

Simple modules of small quantum groups at dihedral groups

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Abstract. Based on previous results on the classification of finite-dimensional Nichols algebras over dihedral groups and the characterization of simple modules of Drinfeld doubles, we compute the irreducible characters of the Drinfeld doubles of bosonizations of finite-dimensional Nichols algebras over the dihedral groups \mathbb{D}_{4t} with $t \geq 3$. To this end, we develop new techniques that can be applied to Nichols algebras over any Hopf algebra. Namely, we explain how to construct recursively irreducible representations when the Nichols algebra is generated by a decomposable module, and show that the highest-weight of minimum degree in a Verma module determines its socle. We also prove that tensoring a simple module by a rigid simple module gives a semisimple module.

1. Introduction

This paper is devoted to study the representations of certain families of Hopf algebras $\mathcal{D}(V, \mathbb{D}_m)$, which are given by Drinfeld doubles of bosonizations of finite-dimensional Nichols algebras $\mathfrak{B}(V)$ over dihedral groups \mathbb{D}_m of order $2m$ with $m = 4t \geq 12$. The Hopf algebras $\mathcal{D}(V, \mathbb{D}_m)$ might be considered as analogs of small quantum groups but with non-abelian torus. This election is based on the classification result in [12] of all finite-dimensional Nichols algebras over \mathbb{D}_m . In particular, V belongs to an infinite family of reducible Yetter–Drinfeld modules over \mathbb{D}_m and $\mathfrak{B}(V) \simeq \bigwedge V$.

The small quantum groups or Frobenius–Lusztig kernels $u_q(\mathfrak{g})$ are finite-dimensional quotients of quantum universal enveloping algebras $U_q(\mathfrak{g})$ at a root of unity q for \mathfrak{g} a semisimple complex Lie algebra [19], with some restrictions on the order ℓ of q depending on the type of \mathfrak{g} . As it is well known, $U_q(\mathfrak{g})$ can be described as a quotient of the Drinfeld double of the quantum group $U_q(\mathfrak{b})$ associated with a standard Borel subalgebra \mathfrak{b} of \mathfrak{g} . Consequently, $u_q(\mathfrak{g})$ can also be described as a quotient of the Drinfeld double of the small quantum group $u_q(\mathfrak{b})$. The latter is a pointed Hopf algebra over the abelian group \mathbb{Z}_ℓ^n with $n = \text{rk } \mathfrak{g}$. As such, it is isomorphic to the smash product or *bosonization* $u_q(\mathfrak{b}) \simeq u_q(\mathfrak{n}) \# \mathbb{C} \mathbb{Z}_\ell^n$ and $u_q(\mathfrak{n})$ is a Nichols algebra of diagonal type [1, 6, 25]. In fact, as a consequence of the classification theorem due to Andruskiewitsch and Schneider [7],

under mild conditions on the order of the groups, all complex finite-dimensional pointed Hopf algebras over abelian groups are variations of $u_q(\mathfrak{b})$. In these notes, we consider objects analogous to the small quantum groups $u_q(\mathfrak{b})$ but containing a non-abelian torus. These pointed Hopf algebras are given by bosonizations of finite-dimensional Nichols algebras over the infinite family of dihedral groups \mathbb{D}_m classified in [12]. These Nichols algebras actually turn out to be exterior algebras over semisimple objects in the braided category ${}^{\mathbb{D}_m}\mathcal{YD}$. See Section 4 for more details.

More generally, one may consider the Drinfeld double $\mathcal{D}(V, H) := \mathcal{D}(\mathfrak{B}(V)\#H)$ of the bosonization of a finite-dimensional Nichols algebra $\mathfrak{B}(V)$ over a finite-dimensional Hopf algebra H . This kind of generalized small quantum group admits a triangular decomposition $\mathfrak{B}(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}(\bar{V})$ in the sense of Holmes and Nakano [9, 16]. Thus, as in the classical context, the simple modules can be obtained as quotients of generalizations of Verma modules and consequently are classified by their *highest-weights*. Here, the *weights* are the simple representations of the Drinfeld double $\mathcal{D}(H)$, which plays the role of the Cartan subalgebra. In case $H = \mathbb{k}\Gamma$ is a group algebra of an abelian group, the weights are one-dimensional and there are several results known, see for example [3, 10, 15, 17, 18]. However, in case Γ is not an abelian group, the weights are not necessarily one-dimensional and the computations turn out to be more involved. In this case, the description of the simple modules is only known for $\mathfrak{B}(V) = \mathcal{E}_3$ being the Fomin–Kirillov algebra over the symmetric group \mathbb{S}_3 [21]. It is worth pointing out that the category of graded modules can be endowed with an structure of *highest-weight category* when H is semisimple.

In conclusion, with the classification of the irreducible representations at hand, the central problem to address is to compute their characters, i.e. their weight decomposition. Our main contribution is the solution to this problem for small quantum groups at the dihedral groups \mathbb{D}_m , see Theorem 5.2. On the way, we develop new techniques and provide results that can be applied to Nichols algebras over any Hopf algebra.

Specifically, we develop a recursive process based on different triangular decompositions and bosonizations when V is reducible: If $V = U \oplus W$, then the Nichols algebra $\mathfrak{B}(V)$ can be described as a *braided* bosonization $\mathfrak{B}(Z)\#\mathfrak{B}(U)$ for certain module Z associated with the adjoint action of U on W , and

$$\mathfrak{B}(V)\#H \simeq \mathfrak{B}(Z)\#(\mathfrak{B}(U)\#H)$$

as Hopf algebras, see [2, 5, 14]. In this situation, we construct first the simple modules of $\mathcal{D}(U, H) = \mathcal{D}(\mathfrak{B}(U)\#H)$ from the simple $\mathcal{D}(H)$ -modules – the weights – and then use the same proceeding to construct those of $\mathcal{D}(V, H)$ from the former. To use this tool, we need first to prove some technical results on composition of certain functors that assure that our recursive process gives the desired answer.

We would like to emphasize other two new results which might help to describe the simple modules of generalized small quantum groups. First, it is known that the simple modules can also be obtained as the socle of the Verma modules [21, Theorem 2]; we

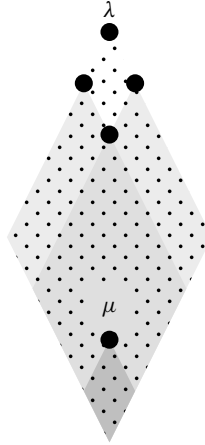


Figure 1. The Verma module $M_H^V(\lambda)$. The (big) dots represent their (highest-)weights. The shadow regions indicate submodules generated by highest-weights. In particular, the region on the bottom is the socle $S_H^V(\lambda)$ which is generated by the highest-weight of minimum degree. The white region on the top depicts its unique simple quotient $L_H^V(\lambda)$.

show in Corollary 3.6 that the socle of a Verma module is generated by its highest-weight of minimum degree, see Figure 1. Second, a simple $\mathcal{D}(V, H)$ -module is said to be *rigid* if it is also simple as $\mathcal{D}(H)$ -module; we prove in Theorem 3.8 that the tensor product between a simple module and a rigid simple module is semisimple.

As the category $\mathcal{D}(V, \mathbb{D}_m)\mathcal{M}$ is non-semisimple, this is a first step towards a complete description of it. Nevertheless, taking into account that the main results needed lots of computations, we prefer to present the result on simple modules first and leave the study of the indecomposable modules, tensor products and extensions for future work. Finally, we point out that the description of the simple modules over $\mathcal{D}(V, H)$ can also be seen as a first step for finding new finite-dimensional Hopf algebras by using the generalized lifting method, see for instance [2].

The paper is mostly self-contained and includes figures to lighten the reading. It is organized as follows. In the preliminaries we collect definitions, notation and basic facts that are used along the paper. In Section 3 we recall the general framework of Hopf algebras with triangular decomposition and present the new results mentioned above, cf. Corollary 3.6 and Theorems 3.8 and 3.11. In Section 4 we summarize some facts about the category of $\mathcal{D}(\mathbb{D}_m)$ -modules. We list the simple $\mathcal{D}(\mathbb{D}_m)$ -modules and compute some tensor products between them. Dealing with these tensor products is one of the main issues when the corresponding weights are not one-dimensional. We also recall the classification due to [12] of the finite-dimensional Nichols algebras $\mathfrak{B}(V)$ over \mathbb{D}_m , cf. Theorem 4.5. We fix a lighter notation to work with their Drinfeld doubles $\mathcal{D}(V, \mathbb{D}_m)$ in the last section and characterize those which are spherical, see Theorem 4.8. Finally, we compute in Section 5

the weight decomposition of the simple $\mathcal{D}(V, \mathbb{D}_m)$ -modules for all V in the classification of [12], see Theorem 5.2; we first consider the case when V is simple in Section 5.1, and then prove the recursive step in Section 5.2.

2. Preliminaries

In this section we introduce notation and recall some basic results that are needed along the paper.

2.1. Conventions

We work over an algebraically closed field \mathbb{k} of characteristic zero. All vector spaces are considered over \mathbb{k} and $\otimes = \otimes_{\mathbb{k}}$. For $n \in \mathbb{N}$, we denote by \mathbb{Z}_n the ring of integers module n . We use the same letter to indicate an integer and its class in \mathbb{Z}_n . Through the work graded means \mathbb{Z} -graded. Let $\mathbb{N} = \bigoplus_{n \in \mathbb{Z}} \mathbb{N}_n$ be a graded vector space. For $i \in \mathbb{Z}$, the shift $\mathbb{N}[i]$ of \mathbb{N} is the same vector space \mathbb{N} but with shifted grading, where $\mathbb{N}[i]_n = \mathbb{N}_{n-i}$ as homogeneous component of degree n .

We work with Hopf algebras H over \mathbb{k} . As usual, we denote the comultiplication by Δ , the antipode by \mathcal{S} and the counit by ε . The comultiplication is written using Sweedler's sigma notation without the summation symbol, i.e.,

$$\Delta(h) = h_{(1)} \otimes h_{(2)} \quad \text{for all } h \in H.$$

Analogously, for a left H -comodule (V, λ) we write $\lambda(v) = v_{(-1)} \otimes v_{(0)} \in H \otimes V$ for all $v \in V$ to denote its coaction. We refer to [24] for basic and well-known results in the theory.

Throughout these notes, we make use of the triangular decomposition associated with finite-dimensional graded algebras as introduced by Holmes and Nakano [16]. Here we apply it to Drinfeld doubles of finite-dimensional Hopf algebras. A graded algebra

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

admits a *triangular decomposition* if there exist graded subalgebras A^- , T and A^+ such that the multiplication $m : A^- \otimes T \otimes A^+ \rightarrow A$ gives a linear isomorphism and

$$(td1) \quad A^\pm \subseteq \bigoplus_{n \in \mathbb{Z}_\pm} A_n \text{ and } T \subseteq A_0;$$

$$(td2) \quad (A^\pm)_0 = \mathbb{k};$$

$$(td3) \quad B^\pm := A^\pm T = T A^\pm.$$

In our situation, this coincides with [9, Definition 3.1] as T is a split \mathbb{k} -algebra because of our assumptions on \mathbb{k} . We denote by ${}_A \mathcal{M}$ the category of finite-dimensional left A -modules and by ${}_A \mathcal{G}$ the category of the graded ones with morphisms preserving the grading. We write ${}_A \mathcal{C}$ to refer to either one of these categories. We denote by $\text{Irr } {}_A \mathcal{C}$ a complete set of non-isomorphic simple objects in ${}_A \mathcal{C}$.

3. The general framework

We outline in this section the general framework of our study. Eventually, we will be more explicit in the successive subsections according to our convenience. For more details we refer the reader to [6, 9, 27].

3.1. Drinfeld doubles of bosonization of Nichols algebras

Fix H a finite-dimensional Hopf algebra.

3.1.1. Nichols algebras. A left Yetter–Drinfeld module over H is a left H -module (V, \cdot) and a left H -comodule (V, λ) that satisfies the compatibility condition

$$\lambda(h \cdot v) = h_{(1)}v_{(-1)}\mathcal{S}(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \quad \text{for all } h \in H, v \in V.$$

The finite-dimensional left Yetter–Drinfeld modules over H together with morphisms of left H -modules and left H -comodules form a braided rigid tensor category denoted by ${}^H_H\mathcal{YD}$. The braiding is given by $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$ for all $v \in V, w \in W$ with V, W objects in ${}^H_H\mathcal{YD}$.

Let $V \in {}^H_H\mathcal{YD}$. Then, the tensor algebra $T(V)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$. Roughly speaking, it satisfies the axioms of a Hopf algebra but the structural maps are morphism in the category. For instance, its comultiplication is determined by setting $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$.

The *Nichols algebra* $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$ of V is the graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ defined by the quotient $\mathfrak{B}(V) = T(V)/\mathcal{I}(V)$, where $\mathcal{I}(V)$ is the largest Hopf ideal of $T(V)$ generated as an ideal by homogeneous elements of degree greater than or equal to 2. By definition, we have that $\mathfrak{B}^0(V) = \mathbb{k}$ and $\mathfrak{B}^1(V) = V$. In case $\mathfrak{B}(V)$ is finite-dimensional, we denote by n_{top} its maximum degree. It is well known that $\lambda_V := \mathfrak{B}^{n_{\text{top}}}(V)$ is one-dimensional (in fact, $\mathfrak{B}(V)$ satisfies the Poincaré duality); in particular, it is a simple H -module and H -comodule. A fixed linear generator $v_{\text{top}} \in \mathfrak{B}^{n_{\text{top}}}(V)$ is usually called a *volume element*. We refer to [1] for more details on Nichols algebras.

3.1.2. Bosonization. The *bosonization* of $\mathfrak{B}(V)$ over H is a usual Hopf algebra whose underlying vector space is

$$\mathfrak{B}(V)\#H := \mathfrak{B}(V) \otimes H$$

and it is endowed with a Hopf algebra structure which is a sort of semidirect product. It is generated by V and H as an algebra, whereas its multiplication and comultiplication are completely determined by

$$hv = (h_{(1)} \cdot v)\#h_{(2)} \quad \text{and} \quad \Delta(v) = v \otimes 1 + v_{(-1)} \otimes v_{(0)}$$

for all $v \in V$ and $h \in H$. In particular, $H = 1\#H$ is a Hopf subalgebra and $\mathfrak{B}(V) = \mathfrak{B}(V)\#1$ is a subalgebra which coincides with the subalgebra of left coinvariants associated with the projection $\mathfrak{B}(V)\#H \rightarrow H$ given by $\pi(b\#h) = \varepsilon(b)h$ for all $b \in \mathfrak{B}(V)$ and

$h \in H$. Note that the adjoint action of H on V coincides with the action as Yetter–Drinfeld module. That is, for all $h \in H$ and $v \in V$ we have that

$$\text{ad}(h)(v) = h_{(1)}v\mathcal{S}(h_{(2)}) = (h_{(1)} \cdot v)\#h_{(2)}\mathcal{S}(h_{(3)}) = h \cdot v.$$

3.1.3. Drinfeld double. The *Drinfeld double* of H is the Hopf algebra defined on the vector space

$$\mathcal{D}(H) := H \otimes H^*$$

in such a way that $H = H \otimes 1$ and $H^{*\text{op}} = 1 \otimes H^{*\text{op}}$ are Hopf subalgebras, and the elements $h \in H$ and $f \in H^*$ obey the multiplication rule

$$fh = \langle f_{(1)}, h_{(1)} \rangle \langle f_{(3)}, \mathcal{S}(h_{(3)}) \rangle h_{(2)} f_{(2)}.$$

By convention we write $hf = h \otimes f$ for all $h \in H$, $f \in H^*$, cf. [20, Theorem 7.1.1].

The Drinfeld double $\mathcal{D}(H)$ is a quasitriangular Hopf algebra with R -matrix given by $R = \sum_i f_i \otimes h_i \in \mathcal{D}(H) \otimes \mathcal{D}(H)$, where $\{h_i\} \subset H$ and $\{f_i\} \subset H^*$ are dual bases of H and H^* , respectively. Also, it holds that $R^{-1} = \sum_i \mathcal{S}(f_i) \otimes h_i$. Thus, the category ${}_{\mathcal{D}(H)}\mathcal{M}$ of $\mathcal{D}(H)$ -modules is braided with braiding

$$c_{M,N}(m \otimes n) = \sum_i (h_i \cdot n) \otimes (f_i \cdot m)$$

for all $m \in M, n \in N$ with $M, N \in {}_{\mathcal{D}(H)}\mathcal{M}$. As the braiding is invertible, one may consider also ${}_{\mathcal{D}(H)}\mathcal{M}$ as braided category with braiding c^{-1} .

It is possible to relate the categories ${}^H_H\mathcal{YD}$, ${}_{\mathcal{D}(H)}\mathcal{M}$ and ${}^{\mathcal{D}(H)}_{\mathcal{D}(H)}\mathcal{YD}$ by means of the following functors:

$$\begin{array}{ccc} {}^H_H\mathcal{YD} & \xrightarrow{F} & ({}_{\mathcal{D}(H)}\mathcal{M}, c) & \xrightarrow{F_R} & {}^{\mathcal{D}(H)}_{\mathcal{D}(H)}\mathcal{YD} \\ & & & \nearrow F_{R^{-1}} & \\ {}^{H^*\text{op}}_{H^*\text{op}}\mathcal{YD} & \xrightarrow{\bar{F}} & ({}_{\mathcal{D}(H)}\mathcal{M}, c^{-1}) & & \end{array}$$

Explicitly, F is the braided equivalence which transforms $M \in {}^H_H\mathcal{YD}$ into a ${}_{\mathcal{D}(H)}\mathcal{M}$ -module with the action

$$(hf) \cdot m = \langle f, m_{(-1)} \rangle h \cdot m_{(0)},$$

for all $h \in H$, $f \in H^*$ and $m \in M$. Analogously, \bar{F} transforms $M \in {}^{H^*\text{op}}_{H^*\text{op}}\mathcal{YD}$ into a ${}_{\mathcal{D}(H)}\mathcal{M}$ -module with the action

$$(hf) \cdot m = \langle (f \cdot m)_{(-1)}, \mathcal{S}(h) \rangle (f \cdot m)_{(0)}.$$

The functors F_R and $F_{R^{-1}}$ are both fully faithful and are defined as follows: For $M \in {}_{\mathcal{D}(H)}\mathcal{M}$, the object $F_R(M)$ (resp. $F_{R^{-1}}(M)$) coincides with M as ${}_{\mathcal{D}(H)}\mathcal{M}$ -module,

whereas the $\mathcal{D}(H)$ -coaction is provided by the action of the R -matrix R (resp. R^{-1}) given by

$$m_{(-1)} \otimes m_{(0)} = \sum_i h_i \otimes (f_i \cdot m) \quad \left(\text{resp.} \quad \sum_i \mathcal{S}(f_i) \otimes (h_i \cdot m) \right) \quad (3.1)$$

for all $m \in M$.

Note that the braidings of V and $F_R \circ F(V) = V$ coincide as linear maps for any $V \in {}^H_H \mathcal{YD}$. This implies that the Nichols algebras $\mathfrak{B}(V)$ and $\mathfrak{B}(F_R \circ F(V))$ are isomorphic as braided Hopf algebras [26]. In particular, they are isomorphic as algebras and coalgebras. In the sequel, we identify both Nichols algebras and consider $\mathfrak{B}(V) \in \frac{\mathcal{D}(H)}{\mathcal{D}(H)} \mathcal{YD}$.

Let V^* be the $\mathcal{D}(H)$ -module dual to V . We define

$$\bar{V} = F_{R^{-1}}(V^*) \in \frac{\mathcal{D}(H)}{\mathcal{D}(H)} \mathcal{YD}.$$

As above, the Nichols algebras $\mathfrak{B}(\bar{V})$ in $\frac{\mathcal{D}(H)}{\mathcal{D}(H)} \mathcal{YD}$ and $\mathfrak{B}(V^*)$ in $(\mathcal{D}(H)\mathcal{M}, c^{-1})$ are isomorphic as braided Hopf algebras. Besides, there is an isomorphism of $\mathcal{D}(H)$ -module algebras

$$\mathfrak{B}(\bar{V}) \simeq \mathfrak{B}(F_R(V^*)), \quad (3.2)$$

where the algebra on the right-hand side is the Nichols algebra of $F_R(V^*)$ in $\frac{\mathcal{D}(H)}{\mathcal{D}(H)} \mathcal{YD}$, or equivalently the Nichols algebra of V^* in $(\mathcal{D}(H)\mathcal{M}, c)$. This follows from [5, Lemma 1.11], see also [27, (4.9)].

Actually, one may consider $V^* \in \frac{H^{*op}}{H^{*op}} \mathcal{YD}$ with action and coaction defined by

$$\langle f \cdot \alpha, x \rangle = \langle f, \mathcal{S}^{-1}(x_{(-1)}) \rangle \langle \alpha, x_{(0)} \rangle \quad \text{and} \quad \langle \alpha, h \cdot x \rangle = \langle \alpha_{(-1)}, h \rangle \langle \alpha_{(0)}, x \rangle$$

for all $f \in H^{*op}$, $h \in H$, $x \in V$ and $\alpha \in V^*$. Then $\bar{F}(V^*)$ is the dual object of V as $\mathcal{D}(H)$ -module and hence

$$\bar{V} = F_{R^{-1}} \circ \bar{F}(V^*).$$

The Nichols algebras $\mathfrak{B}(\bar{V})$ and $\mathfrak{B}(V)$ in $\frac{\mathcal{D}(H)}{\mathcal{D}(H)} \mathcal{YD}$ play a central role in the rest of the paper. In case they are finite-dimensional, they are related by the fact that there is an isomorphism of Hopf algebras

$$(\mathfrak{B}(V)\#H)^{*op} \simeq \mathfrak{B}(\bar{V})\#H^{*op}$$

(adapt the proof of [21, Lemma 5]).

3.1.4. A generalized quantum group. From now on, we denote by $\mathcal{D}(V, H)$ the Drinfeld double of $\mathfrak{B}(V)\#H$ with $\mathfrak{B}(V)$ a finite-dimensional Nichols algebra in ${}^H_H \mathcal{YD}$. We describe its Hopf algebra structure following [27, Lemma 4.3].

From the very definition of the Drinfeld double, it is possible to show that

$$\begin{aligned} \mathcal{D}(V, H) &\simeq \mathfrak{B}(V) \otimes H \otimes (\mathfrak{B}(V) \otimes H)^* \\ &\simeq \mathfrak{B}(V) \otimes H \otimes \mathfrak{B}(\bar{V}) \otimes H^* \end{aligned}$$

as vector spaces. Via this isomorphism, we may assume that $\mathcal{D}(V, H)$ is generated as an algebra by the elements of V, \bar{V}, H and H^* . Henceforth, the Hopf algebra structure of $\mathcal{D}(V, H)$ is completely determined by the following features:

- (a) The subalgebra generated by H and H^* is a Hopf subalgebra isomorphic to $\mathcal{D}(H)$.
- (b) The subalgebra generated by V (resp. \bar{V}) and $\mathcal{D}(H)$ is isomorphic to the bosonization

$$\mathcal{D}^{\leq 0}(V, H) := \mathfrak{B}(V) \# \mathcal{D}(H) \quad (\text{resp.} \quad \mathcal{D}^{\geq 0}(V, H) := \mathfrak{B}(\bar{V}) \# \mathcal{D}(H)).$$

In particular, both $\mathcal{D}^{\leq 0}(V, H)$ and $\mathcal{D}^{\geq 0}(V, H)$ are Hopf subalgebras.

- (c) If $v \in V$ and $\alpha \in \bar{V}$, then

$$\alpha v = (\alpha_{(-1)} \cdot v) \alpha_{(0)} + \Phi_{\alpha, v}, \quad (3.3)$$

where $\Phi_{\alpha, v} = \langle \alpha, v \rangle - \alpha_{(-1)} v_{(-1)} \langle \alpha_{(0)}, v_{(0)} \rangle \in \mathcal{D}(H)^1$.

It turns out that $\mathcal{D}(V, H)$ is a graded Hopf algebra with grading determined by $\deg V = -1$, $\deg \bar{V} = 1$, and $\deg \mathcal{D}(H) = 0$. Moreover, there is an isomorphism of graded vector spaces induced by the multiplication:

$$\mathcal{D}(V, H) \simeq \mathfrak{B}(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}(\bar{V}).$$

In conclusion, $\mathcal{D}(V, H)$ admits a triangular decomposition with $T = \mathcal{D}(H)$, $A^- = \mathfrak{B}(V)$ and $A^+ = \mathfrak{B}(\bar{V})$.

3.2. Simple $\mathcal{D}(V, H)$ -modules

In this subsection we explain how to compute the simple modules of $\mathcal{D}(V, H)$ by exploiting its triangular decomposition. For more details we refer to [9, 21, 27].

We begin by constructing the proper standard modules which are the images of the composition of the functors

$$\mathbb{M}_H^V : \mathcal{D}(H)\mathcal{C} \xrightarrow{\text{Inf}_{\mathcal{D}(H)}^{\mathcal{D}^{\geq 0}(V, H)}} \mathcal{D}^{\geq 0}(V, H)\mathcal{C} \xrightarrow{\text{Ind}_{\mathcal{D}^{\geq 0}(V, H)}^{\mathcal{D}(V, H)}} \mathcal{D}(V, H)\mathcal{C},$$

where $\text{Inf}_{\mathcal{D}(H)}^{\mathcal{D}^{\geq 0}(V, H)}$ is given by the canonical (graded) Hopf algebra epimorphism

$$\mathcal{D}^{\geq 0}(V, H) \twoheadrightarrow \mathcal{D}(H)$$

and $\text{Ind}_{\mathcal{D}^{\geq 0}(V, H)}^{\mathcal{D}(V, H)}$ by the inclusion $\mathcal{D}^{\geq 0}(V, H) \hookrightarrow \mathcal{D}(V, H)$. More explicitly, the *proper standard module* of $N \in \mathcal{D}(H)\mathcal{C}$ is

$$\mathbb{M}_H^V(N) = \mathcal{D}(H, V) \otimes_{\mathcal{D}^{\geq 0}(V, H)} \text{Inf}_{\mathcal{D}(H)}^{\mathcal{D}^{\geq 0}(V, H)}(N) = \mathcal{D}(H, V) \otimes_{\mathcal{D}^{\geq 0}(V, H)} N.$$

Notice that \bar{V} acts by zero on $N = 1 \otimes N \subset \mathbb{M}_H^V(N)$.

¹This formula is a revised version of [27, (4.5)], see equation (4.5) of arXiv:1808.03799v2.

Composing M_H^V with the endofunctor given by taking the head, one has the functor

$$\mathcal{D}(H)\mathcal{C} \xrightarrow{L_H^V} \mathcal{D}(V,H)\mathcal{C}, \quad L_H^V(N) = \text{top}(M_H^V(N)),$$

whose image is the maximal semisimple quotient of $M_H^V(N)$. Note that both M_H^V and L_H^V commute with the shift-of-grading functors.

The proper standard modules of simple objects in $\mathcal{D}(H)\mathcal{C}$ will play a special role in the classification of those in $\mathcal{D}(V,H)\mathcal{C}$. For that reason, we introduce a particular terminology. We call *weights* the elements in $\text{Irr } \mathcal{D}(H)\mathcal{C}$. If $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{C}$ is a subobject of $N \in \mathcal{D}(V,H)\mathcal{C}$ such that $\bar{V} \cdot \lambda = 0$, we say that λ is a *highest-weight (of N)*. An object in $\mathcal{D}(V,H)\mathcal{C}$ generated by a highest-weight is called a *highest-weight module*. We define *lowest-weight (modules)* analogously using V instead of \bar{V} . Given $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{C}$, we call *Verma module*² the proper standard module

$$M_H^V(\lambda) = \mathcal{D}(V, H) \otimes_{\mathcal{D}^{\geq 0}(V,H)} \lambda. \quad (3.4)$$

Thus, any highest-weight module is a quotient of a Verma module.

A classification of the simple modules over algebras with triangular decomposition is well known, see for instance [9, 16]. In the case of $\mathcal{D}(V, H)$ this is given as follows.

Theorem 3.1. *Keeping the notation above, we have:*

- (a) $L_H^V(\lambda)$ is a simple highest-weight module for all $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{C}$.
- (b) Any simple object in $\mathcal{D}(V,H)\mathcal{C}$ is isomorphic to $L_H^V(\lambda)$ for a unique weight $\lambda \in \text{Irr } \mathcal{D}(V,H)\mathcal{C}$.

We list some general remarks about simple modules that will be useful later.

Remark 3.2. If N is a highest-weight module of weight λ , then N is a quotient of $M_H^V(\lambda)$ and the head of N is isomorphic to $L_H^V(\lambda)$. This is a direct consequence of the description above.

Remark 3.3. Let B and \bar{B} be linear bases of V and \bar{V} , respectively. As the elements of V and \bar{V} act nilpotently, it holds that $L_H^V(\lambda) = \lambda$, with V and \bar{V} acting trivially, if and only if $\Phi_{\alpha,v}$ acts by zero on λ for all $v \in B$ and $\alpha \in \bar{B}$. These simple modules are called *rigid* [9].

Remark 3.4. Let $v_{\text{top}} \in \mathfrak{B}^{n_{\text{top}}}(V)$ and $\alpha_{\text{top}} \in \mathfrak{B}^{n_{\text{top}}}(\bar{V})$ be non-zero homogeneous elements; note the maximum degree of these Nichols algebras is the same. By the triangular decomposition of $\mathcal{D}(V, H)$, there exists $\Theta \in \mathcal{D}(H)$ such that

$$\alpha_{\text{top}} v_{\text{top}} - \Theta \in \bigoplus_{n>0} \mathfrak{B}^n(V) \otimes \mathcal{D}(H) \otimes \mathfrak{B}^n(\bar{V}). \quad (3.5)$$

It holds that³ $L_H^V(\lambda) = M_H^V(\lambda)$ if and only if $\Theta \cdot (1 \otimes m) \neq 0$ for some $m \in \lambda$. In such case, these Verma modules are also projective, see [27, Corollary 5.12].

²We change slightly the notation of the Verma modules with respect to [27] to put the emphasis on V and H as this will be useful for our recursive argument for V decomposable.

³This follows from the proof of [21, Corollary 15], as it holds for any Hopf algebra H .

3.2.1. The character of the simple modules. For any $N \in \mathcal{D}(H)\mathcal{M}$, the proper standard module $M_H^V(N)$ in the category $\mathcal{D}(V,H)\mathcal{M}$ inherits the grading afforded by $\mathcal{D}(V,H)$. Explicitly,

$$M_H^V(N) = \bigoplus_{k=0}^{-n_{\text{top}}} (M_H^V(N))_k \quad \text{with } (M_H^V(N))_k = \mathfrak{B}^{-k}(V) \otimes N.$$

It follows from [13, Proposition 3.5] that the head of $M_H^V(N)$ in $\mathcal{D}(V,H)\mathcal{M}$ is a graded quotient. Moreover, L_H^V commutes with the grading-forgetful functors. In particular, for any $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{M}$ we have a decomposition

$$L_H^V(\lambda) = \bigoplus_{n \leq 0} L_H^V(\lambda)_n$$

making it an object in $\mathcal{D}(V,H)\mathcal{G}$.

On the other hand, since $\mathcal{D}(H)$ is concentrated in degree 0, we can consider each $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{M}$ inside $\mathcal{D}(H)\mathcal{G}$ as an object concentrated in degree 0. Thus, $\text{Irr } \mathcal{D}(H)\mathcal{M} \times \mathbb{Z}$ is in bijection with $\text{Irr } \mathcal{D}(H)\mathcal{G}$ via $(\lambda, n) \leftrightarrow \lambda[n]$.

Assume now H is semisimple; hence $\mathcal{D}(H)$ also is. Then for each n ,

$$L_H^V(\lambda)_n \simeq \bigoplus_{\mu \in \text{Irr } \mathcal{D}(H)\mathcal{M}} \mu^{\oplus t_{\mu,n}}$$

as $\mathcal{D}(H)$ -modules. We call *the character of $L_H^V(\lambda)$* the graded $\mathcal{D}(H)$ -module

$$\text{Res}(L_H^V(\lambda)) := \bigoplus_{\substack{n \leq 0 \\ \mu \in \text{Irr } \mathcal{D}(H)\mathcal{M}}} \mu[n]^{\oplus t_{\mu,n}}.$$

This gives us good information about the simple modules as it is a complete invariant, since $\text{Res}(L_H^V(\lambda)) = \lambda \oplus$ (weights in degree < 0).

3.2.2. The action on the Verma modules. By the paragraphs above, one immediately realizes that to describe a simple module $L_H^V(\lambda)$ one has to deal with the submodules of the Verma module $M_H^V(\lambda)$. For explicit computations, it is convenient to keep in mind the following key facts.

(a) The action of $\mathfrak{B}(V)$ and $\mathcal{D}(H)$ on $M_H^V(\lambda)$ is given by

$$z \cdot (v \otimes m) = (zv) \otimes m \quad \text{and} \quad h \cdot (v \otimes m) = (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot m)$$

for all $z, v \in \mathfrak{B}(V)$, $m \in \lambda$ and $h \in \mathcal{D}(H)$. In particular,

$$M_H^V(\lambda) \simeq \mathfrak{B}(V) \otimes \lambda = \bigoplus_{n=0}^{n_{\text{top}}} \mathfrak{B}^n(V) \otimes \lambda \quad (3.6)$$

is an isomorphism and a decomposition as $\mathcal{D}(H)$ -modules, respectively.

(b) Let $M_H^V(\lambda)_k = \mathfrak{B}^{-k}(V) \otimes \lambda$. Then

$$VM_H^V(\lambda)_k = M_H^V(\lambda)_{k-1} \quad \text{and} \quad \bar{V}M_H^V(\lambda)_k \subseteq M_H^V(\lambda)_{k+1}.$$

- (c) To compute the action of $\mathfrak{B}(\bar{V})$ we use the commutation rule (3.3).
 (d) The action of \bar{V} on a $\mathcal{D}(V, H)$ -module N is a morphism of $\mathcal{D}(H)$ -modules:

$$h(\alpha m) = (h_{(1)} \cdot \alpha)(h_{(2)} m) \quad (3.7)$$

for all $\alpha \in \bar{V}$, $m \in N$ and $h \in \mathcal{D}(H)$, cf. [21, (31)].

3.2.3. The simple $\mathcal{D}(V, H)$ -modules as socles of the Verma modules. We introduce now the functor

$$\mathcal{D}(H)\mathcal{C} \xrightarrow{S_H^V} \mathcal{D}(V, H)\mathcal{C}, \quad S_H^V(N) = \text{soc}(M_H^V(N)),$$

i.e. $S_H^V(N)$ is the maximal semisimple submodule of $M_H^V(N)$. As in the case of the head, it follows from [13, Proposition 3.5] that the socle of a standard module is a graded submodule even if $N \in \mathcal{D}(H)\mathcal{M}$, considered as a graded module concentrated in degree 0. Also, S_H^V commutes with the grading-forgetful functor and the shift-of-grading functor.

The socles of the Verma modules give us another classification of the simple modules over $\mathcal{D}(V, H)$. The next result for H being a group algebra is in [21].

Theorem 3.5. *Keeping the notation above, we have:*

- (a) $S_H^V(\lambda)$ is a simple lowest-weight module with lowest-weight is $\lambda_V \lambda$ for all $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{C}$
 (b) Any simple object in $\mathcal{D}(V, H)\mathcal{C}$ is isomorphic to $S_H^V(\lambda)$ for a unique weight $\lambda \in \text{Irr}_{\mathcal{D}(V, H)}\mathcal{C}$.

Proof. From Section 3.2.2 (a), we see that the socle of $M_H^V(\lambda)$ in $\mathcal{D}^{\leq 0}(V, H)\mathcal{C}$ is

$$\mathfrak{B}^{\text{top}}(V) \otimes \lambda$$

and this is simple and isomorphic to $\text{Inf}_{\mathcal{D}(H)}^{\mathcal{D}^{\leq 0}(V, H)}(\lambda_V \lambda)$. This implies (a).

For (b), we first consider the category $\mathcal{D}(V, H)\mathcal{M}$ in which the number of non-isomorphic simple modules is $\#\text{Irr}_{\mathcal{D}(H)}\mathcal{M}$ by Theorem 3.1. In (a) we have found the same number of non-isomorphic simple modules as tensoring by λ_V gives a bijection on $\text{Irr}_{\mathcal{D}(H)}\mathcal{M}$. Therefore any simple module in $\mathcal{D}(V, H)\mathcal{M}$ is isomorphic to $S_H^V(\lambda)$ for a unique $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{M}$.

Let now $S \in \mathcal{D}(V, H)\mathcal{G}$ be a simple object and F the grading-forgetful functor. Then $FS \simeq S_H^V(\lambda)$ for a unique $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{M}$ by the above paragraph and hence

$$S \simeq S_H^V(\lambda)[n] \simeq S_H^V(\lambda[n])$$

for some $n \in \mathbb{Z}$; the first isomorphism is consequence of [13, Theorem 4.1]. This proves (b) for $\mathcal{D}(V, H)\mathcal{G}$ and completes the proof. \blacksquare

Naturally, the socle of a Verma module is isomorphic to a simple highest-weight module. We can determine its highest-weight as follows. The next result is very useful to compute the simple modules.

Corollary 3.6. *Let $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{C}$ be a weight and*

$$n = \min\{k \in \mathbb{Z} \mid \text{there is a highest-weight } \mu \text{ in } (\mathbb{M}_H^V(\lambda))_k\}.$$

Then $\mathbb{M}_H^V(\lambda)$ has a unique highest-weight μ in degree n and $\mathbb{S}_H^V(\lambda) \simeq \mathbb{L}_H^V(\mu)[n]$.

Proof. As we mentioned, the socle of $\mathbb{M}_H^V(\lambda)$ is a graded submodule, then

$$\mathbb{S}_H^V(\lambda) \simeq \mathbb{L}_H^V(\mu)[k]$$

for some homogeneous highest-weight $\mu \in (\mathbb{M}_H^V(\lambda))_k$. Let $\nu \in (\mathbb{M}_H^V(\lambda))_j$ be another homogeneous highest-weight. The submodule generated by ν contains the socle. In particular, $\mu \in \mathcal{D}(V, H)\nu$. Also, by the triangular decomposition,

$$\mathcal{D}(V, H)\nu = \mathfrak{B}(V)\nu = \nu + \sum_{i>0} \mathfrak{B}^i(V)\nu.$$

Then Section 3.2.2 (b) implies that either $j > k$ or $j = k$ and $\nu = \mu$, and the corollary follows. \blacksquare

3.2.4. Example. Exterior algebras are examples of Nichols algebras. They arise when the braiding of V is *flip*, that is $c_{V,V}(v \otimes w) = -w \otimes v$ for all $v, w \in V$. We explain here a general strategy that applies to exterior algebras of two-dimensional vector spaces. In Section 3.3.1, we consider any even dimensional vector space, as the Nichols algebras appearing in the context of the dihedral groups \mathbb{D}_m are all exterior algebras of vector spaces of even dimension.

Fix $V \in {}^H_H\mathcal{YD}$ a two-dimensional module with basis $\{v_+, v_-\}$ and braiding *flip*. Then

$$\mathfrak{B}(V) \simeq \bigwedge V = \mathbb{k} \oplus V \oplus \mathbb{k}\{v_{\text{top}}\}$$

with $v_{\text{top}} = v_+v_-$. Let λ be a weight with $V \otimes \lambda$ semisimple and let μ be the highest-weight of minimum degree in $\mathbb{M}_H^V(\lambda)$, recall Corollary 3.6. We have the following three possibilities:

- If $\deg \mu = 0$, then $\mu = \lambda$ and $\mathbb{M}_H^V(\lambda) = \mathbb{L}_H^V(\lambda) = \mathbb{S}_H^V(\lambda)$ is simple projective. This occurs when Θ acts non-trivially on λ , see Remark 3.4. One can compute Θ using v_{top} and $\alpha_{\text{top}} = \alpha_+\alpha_-$ where $\{\alpha_+, \alpha_-\}$ is a basis of \bar{V} .
- If $\deg \mu = -1$, then $\text{Res}(\mathbb{S}_H^V(\mu)) = \mu[-1] \oplus \lambda_V \lambda[-2]$. We depict this situation in Figure 2. To find μ , first one has to decompose $V \otimes \lambda$ as a direct sum of weights and then determine which is annihilated by \bar{V} .
- If $\deg \mu = -2$, then $\mu = \lambda_V \lambda$ and $\mathbb{S}_H^V(\lambda) = \mathbb{L}_H^V(\lambda_V \lambda)[-2]$ is a rigid module. To find the rigid module, one can use Remark 3.3. For that, one should compute the four elements $\Phi_{\pm, \pm}$ associated with the elements v_{\pm} and α_{\pm} .

3.2.5. Tensoring by rigids. We observe that any semisimple object in $\mathcal{D}(V, H)\mathcal{G}$ is of the form

$$\mathbb{L}_H^V(M) := \bigoplus_i \mathbb{L}_H^V(\lambda_i[n_i])$$

for some semisimple object $M = \bigoplus_i \lambda_i[n_i]$ in $\mathcal{D}(H)\mathcal{G}$.

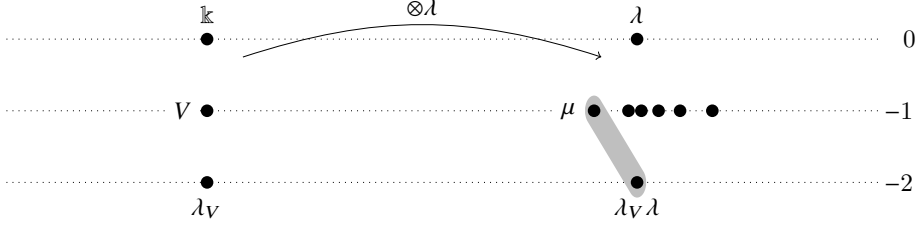


Figure 2. The big dots represent the weights of $\mathfrak{B}(V)$ and $M_H^V(\lambda)$. Their degrees are indicated on the right. Those in the shadow region form the socle $S_H^V(\lambda)$ when $\deg \mu = -1$.

Proposition 3.7. *Let $N = \bigoplus_{i \leq t} N_i$ be an object in $\mathcal{D}(V, H)\mathcal{G}$ with $t \in \mathbb{Z}$. If N_t is a semisimple $\mathcal{D}(H)$ -module and generates N as $\mathcal{D}(V, H)$ -module, then*

$$\text{top}(N) \simeq L_H^V(N_t).$$

Proof. We have that $\bar{V} \cdot N_t = 0$ by the grading assumption on N . Then there is an epimorphism $p : M_H^V(N_t) \rightarrow N$. Let R be the Jacobson radical of $M_H^V(N_t)$, that is,

$$M_H^V(N_t)/R \simeq L_H^V(N_t).$$

Then p induces a projection $\bar{p} : L_H^V(N_t) \rightarrow N/p(R)$ and hence $N/p(R)$ is semisimple. Also, the homogeneous component of R of degree t is zero. Then $\bar{p}|_{N_t}$ is injective and therefore \bar{p} so is. This implies $L_H^V(N_t)$ is a direct summand of $\text{top}(N)$. On the other hand, $\text{top}(N)$ is a semisimple quotient of $M_H^V(N_t)$. Then $\text{top}(N)$ is a direct summand of $L_H^V(N_t)$. Thus we obtain the desired isomorphism. ■

We now prove that tensoring a simple module by a rigid module yields a semisimple module when H is semisimple.

Theorem 3.8. *Let $\lambda, \mu \in \mathcal{D}(H)\mathcal{M}$ with $L_H^V(\mu) = \mu$ rigid. If $\mu \otimes \lambda$ is a semisimple $\mathcal{D}(H)$ -module, then*

$$L_H^V(\mu) \otimes L_H^V(\lambda) \simeq L_H^V(\mu \otimes \lambda) \simeq L_H^V(\lambda \otimes \mu) \simeq L_H^V(\lambda) \otimes L_H^V(\mu).$$

In particular, $L_H^V(\mu) \otimes L_H^V(\lambda)$ is a semisimple $\mathcal{D}(V, H)$ -module. Moreover, it is a direct sum of simple rigid modules if λ is also rigid.

Proof. We prove only the last isomorphism, the others follow from the fact that $\mathcal{D}(H)\mathcal{M}$ and $\mathcal{D}(V, H)\mathcal{M}$ are braided categories.

We start by pointing out that, as $L_H^V(\lambda) \otimes L_H^V(\mu) = L_H^V(\lambda) \otimes \mu$ is a graded $\mathcal{D}(V, H)$ -module, any simple $\mathcal{D}(V, H)$ -submodule $L_H^V(\nu)$ is a graded submodule by [13, Proposition 3.5]. Without loss of generality, we assume that λ and μ are concentrated in degree 0, since the functors involved commute with the shift of grading. Then the homogeneous components are $L_H^V(\lambda)_n \otimes \mu$ for all $n \leq 0$; in particular, its homogeneous component of degree 0 is $\lambda \otimes \mu$.

We prove first that all highest-weights are in degree 0. Let $L_H^V(v)$ be a simple graded $\mathcal{D}(V, H)$ -submodule of $L_H^V(\lambda) \otimes \mu$ and assume $v \subset L_H^V(\lambda)_n \otimes \mu$. We claim that $n = 0$. Indeed, we pick $\sum_k u_k \otimes n_k \in v$ with $\{n_k\}_{k \in K}$ linearly independent. For $\alpha \in \bar{V}$, we have

$$\begin{aligned} 0 &= \alpha \cdot \left(\sum_k u_k \otimes n_k \right) = \sum_k ((\alpha \cdot u_k) \otimes n_k + (\alpha_{(-1)} \cdot u_k) \otimes (\alpha_{(0)} \cdot n_k)) \\ &= \sum_k (\alpha \cdot u_k) \otimes n_k; \end{aligned}$$

where the first and the last equality hold because v and μ are highest-weights, respectively. Hence $\bar{V} \cdot u_k = 0$ for all k and therefore

$$\bar{V} \cdot (\mathcal{D}(H) \cdot u_k) = \mathcal{D}(H) \cdot (\bar{V} \cdot u_k) = 0, \quad \text{for all } k.$$

Being $\mathcal{D}(H) \cdot u_k$ an graded $\mathcal{D}(H)$ -submodule of $L_H^V(\lambda)$ which is annihilated by the action of \bar{V} , it must be λ . In particular, $u_k \in \lambda$ and v is in degree $n = 0$.

Let $v \subset L_H^V(\lambda) \otimes L_H^V(\mu)$ be a weight in degree 0. Then it is a highest-weight. Moreover, the $\mathcal{D}(V, H)$ -submodule N generated by v is isomorphic to $L_H^V(v)$. Indeed, if $N = v \oplus_{i < 0} N_i$ is not simple, there should be a highest-weight in degree < 0 which is not possible by the paragraph above.

In conclusion, the $\mathcal{D}(V, H)$ -submodule generated by $\lambda \otimes \mu$ is semisimple and isomorphic to $L_H^V(\lambda \otimes \mu)$. We prove next that $\lambda \otimes \mu$ actually generates the whole module

$$L_H^V(\lambda) \otimes L_H^V(\mu)$$

and hence $L_H^V(\lambda) \otimes L_H^V(\mu) \simeq L_H^V(\lambda \otimes \mu)$ as we wanted. For that, we fix bases $\{m_j\}_{j \in J}$ and $\{n_k\}_{k \in K}$ of the weights λ and μ , respectively. Since μ is rigid, we have that $x \cdot n_k = 0$ for any homogeneous element $x \in \mathfrak{B}(V)$ of degree ≥ 1 . Also, because $\mathfrak{B}(V)\#H$ is a graded coalgebra, for such an element x one may write its comultiplication by

$$\Delta(x) = x \otimes 1 + \sum_t y_t \otimes z_t,$$

where the elements z_t are homogeneous of degree ≥ 1 for all t . Hence, for all $j \in J$ and $k \in K$ we have that

$$x \cdot (m_j \otimes n_k) = (x \cdot m_j) \otimes n_k + \sum_t (y_t \cdot m_j) \otimes (z_t \cdot n_k) = (x \cdot m_j) \otimes n_k.$$

Since $L_H^V(\lambda)$ is generated as a $\mathcal{D}(V, H)$ -module by the action of $\mathfrak{B}(V)$ on $\{m_j\}_{j \in J}$, our assertion follows.

Lastly, if λ is also rigid, then $L_H^V(\lambda \otimes \mu)$ is concentrated in degree 0. This implies that $L_H^V(v) = v$ for every weight v of $\lambda \otimes \mu$ and hence v is rigid. \blacksquare

Remark 3.9. The hypothesis of $L_H^V(\mu)$ being rigid is necessary. Otherwise, the tensor product might neither be generated in degree zero nor all its highest-weights be in degree zero. See for instance [11, Theorem 4.1] or [22, Proposition 4.3].

3.3. A recursive strategy for V decomposable

We assume here that $V = W \oplus U$ is decomposable as $\mathcal{D}(H)$ -module with $W \neq 0 \neq U$. This situation arises when H is the group algebra of the dihedral group \mathbb{D}_m . In particular, $\mathfrak{B}(W)$ and $\mathfrak{B}(U)$ are braided graded Hopf subalgebras of $\mathfrak{B}(V)$. Following [2, Section 2.3], we set

$$Z = \text{ad}_c \mathfrak{B}(U)(W) \subset \mathfrak{B}(V),$$

where ad_c is the braided adjoint action of a Hopf algebra in $\mathcal{D}(H)\text{-}\mathcal{M}$. Notice that $W \subseteq Z$. It holds that Z is a Yetter–Drinfeld module over $\mathfrak{B}(U)\#H$ via the adjoint action and the coaction $(\pi_{\mathfrak{B}(U)\#H} \otimes \text{id}) \circ \Delta_{\mathfrak{B}(V)\#H}$, where $\pi_{\mathfrak{B}(U)\#H}$ is the natural projection on $\mathfrak{B}(U)\#H$. Besides, it turns out that

$$\mathfrak{B}(V)\#H \simeq \mathfrak{B}(Z)\#(\mathfrak{B}(U)\#H), \quad (3.8)$$

as Hopf algebras, see *loc. cit.* or [14, Section 8] for details and references.

Naturally, we can apply the techniques described in the previous sections to the bosonization on the right-hand side of (3.8), i.e. $\mathfrak{B}(U)\#H$ and Z playing the role of H and V , respectively. This gives us a new description of $\mathcal{D}(V, H)$ and its simple modules in terms of those over $\mathcal{D}(U, H)$, the Drinfeld double of $\mathfrak{B}(U)\#H$. In this sense, we have

$$\mathcal{D}(Z, U, H) := \mathcal{D}(\mathfrak{B}(Z)\#(\mathfrak{B}(U)\#H)) \simeq \mathcal{D}(\mathfrak{B}(V)\#H) = \mathcal{D}(V, H).$$

Namely, one may consider another \mathbb{Z} -grading on the Drinfeld double $\mathcal{D}(Z, U, H)$ given by $-\deg Z = \deg \bar{Z} = 1$ and $\deg \mathcal{D}(U, H) = 0$. Then

$$\mathcal{D}(Z, U, H) \simeq \mathfrak{B}(Z) \otimes \mathcal{D}(U, H) \otimes \mathfrak{B}(\bar{Z}) \quad (3.9)$$

yields a new triangular decomposition on $\mathcal{D}(V, H)$. Thus, the simple $\mathcal{D}(Z, U, H)$ -modules can be constructed from the simple $\mathcal{D}(U, H)$ -modules as before. Of course, the latter can also be described by the same proceeding. Then, we have the functors

$$\mathcal{D}(H)\text{-}\mathcal{M} \begin{array}{c} \xrightarrow{M_H^U} \\ \xrightarrow{L_H^U} \end{array} \mathcal{D}(U, H)\text{-}\mathcal{M} \begin{array}{c} \xrightarrow{M_{\mathfrak{B}(U)\#H}^Z} \\ \xrightarrow{L_{\mathfrak{B}(U)\#H}^Z} \end{array} \mathcal{D}(Z, U, H)\text{-}\mathcal{M}.$$

For instance, the Verma module associated with the simple module $L_H^U(\lambda)$ is

$$M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)) = \mathcal{D}(Z, U, H) \otimes_{\mathcal{D}(U, H) \otimes \mathfrak{B}(\bar{Z})} L_H^U(\lambda), \quad (3.10)$$

where we consider \bar{Z} acting by zero on $L_H^U(\lambda)$. Observe that $M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ is not necessarily isomorphic to the Verma module $M_H^V(\lambda)$ defined in (3.4). Nevertheless, we show that their heads are isomorphic. This allows us to construct the simple modules in a recursive way.

Lemma 3.10. *Keeping the notation above, we have $M_{\mathfrak{B}(U)\#H}^Z \circ M_H^U \simeq M_H^V$.*

Proof. Let N be a finite-dimensional $\mathcal{D}(H)$ -module. Then both $M_{\mathfrak{B}(U)\#H}^Z(M_H^U(N))$ and $M_H^V(N)$ are generated by N as $\mathcal{D}(V, H)$ -modules. Besides, by definition \bar{Z} and \bar{U} act trivially on N inside $M_{\mathfrak{B}(U)\#H}^Z(M_H^U(N))$. Since \bar{Z} contains \bar{W} as a $\mathcal{D}(H)$ -submodule, it follows that $\bar{V} = \bar{W} \oplus \bar{U}$ also acts trivially on it. Thus, there exists a $\mathcal{D}(V, H)$ -module epimorphism

$$\eta_N : M_H^V(N) \twoheadrightarrow M_{\mathfrak{B}(U)\#H}^Z(M_H^U(N)),$$

which is the identity on N . By Section 3.2.2 (a) and (3.8), we have that

$$\text{Res}(M_{\mathfrak{B}(U)\#H}^Z(M_H^U(N))) \simeq \mathfrak{B}(Z) \otimes \mathfrak{B}(U) \otimes N \simeq \mathfrak{B}(V) \otimes N \simeq \text{Res}(M_H^V(N)),$$

that is both objects have the same dimension. This implies that η_N is in fact an isomorphism. Moreover, as $\eta|_N = \text{id}_N$, we see that

$$\eta_Y \circ M_H^V(f) = M_{\mathfrak{B}(U)\#H}^Z(M_H^U(f)) \circ \eta_X$$

for any morphism $f : X \rightarrow Y$ in $\mathcal{D}(H)\mathcal{M}$. Hence, η defines a natural isomorphism between both functors. \blacksquare

Since $\mathcal{D}(Z, U, H) \simeq \mathcal{D}(V, H)$ as Hopf algebras, we may consider the Verma module $M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ and its head $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ as graded $\mathcal{D}(V, H)$ -modules with the unique grading satisfying that $\deg \lambda = 0$ thanks to [13]. We prove next that there is a commutative diagram

$$\begin{array}{ccc} M_H^V(\lambda) & \longrightarrow & L_H^V(\lambda) \\ \downarrow & & \downarrow \simeq \\ M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)) & \longrightarrow & L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)) \end{array}$$

whose arrows are epimorphisms of $\mathcal{D}(V, H)$ -modules.

Theorem 3.11. *Let $\lambda \in \text{Irr } \mathcal{D}(H)\mathcal{M}$. Then $M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ and $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ are both highest-weight $\mathcal{D}(V, H)$ -modules with highest-weight λ . Moreover,*

$$L_H^V(\lambda) \simeq L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$$

and the homogeneous components of $L_H^V(\lambda)$ and $L_H^U(\lambda)$ satisfy as $\mathcal{D}(H)$ -modules

$$L_H^V(\lambda)_n = \bigoplus_{\substack{n=-i-k(j+1) \\ i, j, k \geq 0}} \mathfrak{B}^k(\text{ad}_c \mathfrak{B}^j(U)(W)) L_H^U(\lambda)_{-i}.$$

Proof. By definition, we know that $\bar{W} \subseteq \bar{Z}$ and \bar{U} act by zero on $\lambda = 1 \otimes \lambda$ inside

$$M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)).$$

That is, λ is a highest-weight. Besides, this weight generates $M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$, and hence also $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$, as $\mathcal{D}(V, H)$ -module. For,

$$\begin{aligned} \mathcal{D}(V, H)\lambda &= \mathfrak{B}(Z)\mathcal{D}(U, H)\mathfrak{B}(\bar{Z})\lambda = \mathfrak{B}(Z)\mathcal{D}(U, H)\lambda \\ &= \mathfrak{B}(Z)L_H^U(\lambda) = M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)). \end{aligned}$$

Then, by the characterization of the highest-weight modules,

$$M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$$

and $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ are quotients of $M_H^V(\lambda)$. Since $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ is a simple module, we get $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda)) \simeq L_H^V(\lambda)$ by Remark 3.2. Finally, by looking at the associated gradation, we deduce the second assertion. ■

We have an analogous result for the socle.

Proposition 3.12. *Let $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{M}$. Then $S_H^V(\lambda) \simeq S_{\mathfrak{B}(U)\#H}^Z(S_H^U(\lambda))$ as $\mathcal{D}(V, H)$ -modules.*

Proof. Let $\lambda_Z = \mathfrak{B}^{n_Z}(Z)$ and $\lambda_U = \mathfrak{B}^{n_U}(U)$ be the homogeneous components of maximum degree of $\mathfrak{B}(Z)$ and $\mathfrak{B}(U)$, respectively. These are one-dimensional and simple as modules over $\mathcal{D}(U, H)$ and $\mathcal{D}(H)$, respectively, recall Section 3.1.1. In particular, $U \cdot \lambda_Z = 0$. Moreover, $\lambda_Z \lambda_U \simeq \lambda_V$ as $\mathcal{D}(H)$ -modules by (3.8).

Using the triangular decomposition (3.9), Theorem 3.5 says that $S_{\mathfrak{B}(U)\#H}^Z(S_H^U(\lambda))$ is the unique simple lowest-weight $\mathcal{D}(Z, U, H)$ -module with lowest-weight $\lambda_Z \otimes S_H^U(\lambda)$. That is, $\lambda_Z \otimes S_H^U(\lambda)$ is a simple $\mathcal{D}(U, H)$ -submodule generating $S_{\mathfrak{B}(U)\#H}^Z(S_H^U(\lambda))$ and $Z \cdot (\lambda_Z \otimes S_H^U(\lambda)) = 0$. In particular, $W \cdot (\lambda_Z \otimes S_H^U(\lambda)) = 0$.

Also by Theorem 3.5, $S_H^U(\lambda)$ is the unique simple lowest-weight $\mathcal{D}(U, H)$ -module with lowest-weight $\lambda_U \lambda$. That is, $\lambda_U \lambda$ is a simple $\mathcal{D}(H)$ -submodule generating $S_H^U(\lambda)$ and $U \cdot \lambda_U \lambda = 0$. Hence $U \cdot (\lambda_Z \otimes \lambda_U \lambda) = 0$. In fact, if $u \in U$, then

$$u \cdot (\lambda_Z \otimes \lambda_U \lambda) = u \cdot \lambda_Z \otimes \lambda_U \lambda + u_{(-1)} \cdot \lambda_Z \otimes u_{(0)} \cdot (\lambda_U \lambda) = 0;$$

for the first equality recall (3.6) and the second one follows from the first paragraph.

In conclusion, $S_{\mathfrak{B}(U)\#H}^Z(S_H^U(\lambda))$ is a simple $\mathcal{D}(V, H)$ -module with lowest-weight

$$\lambda_Z \otimes \lambda_U \lambda \simeq \lambda_V \lambda.$$

Again by Theorem 3.5 it should be isomorphic to $S_H^V(\lambda)$ as desired. ■

We stress that Theorems 3.1 and 3.5, Corollary 3.6, Remarks 3.2, 3.3 and 3.4 and Section 3.2.2 also apply to $L_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$ and $M_{\mathfrak{B}(U)\#H}^Z(L_H^U(\lambda))$, since one may take $\mathfrak{B}(U)\#H$ and Z to play the role of H and V , respectively. We will make use of these remarks under these generalized hypotheses when we consider Nichols algebras over the dihedral groups \mathbb{D}_m . In such a case we refer to them as *the recursive version*.

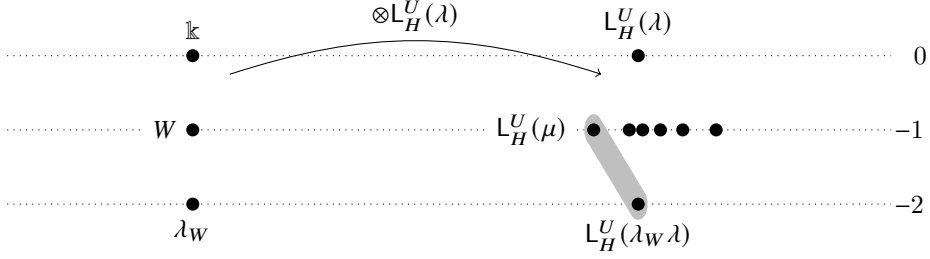


Figure 3. The dots represent the simple $\mathcal{D}(U, H)$ -summands of $\mathfrak{B}(W)$ and $M_{\mathfrak{B}(U)\#H}^W(L_H^U(\lambda))$. Their degrees are indicated on the right. Those in the shadow region form its socle in the case that $\deg L_H^U(\mu) = -1$.

Remark 3.13. The coaction $(\pi_{\mathfrak{B}(U)\#H} \otimes \text{id}) \circ \Delta_{\mathfrak{B}(V)\#H}$ and the adjoint action of H on $W \subset Z$ coincide with its structure in ${}^H_H\mathcal{YD}$. Moreover, if $c_{U,W} \circ c_{W,U} = \text{id}_{W \otimes U}$, then $Z = W$ and hence $\mathfrak{B}(Z) = \mathfrak{B}(W)$ [2, Remark 2.5]. In this case, we have

$$\mathfrak{B}(V)\#H \simeq \mathfrak{B}(W)\#(\mathfrak{B}(U)\#H).$$

On the other hand, if W is a simple $\mathcal{D}(H)$ -module, then $Z \simeq L_H^U(W)$ by [2, Proposition 2.10]. Thus, W is a rigid $\mathcal{D}(U, H)$ -module if $c_{U,W} \circ c_{W,U} = \text{id}_{W \otimes U}$.

Remark 3.14. Assume that $Z = W = L_H^U(W)$ is simple and rigid; e.g. when $c_{U,W} \circ c_{W,U} = \text{id}_{W \otimes U}$ by Remark 3.13. The Nichols algebras appearing in the present work satisfy this property. By applying Theorem 3.8, we have that $W^{\otimes k}$ is a direct sum of simples rigid modules and hence so is $\mathfrak{B}(W) = \mathfrak{B}(Z)$. Therefore $M_{\mathfrak{B}(U)\#H}^W(L_H^U(\lambda))$ and its head are semisimple as $\mathcal{D}(U, H)$ -modules for any $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{M}$. The homogeneous components of its head satisfy for all $n \leq 0$ that

$$L_H^V(\lambda)_n = \bigoplus_{k=0}^{-n} \mathfrak{B}^k(W) L_H^U(\lambda)_{n+k}.$$

Finally, we point out that the Hopf subalgebra generated by W and H is $\mathcal{D}(W, H)$.

3.3.1. A recursive example. Keep the notation and the assumptions of Remark 3.14. Assume further that $W \in {}^H_H\mathcal{YD}$ is a simple two-dimensional module with basis $\{w_+, w_-\}$ and braiding *-flip*. For instance, these hypotheses are satisfied by exterior algebras of vector spaces of even dimension; such is the case for $H = \mathbb{k}\mathbb{D}_m$. Then we have that $L_H^U(W) = W$ is rigid and $\mathfrak{B}(W) = \bigwedge W = \mathbb{k} \oplus W \oplus \mathbb{k}\{w_+w_-\}$.

Let $\lambda \in \text{Irr}_{\mathcal{D}(H)}\mathcal{M}$ be such that $W \otimes \lambda$ is semisimple. We explain below how to describe the socle of $M_{\mathfrak{B}(U)\#H}^W(L_H^U(\lambda))$ using the recursive version of Corollary 3.6. See Figure 3.

First, we observe that $W \otimes L_H^U(\lambda) \simeq L_H^U(W \otimes \lambda)$ by Theorem 3.8. Hence, as $\mathcal{D}(U, H)$ -module, $M_{\mathfrak{B}(U)\#H}^W(L_H^U(\lambda)) \simeq L_H^U(\lambda) \oplus L_H^U(W \otimes \lambda) \oplus L_H^U(\lambda_W \lambda)$ is semisimple. Pick the

homogeneous simple $\mathcal{D}(U, H)$ -module $\mathbb{L}_H^U(\mu)$ of $M_{\mathfrak{B}(U)\#H}^W(\mathbb{L}_H^U(\lambda))$ of minimum degree such that $\bar{W} \cdot \mathbb{L}_H^U(\mu) = 0$. Note that it is enough to check for which homogeneous summand $\mathbb{L}_H^U(\mu)$ it holds that $\bar{W} \cdot \mu = 0$, since the Verma module is semisimple and the action of \bar{W} is a morphism of $\mathcal{D}(U, H)$ -modules, by the recursive version of Section 3.2.2. A similar reasoning can be made using the recursive versions of Remarks 3.3 and 3.4. That is, it is enough to check that the elements Φ act trivially on λ (resp., Θ acts non trivially on λ) to conclude that they also act trivially (resp., non-trivially) on all $\mathbb{L}_H^U(\lambda)$. Thus, as in Section 3.2.4, we have three possibilities:

- If $\deg \mathbb{L}_H^U(\mu) = 0$, then $\mu = \lambda$ and $\mathbb{L}_H^V(\lambda) = M_{\mathfrak{B}(U)\#H}^W(\mathbb{L}_H^U(\lambda))$ is simple and projective as $\mathcal{D}(V, H)$ -module.
- If $\deg \mathbb{L}_H^U(\mu) = -1$, then the socle of $M_{\mathfrak{B}(U)\#H}^W(\mathbb{L}_H^U(\lambda))$ decomposes into $\mathbb{L}_H^U(\mu)[-1] \oplus \mathbb{L}_H^U(\lambda_W \lambda)[-2]$ as $\mathcal{D}(U, H)$ -module. To find μ , one has to decompose $W \otimes \lambda$ as a direct sum of weights and then determine the one that is annihilated by \bar{W} . In this case, one may deduce that $W \otimes \lambda = \mu \oplus \bar{\lambda}$ for some weight $\bar{\lambda}$ and hence

$$\mathbb{L}_H^V(\lambda) = \mathbb{L}_H^U(\lambda) \oplus \mathbb{L}_H^U(\bar{\lambda})[-1]$$

as $\mathcal{D}(U, H)$ -modules. We leave the computation for the interested reader.

- If $\deg \mathbb{L}_H^U(\mu) = -2$, then $\mathbb{L}_H^U(\lambda_W \lambda)[-2]$ is the socle of $M_{\mathfrak{B}(U)\#H}^W(\mathbb{L}_H^U(\lambda))$ and hence it is a simple $\mathcal{D}(V, H)$ -module over which W and \bar{W} act trivially.

4. The dihedral groups framework

From now on, we fix a natural number $m \geq 12$ divisible by 4 and an m -th primitive root of unity ω . We also set $n = \frac{m}{2}$.

The dihedral group of order $2m$ is presented by generators and relations by

$$\mathbb{D}_m = \langle x, y \mid x^2, y^m, xyxy \rangle.$$

It has $n + 3$ conjugacy classes: $\mathcal{O}_e = \{e\}$ with e the identity, $\mathcal{O}_{y^n} = \{y^n\}$, $\mathcal{O}_x = \{xy^j : j \text{ even}\}$, $\mathcal{O}_{xy} = \{xy^j : j \text{ odd}\}$ and $\mathcal{O}_{y^i} = \{y^i, y^{-i}\}$ for $1 \leq i \leq n - 1$.

The algebra of functions $\mathbb{k}^{\mathbb{D}_m}$ is the dual Hopf algebra of $\mathbb{k}\mathbb{D}_m$. We denote by $\{\delta_t\}_{t \in \mathbb{D}_m}$ the dual basis of the basis of $\mathbb{k}\mathbb{D}_m$ given by the group-like elements, i.e. $\delta_t(s) = \delta_{t,s}$ for all $t, s \in \mathbb{D}_m$. The comultiplication and the counit of these elements are

$$\Delta(\delta_t) = \sum_{s \in \mathbb{D}_m} \delta_s \otimes \delta_{s^{-1}t} \quad \text{and} \quad \varepsilon(\delta_t) = \delta_{t,e} \quad \text{for all } t \in \mathbb{D}_m,$$

respectively.

We denote by $\mathcal{D}\mathbb{D}_m$ the Drinfeld double of $\mathbb{k}\mathbb{D}_m$. Since $\mathbb{k}^{\mathbb{D}_m}$ is a commutative algebra, $\mathbb{k}^{\mathbb{D}_m} = (\mathbb{k}\mathbb{D}_m)^{\text{op}}$ and consequently $\mathbb{k}^{\mathbb{D}_m}$ and $\mathbb{k}\mathbb{D}_m$ are Hopf subalgebras of $\mathcal{D}\mathbb{D}_m$. Thus, the algebra structure of $\mathcal{D}\mathbb{D}_m$ is completely determined by the equality

$$\delta_{tst^{-1}} = t\delta_s \quad \text{for all } s, t \in \mathbb{D}_m.$$

In this case, the R -matrix reads $R = \sum_{t \in \mathbb{D}_m} \delta_t \otimes t \in \mathcal{D}\mathbb{D}_m \otimes \mathcal{D}\mathbb{D}_m$.

4.1. The weights of \mathcal{DD}_m

It is well known that the simple modules over the Drinfeld double of a group algebra are classified by the conjugacy classes of the group and irreducible representations of their centralizers, cf. [4] and references therein. Namely, for $g \in \mathbb{D}_m$, write \mathcal{O}_g for its conjugacy class and \mathcal{C}_g for its centralizer in \mathbb{D}_m . Let (U, ϱ) be an irreducible representation of \mathcal{C}_g . The $\mathbb{k}\mathbb{D}_m$ -module induced by (U, ϱ) ,

$$M(g, \varrho) = \text{Ind}_{\mathcal{C}_g}^{\mathbb{D}_m} U = \mathbb{k}\mathbb{D}_m \otimes_{\mathbb{k}\mathcal{C}_g} U,$$

is a \mathcal{DD}_m -module with the $\mathbb{k}\mathbb{D}_m$ -action defined by

$$f \cdot (t \otimes_{\mathbb{k}\mathcal{C}_g} u) = \langle f, tgt^{-1} \rangle t \otimes_{\mathbb{k}\mathcal{C}_g} u, \quad \text{for all } f \in \mathbb{k}\mathbb{D}_m, t \in \mathbb{D}_m \text{ and } u \in U.$$

Then the set Λ consisting of the modules $M(g, \varrho)$'s is a set of representative of simple \mathcal{DD}_m -modules up to isomorphism, that is

$$\Lambda = \text{Irr}_{\mathcal{DD}_m} \mathcal{M}.$$

It is worth noting that a $\mathbb{k}\mathbb{D}_m$ -action on a vector space V is the same as a \mathbb{D}_m -grading. In this sense, a left \mathcal{DD}_m -module V (or equivalently a left Yetter–Drinfeld module over \mathbb{D}_m) is a $\mathbb{k}\mathbb{D}_m$ -module with a \mathbb{D}_m -grading that is compatible with the conjugation in \mathbb{D}_m . In our example, we have that U is concentrated in degree g and the \mathbb{D}_m -degree of $t \otimes_{\mathbb{k}\mathcal{C}_g} u$ in $M(g, \varrho)$ is tgt^{-1} . The action of $f \in \mathbb{k}\mathbb{D}_m$ is performed via the evaluation on the degree. We will denote by $M[s]$ the homogeneous component of degree $s \in \mathbb{D}_m$ of a \mathcal{DD}_m -module. Although this notation coincides with the shift of a grading, we believe that this would not confuse the reader since in the latter case s is an integer and here is an element of \mathbb{D}_m .

In the following we recall the description of the simple \mathcal{DD}_m -modules according to the set of conjugacy classes. We present them by fixing a basis and by describing the action of x, y and the \mathbb{D}_m -grading. We use symbols like $|\mathbf{w}\rangle$ to denote elements of a particular basis for each simple module. For more details, see [12].

4.1.1. The modules $M(e, \varrho)$. Let e be the identity element in \mathbb{D}_m . Since $\mathcal{C}_e = \mathbb{D}_m$, we use the simple representations of \mathbb{D}_m to describe the simple \mathcal{DD}_m -modules. According to the amount of conjugacy classes of \mathbb{D}_m , these are 4 one-dimensional, say $\chi_1, \chi_2, \chi_3, \chi_4$, and $n - 1$ two-dimensional, which we denote by ρ_ℓ , $1 \leq \ell < n$. For $1 \leq i \leq 4$ and $1 \leq \ell \leq n - 1$, the simple \mathcal{DD}_m -modules are:

▷ $M(e, \chi_i) = \mathbb{k}\{|\mathbf{u}_i\rangle\}$ with $M(e, \chi_i) = M(e, \chi_i)[e]$ and

$$\begin{aligned} x \cdot |\mathbf{u}_1\rangle &= |\mathbf{u}_1\rangle, & y \cdot |\mathbf{u}_1\rangle &= |\mathbf{u}_1\rangle; \\ x \cdot |\mathbf{u}_2\rangle &= -|\mathbf{u}_2\rangle, & y \cdot |\mathbf{u}_2\rangle &= |\mathbf{u}_2\rangle; \\ x \cdot |\mathbf{u}_3\rangle &= |\mathbf{u}_3\rangle, & y \cdot |\mathbf{u}_3\rangle &= -|\mathbf{u}_3\rangle; \\ x \cdot |\mathbf{u}_4\rangle &= -|\mathbf{u}_4\rangle, & y \cdot |\mathbf{u}_4\rangle &= -|\mathbf{u}_4\rangle. \end{aligned}$$

▷ $M(e, \rho_\ell) = \mathbb{k}\{|\pm, \ell\rangle, |-, \ell\rangle\}$ with $M(e, \rho_\ell) = M(e, \rho_\ell)[e]$ and

$$x \cdot |\pm, \ell\rangle = |\mp, \ell\rangle, \quad y \cdot |\pm, \ell\rangle = \omega^{\pm\ell} |\pm, \ell\rangle.$$

Note that $M(e, \chi_1)$ is given by the counit of $\mathcal{D}\mathbb{D}_m$. To shorten notation, we write

$$|\pm\rangle = |\pm, \ell\rangle$$

when the parameter ℓ is clear from the context.

4.1.2. The modules $M(y^n, \varrho)$. Since $m = 2n$, the element y^n is central in \mathbb{D}_m . Therefore the simple $\mathcal{D}\mathbb{D}_m$ -modules associated with y^n are given by the simple representations of \mathbb{D}_m . As \mathbb{D}_m -modules they coincide with the ones given in Section 4.1.1, but these are concentrated in degree y^n instead of e . Explicitly, for $1 \leq i \leq 4$ and $1 \leq \ell \leq n-1$ these are

▷ $M(y^n, \chi_i) = \mathbb{k}\{|\mathbf{u}_i, \mathbf{n}\rangle\}$ with $M(y^n, \chi_i) = M(y^n, \chi_i)[y^n]$ and $M(y^n, \chi_i) \simeq M(e, \chi_i)$ as \mathbb{D}_m -modules via $|\mathbf{u}_i, \mathbf{n}\rangle \mapsto |\mathbf{u}_i\rangle$.

▷ $M_\ell := M(y^n, \rho_\ell) = \mathbb{k}\{|\pm, \mathbf{n}, \ell\rangle, |-, \mathbf{n}, \ell\rangle\}$ with $M(y^n, \rho_\ell) = M(y^n, \rho_\ell)[y^n]$,

$$x \cdot |\pm, \mathbf{n}, \ell\rangle = |\mp, \mathbf{n}, \ell\rangle \quad \text{and} \quad y \cdot |\pm, \mathbf{n}, \ell\rangle = \omega^{\pm\ell} |\pm, \mathbf{n}, \ell\rangle.$$

Again, we write $|\pm\rangle = |\pm, \mathbf{n}, \ell\rangle$ when both the parameters ℓ and n are clear from the context. The latter are the modules M_ℓ of [12, Section 2A1].

4.1.3. The modules $M(y^i, \varrho)$. Let $1 \leq i \leq n-1$. The conjugacy class of y^i is $\{y^i, y^{-i}\}$ and its centralizer \mathcal{C}_{y^i} is the subgroup $\langle y \rangle \simeq \mathbb{Z}_m$ whose simple representations are given by the characters

$$\chi_{(k)}(y) = \omega^k$$

for $0 \leq k \leq m-1$. So, the simple $\mathcal{D}\mathbb{D}_m$ -modules associated with y^i are

▷ $M_{i,k} := M(y^i, \chi_{(k)}) = \mathbb{k}\{|\pm, \mathbf{i}, \mathbf{k}\rangle, |-, \mathbf{i}, \mathbf{k}\rangle\}$ with $|\pm, \mathbf{i}, \mathbf{k}\rangle \in M_{i,k}[y^{\pm i}]$,

$$x \cdot |\pm, \mathbf{i}, \mathbf{k}\rangle = |\mp, \mathbf{i}, \mathbf{k}\rangle \quad \text{and} \quad y \cdot |\pm, \mathbf{i}, \mathbf{k}\rangle = \omega^{\pm k} |\pm, \mathbf{i}, \mathbf{k}\rangle.$$

Note that here the simple module is not concentrated in a single degree, in fact

$$\dim M_{i,k} = 2 \quad \text{and} \quad M_{i,k} = M_{i,k}[y^{-i}] \oplus M_{i,k}[y^{+i}].$$

These are the modules $M_{i,k}$ of [12, Section 2A2]. As before, we simply write

$$|\pm\rangle = |\pm, \mathbf{i}, \mathbf{k}\rangle$$

when the context allows us to simplify notation. Note that the simple modules M_ℓ can be describe as $M_{n,\ell}$, where the elements are concentrated in degree

$$y^n = y^{-n}.$$

Notation 4.1. Given $1 \leq i \leq n$ and $0 \leq k \leq m-1$, we set $M_{i,k} = M(y^i, \chi_{(k)})$ for $i \neq n$, and $M_{n,k} = M(y^n, \rho_k)$.

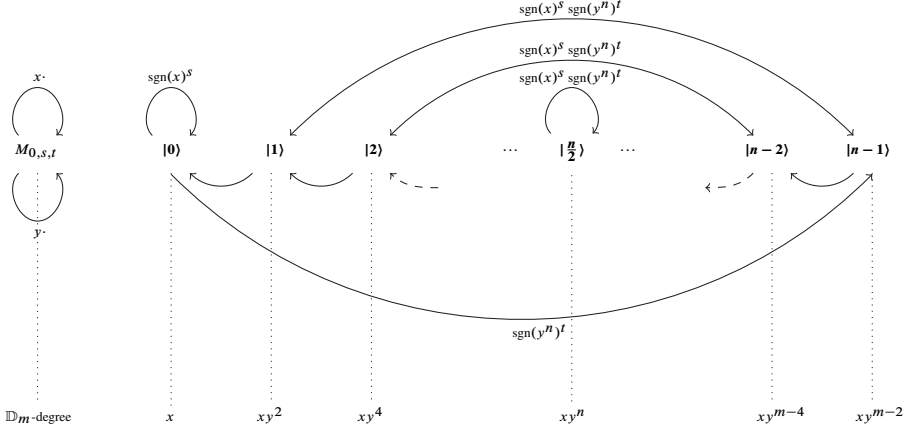


Figure 4. The simple module $M_{0,s,t}$ associated with the conjugacy class of x .

4.1.4. The modules $M(x, \varrho)$. The conjugacy class of x is $\{xy^{2j} \mid j \in \mathbb{Z}_n\}$ and its centralizer is given by the subgroup

$$\langle x \rangle \oplus \langle y^n \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

The irreducible representations are given by the characters $\text{sgn}^s \otimes \text{sgn}^t$, $s, t \in \mathbb{Z}_2$, where

$$\text{sgn}(x) = \text{sgn}(y^n) = -1$$

are the corresponding sgn representation of the \mathbb{Z}_2 summand. Hence, the simple $\mathcal{D}\mathbb{D}_m$ -modules are

▷ $M_{0,s,t} := M(x, \text{sgn}^s \otimes \text{sgn}^t) = \mathbb{k}\{|j, \mathbf{0}, s, t\} \mid j \in \mathbb{Z}_n\}$, see Figure 4, with

$$\begin{aligned} x \cdot |0, \mathbf{0}, s, t\rangle &= \text{sgn}^s(x)|0, \mathbf{0}, s, t\rangle, \\ x \cdot |j, \mathbf{0}, s, t\rangle &= \text{sgn}^s(x) \text{sgn}^t(y^n)|n-j, \mathbf{0}, s, t\rangle, \quad \text{for all } j \neq 0, \\ y \cdot |0, \mathbf{0}, s, t\rangle &= \text{sgn}^t(y^n)|n-1, \mathbf{0}, s, t\rangle, \\ y \cdot |j, \mathbf{0}, s, t\rangle &= |j-1, \mathbf{0}, s, t\rangle, \quad \text{for all } j \neq 0, \end{aligned}$$

and

$$|j, \mathbf{0}, s, t\rangle \in M(x, \text{sgn}^s \otimes \text{sgn}^t)[xy^{2j}], \quad \text{for all } j \in \mathbb{Z}_n.$$

In particular, $\dim M_{0,s,t} = n$ and $M_{0,s,t} = \bigoplus_{j \in \mathbb{Z}_n} M_{0,s,t}[xy^{2j}]$ as \mathbb{D}_m -graded module. We write $|j\rangle = |j, \mathbf{0}, s, t\rangle$ when the notation is clear from the context.

4.1.5. The modules $M(xy, \varrho)$. The conjugacy class of xy is $\{xy^{2j+1} \mid j \in \mathbb{Z}_n\}$. Its centralizer is the subgroup

$$\langle xy \rangle \oplus \langle y^n \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

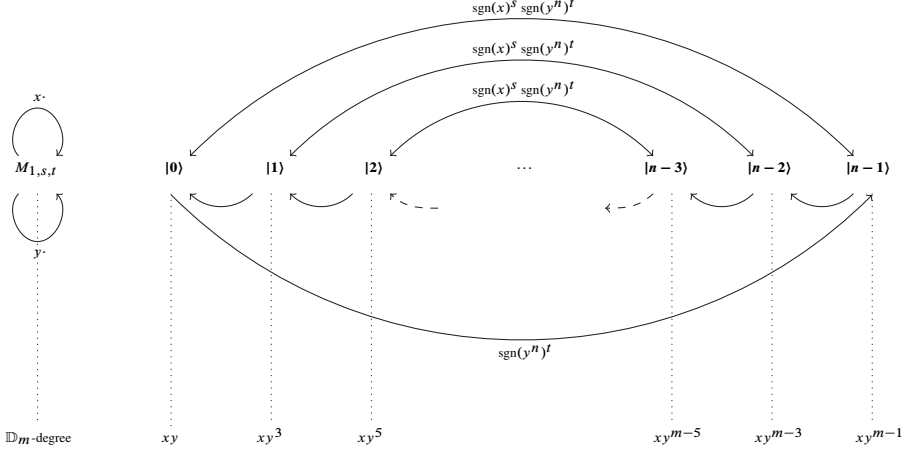


Figure 5. The simple module $M_{1,s,t}$ associated with the conjugacy class of xy .

whose simple representations are given by the characters $\text{sgn}^s \otimes \text{sgn}^t$, $s, t \in \mathbb{Z}_2$, where

$$\text{sgn}(xy) = \text{sgn}(y^n) = -1$$

are the corresponding sgn representation of the \mathbb{Z}_2 summand. Thus, the simple \mathcal{DD}_m -modules are

▷ $M_{1,s,t} := M(xy, \text{sgn}^s \otimes \text{sgn}^t) = \mathbb{k}\{|j, \mathbf{1}, s, t\rangle \mid j \in \mathbb{Z}_n\}$, see Figure 5, with

$$y \cdot |0, \mathbf{1}, s, t\rangle = \text{sgn}^t(y^n)|n-1, \mathbf{1}, s, t\rangle,$$

$$y \cdot |j, \mathbf{1}, s, t\rangle = |j-1, \mathbf{1}, s, t\rangle,$$

$$x \cdot |j, \mathbf{1}, s, t\rangle = \text{sgn}^s(xy) \text{sgn}^t(y^n)|n-j-1, \mathbf{1}, s, t\rangle, \quad \text{for all } j \in \mathbb{Z}_n,$$

and

$$|j, \mathbf{1}, s, t\rangle \in M(xy, \text{sgn}^s \otimes \text{sgn}^t)[xy^{2j+1}], \quad \text{for all } j \in \mathbb{Z}_n.$$

Here, $\dim M_{1,s,t} = n$ and $M_{1,s,t} = \bigoplus_{j \in \mathbb{Z}_n} M_{1,s,t}[xy^{2j+1}]$ as \mathbb{D}_m -graded module. Eventually, we might simply write $|j\rangle = |j, \mathbf{1}, s, t\rangle$.

4.2. Some tensor products of weights

The category of \mathcal{DD}_m -modules is semisimple. As such, any tensor product of two weights can be written as a direct sum of weights. In order to perform our study on simple modules over doubles of bosonizations of Nichols algebras, which is carried out in Section 5, by dealing with Verma modules, we need to know the direct summands of the following products of simple \mathcal{DD}_m -modules.

In the following two lemmata, we decompose the tensor product of the simple modules $M_{r,s,t}$ with $r, s, t \in \mathbb{Z}_2$ as in Sections 4.1.4 and 4.1.5 with some other families of simple modules.

Lemma 4.2. *Let $M_{i,k}$ be a simple \mathcal{DD}_m -module with $1 \leq i \leq n$, $0 \leq k \leq m-1$ as in Notation 4.1. Then*

$$M_{i,k} \otimes M_{r,s,t} \simeq M_{r+i,s+1+\delta_{i,n}t,t+k} \oplus M_{r+i,s+\delta_{i,n}t,t+k},$$

as \mathcal{DD}_m -modules; here we write $r+i$, $s+1+\delta_{i,n}t$, and $t+k$ for their classes in \mathbb{Z}_2 . Moreover, the simple submodules inside the tensor product are given by

$$N_{\pm} = \mathbb{k}\{y^a \cdot n_{\pm} : 0 \leq a \leq n-1\},$$

where $n_{\pm} = \omega^{rk}|-) \otimes |\mathbf{0}\rangle \pm |+\rangle \otimes |i\rangle$, with $|n\rangle = |\mathbf{0}\rangle$ if $i = n$, and

$$N_+ \simeq M_{r+i,s+1+\delta_{i,n}t,t+k} \quad \text{and} \quad N_- \simeq M_{r+i,s+\delta_{i,n}t,t+k}.$$

Proof. We prove first the case $i < n$ and $r = 0$, i.e. $M_{0,s,t}$ is as in Section 4.1.4. Let us show that the subspaces N_{\pm} are simple \mathcal{DD}_m -modules. The elements n_{\pm} are eigenvectors of y^n with eigenvalue $\text{sgn}^{t+k}(y^n)$ because

$$y^n \cdot (|\pm\rangle \otimes |\mathbf{0}\rangle) = (-1)^k \text{sgn}^t(y^n)|\pm\rangle \otimes |\mathbf{0}\rangle$$

and $\omega^n = \omega^{-n} = -1$. Also, it is straightforward to check that

$$x \cdot n_{\pm} = \mp \text{sgn}^s(x)y^i \cdot n_{\pm}. \quad (4.1)$$

This implies that

$$N_{\pm} = \mathbb{k}\{y^a \cdot n_{\pm} : 0 \leq a \leq n-1\}$$

are \mathbb{D}_m -submodules of $M_{i,k} \otimes M_{r,s,t}$. Moreover, the elements n_{\pm} are homogeneous of the same degree for

$$\begin{aligned} \deg(|-\rangle \otimes |\mathbf{0}\rangle) &= \deg|-\rangle \deg|\mathbf{0}\rangle = y^{-i}x = xy^i, \\ \deg(|+\rangle \otimes |i\rangle) &= \deg|+\rangle \deg|i\rangle = y^i xy^{2i} = xy^i. \end{aligned}$$

Hence $\deg y^a \cdot n_{\pm} = y^a x y^i y^{-a} = x y^{i-2a}$ for all $0 \leq a \leq n-1$. This implies that in fact N_{\pm} are \mathcal{DD}_m -modules with $\dim N_{\pm} = n$. Moreover, a direct check shows that they are isomorphic to the simple \mathcal{DD}_m -modules displayed in Sections 4.1.4 and 4.1.5, depending on the parity of i . One way to distinguish these modules is by looking at the eigenvalues of the action of y^n and x on the homogeneous component of \mathbb{D}_m -degree x if $i = 2z$ is even, or the action of y^n and xy on the homogeneous component of \mathbb{D}_m -degree xy if $i = 2z + 1$ is odd. In the case of N_{\pm} , these homogeneous components are spanned by $y^z \cdot n_{\pm}$, respectively.

If $i = 2z$ is even, then $x \cdot (y^z \cdot n_{\pm}) = \mp \text{sgn}^s(x)(y^z \cdot n_{\pm})$ by (4.1). Hence

$$M_{0,s,t+k} \simeq N_- \quad \text{and} \quad M_{0,s+1,t+k} \simeq N_+.$$

If $i = 2z + 1$ is odd, then $(xy) \cdot (y^z \cdot n_{\pm}) = \mp \text{sgn}^s(x)(y^z \cdot n_{\pm})$ by (4.1). Hence

$$M_{1,s,t+k} \simeq N_- \quad \text{and} \quad M_{1,s+1,t+k} \simeq N_+.$$

In both cases the submodules are simple and non-isomorphic. Therefore $N_+ \cap N_- = \{0\}$ and consequently, $M_{i,k} \otimes M_{0,s,t} = N_- \oplus N_+ \simeq M_{i,s,t+k} \oplus M_{i,s+1,t+k}$.

The strategy to prove the case $i < n$ and $r = 1$ is similar. We still have that $y^n \cdot n_{\pm} = \text{sgn}^{t+k}(y^n)n_{\pm}$ and, instead of (4.1), we have that $x \cdot n_{\pm} = \mp \text{sgn}^s(xy)(y^{i+1} \cdot n_{\pm})$. In this case, both n_{\pm} are homogeneous of degree xy^{i+1} . For i even, we have $N_- \simeq M_{1,s,t+k}$ and $N_+ \simeq M_{1,s+1,t+k}$, meanwhile for i odd, we have $N_- \simeq M_{0,s,t+k}$ and $N_+ \simeq M_{0,s+1,t+k}$. We leave the details for the reader.

The proof for $i = n$ follows *mutatis mutandis* from the paragraphs above. \blacksquare

We end this subsection with the following lemma.

Lemma 4.3. *Let $M(e, \chi_2)$ be a simple $\mathcal{D}\mathbb{D}_m$ -modules as in Section 4.1.1. Then*

$$M(e, \chi_2) \otimes M_{r,s,t} \simeq M_{r,s+1,t},$$

as $\mathcal{D}\mathbb{D}_m$ -modules, where we write $s + 1$ for its class in \mathbb{Z}_2 .

Proof. Straightforward. For instance, since $M(e, \chi_2) = M(e, \chi_2)[e]$ we have that

$$(M(e, \chi_2) \otimes M_{r,s,t})[xy^{2j+r}] = M(e, \chi_2) \otimes (M_{r,s,t}[xy^{2j+r}])$$

for all $0 \leq j \leq n - 1$. Also, as the action on $M(e, \chi_2)$ is given by $x \cdot |\mathbf{u}_2\rangle = -|\mathbf{u}_2\rangle$ and $y \cdot |\mathbf{u}_2\rangle = |\mathbf{u}_2\rangle$, the lemma follows easily by the definition of the action on the tensor product. \blacksquare

4.3. Finite-dimensional Nichols algebras over Dihedral groups

Here we recall the classification of finite-dimensional Nichols algebras in ${}_{\mathbb{k}\mathbb{D}_m}^{\mathbb{k}\mathbb{D}_m}\mathcal{Y}\mathcal{D}$, or equivalently in $\mathcal{D}\mathbb{D}_m\mathcal{M}$. Roughly speaking, they are all given by exterior algebras of direct sums of some families of simple $\mathcal{D}\mathbb{D}_m$ -modules $M_{i,k}$, recall Notation 4.1.

The classification in [12, Theorem A] is given in terms of direct sums of three families of simple modules. To shorten notation, we present them below in just one family by changing slightly the description.

Notation 4.4. Let \mathcal{I} be the family of all finite multisets $\{(i_1, k_1), \dots, (i_r, k_r)\}$ of pairs such that $1 \leq i_s \leq n$, $0 \leq k_s \leq m - 1$ and $\omega^{i_s k_t} = -1$ for all $1 \leq s, t \leq r$. For $I \in \mathcal{I}$, we define

$$M_I = \bigoplus_{(i,k) \in I} M_{i,k}.$$

Observe that the families \mathcal{I} , \mathcal{L} and \mathcal{K} defined in [12] fit in the description above. Indeed, if there is a pair (n, ℓ) in a sequence $I \in \mathcal{I}$ and $(i, k) \in I$, then ℓ and k must be odd because $\omega^{n\ell} = (-1)^\ell = -1 = \omega^{nk} = (-1)^k$.

Theorem 4.5. [12] *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in $\mathcal{D}\mathbb{D}_m\mathcal{M}$. Then $M \simeq M_I$ for some $I \in \mathcal{I}$ and $\mathfrak{B}(M) \simeq \bigwedge M$.*

Remark 4.6. We stress that if $V = M_I = W \oplus U$ is decomposable, then it satisfies Remarks 3.13 and 3.14. In particular,

$$\mathfrak{B}(V)\#\mathbb{k}\mathbb{D}_m \simeq \mathfrak{B}(W)\#(\mathfrak{B}(U)\#\mathbb{k}\mathbb{D}_m) \quad \text{and} \quad Z = W = \mathbb{L}_H^U(W).$$

4.4. Drinfeld doubles of bosonizations of Nichols algebras over \mathbb{D}_m

Here we present by generators and relations the Drinfeld double of the bosonization of a finite-dimensional Nichols algebra over \mathbb{D}_m by specifying the recipe given in Section 3.1.4. For that purpose, we need to set up some notation.

Let V be a $\mathcal{D}\mathbb{D}_m$ -module with $\dim \mathfrak{B}(V) < \infty$. We write

$$\mathcal{D}(V) = \mathcal{D}(\mathfrak{B}(V)\#\mathbb{k}\mathbb{D}_m)$$

and \bar{V} for the dual object of V as in Section 3.1.3. By Theorem 4.5, we can fix a decomposition $V = \bigoplus_{(i,k) \in I} M_{i,k}$ and the orthogonal decomposition

$$\bar{V} = \bigoplus_{(i,k) \in I} \overline{M_{i,k}}.$$

Given a (two-dimensional) direct summand $M_{i,k}$, we write v_+ and v_- the elements of the basis $\{|+\rangle, |-\rangle\}$ given in Section 4.1.2 or Section 4.1.3, as appropriate. So, we have

$$x \cdot v_{\pm} = v_{\mp}, \quad y \cdot v_{\pm} = \omega^{\pm k} v_{\pm} \quad \text{for } v_{\pm} \in M_{i,k}[y^{\pm i}].$$

Also, we denote by α_+ and α_- the elements in $\overline{M_{i,k}}$ satisfying $\langle \alpha_{\bullet}, v_{\circ} \rangle = \delta_{\bullet, \circ}$ where $\bullet, \circ \in \{+, -\}$. That is, $\{\alpha_{\pm}\}$ and $\{v_{\pm}\}$ are dual bases. Then the action of $\mathcal{D}\mathbb{D}_m$ on these elements is determined by

$$x \cdot \alpha_{\pm} = \alpha_{\mp}, \quad y \cdot \alpha_{\pm} = \omega^{\mp k} \alpha_{\pm} \quad \text{for } \alpha_{\pm} \in \overline{M_{i,k}}[y^{\mp i}].$$

Thus, $\overline{M_{i,k}} \simeq M_{i,k}$ as $\mathcal{D}\mathbb{D}_m$ -modules via the assignment $\alpha_{\pm} \mapsto v_{\mp}$. The $\mathcal{D}\mathbb{D}_m$ -coactions defined by the functors F_R and $F_{R^{-1}}$ on $M_{i,k}$ and $\overline{M_{i,k}}$, recall (3.1), are

$$(v_{\pm})_{(-1)} \otimes (v_{\pm})_{(0)} = y^{\pm i} \otimes v_{\pm} \tag{4.2}$$

and

$$(\alpha_{\pm})_{(-1)} \otimes (\alpha_{\pm})_{(0)} = \sum_{s=0}^{m-1} \omega^{\mp sk} \delta_{y^{-s}} \otimes \alpha_{\pm} + \omega^{\mp sk} \delta_{y^{-s}x} \otimes \alpha_{\mp}.$$

Proposition 4.7. *As an algebra, $\mathcal{D}(V)$ is generated by the elements of V , \bar{V} , \mathbb{D}_m and $\mathbb{k}\mathbb{D}_m$ subject to relations (4.3)–(4.9) below.*

- For $s, t \in \mathbb{D}_m$ and $z \in V \cup \bar{V}$,

$$\delta_{tst^{-1}t} = t\delta_s \tag{4.3}$$

$$tz = (t \cdot z)t, \quad \delta_t z = \sum_{s \in \mathbb{D}_m} (\delta_s \cdot z)\delta_{s^{-1}t}. \tag{4.4}$$

- For $v_{\pm} \in M_{i,k}$ and $\alpha_{\pm} \in \overline{M_{i,k}}$,

$$\alpha_+ v_+ = -v_+ \alpha_+ + \Phi_{++} \quad \text{with } \Phi_{++} = 1 - \sum_{s=0}^{m-1} \omega^{sk} \delta_{y^s} y^i, \quad (4.5)$$

$$\alpha_+ v_- = -v_- \alpha_+ + \Phi_{+-} \quad \text{with } \Phi_{+-} = - \sum_{s=0}^{m-1} \omega^{-sk} \delta_{x^s} y^{-i}, \quad (4.6)$$

$$\alpha_- v_+ = -v_+ \alpha_- + \Phi_{-+} \quad \text{with } \Phi_{-+} = - \sum_{s=0}^{m-1} \omega^{sk} \delta_{x^s} y^i, \quad (4.7)$$

$$\alpha_- v_- = -v_- \alpha_- + \Phi_{--} \quad \text{with } \Phi_{--} = 1 - \sum_{s=0}^{m-1} \omega^{-sk} \delta_{y^s} y^{-i}. \quad (4.8)$$

- If $z, w \in V$, $z, w \in \bar{V}$ or $w \in V$ and $z \in \bar{V}$ are in orthogonal direct summands,

$$zw = -wz. \quad (4.9)$$

Proof. We briefly explain why these relations hold. Relation (4.3) is the commutation rule in $\mathcal{D}\mathbb{D}_m$. The commutation rules (4.4) are given by the bosonizations $\mathfrak{B}(V)\#\mathcal{D}\mathbb{D}_m$ and $\mathfrak{B}(\bar{V})\#\mathcal{D}\mathbb{D}_m$. The relations (4.5)–(4.8) and (4.9) for generators in orthogonal direct summands follow from (3.1) and (3.3) by using (4.2). By (3.2), $\mathfrak{B}(\bar{V})$ is isomorphic as an algebra to a finite-dimensional Nichols algebra over \mathbb{D}_m . Then it as an exterior algebra like $\mathfrak{B}(V)$ and hence (4.9) holds. ■

Later on, in the upcoming section, we describe the simple $\mathcal{D}(V)$ -modules using the strategy developed in Sections 3.2 and 3.3. Among all the relations above, we use only those involving v_{\pm} and α_{\pm} . Besides, the following elements of $\mathcal{D}(V)$ are going to be useful: For a fixed a summand $M_{i,k}$, we set $v_{\text{top}} = v_+ v_-$, $\alpha_{\text{top}} = \alpha_+ \alpha_-$ and define

$$\Theta = -\Phi_{++}\Phi_{--} + \Phi_{+-}\Phi_{-+} \in \mathcal{D}\mathbb{D}_m. \quad (4.10)$$

Using (4.5)–(4.8), a straightforward computation shows that these elements satisfy (3.5) or its recursive version, as appropriate. Explicitly,

$$\alpha_{\text{top}} v_{\text{top}} - \Theta \in \bigoplus_{n>0} \mathfrak{B}^n(W) \otimes \mathcal{D}(U) \otimes \mathfrak{B}^n(\bar{W}), \quad (4.11)$$

where $W = M_{i,k}$ and $V = W \oplus U$, and $\mathcal{D}(U) = \mathcal{D}\mathbb{D}_m$ if $V = W$.

4.4.1. Spherical. We finish this section by characterizing those Drinfeld doubles $\mathcal{D}(V)$ which are *spherical Hopf algebras*. This means by [8, Definition 3.1] that $\mathcal{D}(V)$ has a group-like element ϖ such that

$$S^2(h) = \varpi h \varpi^{-1} \quad \text{and} \quad \text{tr}_{\mathbb{N}}(\vartheta \varpi) = \text{tr}_{\mathbb{N}}(\vartheta \varpi^{-1})$$

for all $h \in \mathcal{D}(V)$, $\mathbb{N} \in {}_{\mathcal{D}(V)}\mathcal{M}$ and $\vartheta \in \text{End}_{\mathcal{D}(V)}(\mathbb{N})$. A group-like element satisfying the first condition is called *pivot* and, if it fulfills both conditions, it is called *spherical*. An

involutive pivot, i.e. $\varpi^2 = 1$, is clearly an spherical element. The pivot is unique up to multiplication by a central group-like element.

Theorem 4.8. *The Drinfeld double $\mathcal{D}(V)$ is spherical if and only if V does not contain a direct summand isomorphic to $M_{i,k}$ with both i and k even. In such a case, we may choose $\varpi = y^n \chi_3$ as the involutive spherical element.*

Proof. By [23, Proposition 9], we know that the group of group-like elements of $\mathcal{D}(V)$ equals $\mathbb{D}_m \times \{\chi_1, \chi_2, \chi_3, \chi_4\}$. Since S^2 is the identity on $\mathcal{D}\mathbb{D}_m$, a pivot element has to belong to the subset $\{y^n\} \times \{\chi_1, \chi_2, \chi_3, \chi_4\}$, which consist only of involutive elements. Then, in order to prove the statement, it is enough to analyse the existence of the pivot for V simple.

Assume $V = M_{i,k}$ for some $1 \leq i \leq n$ and $0 \leq k < m$. Then

$$\begin{aligned} \chi_3 v_{\pm} \chi_3 &= (-1)^{\pm i} v_{\pm}, & \chi_3 \alpha_{\pm} \chi_3 &= (-1)^{\mp i} \alpha_{\pm}, & S^2(v_{\pm}) &= -v_{\pm}, \\ y^n v_{\pm} y^n &= (-1)^{\pm k} v_{\pm}, & y^n \alpha_{\pm} y^n &= (-1)^{\mp k} \alpha_{\pm}, & S^2(\alpha_{\pm}) &= -\alpha_{\pm}, \end{aligned}$$

for the generators $v_{\pm} \in M_{i,k}$ and $\alpha_{\pm} \in \overline{M_{i,k}}$. Indeed, the formulas for the conjugation by χ_3 and y^n follow from (4.4). The formulas for S^2 are deduced using (4.2) and the definition of the coaction in a bosonization. Similarly, one can see that χ_1 is central, χ_2 commutes with v_{\pm} , and α_{\pm} and $\chi_3 \chi_2 = \chi_4$. We deduce then that $y^n \chi_3$ is a pivot if $i + k$ is odd and that there is no pivot when i and k are even. The case i and k both odd cannot occur because by assumption $\omega^{ik} = -1$. ■

Remark 4.9. The quantum dimension of any simple module in $\mathcal{D}(V, \mathbb{D}_m)\mathcal{M}$ is zero, except for those simple modules that are rigid.

5. Characters of simple $\mathcal{D}(V)$ -modules

In this section we follow the strategy summarized in Sections 3.2 and 3.3 to describe the simple modules over $\mathcal{D}(M_I)$ for $V = M_I = \bigoplus_{(i,k) \in I} M_{i,k}$ with I as in Notation 4.4. Recall the set of weight Λ in Section 4.1. For $\lambda \in \Lambda$, we set

$$\mathbb{L}^I(\lambda) := \mathbb{L}_{\mathbb{k}\mathbb{D}_m}^{M_I}(\lambda),$$

the simple highest-weight module over $\mathcal{D}(M_I)$ associated with λ . The appearance of $\mathbb{L}^I(\lambda)$ depends on certain subsets of Λ where the weight λ belongs. We present first these subsets and then state the results.

First, for $(i, k) \in I$, we fix the partition $\Lambda = \Lambda_{i,k}^r \cup \Lambda_{i,k}^p \cup \Lambda^o$ given in Table 1. The subset $\Lambda_{i,k}^r$ corresponds to the rigid simple modules when $I = \{(i, k)\}$, that is, those weights that satisfy $\mathbb{L}^{(i,k)}(\lambda) = \lambda$ as $\mathcal{D}(M_{i,k})$ -modules, see Lemma 5.4. The subset $\Lambda_{i,k}^p$ corresponds to the simple projective modules, that is, those that satisfy

$$\mathbb{L}^{(i,k)}(\lambda) = \mathbb{M}^{(i,k)}(\lambda) := \mathbb{M}_{\mathbb{k}\mathbb{D}_m}^{M_{i,k}}(\lambda)$$

as $\mathcal{D}(M_{i,k})$ -modules, see Lemma 5.4.

	$\Lambda_{i,k}^r$	$\Lambda_{i,k}^p$	Λ°
$M(e, \chi_j)$	$j = 1, 2,$ $j = 3, 4$ if i even	$j = 3, 4$ if i odd	—
$M(e, \rho_\ell)$	if $\omega^{i\ell} = 1$	if $\omega^{i\ell} \neq 1$	—
$M(y^n, \chi_j)$	$j = 1, 2$ if k even, $j = 3, 4$ if $i + k$ even	$j = 1, 2$ if k odd, $j = 3, 4$ if $i + k$ odd	—
$M_{p,q}$	if $\omega^{iq+pk} = 1$	if $\omega^{iq+pk} \neq 1$	—
$M_{r,s,t}$	—	—	all

Table 1. Partition of the sets of weights with respect to $M_{i,k}$.

The rigidity or projectivity of $L^I(\lambda)$ when $|I| > 1$ is determined by the subsets defined below.

Definition 5.1. For each $\lambda \in \Lambda$, we define

$$I_\lambda^r = \{(i, k) \in I \mid \lambda \in \Lambda_{i,k}^r\} \quad \text{and} \quad I_\lambda^p = \{(i, k) \in I \mid \lambda \in \Lambda_{i,k}^p\}.$$

We also set $M_{I_\lambda^r} = \bigoplus_{(i,k) \in I_\lambda^r} M_{i,k}$ and $M_{I_\lambda^p} = \bigoplus_{(i,k) \in I_\lambda^p} M_{i,k}$.

For $\lambda \in \Lambda \setminus \Lambda^\circ$, we have that $M_I = M_{I_\lambda^p} \oplus M_{I_\lambda^r}$ as $\mathcal{D}\mathbb{D}_m$ -modules, because $I = I_\lambda^p \cup I_\lambda^r$ and $\Lambda_{i,k}^r \cap \Lambda_{i,k}^p = \emptyset$ for all $(i, k) \in I$. In particular, we are under the hypothesis of Section 3.3, with $U = M_{I_\lambda^r}$ and $W = M_{I_\lambda^p}$.

Here is our main result which, in particular, gives the characters of the simple $\mathcal{D}(M_I)$ -modules. To simplify the notation, we write the associated functors

$$L^J = L_{\mathbb{k}\mathbb{D}_m}^{M_J} \quad \text{and} \quad M_J^K = M_{\mathfrak{B}(M_J)\#\mathbb{k}\mathbb{D}_m}^{M_K} \quad \text{if } M_I = M_J \oplus M_K.$$

Theorem 5.2. *Let $\lambda \in \Lambda$ and $M_I = \bigoplus_{(i,k) \in I} M_{i,k} \in \mathcal{D}\mathbb{D}_m\mathcal{M}$ with $I \in \mathcal{I}$. The simple highest-weight $\mathcal{D}(M_I)$ -modules are described as follows:*

- (a) *If $\lambda \in \Lambda \setminus \Lambda^\circ$, then $L^{I_\lambda^r}(\lambda) = \lambda$ is rigid as $\mathcal{D}(M_{I_\lambda^r})$ -modules and*

$$L^I(\lambda) \simeq M_{I_\lambda^p}^{I_\lambda^r}(L^{I_\lambda^r}(\lambda))$$

as $\mathcal{D}(M_I)$ -modules. In particular, $\text{Res}(L^I(\lambda)) = \mathfrak{B}(M_{I_\lambda^p}) \otimes \lambda$ and $\dim L^I(\lambda) = 4^{|I_\lambda^p|} \dim \lambda$.

- (b) *If $\lambda = M_{r,s,t} \in \Lambda^\circ$, then as graded $\mathcal{D}\mathbb{D}_m$ -modules,*

$$\text{Res}(L^I(M_{r,s,t})) \simeq \bigoplus_{J \subseteq I} M_{r+i_J, s+\ell_J+\epsilon_J, t+k_J}[-|J|],$$

where $i_J = \sum_{(i,k) \in J} i$, $k_J = \sum_{(i,k) \in J} k$, $\ell_J = \sum_{(n,\ell) \in J} \ell$ and $\epsilon_J = 1$ if $\ell_J \neq 0$ and zero otherwise; $i_J = k_J = \ell_J = 0$ if $J = \emptyset$. In particular, $\dim L^I(M_{r,s,t}) = 2^{|I|}n$.

Proof. With the aim of giving a clear exposition, we prove in detail the case in which I does not contain pairs (n, ℓ) . In particular, $\ell_J = 0$ and $\epsilon_J = 0$. The other case follows *mutatis mutandis*.

We proceed by induction on the cardinal of I . The case $|I| = 1$ is considered in Section 5.1, see Lemmata 5.4 and 5.5. The inductive step is then proved in Section 5.2, see Lemma 5.8 for part (a) and Lemma 5.7 for part (b). ■

As a direct consequence, one gets the description of the rigid and simple projective modules over $\mathcal{D}(M_I)$.

Corollary 5.3. *Let $\lambda \in \Lambda$. Then*

- (a) $\mathsf{L}^I(\lambda) = \lambda$ if and only if $I = I_\lambda^r$.
- (b) $\mathsf{L}^I(\lambda) = \mathsf{M}^I(\lambda)$ if and only if $I = I_\lambda^p$.

5.1. The singleton case

We assume here that $I = \{(i, k)\}$ with $1 \leq i \leq n-1$, $0 \leq k \leq m-1$ and $\omega^{ik} = -1$. We keep the notation of Section 4.4. In particular, $V = M_{i,k}$, v_\pm and α_\pm are the generators of $\mathcal{D}(M_{i,k})$ that belong to $M_{i,k}$ and its dual, respectively. The elements $\Phi_{\bullet, \circ}$ with $\bullet, \circ \in \{+, -\}$ and Θ defined in (4.5)–(4.8) and (4.10) are instrumental to determine which simple modules are rigid or projective.

Lemma 5.4. *Let $\lambda \in \Lambda$ and $\mathsf{L}^I(\lambda)$ be a simple $\mathcal{D}(M_{i,k})$ -module. Then*

- (a) $\mathsf{L}^I(\lambda) = \lambda$ if and only if $\lambda \in \Lambda_{i,k}^r$.
- (b) $\mathsf{L}^I(\lambda) = \mathsf{M}^I(\lambda)$ if and only if $\lambda \in \Lambda_{i,k}^p$.

Proof. By looking at the \mathbb{D}_m -degree, one easily sees that $\Phi_{\pm, \mp} \lambda = 0$ for $\lambda \in \Lambda_{i,k}^r \cup \Lambda_{i,k}^p$, but it is non-zero for $\lambda \in \Lambda^\circ$. Besides, one may check that $\Phi_{\pm, \pm} \lambda = 0$ for $\lambda \in \Lambda_{i,k}^r$, meanwhile it is non-zero for $\lambda \in \Lambda_{i,k}^p \cup \Lambda^\circ$. Hence (a) follows from Remark 3.3 for $V = M_{i,k}$.

Analogously, through a sheer calculation one can show that $\Theta \lambda \neq 0$ for $\lambda \in \Lambda_{i,k}^p$ and $\Theta \lambda = 0$ for $\lambda \in \Lambda^\circ$. Thus, (b) follows from Remark 3.4 for $V = M_{i,k}$. ■

For the remaining simple modules, we proceed as in Section 3.2.4.

Lemma 5.5. *Let $\lambda = M_{r,s,t} \in \Lambda^\circ$. Then, as graded $\mathcal{D}\mathbb{D}_m$ -modules,*

$$\begin{aligned} \text{Res}(\mathsf{L}^I(M_{r,s,t})) &\simeq M_{r,s,t}[0] \oplus M_{r+i,s,t+k}[-1], \\ \text{Res}(\mathsf{S}^I(M_{r,s,t})) &\simeq M_{r+i,s+1,t+k}[-1] \oplus M_{r,s+1,t}[-2]. \end{aligned}$$

Moreover, $\mathsf{L}^I(M_{r,s,t}) \simeq \mathsf{M}^I(M_{r,s,t})/\mathsf{S}^I(M_{r,s,t})$.

Proof. Taking into account (see Figure 6) that

$$\mathfrak{B}(M_{i,k}) = \bigwedge M_{i,k} = \mathbb{k} \oplus M_{i,k} \oplus \mathbb{k}v_{\text{top}}$$

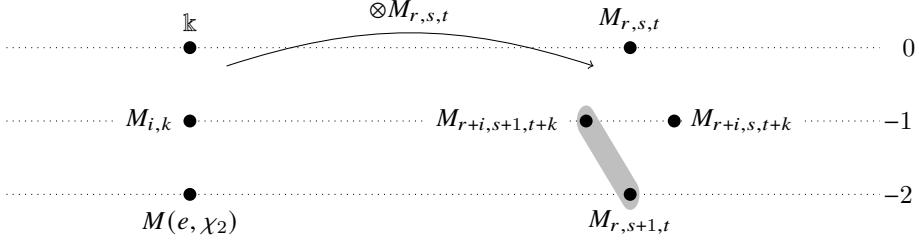


Figure 6. The big dots represent the weights of $\mathfrak{B}(M_{i,k})$ and $\mathfrak{M}^{(i,k)}(M_{r,s,t})$. Their degrees are indicated on the right. Those in the shadow region form the socle $\mathfrak{S}^{(i,k)}(M_{r,s,t})$; the others the head $\mathfrak{L}^{(i,k)}(M_{r,s,t})$.

with $\mathfrak{B}^0(M_{i,k}) = \mathbb{k}$, $\mathfrak{B}^1(M_{i,k}) = M_{i,k}$, $\mathfrak{B}^2(M_{i,k}) = \mathbb{k}v_{\text{top}}$, and $\mathbb{k}v_{\text{top}} \simeq M(e, \chi_2)$ as \mathcal{DD}_m -modules, we have that $\text{Res}(\mathfrak{M}^I(M_{r,s,t}))$ is the direct sum of four weights. Indeed,

$$\text{Res}(\mathfrak{M}^I(M_{r,s,t})_0) = \mathbb{k} \otimes M_{r,s,t} = M_{r,s,t},$$

$$\text{Res}(\mathfrak{M}^I(M_{r,s,t})_{-1}) = M_{i,k} \otimes M_{r,s,t} \simeq M_{i+r,s+1,t+k} \oplus M_{i+r,s,t+k} \quad \text{by Lemma 4.2,}$$

$$\text{Res}(\mathfrak{M}^I(M_{r,s,t})_{-2}) = \mathbb{k}v_{\text{top}} \otimes M_{r,s,t} \simeq M(e, \chi_2) \otimes M_{r,s,t} \simeq M_{r,s+1,t} \quad \text{by Lemma 4.3.}$$

Note that all these weights are in Λ° . Then $\mathfrak{M}^I(M_{r,s,t})$ is not simple and its composition factors are not concentrated in a single degree by Lemma 5.4, as they are not rigid and consist of more than a weight.

We deduce then that $\mathfrak{M}^I(M_{r,s,t})$ has exactly two composition factors, each of them has to be the direct sum of two weights. One must be the socle $\mathfrak{S}^I(M_{r,s,t})$ with

$$\text{Res}(\mathfrak{S}^I(M_{r,s,t})) \simeq \mu \oplus (\mathbb{k}v_{\text{top}} \otimes M_{r,s,t})$$

as \mathcal{DD}_m -modules, where μ is the unique highest-weight in degree -1 and $\mathfrak{S}^I(M_{r,s,t}) \simeq \mathfrak{L}^I(\mu)[-1]$ by Corollary 3.6. The other composition factor is the head $\mathfrak{L}^I(M_{r,s,t})$ with

$$\mathfrak{L}^I(M_{r,s,t}) \simeq \mathfrak{M}^I(M_{r,s,t})/\mathfrak{S}^I(M_{r,s,t})$$

as $\mathcal{D}(M_{i,k})$ -modules. Also,

$$\text{Res}(\mathfrak{L}^I(M_{r,s,t})) \simeq M_{r,s,t} \oplus \bar{\lambda}$$

as \mathcal{DD}_m -modules, where $\bar{\lambda}$ is the complement of μ in degree -1 .

Hence, we should determine which weight in degree -1 is annihilated by

$$\bar{V} = \overline{M_{i,k}}.$$

By Lemma 4.2, we know these weights are generated by

$$n_{\pm} = \omega^{rk} v_{-} \otimes |\mathbf{0}\rangle \pm v_{+} \otimes |i\rangle.$$

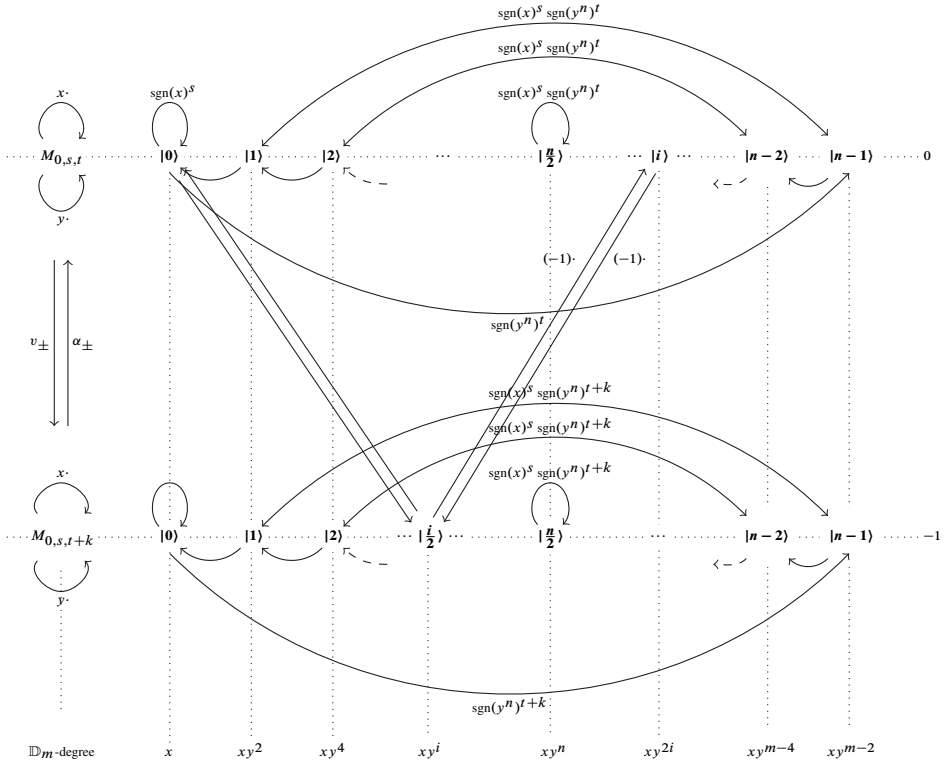


Figure 7. $L^{(i,k)}(M_{0,s,t})$ over $\mathcal{D}(M_{i,k})$ for i even.

Using (4.5)–(4.6), we see that

$$\alpha_+ n_{\pm} = \alpha_+ (\omega^{rk} v_- \otimes |0\rangle \pm v_+ \otimes |i\rangle) = -|i\rangle \pm |i\rangle. \quad (5.1)$$

Then the action of $\overline{M_{i,k}}$ on the weight generated by n_- is non-trivial and hence μ must be the weight generated by n_+ ; that is, $\mu = M_{r+i,s+1,t+k}$. Therefore, we have that

$$\text{Res}(S^I(\lambda)) \simeq M_{r+i,s+1,t+k}[-1] \oplus M_{r,s+1,t}[-2]$$

and

$$\text{Res}(L^I(\lambda)) \simeq M_{r,s,t}[0] \oplus M_{r+i,s,t+k}[-1]. \quad \blacksquare$$

Example 5.6. In Figure 7 above, we depict the simple module $L^I(M_{0,s,t})$ over $\mathcal{D}(M_{i,k})$ for i even. The nodes $|j\rangle$ denote the basis elements of the \mathcal{DD}_m -direct summands $M_{0,s,t}$ and $M_{0,s,t+k}$ of $\text{Res}(L^I(M_{0,s,t}))$. In each node, there should be two arrows going in and two arrows going out, but we only draw those corresponding to $|i/2\rangle$ in level -1 to make

the diagram easy to read. Keep the notation as in the proof of Lemma 5.5, with $n_{\pm} = v_- \otimes |\mathbf{0}\rangle \pm v_+ \otimes |\mathbf{i}\rangle$ and $|\mathbf{0}\rangle, |\mathbf{i}\rangle \in M_{0,s,t}$. Set

$$\overline{n_{\pm}} = v_- |\mathbf{0}\rangle \pm v_+ |\mathbf{i}\rangle$$

for the images of these elements in the quotient $M^I(M_{0,s,t})/S^I(M_{0,s,t})$. Since $\overline{n_+} = 0$, we have that $v_- |\mathbf{0}\rangle = -v_+ |\mathbf{i}\rangle$ as elements of degree -1 . Looking at the $\mathcal{D}\mathbb{D}_m$ -action, one gets that both elements equal (a non-zero scalar multiple of) $|\frac{\mathbf{i}}{2}\rangle$ in $M_{0,s,t+k}$ inside $\text{Res}(L^I(M_{0,s,t}))$. This is depicted by the two arrows arriving at $|\frac{\mathbf{i}}{2}\rangle$. Using (4.5) and (4.8), one may see that $\alpha_+ |\frac{\mathbf{i}}{2}\rangle = -|\mathbf{i}\rangle$ and $\alpha_- |\frac{\mathbf{i}}{2}\rangle = |\mathbf{0}\rangle$, respectively; these are the two arrows leaving $|\frac{\mathbf{i}}{2}\rangle$.

5.2. The recursive step

Fix a decomposition $M_I = \bigoplus_{(i,k) \in I} M_{i,k}$ with $|I| \geq 1$ and keep the notation of Section 4.4. As in the preceding subsection, we divide our analysis with respect to a partition on the set Λ . Here, we simply take $\Lambda = \Lambda^\circ \cup (\Lambda \setminus \Lambda^\circ)$.

The first lemma asserts that $\lambda \in \Lambda \setminus \Lambda^\circ$ is a rigid module over $\mathcal{D}(M_{I_\lambda^i})$.

Lemma 5.7. *Let $\lambda \in \Lambda \setminus \Lambda^\circ$. Then $L^{I_\lambda^i}(\lambda) \simeq \lambda$, that is, λ is a simple $\mathcal{D}(M_{I_\lambda^i})$ -module by letting $M_{I_\lambda^i}$ and $\overline{M_{I_\lambda^i}}$ act trivially.*

Proof. This follows by induction on the cardinal of I_λ^i , using Lemma 5.4 (a) and the recursive version of Remark 3.3 with $M_{I_\lambda^i} = M_{i,k} \oplus M_{I_\lambda^i \setminus (i,k)}$. \blacksquare

Let $\lambda \in \Lambda \setminus \Lambda^\circ$. Using the decomposition $M_I = M_{I_\lambda^p} \oplus M_{I_\lambda^i}$ and the triangular decomposition (3.9) associated with it, one may also consider the $\mathcal{D}(M_{I_\lambda^p}, M_{I_\lambda^i}, \mathbb{D}_m)$ -module given by the description (3.10), this is

$$M_{I_\lambda^i}^p(L^{I_\lambda^i}(\lambda)) = M_{I_\lambda^i}^p(\lambda) = \mathcal{D}(M_{I_\lambda^p}, M_{I_\lambda^i}, \mathbb{D}_m) \otimes_{\mathcal{D}(M_{I_\lambda^i}, \mathbb{D}_m) \otimes \mathfrak{B}(\overline{M_{I_\lambda^i}})} \lambda.$$

Then, $M_{I_\lambda^i}^p(\lambda)$ admits a simple quotient $L_{I_\lambda^i}^p(\lambda)$ which is isomorphic to $L^I(\lambda)$ by Theorem 3.11. In particular, there exists an epimorphism of $\mathcal{D}(M_I)$ -modules

$$M_{I_\lambda^i}^p(\lambda) \twoheadrightarrow L^I(\lambda). \quad (5.2)$$

One could use the recursive version of Remark 3.4 to prove that it is actually an isomorphism, by showing that $M_{I_\lambda^i}^p(\lambda) = L_{I_\lambda^i}^p(\lambda)$ is simple. To do so, it would be enough to compute the element Θ for $M_{I_\lambda^p}$. To avoid the long computation, we prove the simplicity of $M_{I_\lambda^i}^p(\lambda)$ by induction on $|I_\lambda^p|$, using the Θ already computed in the singleton case.

Lemma 5.8. *Let $\lambda \in \Lambda \setminus \Lambda^\circ$. Then $L^I(\lambda) \simeq M_{I_\lambda^i}^p(\lambda)$ as $\mathcal{D}(M_I)$ -modules. In particular, $L^I(\lambda) \simeq \mathfrak{B}(M_{I_\lambda^p}) \otimes_{\mathcal{D}\mathbb{D}_m} \lambda$ as $\mathfrak{B}(M_{I_\lambda^p}) \# \mathcal{D}\mathbb{D}_m$ -modules.*

Proof. We proceed by induction on the cardinal $|I_\lambda^p|$. If $I_\lambda^p = \emptyset$, this is Lemma 5.7.

Suppose $|I_\lambda^p| \geq 1$. Let $(i, k) \in I_\lambda^p$ and set $J = I \setminus \{(i, k)\}$. If $J = \emptyset$, then

$$I = \{(i, k)\} = I_\lambda^p,$$

so $M_\lambda^p(\lambda) \simeq M^I(\lambda)$, and by Lemma 5.4 (b) it follows that $M_\lambda^p(\lambda) = \mathbb{L}^I(\lambda)$. Hence, we may assume that $J \neq \emptyset$. Since $|J_\lambda^p| < |I_\lambda^p|$, by the inductive hypothesis we have that

$$\mathbb{L}^J(\lambda) \simeq M_{J_\lambda^p}^{J_\lambda^p}(\lambda) = M_{J_\lambda^p}^{J_\lambda^p}(\mathbb{L}^{J_\lambda^p}(\lambda)) = \mathcal{D}(M_J, \mathbb{D}_m) \otimes_{\mathcal{D}(M_{J_\lambda^p}, \mathbb{D}_m) \otimes \mathfrak{B}(\overline{M_{J_\lambda^p}})} \lambda.$$

Consider the generators $v_\pm \in M_{i,k}$ and $\alpha_\pm \in \overline{M_{i,k}}$ of $\mathcal{D}(M_I, \mathbb{D}_m)$ defined as in Section 4.4. Then, the commuting relations (4.5)–(4.8) hold in $\mathcal{D}(M_I)$, and we may consider the corresponding element Θ as in (4.10) satisfying (4.11).

Furthermore, the action of Θ on $\mathbb{L}^J(\lambda)$ is non-trivial because, by the proof of Lemma 5.4 (b), the action of Θ on $\lambda \subset \mathbb{L}^J(\lambda)$ is not so. Then, by Theorem 3.11 and the recursive version of Remark 3.4 with $W = M_{i,k}$ and $U = M_J$, we have that

$$\mathbb{L}^I(\lambda) \simeq \mathbb{L}_J^{(i,k)}(\mathbb{L}^J(\lambda)) \simeq M_J^{(i,k)}(\mathbb{L}^J(\lambda)) \simeq M_{J_\lambda^p}^{(i,k)}(M_{J_\lambda^p}^p(\lambda)) \simeq M_\lambda^p(\lambda),$$

where the last isomorphism follows by Lemma 3.10. \blacksquare

The following lemma gives the description of $\mathbb{L}^I(\lambda)$ for $\lambda \in \Lambda^\circ$. With it, we finish the proof of Theorem 5.2. Its proof also relies on the recursive argument in Section 3.3. More explicitly, this fits in the situation of Remark 3.14 and Section 3.3.1.

Lemma 5.9. *Let $\lambda = M_{r,s,t} \in \Lambda^\circ$. Then, as graded $\mathcal{D}\mathbb{D}_m$ -modules*

$$\text{Res}(\mathbb{L}^I(\lambda)) \simeq \bigoplus_{J \subseteq I} M_{r+i_J, s+t+k_J}[-|J|],$$

where $i_J = \sum_{(i,k) \in J} i$ and $k_J = \sum_{(i,k) \in J} k$, with $i_J = 0 = k_J$ if $J = \emptyset$.

Proof. We proceed by induction on $|I|$. If $|I| = 1$, this is Lemma 5.5.

Assume now $I = \{(i, k)\} \cup E$, then $M_I = M_{i,k} \oplus M_E$. Following the strategy on Section 3.3.1, taking $U = M_E$ and $W = M_{i,k}$, we describe $\mathbb{L}^I(\lambda)$ as a quotient of the induced module

$$M_E^{(i,k)}(\mathbb{L}^E(\lambda)) = M_{\mathfrak{B}(M_E) \# \mathbb{k}\mathbb{D}_m}^{(i,k)}(\mathbb{L}^E(\lambda)) = \mathcal{D}(M_{i,k} \oplus M_E) \otimes_{\mathcal{D}(M_E) \otimes \mathfrak{B}(\overline{M_{i,k}})} \mathbb{L}^E(\lambda).$$

Moreover, by the recursive version of (3.6), we have an isomorphism of $\mathcal{D}(M_E)$ -modules (here Res is the restriction functor to $\mathcal{D}(M_E)$)

$$\text{Res}(M_E^{(i,k)}(\mathbb{L}^E(\lambda))) \simeq \bigoplus_{j=0}^2 \mathfrak{B}^j(M_{i,k}) \otimes \mathbb{L}^E(\lambda). \quad (5.3)$$

We first show that $\mathbb{L}^I(\lambda)$ is not isomorphic to neither $\mathbb{L}^E(\lambda)$ nor $M_E^{(i,k)}(\mathbb{L}^E(\lambda))$. To do this, we consider the generators $v_\pm \in M_{i,k}$ and $\alpha_\pm \in \overline{M_{i,k}}$ of $\mathcal{D}(M_I)$ as in Section 4.4. Thus, the commuting relations (4.5)–(4.8) hold and the corresponding element Θ

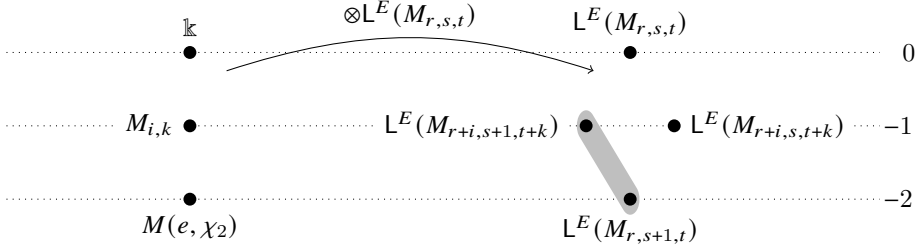


Figure 8. The dots represent the simple $\mathcal{D}(M_E)$ -summands of $\mathfrak{B}(M_{i,k})$ and $M_E^{(i,k)}(L^E(M_{r,s,t}))$. Their degrees are indicated on the right. Those in the shadow region form its socle and the others its head which is isomorphic to $L^{(i,k)}(M_{r,s,t})$.

as in (4.10) satisfies (4.11). Now, the action of some $\Phi_{\pm, \pm}$ is non-zero on λ by the proof of Lemma 5.4 (a) and hence $L^I(\lambda) \not\cong L^E(\lambda)$ by the recursive version of Remark 3.3. Also, by the inductive hypothesis and Lemma 4.2, $L^E(\lambda)$ is the direct sum of weights in Λ° . By Lemma (b), the action of Θ on them is trivial and hence $L^I(\lambda) \not\cong M_E^{(i,k)}(L^E(\lambda))$ by the recursive version of Remark 3.4.

We next analyse the structure of $M_E^{(i,k)}(L^E(\lambda))$ as $\mathcal{D}(M_E)$ -module, similarly as we did in Lemma 5.5. See Figure 8.

Claim 5.10. $M_E^{(i,k)}(L^E(\lambda))$ is semisimple as $\mathcal{D}(M_E)$ -module and

$$\text{Res}(M_E^{(i,k)}(L^E(\lambda))) \simeq L^E(\lambda) \oplus L^E(M_{r+i,s+1,t+k}) \oplus L^E(M_{r+i,s,t+k}) \oplus L^E(M_{r,s+1,t}),$$

where the gradation induced by the new triangular decomposition is

$$\begin{aligned} \text{Res}(M_E^{(i,k)}(L^E(\lambda)))_0 &= L^E(\lambda) = L^E(M_{r,s,t}), \\ \text{Res}(M_E^{(i,k)}(L^E(\lambda)))_{-1} &= L^E(M_{r+i,s+1,t+k}) \oplus L^E(M_{r+i,s,t+k}) \\ \text{Res}(M_E^{(i,k)}(L^E(\lambda)))_{-2} &= L^E(M_{r,s+1,t}). \end{aligned}$$

Indeed, $\mathfrak{B}^0(M_{i,k}) = \mathbb{k} \simeq M(e, \chi_1)$, $\mathfrak{B}^1(M_{i,k}) = M_{i,k}$ and $\mathfrak{B}^2(M_{i,k}) \simeq M(e, \chi_2)$ are all rigid simple $\mathcal{D}(M_E)$ -modules by the inductive hypotheses applied to E . Then Theorem 3.8 implies that

$$\begin{aligned} \mathfrak{B}^0(M_{i,k}) \otimes L^E(\lambda) &\simeq L^E(\lambda), \\ \mathfrak{B}^1(M_{i,k}) \otimes L^E(\lambda) &\simeq L^E(M_{r+i,s+1,t+k}) \oplus L^E(M_{r+i,s,t+k}) \quad (\text{see Lemma 4.2}), \\ \mathfrak{B}^2(M_{i,k}) \otimes L^E(\lambda) &\simeq L^E(M_{r,s+1,t}) \quad (\text{see Lemma 4.3}) \end{aligned}$$

as $\mathcal{D}(M_E)$ -modules. Therefore the claim follows from (5.3).

Being a quotient of $M_E^{(i,k)}(L^E(\lambda))$, $L^I(\lambda)$ is isomorphic to a direct sum of some of the simple $\mathcal{D}(M_E)$ -modules in the claim above.

Claim 5.11. *As graded $\mathcal{D}(M_E)$ -modules, we have that*

$$\text{Res}(\mathbb{L}^I(\lambda)) \simeq \mathbb{L}^E(M_{r,s,t}) \oplus \mathbb{L}^E(M_{r+i,s,t+k})[-1] \quad (5.4)$$

with the gradation induced by the new triangular decomposition.

Indeed, the claim follows by a counting argument similar to the one in the proof of Lemma 5.5. Using the decomposition (5.3), one can deduce that $\text{Res}(\mathbb{L}^I(\lambda))$ is isomorphic to $\mathbb{L}^E(\lambda) \oplus \mathbb{L}^E(M_{r+i,\tilde{s},t+k})[-1]$ as $\mathcal{D}(M_E)$ -module with either $\tilde{s} = s$ or $\tilde{s} = s + 1$. To determine which \tilde{s} is, we can argue as in the proof of Lemma 5.5, see (5.1). Namely, it must be the simple $\mathcal{D}(M_E)$ -module over which the action of $\overline{M}_{i,k}$ is non-trivial. As in (5.1), we see that $\overline{M}_{i,k}$ acts by zero on the weight $M_{r+i,s+1,t+k}$ and its action is non-zero on $M_{r+i,s,t+k}$. Then $\overline{M}_{i,k}$ must act trivially over the simple $\mathcal{D}(M_E)$ -submodule generated by the former weight, that is $\mathbb{L}^E(M_{r+i,s+1,t+k})$, and hence $\tilde{s} = s$.

Finally, the lemma follows by (5.4) and the inductive hypothesis. ■

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