarrow \infty$. We define the *exponential growth number* $\rho(G)$ of a finitely generated pro- p group G , which is a natural invariant associated with its Hilbert series. In this context, there is a natural analogy with the Hilbert series of an algebraic variety. Let $\Omega(G)$ denote the mod- p Iwasawa algebra associated to G , defined as the inverse limit $\varprojlim_U \mathbb{F}_p[G/U]$, where U ranges over all finite index normal subgroups of G . Let I_G be the augmentation ideal of $\Omega(G)$, and for $n \geq 0$, set $c_n(G) := \dim_{\mathbb{F}_p}(I_G^n/I_G^{n+1})$, where it is understood that $c_0(G) := 1$. Let

$$H(G; t) = \sum_{n \geq 0} c_n(G) t^n$$

denote the Hilbert series associated to G , cf. Definition 2.1. If G is finite, then $c_n(G) = 0$ for large enough values of n . When G is a p -adic analytic group of dimension $d > 0$, we find that $c_n(G) = O(n^d)$ (cf. [1, Section 12.3, p. 307]). On the other hand, if a group G has infinite rank, the numbers $c_n(G)$ grow at an exponential rate (cf. Proposition 12.17 of *loc. cit.*). In practice, the Galois groups that arise from infinite p Hilbert class field towers may not be p -adic analytic groups. For many of the Galois groups constructed in this article, the numbers $c_n(G)$ increase at an exponential rate. The radius of convergence of $H(G; t)$ is given by

$$R_G = \rho(G)^{-1},$$

where $\rho(G) := \limsup_{n \rightarrow \infty} |c_n(G)|^{\frac{1}{n}}$. It is understood that $R_G := \infty$ when $\rho(G) = 0$, and in this case, $H(G, t)$ is an *entire function*. The constant $\rho(G)$ measures the exponential growth of subgroups in the Zassenhaus filtration of G . It is clear that if $\rho(G) > 1$, then G is not an analytic pro- p group. It is shown, in various contexts that pro- p Hilbert class field towers are infinite via an application of the Golod–Shafarevich–Vinberg criterion (cf. [4, Theorem 1.2]). This strategy is developed and employed in [4, 5, 7, 8, 15, 16]. The class of pro- p groups for which it can be shown that $\rho(G) > 1$, via the Golod–Shafarevich–Vinberg criterion have special properties, and are known as Golod–Shafarevich groups (cf. Definition 2.4). We refer to [2] for an introduction to the theory of Golod–Shafarevich groups.

Given a number field L , its *root discriminant* is defined to be $D_L^{1/n}$, where D_L is its absolute discriminant, and $n = [L : \mathbb{Q}]$. It is an old question as to whether there exists an infinite tower of number fields, unramified away from a finite set of primes, for which the root discriminant is bounded. Since the root discriminant is constant in unramified extensions, the above question is related to the constants of Martinet [11] and Odlyzko's bounds [14]. In the number field extensions L/\mathbb{L}_n such that $L \subset \mathcal{F}(\mathbb{L}_n)$, the root discriminant remains bounded. Hajir and Maire [3] construct tamely ramified Golod–Shafarevich Galois groups and are able to improve upon Martinet's constants. These root discriminant bounds are further refined by Hajir, Maire and Ramakrishna in [5]. One is thus interested in constructing infinite unramified Galois pro- p groups G , such that $\rho(G)$ is large. In this paper, we study the following question.

Question 1.1. Let \mathbb{L} be a number field and p be a prime number. Given a \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$, what can one say about the growth of $\rho(G_n)$, as $n \rightarrow \infty$?

1.2. Main results

We consider a variant of the above question, for certain natural \mathbb{Z}_p -extensions in which one of the primes above p is infinitely split. For $k \geq 0$, let $\mathcal{F}^{[k]}(\mathbb{L}_n)$ be the maximal pro- p extension of \mathbb{L}_n for which

- all primes $v \nmid p$ are unramified,
- all decomposition groups at primes $v \mid p$ are abelian,
- for all primes $v \mid p$, every element of the decomposition group of v has order dividing p^k .

Note that $\mathcal{F}(\mathbb{L}_n) \supseteq \mathcal{F}^{[0]}(\mathbb{L}_n)$. For $k \geq 1$, there is finite ramification in the extension $\mathcal{F}^{[k]}(\mathbb{L}_n)/\mathbb{L}_n$. As a consequence, there is an intermediate extension $\mathbb{L}_n \subseteq \mathbb{L}'_n \subseteq \mathcal{F}^{[k]}(\mathbb{L}_n)$, such that $\mathbb{L}'_n/\mathbb{L}_n$ is a finite extension, and $\mathcal{F}^{[k]}(\mathbb{L}_n)/\mathbb{L}'_n$ is unramified (cf. the proof of [4, Proposition 1.5]). In particular, for all number fields contained in $\mathcal{F}^{[k]}(\mathbb{L}_n)$, the root discriminant is bounded. We shall set $\mathcal{G}^{[k]}(\mathbb{L}_n) := \text{Gal}(\mathcal{F}^{[k]}(\mathbb{L}_n)/\mathbb{L}_n)$, and $\rho^{[k]}(\mathbb{L}_n) := \rho(\mathcal{G}^{[k]}(\mathbb{L}_n))$.

Let \mathbb{L} be a number field extension of \mathbb{Q} and let p be an odd prime. Let r_1 (resp. r_2) be the number of real embeddings (resp. complex embeddings) of \mathbb{L} . Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_g$ be the primes of \mathbb{L} that lie above p , and assume that $g \geq 2$. Set $e_i := e(\mathfrak{p}_i/p)$ (resp. $f_i := f(\mathfrak{p}_i/p)$) to denote the ramification index (resp. inertial degree) of \mathfrak{p}_i over p . Note that

$$d_i := [\mathbb{L}_{\mathfrak{p}_i} : \mathbb{Q}_p] = e_i f_i.$$

Let $U_i := \mathcal{O}_{\mathbb{L}_{\mathfrak{p}_i}}^\times$ and $U_i^{(1)}$ the principal local units in U_i . We set \mathcal{U} to denote the product $\prod_{i=2}^g U_i^{(1)}$, and \bar{E} to denote the p -adic closure of the image of the group of global units $\mathcal{O}_{\mathbb{L}}^\times$ in \mathcal{U} . Setting $\delta := \text{rank}_{\mathbb{Z}_p}(\bar{E})$, we note that $\delta \leq \text{rank } \mathcal{O}_{\mathbb{L}}^\times = r_1 + r_2 - 1$. We set

$$m := \text{rank}_{\mathbb{Z}_p}(\mathcal{U}/\bar{E}) - 1 = [\mathbb{L} : \mathbb{Q}] - d_1 - \delta - 1,$$

and assume that $m \geq 1$. It follows from a standard application of global class field theory that there exists a \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$ in which \mathfrak{p} is completely split. Note that when \mathbb{L} is totally imaginary, the condition $m \geq 1$ is automatically satisfied when $[\mathbb{L} : \mathbb{Q}] \geq 2(d_1 + 1)$.

Theorem 1.2. Assume that the following conditions are satisfied

- (1) p is odd and there are $g > 1$ primes of \mathbb{L} that lie above p ,
- (2) \mathbb{L} contains μ_p , the p -th roots of unity,
- (3) $[\mathbb{L} : \mathbb{Q}] \geq 2(d_1 + 1)$,
- (4) $([\mathbb{L} : \mathbb{Q}] + 2)^2 > 8(\sum_{i=2}^g d_i^2)$.

Then, there exists a \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$ in which \mathfrak{p}_1 is totally split. Moreover, there exists a constant $C > 0$ (independent of n and k) and $n_0 \in \mathbb{Z}_{\geq 0}$, such that for all $n \geq n_0$ and

$k \geq 1$, we have that

$$\rho^{[k]}(\mathbb{L}_n) \geq Cp^n.$$

Let $\mu(\mathbb{L}_\infty/\mathbb{L})$ denote the Iwasawa μ -invariant for the \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$. Let T_1 be the number of primes of \mathbb{L} that lie above p , that are infinitely decomposed in \mathbb{L}_∞ . The constant C can be chosen to be any value such that

$$0 < C < \frac{2(\sum_{i=2}^g d_i^2)}{(2T_1 + [\mathbb{L} : \mathbb{Q}] + 2\mu(\mathbb{L}_\infty/\mathbb{L}))}.$$

Remark 1.3. Following the above result, we make the following remarks.

- We observe that the hypotheses imply that

$$\frac{2(\sum_{i=2}^g d_i^2)}{(2T_1 + [\mathbb{L} : \mathbb{Q}] + 2\mu(\mathbb{L}_\infty/\mathbb{L}))} < \left(\frac{[\mathbb{L} : \mathbb{Q}] + 2}{4}\right).$$

- Note that condition (2) implies that \mathbb{L} is totally imaginary and (3) implies that $m \geq 1$.

We prove Theorem 1.2 by adapting the strategy of Hajir, Maire and Ramakrishna [4]. The following result illustrates Theorem 1.2.

Corollary 1.4. Let $p \geq 3$ and $\ell \geq 11$ be distinct primes and set $\mathbb{L} := \mathbb{Q}(\mu_{p\ell})$. Let \mathfrak{p} be any prime of \mathbb{L} that lies above p . Assume that there are $g = (\ell - 1)$ primes of \mathbb{L} that lie above p , i.e., $\ell \equiv 1 \pmod p$. Then, there exists a \mathbb{Z}_p -extension $\mathbb{L}_\infty/\mathbb{L}$ in which \mathfrak{p} is totally split. Let T_1 be the number of primes of \mathbb{L} that lie above p , that are infinitely decomposed in \mathbb{L}_∞ . For any constant

$$0 < C < \frac{2(\ell - 2)(p - 1)^2}{(2T_1 + (p - 1)(\ell - 1) + 2\mu(\mathbb{L}_\infty/\mathbb{L}))},$$

we have that

$$\rho^{[k]}(\mathbb{L}_n) \geq Cp^n$$

for all $k \geq 1$, and all large enough values of n .

We also define another notion measuring the size $m(G)$ of a Golod–Shafarevich group G (cf. Definition 2.6). We prove an analogous result for the growth of the numbers

$$m^{[k]}(\mathbb{L}_n) := m(\mathcal{G}^{[k]}(\mathbb{L}_n)),$$

cf. Theorem 3.4.

1.3. Organization

Including the introduction, the manuscript consists of three sections. In Section 2, we introduce preliminary notions. In greater detail, we develop the Golod–Shafarevich theory of pro- p groups. We prove an explicit criterion (cf. Proposition 2.4) which gives an explicit lower bound for the exponential growth number $\rho(G)$ for a Golod–Shafarevich group G . In Section 3, we apply the results from Section 2 to prove Theorem 1.2, Corollary 1.4 and Theorem 3.4.

2. Pro- p groups and their Hilbert series

We review some preliminaries and set up some basic notation in this section. Throughout, p shall be an odd prime. Let G be a finitely generated pro- p group, set $\Omega(G)$ to denote the completed group algebra of G over \mathbb{F}_p . More precisely, $\Omega(G)$ is defined to be the inverse limit

$$\Omega(G) := \varprojlim_U \mathbb{F}_p[G/U],$$

where U runs over all finite index normal subgroups of G . We refer to $\Omega(G)$ as the *mod- p Iwasawa algebra* of G . Many properties of the group G are captured by algebraic properties of $\Omega(G)$. We consider the *Hilbert series* associated with $\Omega(G)$. Given a normal subgroup U of G , there is a natural map $\iota_U : G \rightarrow \mathbb{F}_p[G/U]$, which sends $g \in G$ to the element $\bar{g} \cdot 1$. Here, \bar{g} is the congruence class of g modulo U . The map $\iota : G \rightarrow \Omega(G)$ is the inverse limit $\iota := \varprojlim_U \iota_U$. The augmentation map $\alpha : \Omega(G) \rightarrow \mathbb{F}_p$ maps each element of the form $\iota(g)$ to 1. We shall, by abuse of notation, simply let $g \in \Omega(G)$ simply denote the element $\iota(g)$. Set I_G to denote the augmentation ideal of $\Omega(G)$, i.e., the kernel of the augmentation map. For $n \geq 1$, $\Omega(G)/I_G^n$ is a finite dimensional \mathbb{F}_p -vector space.

Definition 2.1. For $n \geq 1$, let $c_n(G) := \dim_{\mathbb{F}_p}(I_G^n/I_G^{n+1})$, and set $c_0(G) := 1$. The Hilbert series $H(G; t)$ is a formal power series defined by $H(G; t) := \sum_{n \geq 0} c_n(G)t^n$. Setting $\rho(G) := \limsup_{n \rightarrow \infty} |c_n(G)|^{\frac{1}{n}}$, we note that the radius of convergence of $H(G; t)$ is given by

$$R_G := \begin{cases} \rho(G)^{-1} & \text{if } \rho(G) < \infty; \\ 0 & \text{if } \rho(G) = \infty. \end{cases}$$

Given $x \in G$ such that $x \neq 1$, the *depth* of x is defined as follows

$$\omega_G(x) := \max\{n \mid x - 1 \in I_G^n\}.$$

We set $\omega_G(1) := \infty$.

Definition 2.2. The *Zassenhaus filtration* is defined as follows

$$G_n := \{g \in G \mid \omega_G(g) \geq n\}.$$

The sequence G_n is a sequence of open normal subgroups of G . The quotient G_n/G_{n+1} is a finite dimensional \mathbb{F}_p vector space, set $a_n(G) := \dim_{\mathbb{F}_p}(G_n/G_{n+1})$. The quantity $\rho(G)$ measures the order of exponential growth of the quotients (I_G^n/I_G^{n+1}) as $n \rightarrow \infty$. These numbers are related to the growth of groups in the Zassenhaus filtration. There is an explicit relationship between the numbers $\{a_n(G)\}$ and $\{c_n(G)\}$, established by Mináč, Rogelstad and Tân [12].

Assume that G is a finitely presented pro- p group, and let

$$\begin{aligned} d(G) &:= \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p), \\ r(G) &:= \dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p). \end{aligned}$$

Remark 2.1. We identify $H^1(G, \mathbb{F}_p)$ with $\text{Hom}(\frac{G}{[G, G]G^p}, \mathbb{F}_p)$, and thus,

$$d(G) = \dim_{\mathbb{F}_p} \left(\frac{G}{[G, G]G^p} \right).$$

If \mathcal{G} is a quotient of G , then, it is clear that $d(\mathcal{G}) \leq d(G)$.

Consider a minimal presentation

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} G \rightarrow 1 \tag{2.1}$$

of G . Here, $F = \langle \sigma_1, \dots, \sigma_d \rangle$ is a free pro- p group generated by $d = d(G)$ elements. The subgroup R of F is the normal subgroup $\langle \rho_1, \dots, \rho_r \rangle^{\text{Norm}}$ generated by $r = r(G)$ elements. Given a choice of minimal presentation of G , we define the associated Golod–Shafarevich polynomials. Let $\Omega(F)$ be the mod- p Iwasawa algebra associated to F , and I_F be the augmentation ideal of $\Omega(F)$. By a theorem of Lazard [10], the algebra $\Omega(F)$ is isomorphic to the algebra of power series $\mathbb{F}_p \langle\langle u_1, \dots, u_d \rangle\rangle$. Here u_i is identified with the element $(\sigma_i - 1)$. The augmentation ideal I_F is generated by u_1, \dots, u_d . Let ω_F be the depth function on F , defined by setting

$$\omega_F(x) := \max\{n \mid x - 1 \in I_F^n\},$$

for $x \neq 1$, and $\omega_F(1) := \infty$. The map φ induces a surjection $\Omega(F) \rightarrow \Omega(G)$, whose kernel we denote by J . Identify $\Omega(G)$ with the quotient $\Omega(F)/J$ and I_G with I_F/J . The depth filtration ω_G is related to ω_F as follows

$$\omega_G(x) = \max \{ \omega_F(y) \mid \varphi(y) = x \},$$

cf. [10, Lemma 3.4 and Theorem 3.5, pp. 204–205] for further details.

Proposition 2.2. *The depth function $\omega_F : F \rightarrow \mathbb{Z}_{\geq 1} \cup \{\infty\}$ and associated filtration $\{G_n\}_n$ satisfies the following properties*

- (1) for $x \in F$, $\omega_F(x^p) \geq p\omega_F(x)$. Therefore, if $x \in G_n$, then, $x^p \in G_{np}$.
- (2) For $x \in G_n$ and $y \in G_m$, one has $[x, y] \in G_{n+m}$.
- (3) For $x, y \in G$, we have that $\omega_F(xy) \geq \min(\omega_F(x), \omega_F(y))$.

Proof. The above mentioned result is [5, Propositions 1 and 2], also see [9, Section 7.4]. ■

Definition 2.3. The Golod–Shafarevich polynomial associated with the minimal presentation (2.1) is defined as follows

$$P(G; t) := 1 - dt + \sum_{i=1}^r t^{\omega_F(\rho_i)}.$$

Note that since the filtration is minimal, the depth $\omega_F(\rho_i) \geq 2$, cf. [4, p. 3, l.–9].

Theorem 2.3 (Vinberg’s criterion). *Let G be a pro- p group and let $H(G; t)$ be the Hilbert series associated to G . Choose a minimal presentation (2.1) of G and let $P(G; t)$ be the associated Golod–Shafarevich polynomial. Recall that R_G is the radius of convergence of $H(G; t)$, and set $R'_G := \min\{1, R_G\}$. Then, for any value $t \in (0, R'_G)$, the following inequality is satisfied*

$$H(G; t)P(G; t) \geq 1.$$

Proof. This result is well known. We refer to [2, Theorem 2.1] and [17] for a proof of the above statement. ■

The next result shows that if $P(G; t)$ takes a negative value t_0 in the interval $(0, 1)$, then, $\rho(G) > 1$. Furthermore, it gives us a lower bound for $\rho(G)$, and can therefore be viewed as a refinement of the Golod–Shafarevich–Vinberg criterion (cf. [4, Theorem 1.2]).

Proposition 2.4. *Suppose that for some $t_0 \in (0, 1)$, the value $P(G; t_0)$ is negative. Then, the order of growth satisfies the lower bound $\rho(G) \geq \frac{1}{t_0}$.*

Proof. Let R_G be the radius of convergence of the Hilbert series $H(G; t)$ and set $R'_G := \min\{1, R_G\}$. By definition, the coefficients of $H(G; t)$ are all non-negative. Then, for $t \in (0, R'_G)$, the series $H(G; t)$ converges absolutely to a positive value. By Vinberg’s criterion, $P(G; t)H(G; t) \geq 1$ for all $t \in (0, R'_G)$. Since $P(G; t_0) < 0$, it follows that $H(G; t)$ does not converge absolutely at $t = t_0$. In other words, t_0 lies outside the domain of convergence. Therefore, $t_0 \geq R'_G$. Since $t_0 < 1$, it follows that $R_G = R'_G < 1$ and therefore, $t_0 \geq R_G = \rho(G)^{-1}$. Therefore, we conclude that $\rho(G) \geq \frac{1}{t_0}$. ■

Definition 2.4. A pro- p group G is said to be a *Golod–Shafarevich group* if for some minimal presentation

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} G \rightarrow 1,$$

there is a value $t_0 \in (0, 1)$ so that $P(G; t_0) < 0$. Proposition 2.4 shows that if G is a Golod–Shafarevich group with $P(G; t_0) < 0$, then the order of exponential growth satisfies $\rho(G) \geq t_0^{-1} > 1$. In particular G is infinite since the numbers $c_n(G)$ grow at an exponential rate as $n \rightarrow \infty$.

At this point, we introduce a new definition which measures the *size* of a Golod–Shafarevich group. Let G be a Golod–Shafarevich group and

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} G \rightarrow 1$$

be a minimal presentation of G . We consider quotients G' of G such that $d(G') = d(G)$. Suppose that

$$G' = G / \langle x_1, \dots, x_m \rangle^{\text{Norm}}.$$

Then, following [4, p. 4, ll. 10–17], we get a new minimal presentation as follows. Lift each x_i to $y_i \in F$, and let $R' = \langle \rho_1, \dots, \rho_r, y_1, \dots, y_m \rangle^{\text{Norm}}$ to be the normal subgroup of F generated by R and the elements y_1, \dots, y_m . Since it is assumed that $d(G') =$

$d(G)$, this gives a minimal presentation of G' . In particular, note that $\omega_F(y_i) \geq 2$ for all $i = 1, \dots, m$. Following *loc. cit.*, we say that we have *cut* the group G by the elements y_1, \dots, y_m . Let $\varphi' : F \rightarrow G'$ be the composite of φ with the quotient map $G \rightarrow G'$. With respect to the new presentation

$$1 \rightarrow R' \rightarrow F \xrightarrow{\varphi'} G' \rightarrow 1,$$

we find that for $t \in (0, 1)$,

$$P(G'; t) \leq 1 - d(G)t + r(G)t^2 + \sum_{i=1}^m t^{\omega_F(y_i)}.$$

Definition 2.5. Let G be a Golod–Shafarevich group, and choose a minimal presentation for G such that $P(G; t)$ attains a negative value on $(0, 1)$. We say that a set $\{y_1, \dots, y_m\} \subset F$ is an admissible cutting datum if $\omega_F(y_i) \geq 2$ for all i .

Given an admissible cutting datum $\{y_1, \dots, y_m\}$, set $x_i := \varphi(y_i)$ and

$$G' := G / \langle x_1, \dots, x_m \rangle^{\text{Norm}}.$$

Definition 2.6. Let G be a Golod–Shafarevich group, and choose a minimal presentation

$$\mathfrak{M} : 1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

for G such that $P(G; t)$ attains a negative value on $(0, 1)$. We define $m_{\mathfrak{M}}(G)$ to be the smallest value $m \in \mathbb{Z}_{\geq 0}$ such that there exists an admissible cutting datum $\{y_1, \dots, y_{m+1}\}$, such that $P(G'; t) \geq 0$ for all $t \in (0, 1)$. If no such m exists, we set $m_{\mathfrak{M}}(G) := \infty$. We set

$$m(G) := \min \{m_{\mathfrak{M}}(G) \mid \mathfrak{M}\},$$

where the minimum is taken over all minimal presentations \mathfrak{M} of G .

Remark 2.5. We note that the definition of $m(G)$ does not, to our knowledge, appear in the literature prior to this.

Thus, for any admissible cutting datum $\{y_1, \dots, y_k\}$, with $k \leq m(G)$, the associated quotient G' is a Golod–Shafarevich group, and thus in particular is infinite. Like $\rho(G)$, the number $m(G)$ gives one a measure of the size of a Golod–Shafarevich group G .

3. Growth asymptotics for split prime \mathbb{Z}_p -extensions

Let \mathbb{L} be a number field which satisfies the conditions of Theorem 1.2. In this section, S_p (resp. S_∞) denote the primes of \mathbb{L} that lie above p (resp. ∞). Set S to denote the union. Let L/\mathbb{L} be a Galois number field extension and let $S(L)$ (resp. $S_p(L)$) be the set of places of L which lie above S (resp. S_p). We shall denote by L_S the maximal pro- p algebraic

extension of L in which the primes $w \notin S(L)$ are unramified. We denote by $G_S(L)$ the Galois group $\text{Gal}(L_S/L)$. Set $H_S(L)$ to be the maximal abelian unramified extension of L in which the places $v \in S(L)$ are split. The S class group of L is the Galois group $\text{Cl}_S(L) := \text{Gal}(H_S(L)/L)$, and is identified with a quotient of the class group $\text{Cl}(L)$. For $v \in S_p$, let $(e_v(L/\mathbb{L}), f_v(L/\mathbb{L}), g_v(L/\mathbb{L}))$ denote the ramification invariants of v in L . In other words, v splits into $g_v(L/\mathbb{L})$ primes in \mathcal{O}_L , each with ramification index $e_v(L/\mathbb{L})$. The quantity $f_v(L/\mathbb{L}) := [\mathcal{O}_L/w : \mathcal{O}_K/v]$ is independent of the choice of prime $w|v$ of L . We note that $e_v(L/\mathbb{L})f_v(L/\mathbb{L})g_v(L/\mathbb{L}) = [L : \mathbb{L}]$.

Theorem 3.1. *Let L/\mathbb{L} be a finite Galois extension. With respect to the above notation, the following relations hold*

$$\dim_{\mathbb{F}_p} H^1(G_S(L), \mathbb{F}_p) = \sum_{i=1}^g g_{\mathfrak{p}_i}(L/\mathbb{L}) + \frac{[L : \mathbb{Q}]}{2} + \dim(\text{Cl}_S(L) \otimes \mathbb{F}_p),$$

$$\dim_{\mathbb{F}_p} H^2(G_S(L), \mathbb{F}_p) = \sum_{i=1}^g g_{\mathfrak{p}_i}(L/\mathbb{L}) - 1 + \dim(\text{Cl}_S(L) \otimes \mathbb{F}_p).$$

Proof. The above is a special case of [13, Theorem 10.7.3]. ■

We order the set of primes above p such that \mathfrak{p}_i is infinitely decomposed \mathbb{L}_∞ for $i \in [1, T_1]$ and infinitely ramified for $i \in [T_1 + 1, T_2]$. The primes \mathfrak{p}_i for $i \in [T_2 + 1, g]$ are unramified and finitely decomposed in \mathbb{L}_∞ . We shall set $g_i(n)$, $e_i(n)$ and $f_i(n)$ to denote $g_{\mathfrak{p}_i}(\mathbb{L}_n/\mathbb{L})$, $e_{\mathfrak{p}_i}(\mathbb{L}_n/\mathbb{L})$ and $f_{\mathfrak{p}_i}(\mathbb{L}_n/\mathbb{L})$ respectively. We find that

$$\begin{aligned} g_1(n) &= p^n, & e_1(n) &= f_1(n) = 1, \\ g_i(n) &\leq p^n & \text{for } i &\in [2, T_1], \\ g_i(n) &= O(1) & \text{for } i &> T_1. \end{aligned} \tag{3.1}$$

We set

$$d(n) := \dim_{\mathbb{F}_p} H^1(G_S(\mathbb{L}_n), \mathbb{F}_p) \quad \text{and} \quad r(n) := \dim_{\mathbb{F}_p} H^2(G_S(\mathbb{L}_n), \mathbb{F}_p).$$

Before stating the next result, we clarify some standard conventions with regard to our notation. Let $f, g, h : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be non-negative functions. We write $f(n) = g(n) + O(h(n))$ to mean that there is a positive constant $C > 0$, independent of n , such that for all $n \in \mathbb{N}$, $|f(n) - g(n)| \leq Ch(n)$. We write $f(n) \leq g(n) + O(h(n))$ (resp. $f(n) \geq g(n) + O(h(n))$) to mean that $f(n) \leq g(n) + Ch(n)$ (resp. $f(n) \geq g(n) - Ch(n)$) for some absolute constant $C > 0$.

Corollary 3.2. *The following relations hold*

$$\begin{aligned} d(n) &\geq p^n \left(1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} \right), \\ d(n) &\leq p^n \left(T_1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} + \mu(\mathbb{L}_\infty/\mathbb{L}) \right) + o(p^n), \\ r(n) &= O(p^n). \end{aligned}$$

Proof. It follows from Iwasawa’s theorem that

$$\dim_{\mathbb{F}_p} (\text{Cl}(\mathbb{L}_n) \otimes \mathbb{F}_p) \leq \mu(\mathbb{L}_\infty/\mathbb{L})p^n + o(p^n).$$

The result follows therefore as an immediate consequence of Theorem 3.1 and (3.1). ■

For $n \geq 0$ and $v \in S_p(\mathbb{L}_n)$, set $n_v := \dim_{\mathbb{F}_p}(\mathbb{L}_{n,v}^\times \otimes \mathbb{F}_p)$. We choose an embedding of

$$\iota : \bar{\mathbb{L}}_n \hookrightarrow \bar{\mathbb{L}}_{n,v},$$

or equivalently, a prime v' of $\bar{\mathbb{L}}_n$ that lies above v . We shall refer to the decomposition group of $v'|v$ as the decomposition group at v . The inclusion ι prescribes an inclusion of $\text{Gal}(\bar{\mathbb{L}}_{n,v}/\mathbb{L}_{n,v})$ into $G_{\mathbb{L}_n}$, and the image of this embedding is identified with the decomposition group at v . Note that the pro- p completion of the decomposition group at v is generated by n_v elements. For ease of notation, we set $d_i := [\mathbb{L}_{\mathfrak{p}_i} : \mathbb{Q}_p]$. We note that since μ_p is contained in \mathbb{L} ,

$$\begin{aligned} n_v &= \dim_{\mathbb{F}_p}(\mathbb{L}_{n,v}^\times \otimes \mathbb{F}_p) = [\mathbb{L}_{n,v} : \mathbb{Q}_p] + 2, \\ &= (e_i(n) f_i(n) [\mathbb{L}_{\mathfrak{p}_i} : \mathbb{Q}_p] + 2) \quad \text{if } v|\mathfrak{p}_i, \\ &\begin{cases} = (d_1 + 2) & \text{if } v|\mathfrak{p}_1, \\ \leq (p^n d_i + 2) & \text{if } v|\mathfrak{p}_i \text{ for } i \geq 2. \end{cases} \end{aligned} \tag{3.2}$$

Let $\tilde{\mathcal{L}}_n$ be the maximal pro- p extension of \mathbb{L}_n unramified at all primes $v \notin S(\mathbb{L}_n)$. Denote by \mathcal{L}_n the maximal pro- p extension of \mathbb{L}_n unramified at all primes $v \notin S(\mathbb{L}_n)$ and such that for all primes $v \in S_p(\mathbb{L}_n)$, the decomposition group is abelian. For $k \geq 0$, let $\mathcal{F}^{[k]}(\mathbb{L}_n)$ be the maximal subfield of \mathcal{L}_n in which all the elements in the decomposition groups at primes $v \in S_p(\mathbb{L}_n)$ have order dividing p^k . It is thus understood that $\mathcal{F}^{[0]}(\mathbb{L}_n)$ is the maximal unramified pro- p extension of \mathbb{L}_n in which all primes $v \in S(\mathbb{L}_n)$ are completely split. We have the following containments

$$\mathbb{L}_n \subseteq \mathcal{F}^{[0]}(\mathbb{L}_n) \subseteq \mathcal{F}^{[1]}(\mathbb{L}_n) \subseteq \dots \subseteq \mathcal{F}^{[k]}(\mathbb{L}_n) \subseteq \mathcal{F}^{[k+1]}(\mathbb{L}_n) \subseteq \dots \subseteq \mathcal{L}_n \subseteq \tilde{\mathcal{L}}_n.$$

Set $\tilde{\mathcal{G}}_n, \mathcal{G}_n$ and $\mathcal{G}_n^{[k]} = \mathcal{G}^{[k]}(\mathbb{L}_n)$ denote the Galois groups $\text{Gal}(\tilde{\mathcal{L}}_n/\mathbb{L}_n), \text{Gal}(\mathcal{L}_n/\mathbb{L}_n)$ and $\text{Gal}(\mathcal{F}^{[k]}(\mathbb{L}_n)/\mathbb{L}_n)$ respectively. We begin with a minimal filtration

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} \tilde{\mathcal{G}}_n \rightarrow 1 \tag{3.3}$$

of $\tilde{\mathcal{G}}_n$. The group \mathcal{G}_n is obtained from $\tilde{\mathcal{G}}_n$ upon cutting by the commutators of the generators of all decomposition groups for primes $v \in S_p(\mathbb{L}_n)$. In greater detail, for each prime $v \in S_p(\mathbb{L}_n)$, let $z_1^{(v)}, \dots, z_{n_v}^{(v)}$ be a set of elements in F mapping to a set of generators of the decomposition group at v . For $1 \leq i < j \leq n_v$, we let $z_{i,j}^{(v)} \in F$ denote the commutator $[z_i^{(v)}, z_j^{(v)}]$. It follows from Proposition 2.2 that

$$\omega(z_{i,j}^{(v)}) \geq \omega(z_i^{(v)}) + \omega(z_j^{(v)}) \geq 2. \tag{3.4}$$

We represent \mathcal{G}_n as the quotient

$$\mathcal{G}_n = \frac{\tilde{\mathcal{G}}_n}{\langle \varphi(z_{i,j}^{(v)}) \mid 1 \leq i < j \leq n_v, v \in S_p(\mathbb{L}_n) \rangle^{\text{Norm}}},$$

and we obtain a new presentation

$$1 \rightarrow R' \rightarrow F \rightarrow \mathcal{G}_n \rightarrow 1, \tag{3.5}$$

where

$$R' = R\langle z_{i,j}^{(v)} \mid 1 \leq i < j \leq n_v, v \in S_p(\mathbb{L}_n) \rangle^{\text{Norm}}.$$

Going modulo the p^k -th powers of all generators of decomposition groups, we obtain a presentation

$$1 \rightarrow R'' \rightarrow F \rightarrow \mathcal{G}_n^{[k]} \rightarrow 1, \tag{3.6}$$

where

$$R'' = R'\langle (z_i^{(v)})^{p^k} \mid 1 \leq i \leq n_v, v \in S_p(\mathbb{L}_n) \rangle^{\text{Norm}}.$$

Set $\tilde{\mathcal{P}}_n(t) := \mathcal{P}(\tilde{\mathcal{G}}_n, t)$, $\mathcal{P}_n(t) := \mathcal{P}(\mathcal{G}_n, t)$, and $\mathcal{P}_n^{[k]}(t) := \mathcal{P}(\mathcal{G}_n^{[k]}, t)$ to be the Golod–Shafarevich polynomials associated with the minimal filtrations (3.3), (3.5) and (3.6) respectively. It follows from Remark 2.1 that

$$d(n) = d(\tilde{\mathcal{G}}_n) \geq d(\mathcal{G}_n) \geq d(\mathcal{G}_n^{[k]})$$

for all $k \geq 0$. Since we successively cut by elements with depth ≥ 2 , we have that

$$d(n) = d(\tilde{\mathcal{G}}_n) = d(\mathcal{G}_n) = d(\mathcal{G}_n^{[k]}),$$

(cf. [5, Proposition 5]).

Proposition 3.3. For $t \in (0, 1)$, $\mathcal{P}_n(t) \leq 1 - D_n t + R_n t^2$, where D_n, R_n are constants satisfying

$$\begin{aligned} D_n &\geq p^n \left(1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} \right), \\ D_n &\leq p^n \left(T_1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} + \mu(\mathbb{L}_\infty/\mathbb{L}) \right) + o(p^n), \\ R_n &= p^{2n} \left(\frac{\sum_{i=2}^g d_i^2}{2} \right) + O(p^n). \end{aligned}$$

Proof. The group \mathcal{G}_n is obtained from $\tilde{\mathcal{G}}_n$ by quotienting by the commutators of each of the decomposition groups at primes $v \in S_p(\mathbb{L}_n)$. At each prime v , we cut by $\binom{n_v}{2}$ elements

$$\{z_{i,j}^{(v)} \mid 1 \leq i < j \leq n_v, v \in S_p(\mathbb{L}_n)\},$$

where we recall that an upper bound for n_v is given by (3.2). Furthermore, it follows from (3.4) that $\omega(z_{i,j}^{(v)}) \geq 2$. Therefore, we find that for $t \in (0, 1)$,

$$\begin{aligned} \mathcal{P}_n(t) &\leq \tilde{\mathcal{P}}_n(t) + \left(\sum_{v \in S_p(\mathbb{L}_n)} \binom{n_v}{2} \right) t^2 \\ &\leq \tilde{\mathcal{P}}_n(t) + \left(p^n \binom{d_1 + 2}{2} + \sum_{i=2}^g \binom{p^n d_i + 2}{2} \right) t^2 \\ &\leq 1 - d(n)t + \left(r(n) + p^n \binom{d_1 + 2}{2} + \sum_{i=2}^g \binom{p^n d_i + 2}{2} \right) t^2 \\ &\leq 1 - D_n t + R_n t^2, \end{aligned}$$

where

$$\begin{aligned} D_n &:= d(n), \\ R_n &:= r(n) + p^n \binom{d_1 + 2}{2} + \sum_{i=2}^g \binom{p^n d_i + 2}{2} = p^{2n} \left(\frac{\sum_{i=2}^g d_i^2}{2} \right) + O(p^n). \end{aligned}$$

Here, we have invoked the bounds for $d(n)$ and $r(n)$ from Corollary 3.2. ■

Proof of Theorem 1.2. It follows from Proposition 3.3 that for $t \in (0, 1)$,

$$\mathcal{P}_n(t) \leq 1 - D_n t + R_n t^2,$$

where

$$\begin{aligned} D_n &\geq p^n \left(1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} \right), \\ D_n &\leq p^n \left(T_1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} + \mu(\mathbb{L}_\infty/\mathbb{L}) \right) + o(p^n), \\ R_n &= p^{2n} \left(\frac{\sum_{i=2}^g d_i^2}{2} \right) + O(p^n). \end{aligned} \tag{3.7}$$

For $k \geq 1$, the group $\mathcal{G}_n^{[k]}$ is obtained from \mathcal{G}_n by quotienting by the p^k -th powers of the generators of the decomposition groups at the primes $v \in S_p(\mathbb{L}_n)$. The total number of elements we quotient by is

$$\begin{aligned} R'_n &:= \sum_{v \in S_p(\mathbb{L}_n)} n_v \\ &\leq p^n (d_1 + 2) + \sum_{i=2}^g (p^n d_i + 2) = p^n \left(\sum_{i=1}^g d_i + 2 \right) + O(1). \end{aligned} \tag{3.8}$$

Let $y_1, \dots, y_{R'_n}$ be the elements that generate the decomposition groups at the primes $v \in S_p(\mathbb{L}_n)$. By Proposition 2.2, we have that

$$\omega_F(y_i^{p^k}) = p^k \omega_F(y_i) \geq p^k.$$

Therefore, for $t \in (0, 1)$, we find that

$$\mathcal{P}_n^{[k]}(t) \leq \mathcal{P}_n(t) + R'_n t^{p^k} \leq \mathcal{P}_n(t) + R'_n t^p.$$

Consider the polynomial $Q(t) := 1 - D_n t + R_n t^2 + R'_n t^p$. Setting $t_n := \frac{D_n}{2R_n}$, note that the quadratic part $1 - D_n t + R_n t^2$ of $Q(t)$ attains its minimum value at t_n . We find that

$$Q(t_n) = 1 - \frac{D_n^2}{4R_n} + \frac{R'_n D_n^p}{2^p R_n^p}.$$

We require that $Q(t_n) < 0$, i.e.,

$$D_n^2 > 4R_n + \frac{R'_n D_n^p}{2^{p-2} R_n^{p-1}},$$

i.e.,

$$2^{p-2} R_n^{p-1} D_n^2 > 2^p R_n^p + R'_n D_n^p. \tag{3.9}$$

It follows from the bounds (3.7) that as $n \rightarrow \infty$,

$$\begin{aligned} D_n^2 &\geq A p^{2n}, \\ R_n &= B p^{2n} + O(p^n), \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} A &= \left(1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2}\right)^2, \\ B &= \left(\frac{\sum_{i=2}^g d_i^2}{2}\right). \end{aligned}$$

Note that by (3.8), $R'_n = O(p^n)$. We estimate both sides of the inequality (3.9). Therefore, we obtain the following asymptotic estimates

$$\begin{aligned} 2^{p-2} R_n^{p-1} D_n^2 &\geq 2^{p-2} B^{p-1} A p^{2pn} + O(p^{(2p-1)n}), \\ R'_n D_n^p &= O(p^{(p+1)n}), \\ 2^p R_n^p &= 2^p B^p p^{2np} + O(p^{(2p-1)n}). \end{aligned} \tag{3.11}$$

Since $2p - 1 \geq p + 1$, it follows that $R'_n D_n^p = O(p^{(2p-1)n})$. By condition (4) of Theorem 1.2,

$$\frac{2^{p-2} B^{p-1} A}{2^p B^p} = \frac{A}{4B} = \frac{(2 + [\mathbb{L} : \mathbb{Q}])^2}{8(\sum_{i=2}^g d_i^2)} > 1.$$

Therefore, for large enough values of n , we have that $Q(t_n) < 0$. By the estimates in the statement of Proposition 3.3,

$$\begin{aligned} D_n &\leq p^n \left(T_1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2} + \mu(\mathbb{L}_\infty/\mathbb{L}) \right) + o(p^n), \\ R_n &= p^{2n} \left(\frac{\sum_{i=2}^g d_i^2}{2} \right) + O(p^n). \end{aligned} \tag{3.12}$$

Therefore, in particular, we find that $t_n \in (0, 1)$ for all large enough values of n . Since $\mathcal{P}_n^{[k]}(t) \leq Q(t)$ for all $t \in (0, 1)$, it follows that $\mathcal{P}_n^{[k]}(t_n) < 0$ for all large enough values of n , and all values of $k \geq 1$.

With respect to notation above,

$$t_n^{-1} = \frac{2R_n}{D_n}.$$

Let $C > 0$ be a constant for which

$$C < \frac{2(\sum_{i=2}^g d_i^2)}{(2T_1 + [\mathbb{L} : \mathbb{Q}] + 2\mu(\mathbb{L}_\infty/\mathbb{L}))}.$$

Then, we find that for all large enough values of n , we find that $t_n^{-1} > Cp^n$. Since for large enough values of n , $\mathcal{P}_n^{[k]}(t_n) < 0$ and $t_n \in (0, 1)$, it follows from Proposition 2.4 that

$$\rho^{[k]}(\mathbb{L}_n) \geq t_n^{-1},$$

and this proves the result. ■

Proof of Corollary 1.4. We show that the conditions of Theorem 1.2 are satisfied.

- (1) It is assumed that $g = \ell - 1$, in particular, $g > 1$.
- (2) Clearly, $\mathbb{L} := \mathbb{Q}(\mu_{p\ell})$ contains μ_p .
- (3) Note that $d_i = (p - 1)$ for all i . Since $\ell \geq 5$, we find that

$$[\mathbb{L} : \mathbb{Q}] = (\ell - 1)(p - 1) \geq 2(d_1 + 1) = 2p.$$

- (4) We find that $([L : \mathbb{Q}] + 2)^2 = ((\ell - 1)(p - 1) + 2)^2 > (\ell - 1)^2(p - 1)^2$ and that

$$8 \left(\sum_{i=2}^g d_i^2 \right) = 8(\ell - 2)(p - 1)^2.$$

Clearly, $\ell \geq 11$ implies that $(\ell - 1)^2 > 8(\ell - 2)$, and therefore,

$$([L : \mathbb{Q}] + 2)^2 > 8 \left(\sum_{i=2}^g d_i^2 \right). \quad \blacksquare$$

Next, we set $m^{[k]}(\mathbb{L}_n) := m(\mathcal{G}_n^{[k]})$, where we recall that $\mathcal{G}_n^{[k]} := \mathcal{G}^{[k]}(\mathbb{L}_n)$. The following result gives an asymptotic lower bound for the numbers $m^{[k]}(\mathbb{L}_n)$, as $n \rightarrow \infty$.

Theorem 3.4. *Let p be a prime number and \mathbb{L} be a number field for which the conditions of Theorem 1.2 are satisfied. Then, there exists a constant $c > 0$ (independent of n and k) and $n_0 \in \mathbb{Z}_{\geq 0}$, such that for all $n \geq n_0$ and $k \geq 1$, we have that*

$$m^{[k]}(\mathbb{L}_n) \geq cp^n.$$

Proof. We set $m_n := m(\mathcal{G}_n^{[k]})$, and assume without loss of generality that $m_n < \infty$. We obtain a lower bound on m_n . Choose a minimal presentation

$$1 \rightarrow R \rightarrow F \xrightarrow{\varphi} \mathcal{G}_n^{[k]} \rightarrow 1$$

such that there exists $t_0 \in (0, 1)$ for which $\mathcal{P}_n^{[k]}(t) := P(\mathcal{G}_n^{[k]}, t_0) < 0$ for the associated Golod–Shafarevich polynomial. Let $\{y_1, \dots, y_{m+1}\} \subset F$ be an admissible cutting datum. Note that by definition, $\omega_F(y_i) \geq 2$ for all i . Setting $x_i := \varphi(y_i)$, denote by G' the quotient $\mathcal{G}_n^{[k]} / \langle x_1, \dots, x_{m+1} \rangle^{\text{Norm}}$. By definition, the admissible cutting datum $\{y_1, \dots, y_{m+1}\}$ can be chosen so that $P(G', t) \geq 0$ for all $t \in (0, 1)$. From the proof of Theorem 1.2, we have that $\mathcal{P}_n^{[k]}(t) \leq Q(t)$, where $Q(t) := 1 - D_n t + R_n t^2 + R'_n t^{p^k}$. For $t \in (0, 1)$, we find that

$$\begin{aligned} P(G', t) &\leq \mathcal{P}_n^{[k]}(t) + (m + 1)t^2 \\ &\leq Q(t) + (m + 1)t^2 \\ &= 1 - D_n t + (R_n + m + 1)t^2 + R'_n t^{p^k} \\ &\leq 1 - D_n t + (R_n + m + 1)t^2 + R'_n t^p. \end{aligned}$$

In what follows, we set $a_n := \frac{D_n}{2(R_n + m + 1)}$. From the estimates (3.12), it follows that $a_n \in (0, 1)$ for large enough values of n . Therefore, assume that n is large enough so that $a_n \in (0, 1)$. Then, $P(G', a_n) \geq 0$, i.e.,

$$1 - D_n a_n + (R_n + m + 1)a_n^2 + R'_n a_n^p \geq 0.$$

We find that

$$\begin{aligned} &1 - D_n a_n + (R_n + m + 1)a_n^2 + R'_n a_n^p \\ &= \frac{2^p(R_n + m + 1)^p - 2^{p-2}(R_n + m + 1)^{p-1}D_n^2 + R'_n D_n^p}{2^p(R_n + m + 1)^p}. \end{aligned}$$

This implies that

$$\begin{aligned} (R_n + m + 1) &\geq \frac{D_n^2}{4} - \frac{R'_n D_n^p}{2^p(R_n + m + 1)^{p-1}} \\ &\geq \frac{D_n^2}{4} - \frac{R'_n D_n^p}{2^p R_n^{p-1}}, \end{aligned}$$

and therefore, we get the inequality

$$m \geq \frac{D_n^2}{4} - R_n - \frac{R'_n D_n^p}{2^p R_n^{p-1}} - 1. \quad (3.13)$$

By (3.10),

$$\begin{aligned} D_n^2 &\geq Ap^{2n}, \\ R_n &= Bp^{2n} + O(p^n), \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} A &= \left(1 + \frac{[\mathbb{L} : \mathbb{Q}]}{2}\right)^2, \\ B &= \left(\frac{\sum_{i=2}^g d_i^2}{2}\right). \end{aligned}$$

Furthermore, by (3.11),

$$\frac{R'_n D_n^p}{2^p R_n^{p-1}} = O(p^{(3-p)n}) = O(1). \quad (3.15)$$

Therefore, combining (3.13), (3.14) and (3.15), we have that

$$m \geq (A/4 - B)p^{2n} - Cp^n,$$

for some large enough constant $C > 0$. Recall that m_n is the minimum value of m over all possible minimal presentations. We choose a minimal presentation such that $m_n = m$. It follows from the condition (4) of Theorem 1.2 that $A > 4B$. Setting $c := A/4 - B$, the result follows. ■

Acknowledgments. This project was started when the third author was at the Centre de recherches mathématiques, Montreal. At the time, the third author's research was supported by the CRM-Simons fellowship. We are very thankful to Katharina Müller and Ravi Ramakrishna for numerous helpful comments. We thank the anonymous referee for the excellent report.

References

- [1] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, *Analytic pro- p groups*. 2nd edn., Cambridge Stud. Adv. Math. 61, Cambridge University Press, Cambridge, 1999
Zbl 0934.20001 MR 1720368
- [2] M. Ershov, *Golod-Shafarevich groups: a survey*. *Internat. J. Algebra Comput.* **22** (2012), no. 5, article no. 1230001 Zbl 1286.20033 MR 2949205
- [3] F. Hajir and C. Maire, *Tamely ramified towers and discriminant bounds for number fields*. *Compositio Math.* **128** (2001), no. 1, 35–53 Zbl 1042.11072 MR 1847664

- [4] F. Hajir, C. Maire, and R. Ramakrishna, [Infinite class field towers of number fields of prime power discriminant](#). *Adv. Math.* **373** (2020), article no. 107318 Zbl 1469.11440 MR 4130462
- [5] F. Hajir, C. Maire, and R. Ramakrishna, [Cutting towers of number fields](#). *Ann. Math. Qué.* **45** (2021), no. 2, 321–345 Zbl 1483.11246 MR 4308183
- [6] K. Iwasawa, [On \$\mathbb{Z}_l\$ -extensions of algebraic number fields](#). *Ann. of Math. (2)* **98** (1973), 246–326 Zbl 0285.12008 MR 349627
- [7] K. Joshi and C. McLeman, [Infinite Hilbert class field towers from Galois representations](#). *Int. J. Number Theory* **7** (2011), no. 1, 1–8 Zbl 1227.11117 MR 2776006
- [8] H. Kisilevsky and J. Labute, [On a sufficient condition for the \$p\$ -class tower of a CM-field to be infinite](#). In *Théorie des nombres (Quebec, PQ, 1987)*, pp. 556–560, de Gruyter, Berlin, 1989 Zbl 0696.12009 MR 1024586
- [9] H. Koch, [Galois theory of \$p\$ -extensions](#). Springer Monogr. Math., Springer, Berlin, 2002 Zbl 1023.11002 MR 1930372
- [10] M. Lazard, [Groupes analytiques \$p\$ -adiques](#). *Inst. Hautes Études Sci. Publ. Math.* (1965), no. 26, 389–603 Zbl 0139.02302 MR 209286
- [11] J. Martinet, [Tours de corps de classes et estimations de discriminants](#). *Invent. Math.* **44** (1978), no. 1, 65–73 Zbl 0369.12007 MR 460281
- [12] J. Mináč, M. Rogelstad, and N. D. Tân, [Dimensions of Zassenhaus filtration subquotients of some pro- \$p\$ -groups](#). *Israel J. Math.* **212** (2016), 825–855 Zbl 1359.20021 MR 3505404
- [13] J. Neukirch, A. Schmidt, and K. Wingberg, [Cohomology of number fields](#). 2nd edn., Grundlehren Math. Wiss. 323, Springer, Berlin, 2008 Zbl 1136.11001 MR 2392026
- [14] A. M. Odlyzko, [Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results](#). *Sém. Théor. Nombres Bordeaux (2)* **2** (1990), no. 1, 119–141 Zbl 0722.11054 MR 1061762
- [15] R. Schoof, [Infinite class field towers of quadratic fields](#). *J. Reine Angew. Math.* **372** (1986), 209–220 Zbl 0589.12011 MR 863524
- [16] B. B. Venkov and H. Koch, [The \$p\$ -tower of class fields for an imaginary quadratic field](#). *J. Sov. Math.* **9** (1978), 291–299 Zbl 0396.12010 MR 382235
- [17] E. B. Vinberg, [On the theorem concerning the infinite-dimensionality of an associative algebra](#). *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 209–214 Zbl 0171.29401 MR 172892

Communicated by Otmar Venjakob

Received 24 April 2023; revised 19 December 2023.

Arindam Bhattacharyya

Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103, India; arindamb@cmi.ac.in

Vishnu Kadiri

Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103, India; vishnuk@cmi.ac.in

Anwesh Ray

Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam, Siruseri, Tamil Nadu 603103, India; anwesh@cmi.ac.in