Doc. Math. 29 (2024), 191–208
DOI 10.4171/DM/931

Motivic cohomology of the Nisnevich classifying space of even Clifford groups

Fabio Tanania

Abstract. In this paper, we consider the split even Clifford group Γ_n^+ and compute the mod 2 motivic cohomology ring of its Nisnevich classifying space. The description we obtain is quite similar to the one provided for spin groups in [Math. Z. 301 (2022), no. 1, 41–74]. The fundamental difference resides in the behaviour of the second subtle Stiefel–Whitney class that is non-trivial for even Clifford groups, while it vanishes in the spin-case.

1. Introduction

Subtle characteristic classes were introduced by Smirnov and Vishik in [8] to approach the classification of quadratic forms by using motivic homotopical techniques. In particular, these characteristic classes arise as elements of the motivic cohomology ring of the Nisnevich classifying space $\mathrm{B}G$ of a linear algebraic group G over a field k. They naturally provide invariants for Nisnevich locally trivial G-torsors, which take value in the motivic cohomology of the base. What is probably more interesting is that they also provide invariants for étale locally trivial G-torsors, which take value this time in a more complicated and informative object, namely the motivic cohomology of the Čech simplicial scheme of the torsor under study.

In [8], the authors compute the motivic cohomology ring with $\mathbb{Z}/2$-coefficients of BO_n, i.e. the Nisnevich classifying space of the split orthogonal group. Similarly to the topological picture, this cohomology ring is a polynomial algebra over the motivic cohomology of the ground field generated by certain classes $u_1, \dots, u_n$ called subtle Stiefel–Whitney classes. These invariants detect the power I^n of the fundamental ideal of the Witt ring a quadratic form belongs to. In particular, the triviality of all subtle Stiefel–Whitney classes implies the triviality of the quadratic form itself. Besides, from the computation of $H(\mathrm{BO}_n)$ it follows that the mod 2 motivic cohomology of BSO_n is also a polynomial algebra generated by all subtle Stiefel–Whitney classes but the first.

Following [8], we studied the motivic cohomology rings of the Nisnevich classifying spaces of unitary groups in [9], of spin groups in [11] and of projective general linear groups in [10]. This paper is a natural follow-up of [11]. In fact, we focus here

2020 Mathematics Subject Classification. Primary 14F42; Secondary 20G15, 55R40.
Keywords. Clifford groups, motivic cohomology, Nisnevich classifying space, characteristic classes.

on computing the motivic cohomology of the Nisnevich classifying space of even Clifford groups. These algebraic groups are closely related to spin groups. On the level of torsors, this is visible from the fact that a spin-torsor yields a quadratic form in I^3 through a surjective map with trivial kernel, while the torsors of the even Clifford groups are exactly the quadratic forms in I^3.

The topological counterpart of the even Clifford group Γ_n^+ is the Lie group $\mathrm{Spin}^c(n)$. The singular cohomology of the classifying space of $\mathrm{Spin}^c(n)$ was computed by Harada and Kono in [3]. The main result we obtain in this article is a motivic version of [3, Theorem 3.5]. More precisely, we prove the following.

Theorem 1.1. *Let k be a field of characteristic different from 2 containing a square root of -1. Then, for any $n \geq 2$, there exists a cohomology class $e_{2^{l(n)}}$ in bidegree $(2^{l(n)-1})[2^{l(n)}]$ such that the natural homomorphism of H-algebras*

$$H(\mathrm{BSO}_n)/I^{\circ}_{l(n)} \otimes_H H[e_{2^{l(n)}}] \to H(\mathrm{B}\Gamma_n^+)$$

is an isomorphism, where $I^{\circ}_{l(n)}$ is the ideal generated by $\theta_1, \ldots, \theta_{l(n)-1}$ and $l(n) = [\frac{n+1}{2}]$.

The assumption on the characteristic of the ground field is necessary since the mod 2 motivic cohomology of the point and the mod 2 motivic Steenrod algebra are well understood in characteristic different from 2 (see [13]). Moreover, we require also that k contains a square root of -1, since in this case the action of the mod 2 motivic Steenrod algebra on the mod 2 motivic cohomology of the point is trivial, making our computation easier. Anyways, we suspect that a result similar to Theorem 1.1 would still hold after dropping this last assumption, but with more complicated relations involving Steenrod operations and $\rho = Sq^1\tau$, where ρ is the class of -1 in mod 2 Milnor K-theory and τ is the generator of $H^{0,1} \cong \mathbb{Z}/2$.

The similarity between Theorem 1.1 and the computation for the spin-case (see Theorem 2.10) is clear. Nonetheless, a crucial difference is that, while u_2 is trivial in the cohomology ring $H(\mathrm{BSpin}_n)$, it is not in $H(\mathrm{B}\Gamma_n^+)$ where the ideal of relations $I^{\circ}_{l(n)}$ is generated by the action of the motivic Steenrod algebra over $u_3 = Sq^1 u_2$. This also explains the gap between $k(n)$ in Theorem 2.10 and $l(n)$ in Theorem 1.1, which is due to discrepancies in the maximal length of the regular sequences in $H(\mathrm{BSO}_n)$ obtained by applying certain Steenrod operations to u_2 and u_3, respectively.

We conclude by pointing out that understanding the motivic cohomology of Nisnevich classifying spaces also helps in obtaining information about the structure of the Chow ring of étale classifying spaces $\mathrm{B_{\acute{e}t}}G$ (see [12]), which is an interesting object of study that is particularly challenging to fully grasp. For example, the Chow ring of $\mathrm{B_{\acute{e}t}}\Gamma_n^+$ has been recently investigated by Karpenko in [4] where he proves a conjecture that allows him, as a consequence, to compute the exponent indexes of spin grassmannians. In our case, we will show how to apply Theorem 1.1 to compute the subring generated by Chern classes of the Chow ring mod 2 of $\mathrm{B_{\acute{e}t}}\Gamma_n^+$, modulo nilpotents, sheding new light on its complicated structure.

Outline. In Section 2, we report notations and preliminary results that we will use in this paper. In particular, we recall the Thom isomorphism in the triangulated category of motives over a simplicial base, which provides Gysin long exact sequences in motivic cohomology. Besides, we recall definitions and properties of classifying spaces in motivic homotopy theory, as well as the computation of the mod 2 motivic cohomology of BO_n, BSO_n and BSpin_n. In Section 3, we investigate regular sequences in $H(\mathrm{BSO}_n)$ constructed starting from u_3 by acting with specific Steenrod operations. Finally, in Section 4, we exploit the Gysin sequence relating $\mathrm{B}\Gamma_n^+$ and BSpin_n, and the regular sequences studied before, in order to fully compute the mod 2 motivic cohomology of $\mathrm{B}\Gamma_n^+$. After that, we also obtain a complete description of the reduced Chern subring of the Chow ring mod 2 of $\mathrm{B}_{\text{ét}}\Gamma_n^+$.

2. Notation and preliminaries

We start by fixing some notation we will regularly use in this paper.

k	field of characteristic different from 2 containing $\sqrt{-1}$
R	commutative ring with identity
$Y_\bullet$	smooth simplicial scheme over k
$\mathrm{Spc}_*(Y_\bullet)$	category of pointed motivic spaces over $Y_\bullet$
$\mathcal{H}_s(k)$	simplicial homotopy category over k
$\mathcal{DM}^-_{\mathrm{eff}}(k,R)$	triangulated category of effective motives over k with R-coefficients
$\mathcal{DM}^-_{\mathrm{eff}}(Y_\bullet,R)$	triangulated category of effective motives over $Y_\bullet$ with R-coefficients
T	unit object in $\mathcal{DM}^-_{\mathrm{eff}}(k,R)$
$H_{\mathrm{top}}(-)$	singular cohomology with $\mathbb{Z}/2$-coefficients
$H(-)$	motivic cohomology with $\mathbb{Z}/2$-coefficients
H	motivic cohomology with $\mathbb{Z}/2$-coefficients of $\mathrm{Spec}(k)$
$K^M(k)/2$	Milnor K-theory of k mod 2
w_i	ith Stiefel–Whitney class in $H_{\mathrm{top}}(\mathrm{BSO}_n)$
u_i	ith subtle Stiefel–Whitney class in $H(\mathrm{BSO}_n)$
ρ_j	the element $Sq^{2^{j-1}}Sq^{2^{j-2}}\cdots Sq^2Sq^1w_2$ in $H_{\mathrm{top}}(\mathrm{BSO}_n)$
θ_j	the element $Sq^{2^{j-1}}Sq^{2^{j-2}}\cdots Sq^2Sq^1u_2$ in $H(\mathrm{BSO}_n)$
Γ_n^+	even Clifford group

The collection of results [13, Theorem 6.1, Corollaries 6.9 and 7.5] implies that $H \cong K^M(k)/2[\tau]$, where τ is the non-trivial class in $H^{0,1} \cong \mathbb{Z}/2$ and $H^{n,n} \cong K^M_n(k)/2$.

Note that, since we are working over a field containing the square root of -1, all Steenrod squares Sq^i, as defined in [14], act trivially on H.

Since we will mainly work in the triangulated category of motives over a simplicial scheme defined by Voevodsky in [15], we recall a few definitions and propositions about it that will be useful later on to prove our main results.

Definition 2.1. For any smooth simplicial scheme $Y_\bullet$ over k, denote by $c : Y_\bullet \to \mathrm{Spec}(k)$ the projection to the base. Then, we can define the Tate objects $T(q)[p]$ in $\mathcal{DM}^-_{\mathrm{eff}}(Y_\bullet, R)$ as $c^*(T(q)[p])$.

Definition 2.2. A smooth morphism of smooth simplicial schemes $\pi : X_\bullet \to Y_\bullet$ is called coherent if there is a cartesian square

$$\begin{tikzcd} X_j \arrow[r, "\pi_j"] \arrow[d, "X_\theta"'] & Y_j \arrow[d, "Y_\theta"] \\ X_i \arrow[r, "\pi_i"'] & Y_i \end{tikzcd}$$

for any simplicial map $\theta : [i] \to [j]$.

Denote by $CC(Y_\bullet)$ the simplicial set obtained from $Y_\bullet$ by applying the functor CC that sends any connected scheme to the point and respects coproducts.

Proposition 2.3. *Let $\pi : X_\bullet \to Y_\bullet$ be a smooth coherent morphism of smooth simplicial schemes over k and A a smooth k-scheme such that:*

(1) *X_0 is isomorphic to $Y_0 \times A$ and, under this isomorphism, π_0 becomes equal to the projection map $Y_0 \times A \to Y_0$;*

(2) *$H^1(CC(Y_\bullet), R^\times) \cong 0$;*

(3) *$M(A) \cong T \oplus T(r)[s-1]$ in $\mathcal{DM}^-_{\mathrm{eff}}(k, R)$ for arbitrary integers r and s.*

Then, $M(\mathrm{Cone}(\pi)) \cong T(r)[s]$ in $\mathcal{DM}^-_{\mathrm{eff}}(Y_\bullet, R)$ where $\mathrm{Cone}(\pi)$ *is the cone of π in* $\mathrm{Spc}_*(Y_\bullet)$. *Hence, we get a Thom isomorphism of $H(Y_\bullet, R)$-modules*

$$H^{*-s,*'-r}(Y_\bullet, R) \to H^{*,*'}\big(\mathrm{Cone}(\pi), R\big).$$

Proof. See [11, Proposition 4.2]. ∎

Definition 2.4. We call Thom class of π and denote by α the image of 1 under the Thom isomorphism of Proposition 2.3.

The following result guarantees that the Thom isomorphism from Proposition 2.3 is functorial.

Proposition 2.5. *Suppose there is a cartesian square*

$$\begin{tikzcd} X_\bullet \arrow[r, "\pi"] \arrow[d, "p_X"'] & Y_\bullet \arrow[d, "p_Y"] \\ X'_\bullet \arrow[r, "\pi'"'] & Y'_\bullet \end{tikzcd}$$

such that Y_0 is connected, p_X and p_Y are smooth, π and π' are smooth coherent with fiber A satisfying all conditions from Proposition 2.3. Then, the induced homomorphism *in motivic cohomology*

$$H\big(\mathrm{Cone}(\pi'), R\big) \to H\big(\mathrm{Cone}(\pi), R\big)$$

maps α' to α, where α' and α are the respective Thom classes.

Proof. See [11, Proposition 4.3 and Corollary 4.4]. ∎

We now recall from [6] the definitions of the Nisnevich and étale classifying spaces of linear algebraic groups.

Let G be a linear algebraic group over k and $\mathrm{E}G$ the simplicial scheme defined by

$$(\mathrm{E}G)_n = G^{n+1},$$

with partial projections as face maps and partial diagonals as degeneracy maps. The space $\mathrm{E}G$ is endowed with a right free G-action provided by the operation in G.

Definition 2.6. The Nisnevich classifying space of G is the quotient $\mathrm{B}G = \mathrm{E}G/G$.

The morphism of sites $\pi : (Sm/k)_{\text{ét}} \to (Sm/k)_{\mathrm{Nis}}$ induces an adjunction between simplicial homotopy categories

$$\begin{gathered}\mathcal{H}_s\big((Sm/k)_{\text{ét}}\big)\\ \pi^* \uparrow\downarrow R\pi_*\\ \mathcal{H}_s\big((Sm/k)_{\mathrm{Nis}}\big).\end{gathered}$$

Definition 2.7. The étale classifying space of G is defined by $\mathrm{B}_{\text{ét}}G = R\pi_*\pi^*\mathrm{B}G$.

Let H be an algebraic subgroup of G. Then, we can define the simplicial scheme $\widehat{\mathrm{B}}H = \mathrm{E}G/H$ with respect to the embedding $H \hookrightarrow G$. Denote by j the induced morphism $\mathrm{B}H \to \widehat{\mathrm{B}}H$.

Proposition 2.8. *Let $H \hookrightarrow G$ be such that all rationally trivial H-torsors and G-torsors are Zariski-locally trivial. If the map*

$$\mathrm{Hom}_{\mathcal{H}_s(k)}\big(\mathrm{Spec}(K), \mathrm{B}_{\text{ét}}H\big) \to \mathrm{Hom}_{\mathcal{H}_s(k)}\big(\mathrm{Spec}(K), \mathrm{B}_{\text{ét}}G\big)$$

has trivial kernel for any finitely generated field extension K of k, then j is an isomorphism in $\mathcal{H}_s(k)$.

Proof. See [11, Proposition 5.1, Corollary 5.2, and Proposition 5.3]. ∎

Remark 2.9. Note that the obvious map $\pi : \widehat{\mathrm{B}}H \to \mathrm{B}G$ is smooth coherent with fiber G/H.

By using the Gysin sequence induced by the Thom isomorphism, one can compute by induction the following motivic cohomology rings.

Theorem 2.10. *The motivic cohomology rings of* BO_n, BSO_n *and* BSpin_n *are, respectively, given by*

$$\begin{aligned} H(\mathrm{BO}_n) &\cong H[u_1, \dots, u_n], \\ H(\mathrm{BSO}_n) &\cong H[u_2, \dots, u_n], \\ H(\mathrm{BSpin}_n) &\cong H(\mathrm{BSO}_n)/I_{k(n)} \otimes_H H[v_{2^{k(n)}}], \end{aligned}$$

where the ith subtle Stiefel–Whitney class u_i is in bidegree $([\frac{i}{2}])[i]$, the class $v_{2^{k(n)}}$ is in bidegree $(2^{k(n)-1})[2^{k(n)}]$, $I_{k(n)}$ is the ideal generated by $\theta_0, \dots, \theta_{k(n)-1}$ and $k(n)$ depends on n as in the following table.

n	$k(n)$
$8l+1$	$4l$
$8l+2$	$4l+1$
$8l+3$	$4l+2$
$8l+4$	$4l+2$
$8l+5$	$4l+3$
$8l+6$	$4l+3$
$8l+7$	$4l+3$
$8l+8$	$4l+3$

Proof. See [8, Theorem 3.1.1] and [11, Proposition 5.6 and Theorem 8.3]. ∎

3. Regular sequences in $H(\mathrm{BSO}_n)$

In this section, we want to use the techniques developed in [11, Section 7] to produce other regular sequences in the motivic cohomology of BSO_n that will be relevant later to deal with the case of even Clifford groups.

Let V be an n-dimensional $\mathbb{Z}/2$-vector space, B a bilinear form over V and ${}^{\perp}V$ its right radical, i.e.

$$^{\perp}V = \{y \in V : B(x, y) = 0 \text{ for any } x \in V\}.$$

Fix a basis $\{e_1, \dots, e_n\}$ for V and let x_i and y_j be the coordinates of x and y in V, respectively. Then,

$$B(x, y) = \sum_{i,j=1}^{n} B(e_i, e_j) x_i y_j$$

is a homogeneous polynomial of degree 2 in $\mathbb{Z}/2[x_1, \dots, x_n, y_1, \dots, y_n]$.

Proposition 3.1. *The sequence $B(x, y), B(x, y^2), \dots, B(x, y^{2^{h-1}})$ is a regular sequence in the polynomial ring $\mathbb{Z}/2[x_1, \dots, x_n, y_1, \dots, y_n]$, where $h = n - dim({}^{\perp}V)$.*

Proof. See [11, Corollary 7.3]. ∎

Recall from [11, Section 7] that there are commutative squares

$$
\begin{tikzcd}
H(\mathrm{BO}_{2m}) \arrow[r, "\alpha_{2m}"] \arrow[d, "\gamma_{2m}"'] & H(\mathrm{BO}_2)^{\otimes m} \arrow[d, "\delta_{2m}"] \\
S_{2m} \arrow[r, "\beta_{2m}"'] & R_{2m}
\end{tikzcd}
\qquad
\begin{tikzcd}
H(\mathrm{BO}_{2m+1}) \arrow[r, "\alpha_{2m+1}"] \arrow[d, "\gamma_{2m+1}"'] & H(\mathrm{BO}_2)^{\otimes m} \otimes H(\mathrm{BO}_1) \arrow[d, "\delta_{2m+1}"] \\
S_{2m+1} \arrow[r, "\beta_{2m+1}"'] & R_{2m+1}
\end{tikzcd}
$$

where

$$
\begin{aligned}
H(\mathrm{BO}_n) &\cong H[u_1, \dots, u_n], \\
H(\mathrm{BO}_2)^{\otimes m} &\cong H[x_1, y_1, \dots, x_m, y_m], \\
H(\mathrm{BO}_2)^{\otimes m} \otimes_H H(\mathrm{BO}_1) &\cong H[x_1, y_1, \dots, x_m, y_m, x_{m+1}], \\
S_n &= \mathbb{Z}/2[u_1, \dots, u_n], \\
R_{2m} &= \mathbb{Z}/2[x_1, y_1, \dots, x_m, y_m], \\
R_{2m+1} &= \mathbb{Z}/2[x_1, y_1, \dots, x_m, y_m, x_{m+1}],
\end{aligned}
$$

x_i is in bidegree (0)[1] and y_i is in bidegree (1)[2] for any i, β_n is obtained from α_n by tensoring with $\mathbb{Z}/2$ over H, γ_n and δ_n are the reduction homomorphisms along $H \to \mathbb{Z}/2$.

In particular the following formulas hold:

$$
\begin{aligned}
\beta_{2m}(u_{2j}) &= \sigma_j(y_1, \dots, y_m), \\
\beta_{2m}(u_{2j+1}) &= \sum_{i=1}^{m} x_i \sigma_j(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_m), \\
\beta_{2m+1}(u_{2j}) &= \sigma_j(y_1, \dots, y_m), \\
\beta_{2m+1}(u_{2j+1}) &= \sum_{i=1}^{m} x_i \sigma_j(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_m) + x_{m+1}\sigma_j(y_1, \dots, y_m),
\end{aligned}
$$

where σ_j is the jth elementary symmetric polynomial.

Lemma 3.2. *Let $f : A = \mathbb{Z}/2[a_1, \dots, a_m] \to B = \mathbb{Z}/2[b_1, \dots, b_n]$ be a ring homomorphism, where $\deg(b_i) = 1$ for any i and $f(a_j)$ is a homogeneous polynomial in B of positive degree α_j for any j. Moreover, let $r_1, \dots, r_k$ be a sequence of elements of A. If $f(r_1), \dots, f(r_k)$ is a regular sequence in B, then $r_1, \dots, r_k$ is a regular sequence in A.*

Proof. See [11, Lemma 7.4]. ■

Theorem 3.3. *The sequence*

$$
\gamma_n(u_1), \gamma_n(u_3), \gamma_n(Sq^2 u_3), \dots, \gamma_n(Sq^{2^{l(n)-2}} Sq^{2^{l(n)-3}} \cdots Sq^2 u_3)
$$

is regular in S_n, where $l(n) = [\frac{n+1}{2}]$.

Proof. By Lemma 3.2 it is enough to show the regularity of the sequence

$$\beta_n\gamma_n(u_1), \beta_n\gamma_n(u_3), \beta_n\gamma_n(Sq^2u_3), \ldots, \beta_n\gamma_n(Sq^{2^{l(n)-2}}Sq^{2^{l(n)-3}}\cdots Sq^2u_3)$$

in R_n.

First, consider the case $n = 2m$. Then, $\beta_{2m}\gamma_{2m}(u_1) = \beta_{2m}(u_1) = \sum_{i=1}^m x_i$. Moreover, since τ is zero in R_{2m}, we have $\beta_{2m}\gamma_{2m}(u_3) = \sum_{i\neq j=1}^m x_i y_j$ and

$$\begin{aligned}
\beta_{2m}\gamma_{2m}(Sq^{2^l}\cdots Sq^2u_3) &= \delta_{2m}\alpha_{2m}(Sq^{2^l}\cdots Sq^2u_3)\\
&= \delta_{2m}\big(Sq^{2^l}\cdots Sq^2\alpha_{2m}(u_3)\big)\\
&= \sum_{i\neq j=1}^{m} \delta_{2m}\big(Sq^{2^l}\cdots Sq^2(x_i y_j)\big)\\
&= \sum_{i\neq j=1}^{m} x_i y_j^{2^l}
\end{aligned}$$

for $l \geq 1$. Modulo $\beta_{2m}\gamma_{2m}(u_1)$, $\beta_{2m}\gamma_{2m}(u_3) = B(x, y) = \sum_{i=1}^{m-1} x_i(y_i + y_m)$ is a bilinear form over an m-dimensional $\mathbb{Z}/2$-vector space V and $\beta_{2m}\gamma_{2m}(Sq^{2^l}\cdots Sq^2u_3) = B(x, y^{2^l})$ for any $l \geq 1$.

From $y_i + y_m = B(e_i, y)$ for any $i \leq m-1$, it follows that ${}^{\perp}V \cong \langle(1,\ldots,1)\rangle$ and Proposition 3.1 implies that the sequence

$$\beta_{2m}\gamma_{2m}(u_1), \beta_{2m}\gamma_{2m}(u_3), \beta_{2m}\gamma_{2m}(Sq^2u_3), \ldots, \beta_{2m}\gamma_{2m}(Sq^{2^{l(2m)-2}}Sq^{2^{l(2m)-3}}\cdots Sq^2u_3)$$

is regular in R_{2m} where $l(2m) = m = [\frac{2m+1}{2}]$.

Now, consider the case $n = 2m+1$. Then,

$$\beta_{2m+1}\gamma_{2m+1}(u_1) = \beta_{2m+1}(u_1) = \sum_{i=1}^{m+1} x_i.$$

Moreover, since τ is zero in R_{2m+1}, we have

$$\beta_{2m+1}\gamma_{2m+1}(u_3) = \sum_{i\neq j=1}^{m} x_i y_j + x_{m+1}\sum_{j=1}^{m} y_j,$$

$$\begin{aligned}
\beta_{2m+1}\gamma_{2m+1}(Sq^{2^l}\cdots Sq^2u_3) &= \delta_{2m+1}\alpha_{2m+1}(Sq^{2^l}\cdots Sq^2u_3)\\
&= \delta_{2m+1}\big(Sq^{2^l}\cdots Sq^2\alpha_{2m+1}(u_3)\big)\\
&= \sum_{i\neq j=1}^{m} \delta_{2m+1}\big(Sq^{2^l}\cdots Sq^2(x_i y_j)\big)\\
&\quad + \sum_{j=1}^{m} \delta_{2m+1}\big(Sq^{2^l}\cdots Sq^2(x_{m+1} y_j)\big)\\
&= \sum_{i\neq j=1}^{m} x_i y_j^{2^l} + \sum_{j=1}^{m} x_{m+1} y_j^{2^l}
\end{aligned}$$

for $l \geq 1$. Modulo $\beta_{2m+1}\gamma_{2m+1}(u_1)$, $\beta_{2m+1}\gamma_{2m+1}(u_3) = B(x, y) = \sum_{i=1}^{m} x_i y_i$ is a bilinear form over an m-dimensional $\mathbb{Z}/2$-vector space V and

$$\beta_{2m+1}\gamma_{2m+1}(Sq^{2^l} \cdots Sq^1 u_2) = B(x, y^{2^l})$$

for any $l \geq 1$. In this case ${}^{\perp}V \cong 0$, since $y_i = B(e_i, y)$ for any $i \leq m$, and Proposition 3.1 implies that the sequence

$$\beta_{2m+1}\gamma_{2m+1}(u_1), \beta_{2m+1}\gamma_{2m+1}(u_3), \beta_{2m+1}\gamma_{2m+1}(Sq^2 u_3), \ldots,$$
$$\beta_{2m+1}\gamma_{2m+1}(Sq^{2^{l(2m+1)-2}} Sq^{2^{l(2m+1)-3}} \cdots Sq^2 u_3)$$

is regular in R_{2m+1} where $l(2m+1) = m+1 = [\frac{2m+2}{2}]$. This completes the proof. ■

Corollary 3.4. *The sequence $\tau, \theta_1, \ldots, \theta_{l(n)-1}$ is regular in $H(\mathrm{BSO}_n)$, where $l(n) = [\frac{n+1}{2}]$.*

Proof. Since θ_j is inductively computed from $\theta_1 = u_3$ by using only Wu formula (see [11, Proposition 5.7]) and Cartan formula, we know that θ_j is an element of $\mathbb{Z}/2[\tau, u_2, \ldots, u_n]$ for any j. The regularity of the sequence in $\mathbb{Z}/2[\tau, u_2, \ldots, u_n]$ follows from Theorem 3.3 by noticing that, modulo τ and u_1, $\theta_j = \gamma_n(Sq^{2^{j-1}} \cdots Sq^1 u_2)$ in S_n. This clearly implies also the regularity of the sequence in $H(\mathrm{BSO}_n)$. ■

Recall from [11, Section 7] the homomorphisms $i : H_{\mathrm{top}}(\mathrm{BSO}_n) \to H(\mathrm{BSO}_n)$, $h : H_{\mathrm{top}}(\mathrm{BSO}_n) \to H(\mathrm{BSO}_n)$ and $t : H(\mathrm{BSO}_n) \to H_{\mathrm{top}}(\mathrm{BSO}_n)$, where i is the ring homomorphism defined by $i(w_i) = u_i$, h is the linear map defined by $h(x) = \tau^{[\frac{p_{i(x)}}{2} - q_{i(x)}]} i(x)$ for any monomial x, where $(q_{i(x)})[p_{i(x)}]$ is the bidegree of $i(x)$, and t is the ring homomorphism defined by $t(u_i) = w_i$, $t(\tau) = 1$ and $t(K_r^M(k)/2) = 0$ for any $r > 0$.

Lemma 3.5. *For any homogeneous polynomials x and y in $H_{\mathrm{top}}(\mathrm{BSO}_n)$, we have that $h(xy) = \tau^{\varepsilon} h(x)h(y)$, where ε is 1 if $p_{i(x)} p_{i(y)}$ is odd and 0 otherwise.*

Proof. See [11, Lemma 7.7]. ■

Lemma 3.6. *For any j, $t(\theta_j) = \rho_j$ and $h(\rho_j) = \theta_j$.*

Proof. See [11, Lemma 7.9]. ■

Definition 3.7. Let I_j° be the ideal in $H(\mathrm{BSO}_n)$ generated by $\theta_1, \ldots, \theta_{j-1}$ and $I_j^{\circ,\mathrm{top}}$ the ideal in $H_{\mathrm{top}}(\mathrm{BSO}_n)$ generated by $\rho_1, \ldots, \rho_{j-1}$.

Theorem 3.8. *The canonical homomorphism*

$$H_{\mathrm{top}}(\mathrm{BSO}_n)/I_{l(n)}^{\circ,\mathrm{top}} \otimes \mathbb{Z}/2\big[e(\Delta_n)\big] \to H_{\mathrm{top}}(\mathrm{BSpin}_n^c)$$

is an isomorphism, where $l(n) = [\frac{n+1}{2}]$ and $e(\Delta_n)$ is the Euler class of the complex spin representation Δ_n.

Proof. See [3, Theorem 3.5]. ■

The following is the main result of this section.

Theorem 3.9. *The sequence* $\theta_1, \dots, \theta_{l(n)-1}$ *is regular in* $H(\mathrm{BSO}_n)$ *and* $\theta_{l(n)} \in I^{\circ}_{l(n)}$, *where* $l(n) = [\frac{n+1}{2}]$.

Proof. Corollary 3.4 immediately implies that $\theta_1, \dots, \theta_{l(n)-1}$ is a regular sequence in $H(\mathrm{BSO}_n)$.

By Theorem 3.8, we know that

$$\rho_{l(n)} = Sq^{2^{l(n)-1}} \rho_{l(n)-1}$$

vanishes in $H_{\mathrm{top}}(\mathrm{BSpin}^c_n)$, and so $\rho_{l(n)} \in I^{\circ,\mathrm{top}}_{l(n)}$. It follows that

$$\rho_{l(n)} = \sum_{i=1}^{l(n)-1} \phi_i \rho_i$$

for some homogeneous $\phi_i \in H_{\mathrm{top}}(\mathrm{BSO}_n)$ and, after applying h, we obtain that

$$\theta_{l(n)} = \sum_{i=1}^{l(n)-1} h(\phi_i)\theta_i$$

by Lemmas 3.5 and 3.6. Thus, $\theta_{l(n)} \in I^{\circ}_{l(n)}$, which completes the proof. ■

Remark 3.10. Note that either $l(n) = k(n)$ or $l(n) = k(n) + 1$. If $l(n) = k(n)$, then

$$\theta_{k(n)} \in I^{\circ}_{k(n)}.$$

On the other hand, if $l(n) = k(n) + 1$, then the sequence $\theta_1, \dots, \theta_{k(n)}$ is regular in $H(\mathrm{BSO}_n)$, and so $\theta_{k(n)} \notin I^{\circ}_{k(n)}$.

4. The motivic cohomology ring of $\mathrm{B}\Gamma^+_n$

In this last section, we prove our main result that describes the structure of the motivic cohomology of the Nisnevich classifying space of even Clifford groups.

Before proceeding, recall from [1, Section 3] that Γ^+_n-torsors are in one-to-one correspondence with quadratic forms with trivial discriminant and Clifford invariant, i.e. quadratic forms in I^3, where I is the fundamental ideal of the Witt ring. Moreover, for any $n \geq 2$, we have the following short exact sequences of algebraic groups (see [5, Chapter VI, Section 23.A])

$$1 \to \mathbb{G}_m \to \Gamma^+_n \to \mathrm{SO}_n \to 1, \quad 1 \to \mathrm{Spin}_n \to \Gamma^+_n \to \mathbb{G}_m \to 1. \tag{4.1}$$

Lemma 4.1. *For any* $n \geq 2$, $\mathrm{BSpin}_n \cong \hat{\mathrm{B}}\,\mathrm{Spin}_n$ *with respect to the embedding* $\mathrm{Spin}_n \hookrightarrow \Gamma^+_n$.

Proof. First, note that, by [2,7], rationally trivial Spin_n-torsors and Γ_n^+-torsors are locally trivial. Moreover, recall from [6, Section 4.1] that

$$\mathrm{Hom}_{\mathcal{H}_s(k)}\big(\mathrm{Spec}(K), \mathrm{B}_{\text{ét}} G\big) \cong H^1_{\text{ét}}(K, G)$$

for any Nisnevich sheaf of groups G. Therefore, it follows from [1, Section 3] that

$$\mathrm{Hom}_{\mathcal{H}_s(k)}\big(\mathrm{Spec}(K), \mathrm{B}_{\text{ét}}\,\mathrm{Spin}_n\big) \to \mathrm{Hom}_{\mathcal{H}_s(k)}\big(\mathrm{Spec}(K), \mathrm{B}_{\text{ét}}\Gamma_n^+\big)$$

is surjective with trivial kernel, for any finitely generated field extension K of k. Hence, we can apply Proposition 2.8 to the case that G and H are, respectively, Γ_n^+ and Spin_n, which provides the aimed result. ■

Proposition 4.2. *For any $n \geq 2$, there exists a Gysin long exact sequence of $H(\mathrm{B}\Gamma_n^+)$-modules*

$$\cdots \to H^{*-1,*'}(\mathrm{BSpin}_n) \xrightarrow{h^*} H^{*-2,*'-1}(\mathrm{B}\Gamma_n^+) \xrightarrow{\cdot u_2} H^{*,*'}(\mathrm{B}\Gamma_n^+) \xrightarrow{g^*} H^{*,*'}(\mathrm{BSpin}_n) \to \cdots$$

such that the homomorphism $H^{2,*'}(\mathrm{BSO}_n) \to H^{2,*'}(\mathrm{B}\Gamma_n^+)$, induced by the map $\Gamma_n^+ \to \mathrm{SO}_n$ in (4.1), is injective.

Proof. Let M be $M(\mathrm{B}\Gamma_n^+ \to \mathrm{BSO}_n)$ and N be $\mathrm{Cone}(M \to T)[-1]$ in $\mathcal{DM}^-_{\mathrm{eff}}(\mathrm{BSO}_n)$. From the motivic Serre spectral sequence [10, Theorem 5.12] associated to the sequence

$$\mathrm{B}\mathbb{G}_m \to \mathrm{B}\Gamma_n^+ \to \mathrm{BSO}_n \tag{4.2}$$

it follows that $H^{1,*'}(N) \cong 0$. Therefore, the homomorphism

$$H^{2,*'}(\mathrm{BSO}_n) \to H^{2,*'}(\mathrm{B}\Gamma_n^+)$$

is injective. In particular, u_2 is non-trivial in $H^{2,1}(\mathrm{B}\Gamma_n^+)$.

Now, consider the sequence

$$\mathbb{G}_m \to \mathrm{BSpin}_n \to \mathrm{B}\Gamma_n^+.$$

By Proposition 2.3, Remark 2.9 and Lemma 4.1, it induces a Gysin long exact sequence of $H(\mathrm{B}\Gamma_n^+)$-modules

$$\cdots \to H^{p-1,q}(\mathrm{BSpin}_n) \xrightarrow{h^*} H^{p-2,q-1}(\mathrm{B}\Gamma_n^+) \xrightarrow{f^*} H^{p,q}(\mathrm{B}\Gamma_n^+) \xrightarrow{g^*} H^{p,q}(\mathrm{BSpin}_n) \to \cdots.$$

Since g^* is an isomorphism in bidegree (1)[1], we have that $f^*(1)$ is the only non-trivial class in $H^{2,1}(\mathrm{B}\Gamma_n^+)$ that vanishes in $H^{2,1}(\mathrm{BSpin}_n)$. It follows that

$$f^*(1) = u_2,$$

which completes the proof. ■

Lemma 4.3. *For any $n \geq 2$, $H^{p,*'}(\mathrm{BSO}_n) \to H^{p,*'}(\mathrm{B}\Gamma_n^+)$ is surjective for $p < 2^{k(n)}$.*

Proof. We proceed by induction on p. For $p = 0$, the Serre spectral sequence associated to (4.2) implies that $H^{0,*'}(\mathrm{BSO}_n) \cong H^{0,*'}(\mathrm{B}\Gamma_n^+)$, which provides the induction basis. Now, suppose that $H(\mathrm{BSO}_n) \to H(\mathrm{B}\Gamma_n^+)$ is surjective in topological degrees less than $p < 2^{k(n)}$, and consider a class x in $H^{p,*'}(\mathrm{B}\Gamma_n^+)$. Since by Theorem 2.10 the homomorphism $H^{p,*'}(\mathrm{BSO}_n) \to H^{p,*'}(\mathrm{BSpin}_n)$ that factors through $H^{p,*'}(\mathrm{B}\Gamma_n^+)$ is surjective for $p < 2^{k(n)}$, we have that

$$g^*(x) = g^*(y)$$

for some y in the image of $H^{p,*'}(\mathrm{BSO}_n) \to H^{p,*'}(\mathrm{B}\Gamma_n^+)$. Hence, by Proposition 4.2, there exists a class z in $H^{p-2,*'-1}(\mathrm{B}\Gamma_n^+)$ such that $x = y + u_2 z$. By induction hypothesis, z is in the image of $H^{p-2,*'-1}(\mathrm{BSO}_n) \to H^{p-2,*'-1}(\mathrm{B}\Gamma_n^+)$, from which it follows that x is in the image of $H^{p,*'}(\mathrm{BSO}_n) \to H^{p,*'}(\mathrm{B}\Gamma_n^+)$ that is what we wanted to show. ■

Definition 4.4. Denote by ω_n the class $h^*(v_{2^{k(n)}})$ in $H^{2^{k(n)}-1,2^{k(n)-1}-1}(\mathrm{B}\Gamma_n^+)$.

Remark 4.5. It follows from Lemma 4.3 that ω_n belongs to the image of $H(\mathrm{BSO}_n) \to H(\mathrm{B}\Gamma_n^+)$. Moreover, Proposition 4.2 implies that $u_2\omega_n = 0$ in $H(\mathrm{B}\Gamma_n^+)$.

Proposition 4.6. *The motivic cohomology ring of $\mathrm{B}\Gamma_2^+$ is given by*

$$H(\mathrm{B}\Gamma_2^+) \cong H[u_2, e_2],$$

where e_2 is a lift of v_2 in $H(\mathrm{BSpin}_2)$ under the homomorphism $H(\mathrm{B}\Gamma_2^+) \to H(\mathrm{BSpin}_2)$.

Proof. Consider the Gysin long exact sequence from Proposition 4.2

$$\cdots \to H^{*-1,*'}(\mathrm{BSpin}_2) \xrightarrow{h^*} H^{*-2,*'-1}(\mathrm{B}\Gamma_2^+) \xrightarrow{\cdot u_2} H^{*,*'}(\mathrm{B}\Gamma_2^+) \xrightarrow{g^*} H^{*,*'}(\mathrm{BSpin}_2) \to \cdots.$$

Since $H(\mathrm{BSpin}_2) \cong H[v_2]$, with v_2 in bidegree $(1)[2]$, g^* is a ring homomorphism and $H^{1,0}(\mathrm{B}\Gamma_2^+) \cong 0$, we have that h^* is zero, the multiplication by u_2 is injective in $H(\mathrm{B}\Gamma_2^+)$ and the quotient of $H(\mathrm{B}\Gamma_2^+)$ modulo the ideal generated by u_2 is $H[v_2]$. This concludes the proof. ■

Lemma 4.7. *For any $n \geq 3$, $u_3 = 0$ in $H(\mathrm{B}\Gamma_n^+)$. Moreover, there exists a unique element x_1 in $H^{2,1}(N)$ that maps to u_3 in $H^{3,1}(\mathrm{BSO}_n)$.*

Proof. By Proposition 4.2 we have a Gysin long exact sequence

$$\cdots \to H^{*-1,*'}(\mathrm{BSpin}_n) \xrightarrow{h^*} H^{*-2,*'-1}(\mathrm{B}\Gamma_n^+) \xrightarrow{\cdot u_2} H^{*,*'}(\mathrm{B}\Gamma_n^+) \xrightarrow{g^*} H^{*,*'}(\mathrm{BSpin}_n) \to \cdots.$$

Since u_3 is trivial in $H(\mathrm{BSpin}_n)$ and $H^{1,0}(\mathrm{B}\Gamma_n^+) \cong 0$, we have that u_3 is trivial also in $H(\mathrm{B}\Gamma_n^+)$. Moreover, note that, for $n \geq 3$,

$$H^{2,1}(\mathrm{B}\Gamma_n^+) \cong \mathbb{Z}/2 \cdot u_2.$$

Now, consider the long exact sequence

$$\cdots \to H^{*-1,*'}(\mathrm{B}\Gamma_n^+) \to H^{*-1,*'}(N) \to H^{*,*'}(\mathrm{BSO}_n) \to H^{*,*'}(\mathrm{B}\Gamma_n^+) \to \cdots .$$

The homomorphism

$$H^{2,1}(\mathrm{BSO}_n) \to H^{2,1}(\mathrm{B}\Gamma_n^+)$$

is bijective. Since u_3 is trivial in $H(\mathrm{B}\Gamma_n^+)$, we deduce that $H^{2,1}(N) \to H^{3,1}(\mathrm{BSO}_n)$ is an isomorphism, which finishes the proof. ■

Definition 4.8. For any $j \geq 2$ and $n \geq 3$, let x_j be the class in $H^{2^j,2^{j-1}}(N)$ defined by $x_j = Sq^{2^{j-1}} \cdots Sq^2 x_1$ and denote by $\langle x_1, \dots, x_{j-1}\rangle$ the $H(\mathrm{BSO}_n)$-submodule of $H(N)$ generated by $x_1, \dots, x_{j-1}$.

Lemma 4.9. *For any $j \geq 2$ and $n \geq 3$, $x_j \notin \langle x_1, \dots, x_{j-1}\rangle$.*

Proof. This follows by noticing that x_j maps to the respective class defined for spin groups in [11, Lemma 8.2]. ■

Proposition 4.10. *Suppose there exists a class e in $H(\mathrm{B}\Gamma_n^+)$ such that $g^*(e)$ is a monic homogeneous polynomial c in $v_{2^{k(n)}}$ with coefficients in $H(\mathrm{BSO}_n)$, and denote by p the obvious homomorphism $H(\mathrm{BSO}_n) \otimes_H H[e] \to H(\mathrm{B}\Gamma_n^+)$.*

(1) *If* $\mathrm{im}(h^*) = \mathrm{im}(p) \cdot \omega_n$, *then*

$$\ker(p) = J^\circ_{k(n)} + (u_2\omega_n),$$

where $J^\circ_{k(n)}$ is $I^\circ_{k(n)} \otimes_H H[e]$.

(2) *If moreover* $\ker(h^*) = \mathrm{im}(g^* p)$, *then there is an isomorphism*

$$H(\mathrm{BSO}_n)/\big(I^\circ_{k(n)} + (u_2\omega_n)\big) \otimes_H H[e] \to H(\mathrm{B}\Gamma_n^+).$$

Proof. We start by proving (1). It immediately follows from Remark 4.5 and Lemma 4.7 that $J^\circ_{k(n)} + (u_2\omega_n) \subseteq \ker(p)$. We show the opposite inclusion by induction on the topological degree. Proposition 4.2 provides the induction basis. Now, suppose that x is in $\ker(p)$ and every class in $\ker(p)$ with topological degree less than the topological degree of x belongs to $J^\circ_{k(n)} + (u_2\omega_n)$. We can write x as $\sum_{j=0}^m \phi_j e^j$ for some $\phi_j \in H(\mathrm{BSO}_n)$. Then,

$$\sum_{j=0}^m \phi_j c^j = g^* p(x) = 0,$$

and so $\phi_j = 0$ in $H(\mathrm{BSpin}_n)$ for any j since by hypothesis c is a monic polynomial in $v_{2^{k(n)}}$. Therefore, $\phi_j \in I_{k(n)} = I^\circ_{k(n)} + (u_2)$ by Theorem 2.10. Hence, there are $\psi_j \in H(\mathrm{BSO}_n)$ such that $\phi_j + u_2\psi_j \in I^\circ_{k(n)}$, from which it follows that $x + u_2 z \in J^\circ_{k(n)}$ where

$$z = \sum_{j=0}^m \psi_j e^j.$$

Thus, $u_2 p(z) = 0$ which implies that $p(z) \in \mathrm{im}(h^*) = \mathrm{im}(p) \cdot \omega_n$, and so there exists an element y in $H(\mathrm{BSO}_n) \otimes_H H[e]$ such that $p(z) = p(y\omega_n)$. Therefore, $z + y\omega_n \in J^\circ_{k(n)} + (u_2\omega_n)$ by induction hypothesis. It follows that $z \in J^\circ_{k(n)} + (\omega_n)$ and $x \in J^\circ_{k(n)} + (u_2\omega_n)$.

We now move to (2). We prove by induction on the topological degree that, if $\ker(h^*) = \mathrm{im}(g^* p)$, then $\mathrm{im}(p) = H(\mathrm{B}\Gamma_n^+)$. Lemma 4.3 provides the induction basis. Let x be a class in $H(\mathrm{B}\Gamma_n^+)$ and suppose that p is an epimorphism in topological degrees less than the topological degree of x. From $g^*(x) \in \ker(h^*) = \mathrm{im}(g^* p)$ it follows that there is an element χ in $H(\mathrm{BSO}_n) \otimes_H H[e]$ such that $g^*(x) = g^* p(\chi)$. Therefore,

$$x + p(\chi) = u_2 z \quad \text{for some } z \in H(\mathrm{B}\Gamma_n^+).$$

By induction hypothesis $z = p(\zeta)$ for some element $\zeta \in H(\mathrm{BSO}_n) \otimes_H H[e]$, hence $x = p(\chi + u_2\zeta)$ that is what we aimed to show. ■

Remark 4.11. Since $H(\mathrm{BSpin}_n)$ is generated by the powers $v^i_{2^{k(n)}}$ as a $H(\mathrm{BSO}_n)$-module, we have that $\mathrm{im}(h^*)$ is generated by $h^*(v^i_{2^{k(n)}})$ as a $H(\mathrm{BSO}_n)$-module.

Lemma 4.12. *For any $m \geq 0$, we have*

$$Sq^m \omega_n \in \langle \omega_n \rangle,$$

where $\langle \omega_n \rangle$ is the $H(\mathrm{BSO}_n)$-submodule of $H(\mathrm{B}\Gamma_n^+)$ generated by ω_n.

Proof. We proceed by induction on m. For $m = 0$ we have that $Sq^0\omega_n = \omega_n$ and for $m > 2^{k(n)} - 1$ we have that $Sq^m\omega_n = 0$ by [11, Corollary 5.8]. Suppose that $Sq^i\omega_n \in \langle \omega_n \rangle$ for $i < m \leq 2^{k(n)} - 1$. Then, by Cartan formula, in $H(\mathrm{BSO}_n)$ we have that

$$Sq^m(u_2\omega_n) = u_2 Sq^m\omega_n + \tau u_3 Sq^{m-1}\omega_n + u_2^2 Sq^{m-2}\omega_n.$$

Therefore, from Remark 4.5 and Lemma 4.7 we deduce that $u_2 Sq^m\omega_n = 0$ in $H(\mathrm{B}\Gamma_n^+)$, since by induction hypothesis $Sq^{m-2}\omega_n \in \langle \omega_n \rangle$. It follows that $Sq^m\omega_n \in \mathrm{im}(h^*)$. By Remark 4.11, we know that $Sq^m\omega_n = \sum_{i\geq 1} \phi_i h^*(v^i_{2^{k(n)}})$ for some $\phi_i \in H(\mathrm{BSO}_n)$. But, for any $i \geq 2$, the topological degree of $h^*(v^i_{2^{k(n)}})$ is

$$i2^{k(n)} - 1 > 2^{k(n)+1} - 2 \geq m + 2^{k(n)} - 1$$

that is the topological degree of $Sq^m\omega_n$. Hence, $Sq^m\omega_n = \phi_1 h^*(v_{2^{k(n)}}) = \phi_1\omega_n$ that is what we aimed to prove. ■

Lemma 4.13. *It exists an element μ_n in $H(\mathrm{B}\Gamma_n^+)$ such that $v^j_{2^{k(n)}} = g^*(\mu_n^{[\frac{j}{2}]}) v^{j-2[\frac{j}{2}]}_{2^{k(n)}}$ in $H(\mathrm{BSpin}_n)$ for any $j \geq 0$.*

Proof. For $j = 0, 1$ the statement is tautological. For $j = 2$, by Cartan formula, we have that

$$h^*(v^2_{2^{k(n)}}) = h^*(Sq^{2^{k(n)}} v_{2^{k(n)}}) = Sq^{2^{k(n)}}(\omega_n\alpha) = \sum_{i=0}^{2^{k(n)}} \tau^{i \bmod 2} Sq^{2^{k(n)}-i}\omega_n Sq^i\alpha,$$

where α is the Thom class of the map $\mathrm{BSpin}_n \to \mathrm{B}\Gamma_n^+$. Note that, by Proposition 4.2 and Lemma 4.3, $H^{p-2,*'-1}(\mathrm{B}\Gamma_n^+) \xrightarrow{\cdot u_2} H^{p,*'}(\mathrm{B}\Gamma_n^+)$ is a monomorphism for $p \leq 2^{k(n)}$. This implies, in particular, that $Sq^1\alpha = 0$ and $Sq^2\alpha = u_2\alpha$. Moreover, $Sq^i\alpha = 0$ for $i \geq 3$, since α is in bidegree (1)[2], and $Sq^{2^{k(n)}}\omega_n = 0$. Therefore, we have that

$$h^*(v_{2^{k(n)}}^2) = Sq^{2^{k(n)}-2}\omega_n u_2\alpha = 0$$

since, by Lemma 4.12, $Sq^{2^{k(n)}-2}\omega_n \in \langle\omega_n\rangle$ and $u_2\omega_n = 0$ by Remark 4.5. Hence, $v_{2^{k(n)}}^2 \in \mathrm{im}(g^*)$.

Let μ_n be a class in $H(\mathrm{B}\Gamma_n^+)$ such that $g^*(\mu_n) = v_{2^{k(n)}}^2$. Suppose the statement is true for $i < j$, then

$$v_{2^{k(n)}}^j = v_{2^{k(n)}}^2 v_{2^{k(n)}}^{j-2} = g^*(\mu_n)g^*(\mu_n^{[\frac{j-2}{2}]})v_{2^{k(n)}}^{j-2-2[\frac{j-2}{2}]} = g^*(\mu_n^{[\frac{j}{2}]})v_{2^{k(n)}}^{j-2[\frac{j}{2}]}$$

that concludes the proof. ∎

Remark 4.14. It immediately follows from Lemma 4.13 that

$$h^*(v_{2^{k(n)}}^j) = \begin{cases} 0, & \text{for } j \text{ even;} \\ \mu_n^{\frac{j-1}{2}}\omega_n, & \text{for } j \text{ odd.} \end{cases}$$

The following is the main result of this paper.

Theorem 4.15. *Let k be a field of characteristic different from 2 containing $\sqrt{-1}$. Then, for any $n \geq 2$, there exists a cohomology class $e_{2^{l(n)}}$ in bidegree $(2^{l(n)-1})[2^{l(n)}]$ such that the natural homomorphism of H-algebras*

$$H(\mathrm{BSO}_n)/I_{l(n)}^\circ \otimes_H H[e_{2^{l(n)}}] \to H(\mathrm{B}\Gamma_n^+)$$

is an isomorphism, where $I_{l(n)}^\circ$ is the ideal generated by $\theta_1, \ldots, \theta_{l(n)-1}$ and $l(n) = [\frac{n+1}{2}]$.

Proof. For $n = 2$ this is given by Proposition 4.6, so suppose from now on that $n \geq 3$.

If $\omega_n = 0$, then there is a class $e_{2^{k(n)}}$ in $H(\mathrm{B}\Gamma_n^+)$ such that $g^*(e_{2^{k(n)}}) = v_{2^{k(n)}}$. Let p be the homomorphism

$$H(\mathrm{BSO}_n) \otimes_H H[e_{2^{k(n)}}] \to H(\mathrm{B}\Gamma_n^+).$$

Then, $\mathrm{im}(h^*) = 0 = \mathrm{im}(p) \cdot \omega_n$ and $\ker(h^*) = H(\mathrm{BSpin}_n) = \mathrm{im}(g^*p)$. Hence, Proposition 4.10 implies that the homomorphism

$$H(\mathrm{BSO}_n)/I_{k(n)}^\circ \otimes_H H[e_{2^{k(n)}}] \to H(\mathrm{B}\Gamma_n^+)$$

is an isomorphism. Since $\theta_{k(n)}$ vanishes in $H(\mathrm{B}\Gamma_n^+)$ we have that $\theta_{k(n)} \in I_{k(n)}^\circ$, which means that $k(n) = l(n)$ by Remark 3.10.

If $\omega_n \neq 0$, then set $p : H(\mathrm{BSO}_n) \otimes_H H[\mu_n] \to H(\mathrm{B}\Gamma_n^+)$ where μ_n is the class from Lemma 4.13. It follows from Remarks 4.11 and 4.14 that $\mathrm{im}(h^*) = \mathrm{im}(p) \cdot \omega_n$. Then, by Proposition 4.10, we obtain that $\ker(p) = J_{k(n)}^\circ + (u_2\omega_n)$.

Since $\omega_n \neq 0$, we can extend the result in Lemma 4.3 to the degree $p = 2^{k(n)}$, i.e. we have that

$$H^{2^{k(n)},2^{k(n)-1}}(\mathrm{BSO}_n) \to H^{2^{k(n)},2^{k(n)-1}}(\mathrm{B}\Gamma_n^+)$$

is surjective. Hence, we deduce that the homomorphism

$$H^{2^{k(n)},2^{k(n)-1}}(\mathrm{B}\Gamma_n^+) \to H^{2^{k(n)},2^{k(n)-1}}(N)$$

is zero and so the homomorphism $H^{2^{k(n)},2^{k(n)-1}}(N) \to H^{2^{k(n)}+1,2^{k(n)-1}}(\mathrm{BSO}_n)$ is injective. It follows that $\theta_{k(n)} \notin I^\circ_{k(n)}$, since $x_{k(n)} \notin \langle x_1, \ldots, x_{k(n)-1}\rangle$ by Lemma 4.9, and $k(n)+1 = l(n)$ by Remark 3.10. Observe that, as we have already shown, $\ker(p) = J^\circ_{k(n)} + (u_2\omega_n)$ and $\theta_{k(n)}$ vanishes in $H(\mathrm{B}\Gamma_n^+)$. Therefore, $\theta_{k(n)} + u_2\omega_n \in I^\circ_{k(n)}$, which implies that $\ker(p) = J^\circ_{k(n)+1} = J^\circ_{l(n)}$.

Now, it remains to prove that $\ker(h^*) = \mathrm{im}(g^*p)$. Obviously, $\mathrm{im}(g^*p) \subseteq \ker(h^*)$, so we only have to prove the other side inclusion. Let x be an element of $\ker(h^*)$. We can write x as $\sum_{j=0}^m \gamma_j v^j_{2^{k(n)}}$ with $\gamma_j \in H(\mathrm{BSO}_n)$. Then, by Remark 4.14, we have that

$$\sum_{j=1,\,\mathrm{odd}}^m \gamma_j \mu_n^{\frac{j-1}{2}} \omega_n = 0.$$

Denote by σ the element $\sum_{j=1,\,\mathrm{odd}}^m \gamma_j \mu_n^{\frac{j-1}{2}}$ in $H(\mathrm{BSO}_n) \otimes_H H[\mu_n]$. From $p(\sigma\omega_n) = 0$ we deduce that $\sigma\omega_n \in J^\circ_{k(n)+1}$, since we have shown that $\ker(p) = J^\circ_{k(n)+1}$. Thus, $\sigma\omega_n = \sum_{j=1}^{k(n)} \sigma_j\theta_j$ for some $\sigma_j \in H(\mathrm{BSO}_n) \otimes_H H[\mu_n]$ and, multiplying by u_2, we obtain that $u_2\sigma\omega_n + u_2\sigma_{k(n)}\theta_{k(n)} \in J^\circ_{k(n)}$. On the other hand, $\theta_{k(n)} + u_2\omega_n \in I^\circ_{k(n)}$, from which it follows by multiplying by σ that $\sigma\theta_{k(n)} + u_2\sigma\omega_n \in J^\circ_{k(n)}$. Hence, $(\sigma + u_2\sigma_{k(n)})\theta_{k(n)} \in J^\circ_{k(n)}$. Theorem 3.9 implies that $\sigma + u_2\sigma_{k(n)} \in J^\circ_{k(n)}$, from which it follows that

$$\sigma \in J^\circ_{k(n)} + (u_2) = J_{k(n)}.$$

Therefore, $g^*p(\sigma) = 0$ in $H(\mathrm{BSpin}_n)$ and by Lemma 4.13

$$\begin{aligned} x &= \sum_{j=1,\,\mathrm{odd}}^m \gamma_j g^*(\mu_n^{\frac{j-1}{2}}) v_{2^{k(n)}} + \sum_{j=0,\,\mathrm{even}}^m \gamma_j g^*(\mu_n^{\frac{j}{2}}) \\ &= g^*p(\sigma)v_{2^{k(n)}} + \sum_{j=0,\,\mathrm{even}}^m \gamma_j g^*(\mu_n^{\frac{j}{2}}) = \sum_{j=0,\,\mathrm{even}}^m \gamma_j g^*(\mu_n^{\frac{j}{2}}) \end{aligned}$$

is an element of $\mathrm{im}(g^*p)$.

Rename the class μ_n by $e_{2^{l(n)}}$. Then, by Proposition 4.10 we have that the homomorphism

$$H(\mathrm{BSO}_n)/I^\circ_{l(n)} \otimes_H H[e_{2^{l(n)}}] \to H(\mathrm{B}\Gamma_n^+)$$

is an isomorphism, and the proof is complete. ∎

Definition 4.16. Denote by $\mathrm{Chern}(\mathrm{B}_{\text{ét}}\Gamma_n^+)$ the subring of the Chow ring $\mathrm{Ch}(\mathrm{B}_{\text{ét}}\Gamma_n^+)$ with $\mathbb{Z}/2$-coefficients generated by the Chern classes of the representation $\Gamma_n^+ \to \mathrm{SO}_n$.

For any $2 \le i \le n$, let $\tilde{w}_i$ be the Stiefel–Whitney class in $H^{i,i}(\mathrm{B_{\acute{e}t}SO}_n)$. Recall from [8, Theorem 3.1.1] that the homomorphism $H(\mathrm{B_{\acute{e}t}SO}_n) \to H(\mathrm{BSO}_n)$, induced by the canonical map $\mathrm{BSO}_n \to \mathrm{B_{\acute{e}t}SO}_n$, sends $\tilde{w}_i$ to $\tau^{[\frac{i+1}{2}]}u_i$.

Lemma 4.17. *The homomorphism $H(\mathrm{B_{\acute{e}t}SO}_n) \to H(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$ maps $Sq^1\tilde{w}_2$ to 0.*

Proof. Note that the homomorphism $H^{3,2}(\mathrm{B_{\acute{e}t}}\Gamma_n^+) \to H^{3,2}(\mathrm{B}\Gamma_n^+)$ is injective, since the change of topology

$$H^{3,2}(\mathrm{B_{\acute{e}t}}\Gamma_n^+) \to H^{3,2}_{\acute{e}t}(\mathrm{B_{\acute{e}t}}\Gamma_n^+) \cong H^{3,2}_{\acute{e}t}(\mathrm{B}\Gamma_n^+),$$

which factors through $H^{3,2}(\mathrm{B}\Gamma_n^+)$, is a monomorphism by [13, Corollary 6.9].

On the other hand, the homomorphism $H^{3,2}(\mathrm{B_{\acute{e}t}SO}_n) \to H^{3,2}(\mathrm{BSO}_n)$ maps $Sq^1\tilde{w}_2$ to $Sq^1(\tau u_2) = \tau u_3$ that vanishes in $H^{3,2}(\mathrm{B}\Gamma_n^+)$. Hence, $Sq^1\tilde{w}_2$ maps to 0 in $H^{3,2}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$ that completes the proof. ∎

Remark 4.18. As noted in [11, Remark 11.3], the class $\tau\theta_i^2$ belongs to the Chern subring $\mathrm{Chern}(\mathrm{B_{\acute{e}t}SO}_n) \cong \mathbb{Z}/2[c_2, \dots, c_n]$, for any $i \ge 1$.

Then, [11, Lemma 11.2] and Lemma 4.17 imply that

$$\tau\theta_i^2 = \tau Sq^1\theta_{i+1} = Sq^1Sq^{2^i}\cdots Sq^1\tilde{w}_2$$

vanishes in $\mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$ for all $i \ge 1$.

The following result provides a complete description of $\mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$ modulo nilpotents.

Corollary 4.19. *There exists a ring isomorphism*

$$\mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)_{\mathrm{red}} \cong \mathbb{Z}/2[c_2, \dots, c_n]/\sqrt{(\tau\theta_1^2, \dots, \tau\theta_{l(n)-1}^2)},$$

where $c_i = \tau^{i \bmod 2}u_i^2$ is the ith Chern class in $H(\mathrm{BSO}_n)$.

Proof. Let $\mathcal{I}_n^\circ$ be the kernel of the epimorphism $\mathbb{Z}/2[c_2, \dots, c_n] \to \mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$. Then, by Remark 4.18 we have that $(\tau\theta_1^2, \dots, \tau\theta_{l(n)-1}^2) \subseteq \mathcal{I}_n^\circ$. On the other hand, since the epimorphism $\mathbb{Z}/2[c_2, \dots, c_n] \to \mathrm{Chern}(\mathrm{B}\Gamma_n^+)$ factors through $\mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)$, Theorem 4.15 implies that $\mathcal{I}_n^\circ \subseteq \iota^{-1}(I_{l(n)}^\circ)$, where $\iota : \mathbb{Z}/2[c_2, \dots, c_n] \to H(\mathrm{BSO}_n)$ is the inclusion of the Chern subring of $H(\mathrm{BSO}_n)$.

Now, observe that $\sqrt{(\tau\theta_1^2, \dots, \tau\theta_{l(n)-1}^2)} = \sqrt{\iota^{-1}(I_{l(n)}^\circ)}$. Therefore, we obtain that

$$\mathrm{Chern}(\mathrm{B_{\acute{e}t}}\Gamma_n^+)_{\mathrm{red}} \cong \mathbb{Z}/2[c_2, \dots, c_n]/\sqrt{\mathcal{I}_n^\circ} \cong \mathbb{Z}/2[c_2, \dots, c_n]/\sqrt{(\tau\theta_1^2, \dots, \tau\theta_{l(n)-1}^2)}$$

that is what we aimed to show. ∎

Remark 4.20. Note that the relations appearing in Corollary 4.19 are also expressible in terms of the action of some Steenrod operations on c_2. More precisely, we have that $\tau\theta_j^2 = Sq^{2^j}Sq^{2^{j-1}}\cdots Sq^4Sq^2c_2$, for any $j \ge 1$.

Acknowledgements. I wish to sincerely thank Nikita Karpenko for encouraging me to write this paper. I am also grateful to the referee for very helpful comments.

References

[1] V. Chernousov and A. Merkurjev, Essential dimension of spinor and Clifford groups. *Algebra Number Theory* **8** (2014), no. 2, 457–472 Zbl 1312.11024 MR 3212863

[2] R. Fedorov and I. Panin, A proof of the Grothendieck–Serre conjecture on principal bundles over regular local rings containing infinite fields. *Publ. Math. Inst. Hautes Études Sci.* **122** (2015), 169–193 Zbl 1330.14077 MR 3415067

[3] M. Harada and A. Kono, Cohomology mod 2 of the classifying space of $\mathrm{Spin}^c(n)$. *Publ. Res. Inst. Math. Sci.* **22** (1986), no. 3, 543–549 Zbl 0627.55011 MR 861783

[4] N. Karpenko, On special Clifford groups and their characteristic classes. 2023, https://sites.ualberta.ca/~karpenko/publ/spind2-16.pdf visited on 12 October 2023

[5] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*. Amer. Math. Soc. Colloq. Publ. 44, American Mathematical Society, Providence, RI, 1998 Zbl 0955.16001 MR 1632779

[6] F. Morel and V. Voevodsky, $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45–143 Zbl 0983.14007 MR 1813224

[7] I. Panin, Proof of Grothendieck–Serre conjecture on principal bundles over regular local rings containing a finite field. 2017, arXiv:1707.01767

[8] A. Smirnov and A. Vishik, Subtle characteristic classes. 2014, arXiv:1401.6661

[9] F. Tanania, Subtle characteristic classes and hermitian forms. *Doc. Math.* **24** (2019), 2493–2523 Zbl 1453.14069 MR 4061057

[10] F. Tanania, A Serre-type spectral sequence for motivic cohomology. 2022, arXiv:2208.03254, to appear in *Algebr. Geom.*

[11] F. Tanania, Subtle characteristic classes for Spin-torsors. *Math. Z.* **301** (2022), no. 1, 41–74 Zbl 1493.14035 MR 4405644

[12] B. Totaro, The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, pp. 249–281, Proc. Sympos. Pure Math. 67, American Mathematical Society, Providence, RI, 1999 Zbl 0967.14005 MR 1743244

[13] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$-coefficients. *Publ. Math. Inst. Hautes Études Sci.* (2003), no. 98, 59–104 Zbl 1057.14028 MR 2031199

[14] V. Voevodsky, Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.* (2003), no. 98, 1–57 Zbl 1057.14027 MR 2031198

[15] V. Voevodsky, Motives over simplicial schemes. *J. K-Theory* **5** (2010), no. 1, 1–38 Zbl 1194.14029 MR 2600283

Communicated by Nikita A. Karpenko

Received 21 June 2023; revised 7 August 2023.

Fabio Tanania
Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany; fabio.tanania@gmail.com