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# Traffic distributions and independence II: Universal constructions for traffic spaces

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**Abstract.** We investigate questions related to the notion of *traffics* introduced by the third author as a non-commutative probability space with additional operations and equipped with the notion of *traffic independence*. We prove that any sequence of unitarily invariant random matrices, that converges in non-commutative distribution, converges as well in traffic distribution whenever it fulfils some factorization property and we provide an explicit description of the limit. We also improve the theory of traffic spaces by considering a positivity axiom related to the notion of *state* in non-commutative probability. We construct the free product of traffic spaces and prove that it preserves the positivity condition. This analysis leads to our main result stating that every non-commutative probability space endowed with a tracial state can be enlarged and equipped with a structure of traffic space.

## Contents



## 1. Introduction

### 1.1. Presentation of the results

**1.1.1. Motivations for traffics.** Thanks to the fundamental work of Voiculescu [31], it is now understood that free probability is a good framework for the study of large

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random matrices. Here are two important considerations which sum up the role of non-commutative probability in the description of the macroscopic behavior of large random matrices:

(1) A large class of families of random matrices $\mathbf{A}_N \in \mathrm{M}_N(\mathbb{C})$ *converge in non-commutative distribution* as $N$ tends to $\infty$ (in the sense that the normalized trace of any polynomial in the matrices converges). The pioneering result of Voiculescu [31] shows the convergence for independent GUE matrices, Haar unitary matrices and certain diagonal matrices having a limiting non-commutative distribution. This result have been furthermore extended for independent Wigner matrices [1, 9], for independent uniformly distributed random permutations matrices and GUE matrices [24, 25], independent matrices with i.i.d. entries having exploding moments [28] for independent Wishart matrices [6], for the dynamical SYK model [27].

(2) If two independent families of random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ converge separately in non-commutative distribution and are invariant in law when conjugating by a *unitary* matrix, then the joint non-commutative distribution of the family $\mathbf{A}_N \cup \mathbf{B}_N$ converges as well. The joint limit can be described from the separate limits thanks to the relation of *free independence* introduced by Voiculescu.

In [16–18], it was pointed out that there are cases where other important macroscopic convergences occur in the study of large random matrices and graphs. One example is the adjacency matrix of the so-called sparse Erdös–Reńyi graph: it is the symmetric real random matrix $X_N$ whose sub-diagonal entries are independent and distributed according to Bernoulli random variable with parameter $\frac{p}{N}$, where $p$ is fixed. Let $Y_N$ be a deterministic matrix bounded in operator norm. Then the possible limiting $^*$-distributions of $(X_N, Y_N)$ depend on more than the limiting $^*$-distribution of $Y_N$ [16].

The notion of non-commutative probability spaces is too restrictive for such examples and should be generalized to get more information about the limit in large dimension. This is precisely the motivation to introduce the concept of *traffic space*, which comes together with its own notions of distribution and independence: a traffic space is a non-commutative probability space where one can consider not only the usual operations of algebras, but also more general $n$-ary operations called *graph operations*. We will introduce those concept in detail, but let us first describe the role of traffics enlightened in [17] for the description of large $N$ asymptotics of random matrices:

(1) A large class of families of random matrices $\mathbf{A}_N \in \mathrm{M}_N(\mathbb{C})$ *converge in traffic distribution* as $N$ tends to $\infty$ (in the sense that the normalized trace of any *graph operation* in the matrices converges), in particular independent Wigner, Haar unitary, uniform permutation matrices and deterministic matrices having a limiting traffic distribution [17], independent random band matrices [2], and (for a slightly restricted class of graph operation) for independent matrices with i.i.d. entries having exploding moments [16] and uniform regular weighted graphs with large degree [18].

(2) If two independent families of random matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ converge separately in traffic distribution, satisfy a factorization property and are invariant in law when conjugating by a *permutation* matrix, then the joint traffic distribution of the family $\mathbf{A}_N \cup \mathbf{B}_N$ converges as well. Moreover, the joint limit can be described from the separate limits thanks to the relation of *traffic independence* introduced in [17].

As a sequel of [17], the purpose of this article is to develop the theory of traffics and provide more examples.

**1.1.2. Limiting traffic distribution of large unitarily invariant random matrices.** For concreteness, we first describe how we encode new operations on matrix spaces and state one example of matrices that are considered in this article.

For all $K \geq 0$, a *$K$-graph operation* is a connected graph $g$ with $K$ oriented and ordered edges, and two distinguished vertices (one input and one output, not necessarily distinct). The set $\mathcal{G}$ of graph operations is the set of all $K$-graph operations for all $K \geq 0$. A $K$-graph operation $g$ has to be thought as an operation that accepts $K$ objects and produces a new one.

For example, it acts on the space $\mathrm{M}_N(\mathbb{C})$ of $N$ by $N$ complex matrices as follows. For each $K$-graph operation $g \in \mathcal{G}$, we define a linear map

$$Z_g : \mathrm{M}_N(\mathbb{C}) \otimes \cdots \otimes \mathrm{M}_N(\mathbb{C}) \to \mathrm{M}_N(\mathbb{C})$$

in the following way. Denoting by

- $V$ the vertex set of $g$,
- $(v_1, w_1), \ldots, (v_K, w_K)$ the ordered edges of $g$,
- *in* and *out* the distinguished vertices of $g$,
- $E_{k,l}$ the matrix unit $(\delta_{ik}\delta_{jl})_{i,j=1}^N \in \mathrm{M}_N(\mathbb{C})$,

we set, for all $A_1, \ldots, A_K \in \mathrm{M}_N(\mathbb{C})$,

$$Z_g(A_1 \otimes \cdots \otimes A_K) = \sum_{\phi : V \to \{1,\ldots,N\}} \left( \prod_{k=1}^K A_k\big(\phi(w_k), \phi(v_k)\big) \right) \cdot E_{\phi(out),\phi(in)}.$$

Those operations appear quite naturally in investigations of random matrices, see for instance [3, Appendix A.4] and [21]. Following [21], we can think of the linear map $\mathbb{C}^N \to \mathbb{C}^N$ associated to $Z_g(A^{(1)} \otimes \cdots \otimes A^{(K)})$ as an algorithm, where we feed a vector into the input vertex and then operate it through the graph, each edge doing some calculation thanks to the corresponding matrix $A^{(i)}$, and each vertex acting like a logic gate, doing some compatibility checks. This description relies only on the so-called *commutative special †-Frobenius comonoid structure* of matrix spaces [7].

The linear maps $Z_g$ encode naturally the product of matrices, but also other natural operations, like the Hadamard (entry-wise) product $(A, B) \mapsto A \circ B$, the real transpose $A \mapsto A^t$ or the degree matrix $\deg(A) = \mathrm{diag}(\sum_{j=1}^N A_{i,j})_{i=1,\ldots,N}$.

Starting from a family $\mathbf{A} = (A_j)_{j\in J}$ of random matrices of size $N \times N$, the smallest algebra close by adjunction and by the action of the $K$-graph operations is the *traffic space* generated by $\mathbf{A}_N$. The *traffic distribution* of $\mathbf{A}_N$ is the data of the non-commutative distribution of the matrices which are in the traffic space generated by $\mathbf{A}_N$. More concretely, it is the collection of the quantities

$$\frac{1}{N}\mathbb{E}\big[\operatorname{Tr}\big(Z_g(A_{j_1}^{\varepsilon_1}\otimes\cdots\otimes A_{j_K}^{\varepsilon_K})\big)\big]$$

for all $K$-graph operations $g\in\mathcal{G}$, indices $j_1,\dots,j_K\in J$ and labels $\varepsilon_1,\dots,\varepsilon_K\in\{1,*\}$.

In this article, we prove the following theorem. It shows that for a general class of unitarily invariant matrices, the convergence of the $*$-distribution is sufficient to deduce the convergence in traffic distribution.

**Theorem 1.1.** *For all $N\geq 1$, let $\mathbf{A}_N=(A_j)_{j\in J}$ be a family of random matrices in $\mathrm{M}_N(\mathbb{C})$. We assume*

(1) *Unitary invariance: for all $N\geq 1$ and all $U\in\mathrm{M}_N(\mathbb{C})$ which is unitary, $U\mathbf{A}_NU^*:=(UA_jU^*)_{j\in J}$ and $\mathbf{A}_N$ have the same law.*
(2) *Convergence in $*$-distribution of $\mathbf{A}_N$: for all indices $j_1,\dots,j_K\in J$ and labels $\varepsilon_1,\dots,\varepsilon_K\in\{1,*\}$, the quantity $(1/N)\mathbb{E}[\operatorname{Tr}(A_{j_1}^{\varepsilon_1}\cdots A_{j_K}^{\varepsilon_K})]$ converges.*
(3) *Factorization property: for all $*$-monomials $m_1,\dots,m_k$, we have the following convergence*

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\big(m_1(\mathbf{A}_N)\big)\cdots\frac{1}{N}\operatorname{Tr}\big(m_k(\mathbf{A}_N)\big)\right]$$
$$=\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\big(m_1(\mathbf{A}_N)\big)\right]\cdots\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\big(m_k(\mathbf{A}_N)\big)\right].$$

*Then $\mathbf{A}_N$ converges in traffic distribution: for every $K$-graph operation $g\in\mathcal{G}$, indices $j_1,\dots,j_K\in J$ and labels $\varepsilon_1,\dots,\varepsilon_K\in\{1,*\}$, the following quantity converges*

$$\frac{1}{N}\mathbb{E}\big[\operatorname{Tr}\big(Z_g(A_{j_1}^{\varepsilon_1}\otimes\cdots\otimes A_{j_K}^{\varepsilon_K})\big)\big].$$

*The limit of the traffic distribution of $\mathbf{A}_N$ is unitarily invariant in a sense defined later in Definition 4.4, and it depends explicitly on the limit of the non-commutative $*$-distribution of $\mathbf{A}_N$.*

Note that the convergence is about macroscopic quantities built from the matrices. However, it contains more information than the convergence in $^*$-moments.

For example, a recent result of Mingo and Popa [20] says that for every sequence of unitarily invariant random matrices $\mathbf{A}_N$ the family $\mathbf{A}_N^t$ of the transposes of $\mathbf{A}_N$ has the same non-commutative $*$-distribution as $\mathbf{A}_N$ and is asymptotically freely independent from $\mathbf{A}_N$ (under assumptions stronger than those of Theorem 1.1 that also imply the asymptotic free independence of second order). Thanks to the description of the limiting traffic distribution of unitarily invariant matrices, we will get that for a family $\mathbf{A}_N=(A_j)_{j\in J}$ as in Theorem 1.1, $\mathbf{A}_N$, $\mathbf{A}_N^t$ and $\deg(\mathbf{A}_N)$ are asymptotically free independent.

Remark that Gabriel [12] proved independently a result similar to Theorem 1.1 about the convergence of permutation invariant observables on random matrices. More generally, up to some conventions the framework developed in [11–13] is equivalent to the framework of traffics. Interestingly, it develops aspects that are not yet considered for traffics, such as the central notion of cumulants.

**1.1.3. Non-commutative probability spaces and traffic spaces.** We now leave the example of random matrices and introduce the abstract notion of traffic spaces. The purpose is to define a structure for the limit of large matrices that captures the limiting traffic distribution, in a similar way the model of non-commutative random variables captures the limiting joint distribution of large matrices in the theory of free probability.

Let us start by recalling the setting of non-commutative probability. A non-commutative probability space is a pair $(\mathcal{A}, \Phi)$, where $\mathcal{A}$ is an algebra and $\Phi$ is linear form. One often assumes

- Unitality: $\Phi(1_{\mathcal{A}}) = 1$,
- Traciality: $\Phi(ab) = \Phi(ba)$ for any $a, b \in \mathcal{A}$,
- Positivity: there is an anti-linear involution $\cdot^*$ satisfying $(ab)^* = b^*a^*$, such that $\Phi$ is positive, that is, $\Phi(a^*a) \geq 0$ for any $a \in \mathcal{A}$.

A non-commutative probability space satisfying the above three properties is called a $^*$-probability space. The distribution of a family $\mathbf{a}$ of elements of a non-commutative probability space is the linear form $\Phi_{\mathbf{a}} : P \mapsto \Phi(P(\mathbf{a}))$ defined for non-commutative polynomials in elements of $\mathbf{a}$. On $^*$-probability spaces, the $^*$-distribution is defined by the same formula for non-commutative polynomials in the elements and their adjoints. The convergence in ($^*$-)distribution of a sequence $\mathbf{a}_N$ is the pointwise convergence of $\Phi_{\mathbf{a}_N}$.

An algebraic traffic space is equivalent to the data of a non-commutative probability space $(\mathcal{A}, \Phi)$ and of a collection of $K$-linear maps from $\mathcal{A}^K$ to $\mathcal{A}$ indexed by the $K$-graph operations satisfying mild assumptions. More precisely, to each $K$-graph operation $g \in \mathcal{G}$ there is a linear map

$$Z_g : \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{K \text{ times}} \to \mathcal{A}$$

subject to some requirements of compatibility. Namely, it should be a so-called operad algebra over the set of graph operations (Definition 1.7). The traffic distribution of a family

$$\mathbf{a} = (a_j)_{j \in J} \in \mathcal{A}^J$$

is equivalent to the collection of the quantities $\Phi[Z_g(a_{\gamma(1)} \otimes \cdots \otimes a_{\gamma(K)})]$ for any $K$ graph operation $g$ and for any map $\gamma : \{1, \ldots, K\} \to J$. Actually, the definition of the traffic spaces will be given as pairs $(\mathcal{A}, \tau)$, where $\tau$ is a combinatorial function, that is, equivalent to the data of $\Phi$, although it is more intrinsic.

Finally, a *traffic* (an element of $\mathcal{A}$) is a non-commutative random variable, albeit coming with more information, as the action of graph operations permits us to consider additional operations, including many operations on matrices such as the Hadamard product, the transpose, and the degree. As an example, let us highlight that if a matrix $A_N$ converges in traffic distribution to $a \in \mathcal{A}$, the joint non-commutative distribution of $A_N, A_N \circ A_N, A_N^t, \deg(A_N), \ldots$ converges to the distribution of $a, a \circ a, a^t, \deg(a), \ldots$ in $(\mathcal{A}, \Phi)$.

**1.1.4. Independence and positivity.** In non-commutative probability theory, it is possible to consider three different products of noncommutative probability spaces, each one corresponding to a particular notion of independence: the tensor independence, the free independence and the Boolean independence. Moreover, these products are the only existing ones in a certain sense (see [14, 29]). Interestingly, all three products preserve the positivity of the linear form.

One important contribution of the present paper is the definition of the free product of traffic spaces which yields to the appropriate notion of independence for traffics defined in [17]. More precisely, in Section 3.1, for any collection $\mathcal{A}_j$, $j \in J$ of algebraic traffic spaces (with traces $\Phi_j$), we define their free product $*_{j \in J} \mathcal{A}_j$, in such a way that the algebras $A_j$ seen as traffic subspaces of $*_{j \in J} \mathcal{A}_j$ are traffic independent with respect to the canonical trace.

It has to be noted that the positivity of the traces $\Phi_j$ on the spaces $\mathcal{A}_j$ is not sufficient to ensure the positivity of the resulting trace on $*_{j \in J} \mathcal{A}_j$. One has to require more positivity conditions on $\Phi_j$ to get positivity at the end. This is one motivation to define the good notion of positivity for traffic spaces. In Definition 1.10 of Section 1.2, we define a *traffic space* as an algebraic traffic space $\mathcal{A}$ with trace $\Phi$ with two additional properties: the compatibility of the involution $\cdot^*$ with graph operations, and a positivity condition on $\Phi$ which is stronger than assuming that $\Phi$ is a state. The main point is to prove the compatibility between traffic independence and the notion of positivity, stated in the following theorem.

**Theorem 1.2.** *The free product of traffic spaces preserves the positivity of traffic spaces, so that the free product of traffic spaces is well defined as a traffic space.*

In particular, for any traffic $a$, there exists a traffic space that contains a sequence of traffic independent variables distributed as $a$. Moreover, a traffic space can always be enlarged in order to introduce traffic independent random variables.

Interestingly, the proof of Theorem 1.2 requires a new characterization of traffic independence (contained in Theorem 2.8) which is much more similar to the usual definition of free independence (the trace of an alternated product of centered elements is centered) than the original one. We deduce from it a simple criterion to characterize the free independence of variables assuming their traffic independence. Part I of the article starts by presenting this aspect.

**1.1.5. Three canonical models of traffics.** We turn now to our last result, which was the first motivation of this article and whose demonstration uses both Theorem 1.1 and Theorem 1.2. It states that there exist three different ways of enlarging a $^*$-probability space into a traffic space, each one related to respectively the tensor, the free and the

Boolean independence. Let us be more explicit, starting with the model related to freeness. As explained, Theorem 1.1 in its full form gives a formula for the limiting traffic distribution of large unitary invariant random matrices which involves only the limiting non-commutative distribution. Replacing in this formula the limiting non-commutative distribution of matrices by an arbitrary distribution, we obtain a traffic distribution which is related to free independence as the following result highlights.

**Theorem 1.3.** *Let* $(\mathcal{A}, \Phi)$ *be a tracial* $^*$*-probability space. There exists a traffic space* $\mathcal{B}$ *such that:*

(1) $\mathcal{A} \subset \mathcal{B}$ *as* $*$*-algebras and the trace induced by* $\mathcal{B}$ *on* $\mathcal{A}$ *is* $\Phi$*;*

(2) *two families* $\mathbf{a}$ *and* $\mathbf{b} \in \mathcal{A} \subset \mathcal{B}$ *are* freely independent *in* $\mathcal{A}$ *if and only if they are traffic independent in* $\mathcal{B}$*.*

The formula for the traffic distribution given, the difficulty consists in proving that this distribution satisfies the positivity condition.

Remark that, as described in [17] and recalled in Section 8, an Abelian non-commutative probability space can be endowed with a structure of traffic space.

**Theorem 1.4.** *Let* $(\mathcal{A}, \Phi)$ *be a Abelian* $^*$*-probability space. There exists a traffic space* $\mathcal{B}$ *such that:*

(1) $\mathcal{A} \subset \mathcal{B}$ *as* $*$*-algebras and the trace induced by* $\mathcal{B}$ *on* $\mathcal{A}$ *is* $\Phi$*;*

(2) *two families* $\mathbf{a}$ *and* $\mathbf{b} \in \mathcal{A} \subset \mathcal{B}$ *are* tensor independent *in* $\mathcal{A}$ *if and only if they are traffic independent in* $\mathcal{B}$*.*

Finally, thanks to Section 8.1, one can produce an analogue construction for Boolean independence. We recall that any traffic space is endowed with two linear forms: a trace and a second linear form called the *anti-trace*.

**Theorem 1.5.** *Let* $(\mathcal{A}, \Psi)$ *be a* $^*$*-probability space. There exists a traffic space* $\mathcal{B}$ *such that:*

(1) $\mathcal{A} \subset \mathcal{B}$ *as* $*$*-algebras and the anti-trace induced by* $\mathcal{B}$ *on* $\mathcal{A}$ *is* $\Psi$*;*

(2) *two families* $\mathbf{a}$ *and* $\mathbf{b} \in \mathcal{A} \subset \mathcal{B}$ *are* Boolean independent *in* $\mathcal{A}$ *if and only if they are traffic independent in* $\mathcal{B}$*.*

This construction comes together with a large model for asymptotically Boolean independent random matrices.

In other words, the free product of traffic space leads to the tensor product, Boolean product or the free product of the probability spaces, depending on the way the $*$-distribution and the traffic distribution of our random variables are linked. It corresponds to three different types of traffic that we will define in Section 8: the traffics of free, tensor, or Boolean types. Interestingly, we also see that the last notions of monotone and anti-monotone independence (see [22, 23]) appear to describe the relations between traffics of different types when they are traffic independent. We sum up the non-commutative independences which follows from traffic independence in Figure 1.

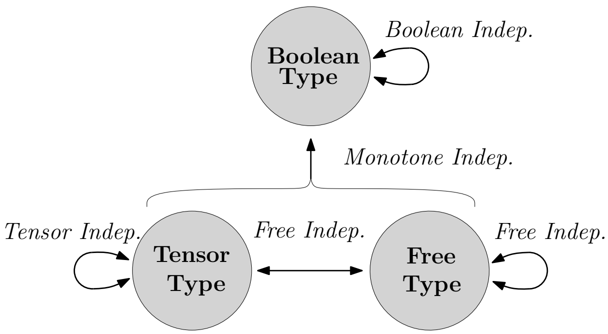


**Figure 1.** The non-commutative independences of traffics of free, tensor, and Boolean types which are traffic independent.

**Organization of the article.** In the rest of this introduction, we first recall the definitions of algebraic traffic spaces and traffic independence. Part I is dedicated to general facts on traffics. In Section 2, we introduce an equivalent definition of traffic independence. In Section 3, we define the free product of traffic spaces and prove Theorem 1.2. Part II is devoted to particular types of traffics, starting with the so-called unitarily invariant traffics that are introduced and described in Sections 4 and 5. Theorem 1.1 on unitarily invariant matrices is proved in Section 6. In Section 7, we prove Theorem 1.3 on the canonical extension of $^*$-probability spaces via traffics of free type. In Section 8, we investigate the canonical extensions of tensor and Boolean type, and prove Theorems 1.4 and 1.5.

## 1.2. Definitions

This section provides basic definitions from [17, Chapter 4] in the theory of traffic spaces.

**1.2.1. Algebras over an operad.** We first make more precise the definition of graph operations given in the introduction.

**Definition 1.6.** For all $K \geq 0$, a $K$*-graph operation* is a finite, connected and oriented graph with $K$ ordered edges, and two particular vertices (one input and one output). The set of $K$-graph operations is denoted by $\mathcal{G}_K$, and we define $\mathcal{G} = \bigcup_{K\geq 0} \mathcal{G}_K$.

A $K$-graph operation can produce a new graph operation from $K$ different graph operations thanks to the following *composition maps*

$$\mathcal{G}_K \times \mathcal{G}_{L_1} \times \cdots \times \mathcal{G}_{L_K} \to \mathcal{G}_{L_1+\cdots+L_K}$$
$$(g, g_1, \ldots, g_K) \mapsto g(g_1, \ldots, g_K),$$

for $K \geq 1$ and $L_i \geq 0$, $i = 1, \ldots, K$ which consist in replacing the $i$-th edge of $g \in \mathcal{G}_K$ by the $L_i$-graph operation $g_i$ (leading at the end to a $(L_1 + \cdots + L_K)$-graph operation), the input and output vertices of $g_i$ being substituted onto the input and output vertices of $e_i$, respectively. Let also consider the *action of the symmetric group*

$$\begin{aligned} S_K \times \mathcal{G}_K &\to \mathcal{G}_K \\ (\sigma, g) &\mapsto g^{(\sigma)}, \end{aligned}$$

for $K \geq 2$ which consists in reordering the edges of $g$ according to $\sigma$: if $e_1, \dots, e_K$ are the ordered edges of $g$, $e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(K)}$ are the ordered edges in $g^{(\sigma)}$. Finally, let us denote by id the graph operation which consists in two vertices and one edge from the input to the output. Endowed with those composition maps and the action of the symmetric groups, the set $\mathcal{G}$ is a *symmetric operad*, in the sense that it satisfies

(1) the *identity* property $g(\mathrm{id}, \dots, \mathrm{id}) = g = \mathrm{id}(g)$,

(2) the *associativity* property

$$\begin{aligned} & g\big(g_1(g_{1,1}, \dots, g_{1,k_1}), \dots, g_K(g_{K,1}, \dots, g_{K,k_K})\big) \\ & \quad = \big(g(g_1, \dots, g_K)\big)(g_{1,1}, \dots, g_{1,k_1}, \dots, g_{K,1}, \dots, g_{K,k_K}), \end{aligned}$$

(3) the *equivariance* properties $(g^{(\sigma)})(g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(K)}) = g(g_1, \dots, g_K)$; and $g(g_1^{(\sigma_1)}, \dots, g_K^{(\sigma_K)}) = (g(g_1, \dots, g_K))^{(\sigma_1 \times \dots \times \sigma_K)}$.

The element $\mathrm{id} \in \mathcal{G}_1$ is called the *identity* of the operad.

Let us now define how a $K$-graph operation can produce a new element from $K$ elements of a vector space in a linear way.

**Definition 1.7.** An *action* of an operad $\mathcal{G} = \bigcup_{K \geq 0} \mathcal{G}_K$ on a vector space $\mathcal{A}$ is the data, for all $K \geq 0$ and $g \in \mathcal{G}_K$, of a linear map $Z_g : \mathcal{A}^{\otimes K} \to \mathcal{A}$ such that: $\forall g \in \mathcal{G}_K$, $g_i \in \mathcal{G}$, $a_i \in \mathcal{A}$, $\sigma \in S_K$,

(1) $Z_{\mathrm{id}}$ is the identity on $\mathcal{A}$, where $\mathrm{id} \in \mathcal{G}_1$ is the identity of the operad,

(2) $Z_g(Z_{g_1} \otimes \dots \otimes Z_{g_K}) = Z_{g(g_1, \dots, g_K)}$,

(3) $Z_g(a_1 \otimes \dots \otimes a_K) = Z_{g_\sigma}(a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(K)})$.

A vector space on which acts $\mathcal{G}$ is called a *$\mathcal{G}$-algebra.* A $\mathcal{G}$-subalgebra is a vector subspace of a $\mathcal{G}$-algebra stable by the action of $\mathcal{G}$. A $\mathcal{G}$-morphism between two $\mathcal{G}$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $f : \mathcal{A} \to \mathcal{B}$ such that $f(Z_g(a_1, \dots, a_K)) = Z_g(f(a_1), \dots, f(a_K))$ for any $K$-graph operation $g$ and $a_1, \dots, a_K \in \mathcal{A}$.

In the following, $\mathcal{G}$ always denotes the operad of graph operations. We now review some linear maps $Z_g$ of particular interest by describing the underlying graphs $g$. At each time, we shall represent $g$ graphically, forgetting the mention of the ordering of edges when it is not relevant, and assuming the input is the rightmost vertex of the graph and the output the leftmost one when they are not equal.

- The only element of $\mathcal{G}_0$ is the graph $(\cdot)$ with a single vertex and no edge. By convention, the map $Z_{(\cdot)}$ is a linear map $\mathbb{C} \to \mathcal{A}$. It is then characterized by the image of $1 \in \mathbb{C}$, that is, denoted by $\mathbb{I} := Z_{(\cdot)}(1)$ and is called the unit of $\mathcal{A}$.

- By definition, $Z_{\cdot\leftarrow\cdot} = \mathrm{id}_{\mathcal{A}}$. The graph $(\cdot \to \cdot) \in \mathcal{G}_1$, which consists in two vertices and one edge from the output to the input, induces another involution on $\mathcal{A}$ which will be denoted by $a \mapsto a^t := Z_{\cdot\rightarrow\cdot}(a)$. We call $a^t$ the transpose of $a$.
- The graph operation $(\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot)$, which consists in three vertices and two successive edges from the input to the output, induces a bilinear map $(a,b) \in \mathcal{A}^2 \mapsto ab := Z_{\cdot\overset{1}{\leftarrow}\cdot\overset{2}{\leftarrow}\cdot}(a \otimes b) \in \mathcal{A}$ which gives to $\mathcal{A}$ a structure of associative algebra over $\mathbb{C}$, with unit $\mathbb{I}$. Hence, every $\mathcal{G}$-algebra is in particular a unital algebra.
- The Hadamard product is the bilinear map $(a,b) \in \mathcal{A}^2 \mapsto a \circ b := Z_{\cdot\leftleftarrows\cdot}(a \otimes b)$, where the graph operation consists in two vertices and two edges from the input to the output. It defines an associative and commutative product.
- The diagonal of an element $a \in \mathcal{A}$ is defined by $\Delta(a) := Z_{\circlearrowleft}(a)$, for the graph $\circlearrowleft$ with one vertex and one edge (which is a self loop). The map $\Delta$ is a projection, and its image $\Delta(\mathcal{A}) := \{\Delta(a),\ a \in \mathcal{A}\}$ is a commutative $\mathcal{G}$-subalgebra of $\mathcal{A}$.
- The degree of an element $a \in \mathcal{A}$ is defined by $\deg(a) := Z_{\downarrow}(a)$, for the graph $\downarrow$ with two vertices, where one is both the input and the output, and an edge from the second vertex to the input/output. The map deg is a projection. As $Z_{\circlearrowleft}(Z_{\downarrow}) = Z_{\downarrow}$ and $Z_{\downarrow}(Z_{\circlearrowleft}) = Z_{\circlearrowleft}$, its image is $\Delta(\mathcal{A})$.

**Example 1.8.** Denote by $\mathrm{M}_N(\mathbb{C})$ the algebra of $N$ by $N$ complex matrices. For any $K \geq 1$ and $g \in \mathcal{G}_K$ with vertex set $V$ and ordered edges $(v_1, w_1), \dots, (v_K, w_K)$, let us define $Z_g$ by setting, for all $A_1, \dots, A_K \in \mathrm{M}_N(\mathbb{C})$, the $(i,j)$-coefficient of $Z_g(A_1 \otimes \dots \otimes A_K)$ as

$$\big[Z_g(A_1 \otimes \dots \otimes A_K)\big](i,j) := \sum_{\substack{\phi: V \to [N] \\ \phi(in)=j,\, \phi(out)=i}} \prod_{k=1}^{K} A_k\big(\phi(w_k), \phi(v_k)\big).$$

This defines an action of the operad $\mathcal{G}$ on $\mathrm{M}_N(\mathbb{C})$. The product $AB = Z_{\cdot\overset{1}{\leftarrow}\cdot\overset{2}{\leftarrow}\cdot}(A \otimes B)$ induced by this action coincides with the classical product of matrices. The Hadamard product $A \circ B = Z_{\cdot\leftleftarrows\cdot}(A \otimes B)$ is the entry-wise product of matrices $(A(i,j)B(i,j))_{i,j=1}^N$. The diagonal of a matrix $\Delta(A) := Z_{\circlearrowleft}(A)$ and the transpose $A^t = Z_{\cdot\rightarrow\cdot}(A)$ are the diagonal $(\delta_{ij} A(i,i))_{i,j=1}^N$ and the transpose $(A(j,i))_{i,j=1}^N$ in the usual sense. The degree[1] $\deg(A) := Z_{\downarrow}(A)$ is the row sum diagonal matrix $(\delta_{ij} \sum_k A(i,k))_{i,j=1}^N$. For more information about the traffic distribution of matrices, see [17, Section 1.2].

The graph operations can be equivalently encoded in terms of analogues of polynomials, using in place of monomials, finite, connected, oriented graphs with edges labeled by variables, and turning the linearity on $\mathcal{A}^{\otimes K}$ into $K$-linearity on $\mathcal{A}$, $K \geq 2$. We also define now a notion with no input and output for the purpose of the next section, and later we will consider a generalization with arbitrary numbers of in/outputs.

[1] This terminology is motivated by the case where $A$ is the adjacency matrix of a graph with $N$ vertices.

**Definition 1.9.** Let $J$ be a labeling set.

- A test graph labeled in $J$ is a collection $T = (V, E, \gamma)$, where $(V, E)$ is a finite, connected and oriented graph and $\gamma : E \to J$ is a labeling of the edges by indices.
- A graph monomial labeled in $J$ is a collection $g = (V, E, \gamma, \mathbf{v})$, where $T = (V, E, \gamma)$ is a test graph and $\mathbf{v} = (in, out)$ is an ordered pair of vertices of $T$, considered respectively as the input and the output of $T$.

We denote by $\mathcal{T}\langle J\rangle$ the set of test graphs labeled in $J$, and by $\mathcal{G}\langle J\rangle$ the set of graph monomials labeled in $J$. We denote by $\mathbb{C}\mathcal{T}\langle J\rangle$ and $\mathbb{C}\mathcal{G}\langle J\rangle$ the vector spaces generated by elements of the respective sets.

The labeling map $\gamma$ of a graph monomial is not a bijection in general, so that a same variable can appear on several edges of the graph.

Let us consider a family $\mathbf{a} = (a_j)_{j\in J} \in \mathcal{A}^J$ of elements of a $\mathcal{G}$-algebra, and consider a graph monomial $t = (V, E, \gamma, \mathbf{v})$ with labels in $J$. Let us list arbitrarily the edges $E = \{e_1, \dots, e_K\}$ and denote by $g$ the $K$-graph operation $(V, E)$ with ordered edges $e_1, \dots, e_K$, and pair of input and output given by $\mathbf{v}$. We set $t(\mathbf{a}) = Z_g(a_{\gamma(e_1)} \otimes \cdots \otimes a_{\gamma(e_K)})$, which does not depend on the choice of the ordering of $e_1, \dots, e_K$, thanks to the equivariance property. For more details about graph polynomials, see [17, Section 4.2.2].

### 1.2.2. Algebraic traffic spaces.

**Definition 1.10.** An *algebraic traffic space* is a couple $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a $\mathcal{G}$-algebra and $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ is a linear functional, called the *combinatorial trace*, defined on the space of test graphs labeled in $\mathcal{A}$, satisfying

- the *unity* property $\tau[(\cdot)] = 1$ for $(\cdot)$ the graph with a single vertex and no edge,
- the *multi-linearity w.r.t. the edges* $\tau[T_{a+\lambda b}] = \tau[T_a] + \lambda\tau[T_b]$, for any test graph $T_{a+\lambda b} \in \mathcal{T}\langle\mathcal{A}\rangle$ having an edge $e_0$ with label $a + \lambda b$, where $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, and for $T_a$ and $T_b$ defined as $T$ with label $a$ and $b$ respectively for the edge $e_0$,
- the *substitution* property $\tau[T] = \tau[T_g]$ for any test graph $T \in \mathcal{T}\langle\mathcal{A}\rangle$ having an edge $e_0$ with label $g(\mathbf{a})$, where $g$ is a graph monomial and $\mathbf{a}$ a family of elements of $\mathcal{A}$, and $T_g$ obtained from $T$ by replacing the edge $e_0$ by the graph $g$ whose edges are labeled by the element of $\mathbf{a}$.

An element of an algebraic traffic space is called a traffic. A homomorphism between two algebraic traffic spaces $(\mathcal{A}, \tau)$ and $(\mathcal{A}', \tau')$ is a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{A}'$ such that

$$\tau'\big[T\big(f(\mathbf{a})\big)\big] = \tau\big[T(\mathbf{a})\big],$$

for any $T \in \mathcal{T}\langle J\rangle$ and $\mathbf{a} = (a_j)_{j\in J} \in \mathcal{A}^J$, where $f(\mathbf{a}) := (f(a_j))_{j\in J}$.

The map $\tau$ takes as input a test graph whose edges are labeled by elements of $\mathcal{A}$ and produces a complex number from it. There is no meaning in the expression $\tau[a]$ for an element $a \in \mathcal{A}$. In particular, $(\mathcal{A}, \tau)$ is not an algebraic non-commutative probability space. Nevertheless it can always be endowed with two different structures of algebraic non-commutative probability spaces.

**Definition 1.11.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space. The *trace* $\Phi : \mathcal{A} \to \mathbb{C}$ and the anti-trace $\Psi : \mathcal{A} \to \mathbb{C}$ are the linear maps given by the application of $\tau$ on a self loop and on a simple edge, namely

$$\Phi : a \mapsto \tau[\circlearrowleft^a], \quad \Psi : a \mapsto \tau[\cdot \overset{a}{\leftarrow} \cdot].$$

Recall that the product of two elements $a, b \in \mathcal{A}$ is defined by $ab := Z_{\cdot \overset{1}{\leftarrow} \cdot \overset{2}{\leftarrow} \cdot}(a \otimes b)$, and that endowed with this product $\mathcal{A}$ is an associative algebra. Then $(\mathcal{A}, \Phi)$ and $(\mathcal{A}, \Psi)$ are two algebraic non-commutative probability spaces. The map $\Phi$ is tracial in the sense that $\Phi(ab) = \Phi(ba)$ for any $a, b \in \mathcal{A}$, and it satisfies $\Phi(\Delta(a)) = \Phi(a)$ for any $a \in \mathcal{A}$. Properties relating the different functionals $\tau$, $\Phi$ and $\Psi$ are explained in [17, Section 4.2.4]

In the following definition, for a test graph $T$ of $\mathcal{T}\langle J \rangle$ and a family $\mathbf{a} \in \mathcal{A}^J$ of elements of a set $\mathcal{A}$, we denote $T(\mathbf{a}) \in \mathcal{T}\langle \mathcal{A} \rangle$ the test graph obtained by replacing labels $j \in J$ of the edges of $T$ by $a_j$. This definition is extended for $T \in \mathbb{C}\mathcal{T}\langle J \rangle$ by linearity.

**Definition 1.12.** Let $(\mathcal{A}, \tau)$ and $(\mathcal{A}_N, \tau_N)$, $N \geq 1$, be algebraic traffic spaces, and $J$ be an index set.

(1) The *traffic distribution* of a family $\mathbf{a} = (a_j)_{j \in J}$ of elements in $\mathcal{A}$ is the linear map $\tau_{\mathbf{a}} : T \in \mathbb{C}\mathcal{T}\langle J \rangle \mapsto \tau[T(\mathbf{a})] \in \mathbb{C}$.

(2) A sequence of families $\mathbf{a}_N \in \mathcal{A}_N^J$ *converges in traffic distribution* to $\mathbf{a}$ if the traffic distribution of $\mathbf{a}_N$ converges pointwise to the traffic distribution of $\mathbf{a}$ on $\mathbb{C}\mathcal{T}\langle J \rangle$.

**Example 1.13** (Example 1.8 continued). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in the classical sense and let us consider the algebra $\mathrm{M}_N(L^{\infty-}(\Omega, \mathbb{C}))$ of matrices whose coefficients are random variables with finite moments of all orders. Endowed with the action of the operad $\mathcal{G}$ described in Example 1.8, it is a $\mathcal{G}$-algebra, and it becomes an algebraic traffic space endowed with the combinatorial trace $\tau_N$ given by: for any test graph $T = (V, E, M)$ labeled in $\mathrm{M}_N(L^{\infty-}(\Omega, \mathbb{C}))$, where $M : E \to \mathrm{M}_N(L^{\infty-}(\Omega, \mathbb{C}))$,

$$\tau_N[T] = \mathbb{E}\Bigg[\frac{1}{N} \sum_{\phi : V \to [N]} \prod_{e=(v,w) \in E} \big(M(e)\big)\big(\phi(w), \phi(v)\big)\Bigg]. \tag{1.1}$$

The trace associated to $\tau_N$ is the usual normalized trace $\Phi_N : A \mapsto \mathbb{E}[\mathrm{Tr}\, A]/N$ and the anti-trace is the map $\Psi_N : A \mapsto \mathbb{E}[\sum_{i,j} A(i,j)/N]$.

**1.2.3. Möbius inversion and injective trace.** For any set $X$, denote by $\mathcal{P}(X)$ the set of partitions of $X$ equipped with refinement order, that is, $\pi \leq \pi'$ if the blocks of $\pi$ are included in blocks of $\pi'$. Let $(\mathcal{A}, \Phi)$ be a non-commutative probability space and denote by $\mathrm{NC}(K) \subset \mathcal{P}(\{1, \ldots, K\})$ the poset of non-crossing partitions of $\{1, \ldots, K\}$ [26, Lecture 9]. We recall that in an algebraic non-commutative probability space $(\mathcal{A}, \Phi)$, the free cumulants are the multi-linear maps $(\kappa)_{L \geq 1}$ on $\mathcal{A}^L$ given implicitly by

$$\Phi(a_1 \times \cdots \times a_K) = \sum_{\pi \in \mathrm{NC}(K)} \underbrace{\prod_{\{i_1 < \cdots < i_L\} \in \pi} \kappa_L(a_{i_1}, \ldots, a_{i_L})}_{=: \kappa(\pi)}.$$

With $\Phi(\pi)$ defined as $\kappa(\pi)$ using $\Phi(a_{i_1} \dots a_{i_L})$ instead of $\kappa_L(a_{i_1}, \dots, a_{i_L})$, we can express $\kappa(\pi)$ in terms of $\Phi(\pi')$ for $\pi' \geq \pi$ thanks to Möbius inversion in the poset on non-crossing partitions.

In order to define traffic independence, we also need to define a transform of combinatorial traffic traces. When $\mathcal{A}$ is a fixed set, let $T = (V, E, \gamma)$ be a test graph in $\mathcal{T}\langle\mathcal{A}\rangle$, with vertex set $V$. For any partition $\pi \in \mathcal{P}(V)$ of $V$, we denote by $T^\pi = (V^\pi, E^\pi, \gamma^\pi)$ the test graph obtained by identifying vertices in a same block of $\pi$. More precisely:

- the vertex set of $T_\pi$ is the set of blocks of $\pi$,
- each edge $e = (v, w)$ of $T$ generates an edge $e^\pi = (B_v, B_w)$, where $B_v$ denotes the block of $\pi$ containing $v$,
- the label of $e^\pi$ is the label of $e$, namely $\gamma^\pi(e^\pi) = \gamma(e)$.

We say that $T^\pi$ is a *quotient* of $T$. Denote $0_V$ the partition of $V$ with singletons only (it then satisfies $T^{0_V} = T$).

**Definition 1.14.** Let $\mathcal{A}$ be a set and let $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ be a linear form. We define the injective version of $\tau$, and denote $\tau^0$, the linear form on $\mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$ implicitly given by the following formula: for any test graph $T \in \mathcal{T}\langle\mathcal{A}\rangle$

$$\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi],$$

in such a way for any test graph $T$ one has

$$\tau^0[T] = \sum_{\pi \in \mathcal{P}(V)} \text{Möb}_{\mathcal{P}(V)}(0_V, \pi) \cdot \tau[T^\pi].$$

The injective version of a combinatorial trace (resp. a traffic distribution) is called the *injective trace* (resp. the *injective distribution*).

**Example 1.15.** The injective version $\text{Tr}^0$ of the trace of test graph in random matrices of $M_N(\mathbb{C})$ defined in (1.1) is given, for $T = (V, E, M)$ a test graph labeled in $\mathrm{M}_N(L^{\infty-}(\Omega, \mathbb{C}))$, by[2]

$$\tau_N^0[T] = \mathbb{E}\Bigg[\frac{1}{N} \sum_{\substack{\phi: V \to [N] \\ \text{injective}}} \prod_{e=(w,v)\in E} \big(M(e)\big)\big(\phi(w), \phi(v)\big)\Bigg].$$

Limiting injective combinatorial distributions of usual matrix models (unitary Haar matrices, uniform permutation matrices, certain Wigner matrices) are proved to exist [17, Chapter 3] and are shown to have simple and natural expressions.

**1.2.4. Traffic independence.** Let $J$ be a fixed index set and, for each $j \in J$, let $\mathcal{A}_j$ be some sets. Given a family of linear maps $\tau_j : \mathbb{C}\mathcal{T}\langle\mathcal{A}_j\rangle \to \mathbb{C}$, $j \in J$, sending the graph with

[2] The following identity is the origin of the name "injective trace".

no edge to one, we shall define a linear map denoted $\star_{j\in J}\tau_j : \mathbb{C}\mathcal{T}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle$ with the same property and called the free product of the $\tau_j$'s. The terminology *free* product should be understood as *canonical* product, and may not be confused with the terminology *free* independence. Therein, $\bigsqcup_{j\in J}\mathcal{A}_j$ denotes the disjoint union of copies of $\mathcal{A}_j$, although the sets $\mathcal{A}_j$ can originally intersect or be equal: it is formally defined as the set of all couples $(j, a)$ where $j \in J$ and $a \in \mathcal{A}_j$.

Let us consider a test graph $T$ in $\mathcal{T}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle$ and introduce an undirected graph as follows. We first call *colored components* of $T$ with respect to the families $(\mathcal{A}_j)_{j\in J}$ the maximal nontrivial connected subgraphs of $T$ whose edges are labeled by elements of $\mathcal{A}_j$ for some $j \in J$ (they are elements of $\mathcal{T}\langle\mathcal{A}_j\rangle$); we then call $j$ the *color* of the component. There is no confusion about the definition of colored components because of the convention for $\bigsqcup_{j\in J}\mathcal{A}_j$. When there is no ambiguity about the collection $(\mathcal{A}_j)_{j\in J}$, we denote by $\mathcal{CC}(T)$ the set of colored components of $T$. We call *connectors* of $T$ the vertices of $T$ belonging to at least two different colored components. The bipartite graph $\mathcal{GCC}(T)$ defined below is called *graph of colored components* of $T$ with respect to $(\mathcal{A}_j)_{j\in J}$:

- the vertices of $\mathcal{GCC}(T)$ are the colored components of $T$ and its connectors;
- there is an edge between a colored component in $\mathcal{CC}(T)$ and a connector if the connector belongs to the component.

The following definition is from [17, Section 2.2.].

**Definition 1.16.**

(1) For each $j \in J$, let $\mathcal{A}_j$ be a set and $\tau_j : \mathbb{C}\mathcal{T}\langle\mathcal{A}_j\rangle \to \mathbb{C}$ be a linear map sending the test graph with no edges to one. The free product of the maps $\tau_j$ is the linear map

$$\star_{j\in J}\tau_j : \mathbb{C}\mathcal{T}\Big\langle\bigsqcup_j \mathcal{A}_j\Big\rangle \to \mathbb{C}$$

whose injective version is given by: for any test graph $T$,

$$(\star_{j\in J}\tau_j)^0[T] = \mathbb{1}\big(\mathcal{GCC}(T) \text{ is a tree}\big) \times \prod_{S\in\mathcal{CC}(T)} \tau^0_{j(S)}[S],$$

where $j(S)$ is the index of the labels of $S$.

(2) Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $J$ be a fixed index set. For each $j \in J$, let $\mathcal{A}_j \subset \mathcal{A}$ be a $\mathcal{G}$-subalgebra. The subalgebras $(\mathcal{A}_j)_{j\in J}$ are called traffic independent whenever the restriction of $\tau$ on the test graphs labeled by elements of $\mathcal{A}_j$, $j \in J$, coincides with $\star_{j\in J}\tau_j$.

(3) Let $X_j$, $j \in J$ be subsets of $\mathcal{A}$ and let $(\mathbf{a}_j)_{j\in J}$ be a family of elements of $\mathcal{A}$. Then $(X_j)_{j\in J}$ (resp. $(\mathbf{a}_j)_{j\in J}$) are called traffic independent whenever the $\mathcal{G}$-subalgebra induced by the $X_j$'s (resp. by the $\mathbf{a}_j$'s) are traffic independent.

The motivation for introducing this definition is, in the context of large matrices, Example 1.13, the asymptotic traffic independence for permutation invariant matrices, see [17, Theorem 1.8].

We end this section with the following elementary property of traffic independence, see [17, Lemma 5.3].

**Lemma 1.17.** *Traffic independence is symmetric and associative, i.e.,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *are independent if and only if* $\mathcal{A}_2$ *and* $\mathcal{A}_1$ *are independent, and* $\mathcal{A}_j, j = 1,2,3$ *are independent if an only if* $\mathcal{A}_1$ *and* $(\mathcal{A}_2, \mathcal{A}_3)$ *are independent and* $\mathcal{A}_2$ *and* $\mathcal{A}_3$ *are independent.*

**Part I**

# General traffic spaces

## Presentation

According to Section 1.2, traffic independence in an algebraic traffic space $(\mathcal{A}, \tau)$ is defined in terms of the injective version $\tau^0$ of $\tau$, thanks to the formula involving the graph of colored components. Such a definition of independence is unusual in non-commutative probability, where the injective trace has no analogue. As a comparison, let us remind the two equivalent definitions of free independence in free probability. It is usually defined by a relation of moments, namely the centering of alternated products of centered elements. The second usual characterization of free independence is the vanishing of mixed free cumulants.

We propose in Theorem 2.8 of Section 2 a characterization of traffic independence in terms of moment functions as the centering of some *generalized alternated products of reduced elements*, in an appropriate sense that we shall make precise. Note that Gabriel proposes in [11] a definition of traffic cumulants, and traffic independence is the vanishing of these mixed traffic cumulants.

In Section 3, we construct the product of traffic spaces: given for each $j \in J$ an algebraic traffic space $(\mathcal{A}_j, \tau_j)$, we construct a new algebraic traffic space $(\mathcal{A}, \tau)$ that contains the $\mathcal{A}_j$ as independent $\mathcal{G}$-subalgebras. The space $\mathcal{A}$ will be made with equivalent classes of graph operations with an input and output whose edges are labeled by the $\mathcal{A}_j$. The combinatorial trace $\tau$ will be the extension to $\mathcal{A}$ of the free product of the combinatorial traces $\tau_j$, $j \in J$.

Positivity of state is another important notion in noncommutative probability. We propose a definition of positivity for combinatorial trace in Section 3.3. We prove that the free product traffic spaces with positive traces also admits a positive trace.

## 2. A natural characterization of traffic independence

### 2.1. Statement

In order to give the characterization of traffic independence which is the analogue of the usual presentation of freeness, we need a generalization of test graphs and graph polynomials with arbitrary numbers of marked vertices. To explain this fact, recall that the

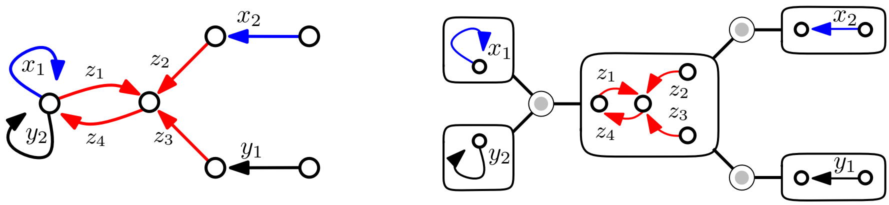


**Figure 2.** Left: a test graph $T$ in three families of traffics $(x_1, x_2)$, $(y_1, y_2)$ and $(z_1, z_2, z_3, z_4)$. Note that $\tau[T] = \Phi[\Delta(x_1)\Delta(y_2)\,(z_4 \circ z_1^t)\,\deg(z_2\,x_2)\deg(z_3\,y_1)\,z_2\,\Delta(y_1)]$. Right: the graph of colored component $\mathcal{GCC}(T)$.

definition of traffic independence involves the graph of colored components. To define correctly the operation which consists in reconstructing a test graph from its colored components and its graph of colored components, we need formal objects that are specified in the two following definitions (see Figure 2).

**Definition 2.1.** A graph monomial of rank $n \geq 1$ (in short a $n$-graph monomial) labeled in $J$ is the data $g = (V, E, \gamma, \mathbf{v})$ of a test graph $T = (V, E, \gamma)$ and of a $n$-tuple $\mathbf{v} = (v_1, \dots, v_n)$ of vertices of $T$, called the *outputs*. We denote by $\mathcal{G}^{(n)}\langle J\rangle$ the set of $n$-graph monomials and by $\mathbb{C}\mathcal{G}^{(n)}\langle J\rangle$ the space of $n$-graph polynomials.

We have $\mathbb{C}\mathcal{G}^{(2)}\langle J\rangle = \mathbb{C}\mathcal{G}\langle J\rangle$ where a graph monomial of rank 2 is identified with the graph monomial whose input is the first output. A test graph is also called a 0-graph monomial and we set $\mathbb{C}\mathcal{G}^{(0)}\langle J\rangle := \mathbb{C}\mathcal{T}\langle J\rangle$. To define generalized products of graph polynomials of arbitrary rank, we use the following objects, drawn in Figure 3.

**Definition 2.2.** A *bigraph operation of rank* $n \geq 1$ (in short an $n$-bigraph operation) in $L \geq 0$ variables is the data of

- a finite, connected, undirected and bipartite graph $g$, endowed with a bipartition of its vertices into two sets $V_{in}(g)$ and $V_{co}(g)$, whose elements are called *inputs* and *connectors*,
- with exactly $L$ ordered inputs, given together with an ordering of its edges around each input,
- and the data of an ordered subset $V_{out}(g)$ consisting in $n$ elements of the connectors $V_{co}(g)$ that we call outputs,

and such that all connectors that are not an output have degree greater than or equal to 2. We denote by $\mathcal{B}^{(n)}$ the set of $n$-bigraph operations. For any $L, n \geq 0$ and any tuple $\mathbf{d} = (d_1, \dots, d_L) \in (\mathbb{N}^*)^L$, we denote by $\mathcal{B}^{(n)}_{L,\mathbf{d}}$ if $L \neq 0$ and by $\mathcal{B}^{(n)}_0$ otherwise the set of $n$-bigraph operations with $L$ inputs such that the $\ell$-th one has degree $d_\ell$.

An $n$-bigraph operation in $L$ variables with degrees $d_1, \dots, d_L$ has to be thought as an operation that accepts $L$ objects with ranks $d_1, \dots, d_L$, and produces a new object of rank $n$. The set of bigraph operations is actually an operad, although we do not use this fact.

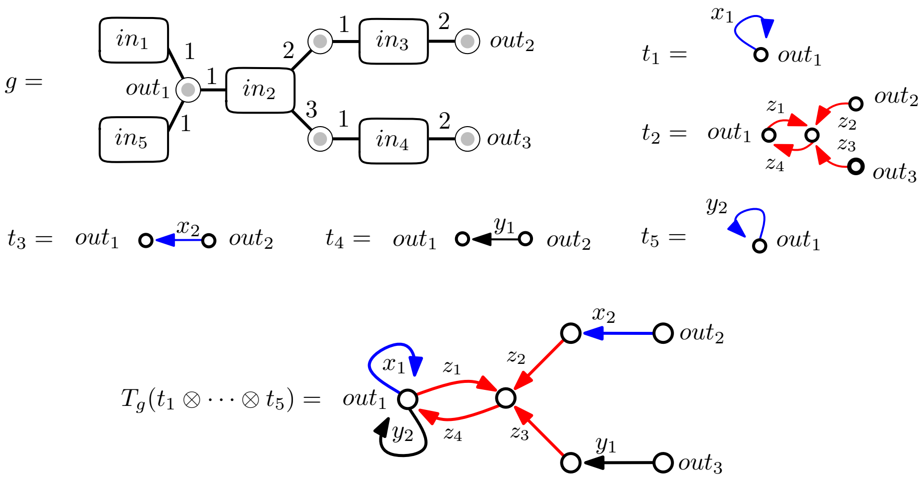


**Figure 3.** A bigraph operation $g$ of order 3 with 5 inputs and 3 outputs with degree sequence $(1,3,2,2,1)$; the numbers in the figure describe the order of the edges around each input. Five graph operations $t_1,\ldots,t_5$ which satisfy that $t_1 \otimes \cdots \otimes t_5$ is $g$-alternated. The graph operation $T_g(t_1 \otimes \cdots \otimes t_5)$.

In particular, an $n$-bigraph operation can produce a new $n$-graph monomial from $L$ different graph monomials in the following way, see Figure 3. Let us consider $L$ graph monomials $t_1,\ldots,t_L$ labeled on some set $\mathcal{A}$, with respective number of outputs given by $\mathbf{d} \in (\mathbb{N}^*)^L$ (that is, $t_\ell \in \mathcal{G}^{(d_\ell)}\langle\mathcal{A}\rangle$), and a bigraph operation $g \in \mathcal{B}^{(n)}_{L,\mathbf{d}}$. Replacing the $\ell$-th input of $g$ and its adjacent ordered edges $(e_1,\ldots,e_{d_\ell})$ by the graph of $t_\ell$, identifying for each $k \in [L]$ the connector attached to $e_k$ with the $k$-th output of $t_\ell$, yields a connected graph. We denote by $T_g(t_1 \otimes \cdots \otimes t_L) \in \mathcal{G}^{(n)}\langle\mathcal{A}\rangle$ the $n$-graph monomial whose labeling is induced by those of $t_1,\ldots,t_L$, and with outputs given by the outputs of $g$. We then define by linear extension

$$\begin{aligned} T_g : \mathbb{C}\mathcal{G}^{(d_1)}\langle\mathcal{A}\rangle \otimes \cdots \otimes \mathbb{C}\mathcal{G}^{(d_L)}\langle\mathcal{A}\rangle &\to \mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle \\ t_1 \otimes \cdots \otimes t_L &\mapsto T_g(t_1 \otimes \cdots \otimes t_L). \end{aligned}$$

**Example 2.3** (Example 1.8 continued). Assume that $\mathcal{A}$ is a subset of $M_N(\mathbb{C})$ given by a family of matrices $\mathbf{A}_N = (A_j)_{j\in J}$. There is a natural way to associate a tensor of order $n$ to an $n$-graph monomial. When $g$ is a bigraph operation, the operator $T_g$ is then compatible with a natural operation of $g$ on tensors explained below.

- Let $\mathbf{A}_N = (A_j)_{j\in J}$ be a family of matrices and $t = (V,E,\gamma,\mathbf{v})$ be a $n$-graph monomial labeled by $J$. We define a tensor $t(\mathbf{A}_N) \in (\mathbb{C}^N)^{\otimes n}$ as follows. Denoting by $\mathbf{v} = (v_1,\ldots,v_n)$ the sequence of outputs of $t$ and by $(\xi_i)_{i=1,\ldots,N}$ the canonical basis of $\mathbb{C}^N$, we set,

$$t(\mathbf{A}_N) = \sum_{\phi:V\to[N]} \prod_{e=(v,w)\in E} A_{\gamma(e)}\big(\phi(w),\phi(v)\big)\xi_{\phi(v_1)} \otimes \cdots \otimes \xi_{\phi(v_n)}.$$

- More generally, let $g$ be an $n$-bigraph operation with $L$ inputs; for $\ell = 1, \dots, L$, let $d_k$ be the degree of the $\ell$-th input of $g$, and let $t_\ell \in (\mathbb{C}^N)^{\otimes d_\ell}$ be a tensor of order $d_\ell$; set $\mathbf{t}_N = (t_\ell)_{1 \le \ell \le L}$. Denote by $(v_1, \dots, v_n)$ the outputs of $g$ and for each $\ell = 1, \dots, L$ denote by $(w_1^\ell, \dots, w_{d_\ell}^\ell)$ the ordered neighborhood connectors of the $\ell$-th input. Then we define an element of $(\mathbb{C}^N)^{\otimes n}$ by

$$\mathfrak{T}_g(\mathbf{t}_N) = \sum_{\phi : V_{\mathrm{co}}(g) \to [N]} \prod_{\ell=1}^{L} t_k\big(\phi(w_1^\ell), \dots, \phi(w_{d_\ell}^\ell)\big) \xi_{\phi(v_1)} \otimes \dots \otimes \xi_{\phi(v_n)}.$$

- When $t_1, \dots, t_L$ are $L$ graph monomials labeled by $J$, with respective number of outputs given by $\mathbf{d} \in (\mathbb{N}^*)^L$, and a bigraph operation $g \in \mathcal{B}_{L,\mathbf{d}}^{(n)}$, it is then elementary to check that

$$T_g(t_1 \otimes \dots \otimes t_L)(\mathbf{A}_N) = \mathfrak{T}_g\big(t_1(\mathbf{A}_N), \dots, t_L(\mathbf{A}_N)\big).$$

**Definition 2.4.** Let $J$ be an index set and $(\mathcal{A}_j)_{j \in J}$ be a family of sets, and let $g \in \mathcal{B}_{L,\mathbf{d}}^{(n)}$ be a bigraph operation with $\mathbf{d} = (d_1, \dots, d_L)$. A tensor product $(t_1 \otimes \dots \otimes t_{L'})$ of graph polynomials labeled by $\bigsqcup_j \mathcal{A}_j$ is alternated along $g$ (in short $g$-*alternated*) whenever

(1) $L' = L$,

(2) $t_i \in \mathbb{C}\mathcal{G}^{(d_i)}\langle \mathcal{A}_{j_i} \rangle$ for each $i = 1, \dots, L$, and

(3) for all $p, q \in [L]$ such that the $p$-th and the $q$-th inputs are neighbors of a same connector, then $j_p \neq j_q$.

Let $T_g$ be a bigraph operation and let $m_1 \otimes \dots \otimes m_L$ be a tensor product of graph monomials, labeled in a set $\bigsqcup_j \mathcal{A}_j$, $j \in J$, alternated along $g \in \mathcal{B}_{L,\mathbf{d}}^{(0)}$. Assume that $T_g$ does not identify any pair of outputs of each $m_\ell$ and that the output vertices of each $m_\ell$ are pairwise distinct. Then $T_g(m_1 \otimes \dots \otimes m_L)$ is a test graph with graph of colored components $g$, and its colored components are $m_1, \dots, m_L$, (considered as graphs with no outputs). Reciprocally, the graph of colored component gives a decomposition of any test graph as an element of the form $T_g(m_1 \otimes \dots \otimes m_L)$. This decomposition is unique up to the symmetry of a certain automorphism group introduced later in Section 3.3.

We shall now define a notion of *reduced* $n$-graph polynomials. For any $n \ge 2$, any partition $\pi \in \mathcal{P}(n)$ of $\{1, \dots, n\}$, and any $n$-graph monomial $g$ with outputs $(v_1, \dots, v_n)$, let us denote by $g^\pi$ the quotient graph obtained by identifying vertices $v_1, \dots, v_n$ that belong to a same block of $\pi$, with outputs given by the images of $(v_1, \dots, v_n)$ by the quotient map, so that the edges of $g^\pi$ can be identified with the edges of $g$. This defines a linear map

$$\Delta_\pi : \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle \to \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$$

such that $\Delta_\pi(g) = g^\pi$ for $n$-graph monomials $g$. The map $\Delta_\pi$ can also be seen as the action of a bigraph operation (see an example in Figure 4). Denote respectively by $0_n$ and $1_n$ the partitions of $\{1, \dots, n\}$ made of $n$ singletons and of one single block respectively. Note that $\Delta_{0_n}(g) = g$ for any $g \in \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$.

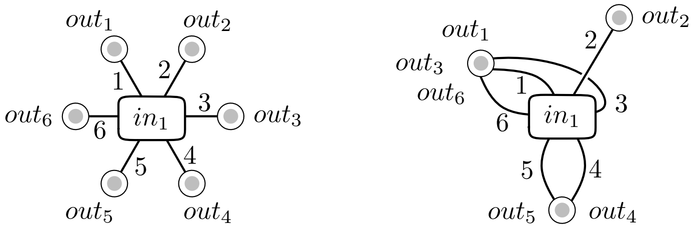


**Figure 4.** The bigraph operations $\Delta_{0_6}$ (left) and $\Delta_{\{\{1,3,6\},\{2\},\{4,5\}\}}$ (right).

**Definition 2.5.** Let $\mathcal{A}$ be a set and $\tau : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ be a linear form. We extend $\tau$ to a linear map

$$\mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \oplus \mathbb{C}\mathcal{G}^{(1)}\langle\mathcal{A}\rangle \to \mathbb{C}$$

by forgetting the position of the output in 1-graph monomials. A $n$-graph polynomial $t \in \mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle$ is called *reduced* with respect to $\tau$, if

- $n \in \{0, 1\}$ and $\tau(t) = 0$, or
- $n \geq 2$ and for any $\pi \in \mathcal{P}(n) \setminus \{0_n\}$ one has $\Delta_\pi(t) = 0$.

Note that the reducedness condition does not depend on $\tau$ when $n \geq 2$.

**Example 2.6.** If $n = 2$, then $\Delta_{1_2}(t) = \Delta(t)$, where we recall that the diagonal operator $\Delta$ is the graph operation with one vertex and one edge. So $t$ is reduced if and only if $\Delta(t) = 0$.

**Example 2.7.** Let $\mathbf{A}_N$ be a family of matrices of size $N$ by $N$ and let $t$ be a $n$-graph polynomial, $n \geq 2$. Consider the $n$-tensor $t(\mathbf{A}_N)$ defined in Example 2.3, and denote by $B_{\mathbf{i}}$, $\mathbf{i} \in [N]^n$, its components in the canonical basis. If $t$ is reduced, then $B_{\mathbf{i}} = 0$ as soon as two indices of $\mathbf{i}$ are equal. In particular for $n = 2$, if $t$ is reduced, then $t(\mathbf{A}_N)$ is a matrix with vanishing diagonal entries.

We can now state the main result of the section.

**Theorem 2.8.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space with trace $\Phi$ and anti-trace $\Psi$. For each $j \in J$, let $\mathcal{A}_j$ be a $\mathcal{G}$-subalgebra. The following properties are equivalent:*

(1) *The $\mathcal{G}$-subalgebras $\mathcal{A}_j$, $j \in J$, are traffic independent (Definition 1.16).*

(2) *One has $\tau[h] = 0$ for any $h = T_g(t_1 \otimes \cdots \otimes t_L)$ in $\mathbb{C}\mathcal{T}\langle\bigsqcup_j \mathcal{A}_j\rangle$ where $g \in \mathcal{B}^{(0)}$ is a bigraph operation and $t_1 \otimes \cdots \otimes t_L$ is a $g$-alternated tensor product of reduced elements with respect to $\tau$.*

(3) *One has $\Phi[h] = 0$ for any $h = T_g(t_1 \otimes \cdots \otimes t_L)$ in $\mathbb{C}\mathcal{G}^{(2)}\langle\bigsqcup_j \mathcal{A}_j\rangle$, where $g \in \mathcal{B}^{(2)}$ is a bigraph operation and $t_1 \otimes \cdots \otimes t_L$ is a $g$-alternated tensor product of reduced elements with respect to $\tau$.*

(4) *One has $\Psi[h] = 0$ for any $h = T_g(t_1 \otimes \cdots \otimes t_L)$ in $\mathbb{C}\mathcal{G}^{(2)}\langle\bigsqcup_j \mathcal{A}_j\rangle$, where $g \in \mathcal{B}^{(2)}$ is a bigraph operation and $t_1 \otimes \cdots \otimes t_L$ is a $g$-alternated tensor product of reduced elements with respect to $\tau$.*

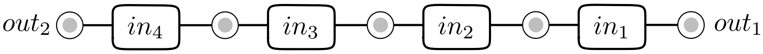


**Figure 5.** The bigraph operations $g$ of the proof of Corollary 2.9 for $n = 4$.

Hence traffic independence is the centering of alternated bigraph operations of reduced elements with respect to either $\tau$, $\Phi$ or $\Psi$. The proof of the theorem is given in the next section.

As a direct application, we get a useful criterion of free independence.

**Corollary 2.9.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space such that $\mathcal{A}$ is a ${}^*$-algebra and the associated trace $\Phi$ is a state. Denote for any $a \in \mathcal{A}$*

$$\eta(a) = \tau[{}^{a}\circlearrowleft \cdot \circlearrowright^{a^*}] - \big|\tau[\circlearrowright^{a}]\big|^2 = \Phi\big(\Delta(a^*)\Delta(a)\big) - \big|\Phi(a)\big|^2 = \Phi(a^* \circ a) - \big|\Phi(a)\big|^2,$$

*where we recall (see Section 1.2.1) that $\Delta = Z_{\circlearrowleft}$ is the diagonal operator and $(a \circ b) = Z_{\cdot\leftleftarrows\cdot}(a \otimes b)$ is the Hadamard product. Let $\mathcal{B} \subset \mathcal{A}$ be a unital ${}^*$-subalgebra such that $\eta(a) = 0$ for any $a \in \mathcal{B}$, and let $\mathcal{B}_j \subset \mathcal{B}$, $j \in J$, be subalgebras. If $(\mathcal{B}_j)_{j \in J}$ are traffic independent in $(\mathcal{A}, \tau)$, then they are freely independent in the ${}^*$-probability space $(\mathcal{B}, \Phi_{|\mathcal{B}})$.*

**Example 2.10.** In [17, Proposition 2.16], it is proved that two independent traffics $a$ and $b$ such that $\eta(a) \neq 0 \neq \eta(b)$ are not free independent with respect to the trace. If $\eta(a) \neq 0$ and $\eta(b) = 0$, both situations can happen as we can see with the limits of Wigner matrices, uniform permutation matrices and diagonal matrices [17]: the map $\eta$ vanishes only for the two first models, a Wigner matrices is asymptotically free from a diagonal matrix, but a uniform permutation matrix is not asymptotically free from a diagonal matrix.

*Proof of Corollary 2.9.* Since the trace defined on $\mathcal{A}$ is a state, the assumption implies, for every $a \in \mathcal{B}$, that $\Delta(a)$ has the same ${}^*$-distribution as $\Phi(a)\mathbb{I}$. Let $(\mathcal{B}_j)_{j \in J}$ be traffic independent ${}^*$-subalgebras of $\mathcal{B}$. Let $a_1, \dots, a_n \in \mathcal{B}$, such that for any $k \in [n]$, $\Phi(a_k) = 0$ and $a_k \in \mathcal{B}_{j_k}$, with $j_1 \neq j_2 \neq \cdots \neq j_n$. Then,

$$\begin{aligned} \Phi\big(\big(a_1 - \Delta(a_1)\big) \dots \big(a_n - \Delta(a_n)\big)\big) &= \Phi\big(\big(a_1 - \Phi(a_1)\big) \dots \big(a_n - \Phi(a_n)\big)\big) \\ &= \Phi(a_1 \dots a_n). \end{aligned}$$

Let $g$ be the bigraph operation with two outputs $out_1$ and $out_2$, $n$ inputs and $n+1$ connectors, whose graph is a directed line from $out_1$ to $out_2$, with input vertices (alternating with the connectors) ordered consecutively from $out_1$ to $out_2$, see Figure 5.

Then, denoting by $t_i$ the graph monomial with two vertices, distinct input and output, and an edge labeled by $a_i$ from the input to the output, one has

$$\Phi\big(\big(a_1 - \Delta(a_1)\big) \dots \big(a_n - \Delta(a_n)\big)\big) = \Phi\Big[T_g\Big(\big(t_1 - \Delta(t_1)\big) \otimes \cdots \otimes \big(t_n - \Delta(t_n)\big)\Big)\Big],$$

and $(t_1 - \Delta(t_1)) \otimes \cdots \otimes (t_n - \Delta(t_n))$ is a $g$-alternated tensor product of reduced graph polynomials, so that by Theorem 2.8 we get $\Phi(a_1 \dots a_n) = 0$. ∎

### 2.2. Proof of Theorem 2.8

**2.2.1. A decomposition of graph polynomials.** We start by stating several preliminary lemmas. The first three statements are about the space of $n$-graph polynomials $\mathbb{C}\mathcal{G}\langle\bigsqcup_j \mathcal{A}_j\rangle$. Note that in these lemmas we only assume that the sets $\mathcal{A}$ and $\mathcal{A}_j$, $j \in J$, are arbitrary ensembles, we do not use their $\mathcal{G}$-algebra structure. The first lemma gives an explicit characterization of reducedness.

**Lemma 2.11.** *Let $\mathcal{A}$ be an ensemble and for $n \geq 0$ let $m$ a $n$-graph monomial labeled in $\mathcal{A}$. Denote by $\mathcal{O}$ the output set of $m$ (empty if $n = 0$). For each partition $\sigma$ of $\mathcal{O}$, recall that $\Delta_\sigma(m) = m^\sigma$ denotes the graph monomial obtained by identifying the outputs of $m$ that belong to a same block of $\sigma$. Let us denote by* Möb *the Möbius function for the poset of partitions of $\mathcal{O}$ (Section 1.2.3) and $0_\mathcal{O}$ the partition of $\mathcal{O}$ made of singletons. Then, with $(\cdot)$ denoting the graph with one vertex, no edges, and $n$ equal outputs,*

$$p(m) := \begin{cases} m - \tau(m) \times (\cdot) & \text{if } n = 1, 0, \\ \sum_{\sigma \in \mathcal{P}(\mathcal{O})} \text{Möb}(0_\mathcal{O}, \sigma) m^\sigma & \text{if } n \geq 2. \end{cases}$$

*is a reduced $n$-graph polynomial with respect to $\tau$. Moreover, extending $p$ by linearity on $n$-graph polynomials, every reduced $n$-graph polynomial $t$ satisfies $t = p(t)$.*

*Proof.* The proposition is clear if $n = 0, 1$. Assume $n \geq 2$ in the following. For any $\nu \in \mathcal{P}(\mathcal{O})$,

$$\begin{aligned} \Delta_\nu\big(p(m)\big) &= \Delta_\nu\Big(\sum_{\sigma \in \mathcal{P}(\mathcal{O})} \text{Möb}(0_\mathcal{O}, \sigma) m^\sigma\Big) \\ &= \sum_{\mu \in \mathcal{P}(\mathcal{O})} \Big(\sum_{\sigma \in \mathcal{P}(\mathcal{O}) : \sigma \vee \nu = \mu} \text{Möb}(0_\mathcal{O}, \sigma)\Big) m^\mu, \end{aligned}$$

where $\sigma \vee \nu$ is the join of the partitions $\sigma$ and $\nu$, i.e. the smallest partition whose blocks contain those of $\sigma$ and $\nu$. Now, for any $\mu \in \mathcal{P}(\mathcal{O})$, by [30, Sections 3.6 and 3.7] for the first and last equalities, one has

$$\begin{aligned} \sum_{\sigma \in \mathcal{P}(\mathcal{O}) : \sigma \vee \nu = \mu} \text{Möb}(0_\mathcal{O}, \sigma) &= \sum_{\sigma \leq \mu} \sum_{\sigma \vee \nu \leq \xi \leq \mu} \text{Möb}(\xi, \mu)\, \text{Möb}(0_\mathcal{O}, \sigma) \\ &= \sum_{\nu \leq \xi \leq \mu} \text{Möb}(\xi, \mu) \Big(\sum_{\sigma \leq \xi} \text{Möb}(0_\mathcal{O}, \sigma)\Big) \\ &= \sum_{\nu \leq \xi \leq \mu} \text{Möb}(\xi, \mu) \delta_{\xi, 0_\mathcal{O}} = \delta_{\nu, 0_\mathcal{O}} \text{Möb}(0_\mathcal{O}, \mu). \end{aligned}$$

Hence we have obtained $\Delta_\nu(p(m)) = \delta_{\nu, 0_\mathcal{O}} p(m)$, that is, $p(m)$ is reduced. To conclude, consider a reduced graph monomial $m$. Then, any partition $\sigma$ of its outputs $\mathcal{O}$ satisfies $m^\sigma = \delta_{\sigma, O_\mathcal{O}} m$. Therefore $p(m) = \text{Möb}(0_\mathcal{O}, 0_\mathcal{O}) m = m$. ∎

**Remark 2.12.** Let $m$ be a $n$-graph monomial with output set $\mathcal{O}$. For any $\eta \in \mathcal{P}(\mathcal{O})$, let us define $p_\eta(m) = \sum_{\pi \geq \eta} \text{Möb}(\eta, \pi) m^\pi$. Extended by linearity, the $p_\sigma$'s define a partition of unity, that is, $t = \sum_{\eta \in \mathcal{P}(O)} p_\eta(t)$ for any $t$. By the same computation as above, one sees that $t$ is reduced if and only if $p_\eta(t) = \delta_{\eta = 0_\mathcal{O}} t$ for any $\eta$.

The second lemma tells that any $n$-graph polynomial in $\mathbb{C}\mathcal{G}\langle \bigsqcup_j \mathcal{A}_j \rangle$ can be written as a linear combination of bigraph operations evaluated in alternated and reduced elements.

**Definition 2.13.** Let $J$ be an index set and, for each $j \in J$, let $\mathcal{A}_j$ be an ensemble.

- A colored bigraph operation with color set $J$ is a couple $(g, \gamma)$ where $g \in \bigcup_{n \geq 0} \mathcal{B}^{(n)}$ is a bigraph operation with $L \geq 1$ inputs and $\gamma : [L] = \{1, \ldots, L\} \to J$ is a map telling that the $\ell$-th input is of color $\gamma(\ell)$. With a small abuse of notation, we still denote $g$ instead of $(g, \gamma)$ the colored bigraph operation with implicit mention of $\gamma$. We say that $g$ is alternated if $\gamma$ associates distinct colors to the neighbours of a same connector. We denote by $\mathcal{B}^{(n)}_{\text{col}}$ the set of colored bigraph operations with $n \geq 0$ outputs and by $\mathcal{B}^{(n)}_{\text{alt}}$ the set of alternated colored bigraph operations.
- Let $t_1, \ldots, t_L$ be graph polynomials of arbitrary ranks in $\bigsqcup_j \mathcal{A}_j$. We say that the tensor product $\mathbf{t} = (t_1 \otimes \cdots \otimes t_L)$ is $g$-colored if $t_\ell \in \mathbb{C}\mathcal{G}^{(d_\ell)}\langle \mathcal{A}_{\gamma(\ell)} \rangle$ for any $\ell = 1, \ldots, L$.

**Lemma 2.14.** *Let $J$ be an index set, let $\mathcal{A}_j$ be an ensemble for each $j \in J$, and let $\tau : \mathbb{C}\mathcal{T}\langle \bigsqcup \mathcal{A}_j \rangle \to \mathbb{C}$ be a unital linear form. Then we have the decomposition*

$$\mathbb{C}\mathcal{G}^{(n)}\Big\langle \bigsqcup_{j \in J} \mathcal{A}_j \Big\rangle = \mathbb{C}\,(\cdot) + \sum_{g \in \mathcal{B}^{(n)}_{\text{alt}}} \mathcal{W}_g$$

*where $\mathcal{W}_g$ is the space generated by $T_g(t_1 \otimes \cdots \otimes t_L)$, for any $(t_1 \otimes \cdots \otimes t_L)$ which is a $g$-colored tensor product of reduced elements with respect to $\tau$, and $\mathbb{C}\,(\cdot)$ denotes the space generated by the graph monomial $(\cdot)$ with a single vertex and no edge in $\mathbb{C}\mathcal{G}^{(n)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$.*

*Proof.* Let us denote by $\mathcal{E}^0$ the vector space on the right hand side, spanned by $(\cdot)$ and the $\mathcal{W}_g$'s. For any $k \geq 1$, let us denote by $\mathcal{E}_k$ the vector space generated by the graph polynomials $T_g(\mathbf{t})$, where $g \in \mathcal{B}_{\text{col}}$ has a number of vertices less than or equal to $k$ and $\mathbf{t}$ is $g$-colored. Let us prove by induction that for any $k \geq 1$, $\mathcal{E}_k \subset \mathcal{E}^0$. Since $\mathbb{C}\mathcal{G}^{(n)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle = \bigcup_{k \geq 0} \mathcal{E}_k$, this shall conclude the proof.

To begin with, note that for any $n \geq 1$ the only element of $\mathcal{E}_1$ is $g = (\cdot)$ consists in a single connector vertex which is the common values of all outputs. Hence $\mathcal{E}_1 = \mathbb{C}(\cdot) \subset \mathcal{E}^0$. If $n = 0$, then $g = (\cdot)$ consists in a single input vertex and $W_{g,\gamma}$ is the linear space generated by the $\mathbb{C}\mathcal{T}\langle \mathcal{A}_j \rangle$, $j \in J$. Every element $T$ in this space can be written $T = \tau[T](\cdot) + (T - \tau[T](\cdot)) \in \mathbb{C}\mathbb{I} \oplus_j W_{(\cdot),j}$.

Let us now assume the claim for $k \in \mathbb{N}$. For any $k' \geq 1$ and any $s \geq 0$ we denote by $\mathcal{E}^s_{k'}$, the vector space spanned by the graph polynomials $T_g(\mathbf{t})$ of $\mathcal{E}_{k'}$ where at most $s$ elements are non reduced in $\mathbf{t}$. Note in particular that $\mathcal{E}^0 = \bigcup_{k' \geq 0} \mathcal{E}^0_{k'}$ and $\mathcal{E}_{k'} = \bigcup_{s \geq 0} \mathcal{E}^s_{k'}$. Let us prove by induction on $s \geq 0$ that $\mathcal{E}^s_{k+1} \subset \mathcal{E}$.

We first assume that $\mathcal{E}_{k+1}^s \subset \mathcal{E}$ for some $s \geq 0$ and consider $T_g(\mathbf{t})$, a bigraph operation $g$ with $k+1$ vertices evaluated in a $g$-colored tensor product $\mathbf{t}$ with $s+1$ non reduced elements. Without loss of generality, we can assume the first graph $t_1$ is not reduced. We will denote $t_1 \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_{\gamma(1)} \rangle$. If the rank $d_1$ of $t_1$ is one, then we can write

$$T_g(\mathbf{t}) = T_g\big((t_1 - \Phi(t_1)) \otimes t_2 \ldots \otimes t_L\big) + \Phi(t_1) T_g\big((\cdot) \otimes t_2 \ldots \otimes t_L\big) =: a + b,$$

where $a \in \mathcal{E}_{k+1}^s$ and $b \in \mathcal{E}_k$, so that $T_g(\mathbf{t}) \in \mathcal{E}$. If the rank of $t_1$ is greater than one, according to Lemma 2.11 we can write

$$t_1 = r + \sum_{i=1}^m x_i,$$

where $r \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_{\gamma(1)} \rangle$ is a reduced graph polynomial and $x_1, \ldots, x_m \in \mathbb{C}\mathcal{G}^{(d_1)}\langle \mathcal{A}_{\gamma(1)} \rangle$ are graph monomials having at least two outputs equal to the same vertex. Then, for any $i = 1, \ldots, m$, $T_g(x_i \otimes t_2 \ldots \otimes t_L) \in \mathcal{E}_k$ and $T_g(r \otimes t_2 \otimes \cdots \otimes t_L) \in \mathcal{E}_{k+1}^s$, so that

$$T_g(\mathbf{t}) \in \mathcal{E}. \qquad \blacksquare$$

Below, $p$ denotes the operator defined in Lemma 2.11.

**Corollary 2.15.** *In the setting of Lemma 2.14, the linear space $\mathbb{C}\mathcal{G}^{(n)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ is generated by the $n$-graph polynomials of the form $T_g(p(m_1) \otimes \cdots \otimes p(m_L))$, where $g \in \mathcal{B}_{\mathrm{alt}}^{(n)}$ and $m_1 \otimes \cdots \otimes m_L$ is a $g$-colored tensor product of monomials, such that outputs of the $m_\ell$'s are pairwise distinct and $T_g$ does not identify any pair of outputs of each input.*

*Proof.* Let $\mathbf{t}' = (t_1, \ldots, t_L)$ be an arbitrary sequence of $g$-alternated, reduced graph polynomials and denote $t_\ell = \sum_i \alpha_i^{(\ell)} m_{i,\ell}$ where the $m_{i,\ell}$'s are graph monomials. Then we have

$$\begin{aligned} T_g(t_1 \otimes \cdots \otimes t_L) &= T_g\big(p(t_1) \otimes \cdots \otimes p(t_L)\big) \\ &= \sum_{i_1, \ldots, i_L} \left( \prod_{\ell=1}^{L} \alpha_{i_\ell}^{(\ell)} \right) \times T_g\big(p(m_{i_1,1}) \otimes \cdots \otimes p(m_{i_L,L})\big). \end{aligned}$$

By Lemma 2.14, we get that $\mathbb{C}\mathcal{G}^{(n)}\langle \bigsqcup_{j \in J} \mathcal{A}_j \rangle$ is generated by the elements of the form $T_g(p(m_1) \otimes \cdots \otimes p(m_L))$, where $m_1 \otimes \cdots \otimes m_L$ is a $g$-alternated tensor product of monomials. Moreover, if $m$ has two outputs that are equal, then $p(m) = 0$. Hence one can assume that the outputs are pairwise distinct for each $m_\ell$. $\blacksquare$

**2.2.2. Solidity, validity and primitivity.** This section contains most of the arguments of the proof of Theorem 2.8 and it introduces tools that will be used later, in particular in Section 3.3 to prove the positivity of the free product.

In the first statement, we see how the reducedness of $n$-graph polynomials for $n \geq 2$ simplifies the computation of combinatorial traces (reducedness when $n = 1$ plays a role at the last stage of the proof). We shall need the following definition.

**Definition 2.16.** Let $\mathcal{A}$ be a set and let $T = T_g(m_1 \otimes \cdots \otimes m_L)$ be a test graph in $\mathcal{T}\langle\mathcal{A}\rangle$, where $g$ is a bigraph operation and $m_1 \otimes \cdots \otimes m_L$ is a tensor product of graph monomials, such that outputs of a same $m_\ell$ are pairwise distinct and the operation $T_g$ does not identify any pair of outputs of each $m_\ell$. Consider the graphs of the $m_\ell$'s as subgraphs of $T$ and denote

- by $V$ the vertex set of $T$,
- by $\mathcal{O}_\ell \subset V$ the set of outputs of $m_\ell$,
- by $\pi_{|\mathcal{O}_\ell}$ the restriction of $\pi \in \mathcal{P}(V)$ on $\mathcal{O}_\ell$, namely $\{B \cap \mathcal{O}_\ell, B \in \pi\}$,
- by $0_{\mathcal{O}_\ell}$ the partition of $\mathcal{O}_\ell$ made of singletons.

Consider a partition $\pi \in \mathcal{P}(V)$. For each $\ell$ in $\{1, \ldots, L\}$, we say that $m_\ell$ is *solid* for $\pi$ whenever $\pi_{|\mathcal{O}_\ell} = 0_{\mathcal{O}_\ell}$. In other words, in $T^\pi$ there is no identification of outputs of the graph $m_\ell$. In a context where there is no confusion about $m_1, \ldots, m_L$, we simply say that $\pi$ is solid, when $m_\ell$ is solid for $\pi$ for any $\ell = 1, \ldots, L$.

Beware that there is here no uniqueness in the decomposition[3]

$$T = T_g(m_1 \otimes \cdots \otimes m_L).$$

**Lemma 2.17.** *Let $\mathcal{A}$ be a set and let $h = T_g(t_1 \otimes \cdots \otimes t_L) \in \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$ be a 0-graph polynomial, where $g \in \mathcal{B}^{(0)}_{L,\mathbf{d}}$ is a bigraph operation with $\mathbf{d} = (d_1, \ldots, d_L)$, and*

- *$t_\ell = m_\ell$ is a monomial if $d_\ell = 1$,*
- *$t_\ell = p(m_\ell)$ where $m_\ell$ is a monomial with pairwise distinct outputs if $d_\ell \geq 2$.*

*Let $T$ denote the test graph $T_g(m_1 \otimes \cdots \otimes m_L) \in \mathcal{T}\langle\mathcal{A}\rangle$. Then the trace of $h$ is the sum of the injective traces of quotient graphs of $T$ by solid partitions: with notations of Definition 2.16, one has*

$$\tau[h] = \sum_{\substack{\pi \in \mathcal{P}(V)\\ \text{solid}}} \tau^0[T^\pi].$$

*Proof.* Without loss of generality, we can assume that the indices $\ell \in \{1, \ldots, L\}$ such that $d_\ell \geq 2$ are $1, \ldots, K$ for $K \leq L$. Let us denote $c_{\sigma_k} = \text{Möb}(0_{\mathcal{O}_k}, \sigma_k)$ for any $\sigma_k \in \mathcal{P}(\mathcal{O}_\ell)$ and any $k = 1, \ldots, K$. Consider the graph $T_\sigma = T_g(m_1^{\sigma_1} \otimes \cdots \otimes m_L^{\sigma_L})$, with the convention that $m_\ell^{\sigma_\ell} = m_\ell$ if $\ell > K = 1$. The definition of $p$ in Lemma 2.11 allows to write

$$\tau[h] = \sum_{\substack{\sigma_\ell \in \mathcal{P}(\mathcal{O}_\ell)\\ \forall \ell = 1, \ldots, K}} \left( \prod_{k=1}^{K} c_{\sigma_k} \right) \tau[T_\sigma].$$

[3] You can for instance consider a test graph given by a cycle and group consecutive edges in different ways.

Denoting by $V_\sigma$ the vertex set of $T_\sigma$, the linearity of $\tau$ and the definition of the injective trace lead to

$$\tau[h] = \sum_{\substack{\sigma_\ell \in \mathcal{P}(\mathcal{O}_\ell) \\ \forall \ell = 1,\dots,K}} \left( \prod_{k=1}^{K} c_{\sigma_k} \right) \sum_{\pi \in \mathcal{P}(V_\sigma)} \tau^0[T_\sigma^\pi].$$

Recall that for two partitions $\pi$ and $\pi'$ of some set, $\pi \le \pi'$ means that the blocks of $\pi$ are included in blocks of $\pi'$. Given $\sigma_1, \dots, \sigma_L$ as above, forming a graph $T_\sigma^\mu$ with a choice of a partition $\mu$ of $V_\sigma$ is equivalent to forming a graph $T^\pi$ with a choice of a partition $\pi$ of $V$ with the restrictions below.

(1) We consider firstly for each $\ell = 1, \dots, L$ a partition $\pi_\ell$ of the vertex set $V_\ell$ of $m_\ell$. We assume that $\pi_\ell$ does more identifications of outputs of $m_\ell$ than $\sigma_\ell$: for any $\ell = 1, \dots, K$, one has $\pi_{\ell|\mathcal{O}_\ell} \ge \sigma_\ell$.

(2) Given a collection $\Pi = (\pi_1, \dots, \pi_L) \in \prod_{\ell=1}^{L} \mathcal{P}(V_\ell)$ of partitions as in the previous point, we consider a partition $\pi$ of $V$ with same identification as the $\pi_\ell$ for vertices of the monomials: for any $\ell = 1, \dots, L$, one has $\pi_{V_\ell} \ge \pi_\ell$. We denote by $\mathcal{P}_\Pi(V)$ the set of partitions $\pi \in \mathcal{P}(V)$ with this condition.

We then obtain as expected, using the property of the Möbius map [30, Sections 3.6 and 3.7] in the third identity,

$$\begin{aligned} \tau[h] &= \sum_{\substack{\sigma_\ell \in \mathcal{P}(\mathcal{O}_\ell) \\ \forall \ell = 1,\dots,K}} \left( \prod_{k=1}^{K} c_{\sigma_k} \right) \sum_{\substack{\pi_\ell \in \mathcal{P}(V_\ell) \\ \forall \ell = 1,\dots,L \\ \text{s.t. } \sigma_\ell \le \pi_{\ell|\mathcal{O}_\ell} \\ \forall \ell = 1,\dots,K}} \sum_{\pi \in \mathcal{P}_\Pi(V)} \tau^0[T^\pi] \\ &= \sum_{\substack{\pi_\ell \in \mathcal{P}(V_\ell) \\ \forall \ell = 1,\dots,L}} \left( \prod_{k=1}^{K} \sum_{\substack{\sigma_k \in \mathcal{P}(\mathcal{O}_\ell) \\ \text{s.t. } \sigma_k \le \pi_{k|\mathcal{O}_k}}} c_{\sigma_k} \right) \sum_{\pi \in \mathcal{P}_\Pi(V)} \tau^0[T^\pi] \\ &= \sum_{\substack{\pi_\ell \in \mathcal{P}(V_\ell) \\ \pi_{\ell|\mathcal{O}_\ell} = 0_{\mathcal{O}_\ell} \\ \forall \ell = 1,\dots,L}} \sum_{\pi \in \mathcal{P}_\Pi(V)} \tau^0[T^\pi] = \sum_{\substack{\pi \in \mathcal{P}(V) \\ \pi_{|\mathcal{O}_\ell} = 0_{\mathcal{O}_\ell} \\ \forall \ell = 1,\dots,L}} \tau^0[T^\pi]. \qquad \blacksquare \end{aligned}$$

The next lemma highlights an elementary property of the graph of colored components that we will use several times. We use the following terminology.

**Definition 2.18.** We say that a partition $\pi$ of the vertex set of $T \in \mathcal{T}\langle \bigsqcup \mathcal{A}_j \rangle$ is *valid* whenever $\mathcal{GCC}(T^\pi)$ is a tree.

**Lemma 2.19.** *Let $J$ be an index set and, for each $j \in J$, let $\mathcal{A}_j$ be an ensemble. Let us consider the data of*

- *a test graph $T \in \mathcal{T}\langle \bigsqcup_j \mathcal{A}_j \rangle$ such that $\mathcal{GCC}(T)$ is not a tree,*
- *a valid partition $\pi$ of the vertex set of $T$,*

- *a simple cycle $\mathcal{C} : o_1, S_1, o_2, S_2, \dots, o_K, S_K$ of $\mathcal{GCC}(T)$, $K \geq 2$, where $S_k$ is a colored component of $T$ attached to the connectors $o_k$ and $o_{k+1}$, with indices $k$ modulo $K$ (the $o_k$'s and $S_k$'s are pairwise distinct).*

*Then, identifying connectors $o_k$'s with their image in $T$, there exist at least two indices $k, k' \in \{1, \dots, K\}$ such that $o_k \sim_\pi o_{k+1}$ and $o_{k'} \sim_\pi o_{k'+1}$, with indices modulo $K$. If moreover $K \geq 4$ then there exist non consecutive such $k, k'$, i.e. one can choose $|k - k'| \geq 2$ with distance in $\mathbb{Z}/K\mathbb{Z}$.*

In other words, when identifying vertices of a test graph, if the graph of colored components has a cycle before identification, and has none afterwards, then at least two different colored components along the cycle must get "pinched".

*Proof.* Given $\pi \in \mathcal{P}(V)$, the cycle $\mathcal{C}$ of $\mathcal{GCC}(T)$ induces a closed path of $\mathcal{GCC}(T^\pi)$. Since any connector in $T$ is mapped under the quotient of $T$ to a connector in $T^\pi$, and since $\mathcal{C}$ is simple, we can assume this closed path to be of the form

$$o'_1, S'_1, o'_2, S'_2, \dots, o'_K, S'_K,$$

where $o'_k$ is the image of $o_k$, and $S'_k$ is a colored component of $T^\pi$ attached to the connectors $o'_k$ and $o'_{k+1}$, having a color distinct from the one of $S_{k+1}$, with indices $k$ modulo $K$. Since $\mathcal{GCC}(T^\pi)$ is a tree, the closed path visits a subtree of $\mathcal{GCC}(T^\pi)$. This subtree has at least two leaves (vertices of degree one). They do not consist in connectors, since colors are alternated along this path. Hence the two leaves are equal to $S'_k$ and $S'_{k'}$ for some $k \neq k' \in \{1, \dots, K\}$ for which we have $o'_k = o'_{k+1}$ and $o'_{k'} = o'_{k'+1}$. When $K \geq 4$ and $k' = k + 1$, applying the same argument to the closed path $o'_k, S'_{k+2}, o'_{k+3}, S'_{k+2}, \dots, S'_K, o'_1, S'_1, \dots, o'_{k-1}, S'_{k-1}$ (where indices are considered modulo $K$), there is $k''$ with $o'_{k''} = o'_{k''+1}$ and $|k + 1 - k''|, K - |k + 1 - k''| \geq 2$. Then the elements of the pair $(a, b) = (k + 1, k'')$ are at distance larger than 2 in $\mathbb{Z}/K\mathbb{Z}$, with $o'_a = o'_{a+1}, o'_b = o'_{b+1}$. Hence the result. ■

We deduce the following corollary which implies that traces of alternated bigraph operations in reduced elements vanish, by a simple argument of linearity, that is, given explicitly in next section.

**Corollary 2.20.** *Let $j$ be an index set and, for each $j \in J$, let $\mathcal{A}_j$ be a set. Let $\tau : \mathbb{C}\mathcal{T}\langle \bigsqcup_j \mathcal{A}_j \rangle \to \mathbb{C}$ be a unital linear form such that $\tau$ is the free product of its restrictions on test graphs labeled in $\mathcal{A}_j$, $j \in J$. Let $h = T_g(t_1 \otimes \cdots \otimes t_L)$ in $\mathbb{C}\mathcal{T}\langle \bigsqcup_j \mathcal{A}_j \rangle$ where $g \in \mathcal{B}^{(0)}_{\text{alt}}$ and $(t_1 \otimes \cdots \otimes t_L)$ is $g$-colored and satisfies $t_\ell = m_\ell$ for $d_\ell = 1$ and $t_\ell = p(m_\ell)$ for $d_\ell \geq 2$ as in Lemma 2.17. Then if $g$ is not a tree, $\tau[h] = 0$, and otherwise*

$$\tau[h] = \prod_{\ell=1}^{L} \tau[t_\ell],$$

*where in the above formula we extend $\tau$ as a linear map $\tau : \bigoplus_{n \geq 0} \mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$ by forgetting the position of the outputs.*

The proof of the corollary can be summarised as follows. Let $h = T_g(t_1 \otimes \cdots \otimes t_L)$ and $T = T_g(m_1 \otimes \cdots \otimes m_L)$ be as in the above corollary. By Lemma 2.17 and since $\tau$ is the free product of its restriction on the $\mathcal{A}_j$'s, one has $\tau[h] = \sum_\pi \tau^0[T^\pi]$, where the sum is over valid and solid partitions $\pi$. By Lemma 2.19 the set of such partitions is empty if $g$ is not a tree. The first part of the corollary is then a direct consequence of the lemmas. It will be enough to prove the following.

**Lemma 2.21.** *Let $\pi$ be a partition of the vertex set of $T \in \mathcal{T}\langle \bigsqcup_j \mathcal{A}_j,\ j \in J\rangle$. The following statement are equivalent:*

(1) *the graph of colored components is preserved after a quotient by $\pi$: $\mathcal{GCC}(T^\pi) = \mathcal{GCC}(T)$;*

(2) *for any vertices $v, w$ of $T$ belonging to different colored components such that $v \sim_\pi w$, the components of $v$ and $w$ in $T$ have exactly one connector $o$ in common and $v \sim_\pi o \sim_\pi w$;*

(3) *the colored components $m_1, \ldots, m_L$ of $T$ are solid for $\pi$ and given its restriction $\Pi = (\pi_{|V_1}, \ldots, \pi_{|V_L})$ to the vertex sets of the $m_\ell$'s, it is the smallest partition of the poset $\mathcal{P}_\Pi(V)$ partially ordered by refinement and constructed in the proof of Lemma 2.17.*

*We say that $\pi$ is* primitive *whenever it satisfies one of the equivalent properties above.*

*Let $T$ be a test-graph such that $\mathcal{GCC}(T)$ is a tree and let $\pi$ be a valid partition which is solid for the colored components of $T$. Then $\pi$ is primitive.*

The lemma implies that the trace of $h$ is the sum of the injective trace of quotient graphs of $T$ by primitive partitions. Denote by $T_\ell$ the test graph of $m_\ell$ and $V_\ell$ its vertex set. Assume that $g$ is a tree. By multiplicativity with respect to the colored components in the definition of traffic independence (Definition 1.16 above), for any $\pi$ primitive we have $\tau^0[T^\pi] = \prod_{\ell=1}^L \tau^0[T_\ell^{\pi_\ell}]$ where $\pi_\ell$ is the restriction of $\pi$ to $V_\ell$. Since $g$ is a tree, primitive partitions of $T$ are in bijection with tuples $(\pi_1, \ldots, \pi_\ell)$ of solid partitions of $V_1, \ldots, V_\ell$, and $\tau[h] = \prod_{\ell=1}^L \sum_{\pi_\ell} \tau^0[T_\ell^{\pi_\ell}]$ where the sums are over the solid partitions $\pi_\ell$ of $V_\ell$ with respect to $m_\ell$. By Lemma 2.17 again, the sum of quotients of $T_\ell$ by solid partitions is $\tau[p(T_\ell)]$. Hence Lemma 2.21 implies Corollary 2.20.

*Proof of Lemma* 2.21. Let us prove the equivalence between the first and the third properties. The equivalence between the first and the second is left as an exercise. For a test graph $T$, denote by $\mathcal{O}(T)$ the set of vertices of $T$ identified with connectors of $\mathcal{GCC}(T)$, and recall that $\mathcal{CC}(T)$ denotes the set of its colored components. Let $\pi$ be a partition of the vertices of $V$. Consider the vertices of $\mathcal{GCC}(T^\pi)$. On the one hand, the image $\mathcal{O}(T)^\pi$ of $\mathcal{O}(T)$ under the quotient map is a subset of $\mathcal{O}(T^\pi)$. Denote by $\mathcal{O}_p(T^\pi) = \mathcal{O}(T^\pi) \setminus \mathcal{O}(T)^\pi$ its complement. On the other hand, there is an equivalence relation $\sim_\pi$ on $\mathcal{CC}(T)$, such that $\mathcal{CC}(T^\pi)$ is the quotient image of $\mathcal{CC}(T)$ under $\sim_\pi$: two components $A, B \in \mathcal{CC}(T)$ satisfy $A \sim_\pi B$ if and only if $A$ and $B$ have the same color, and there are $a \in A$ and $b \in B$ with $a \sim_\pi b$. Consider now the third property. Note that a partition $\pi$ is minimal in $\mathcal{P}_\Pi(V)$ if and only if there are no $a, b \in V$ with $a \sim_\pi b$ that belong to two distinct colored components of $T$. Recall that $\pi$ is solid if and only if there are no outputs $a, b$ within a same colored component, with $a \sim_\pi b$. We conclude that:

(1) When $\pi$ is a minimal partition of $\mathcal{P}_\Pi(V)$, $\pi$ is moreover solid if and only if $\#\mathcal{O}(T)^\pi = \#\mathcal{O}(T)$.

(2) When $\pi$ is solid, $\pi$ is moreover minimal if and only if $\mathcal{O}_p(T^\pi)$ is empty and $\#\mathcal{CC}(T) = \#\mathcal{CC}(T^\pi)$.

(3) If $\mathcal{O}_p(T^\pi)$ is empty, $\#\mathcal{O}(T)^\pi = \#\mathcal{O}(T)$ and $\#\mathcal{CC}(T) = \#\mathcal{CC}(T^\pi)$, then $\pi$ is solid and minimal.

It easily follows that $\pi$ is both minimal and solid if and only $\mathcal{GCC}(T)$ can be identified with $\mathcal{GCC}(T^\pi)$.

We now prove that a solid partition $\pi$ which is not primitive is not valid, that is, if $\pi$ does not identify outputs of a same $m_\ell$ but identifies vertices of different colored components in a nontrivial way, then $\mathcal{GCC}(T^\pi)$ is not a tree. So let $v$ and $w$ be two vertices in different colored components that are identified by $\pi$ and denote by $T_{v\sim w}$ the graph obtained from $T$ by identifying $v$ and $w$. Then $T^\pi$ is a quotient of $T_{v\sim w}$. We denote by $S_1$ the colored component of $v \sim w$ in $\mathcal{GCC}(T_{v\sim w})$. The path between $v$ and $w$ yields a simple cycle $S_1, o_2, S_2, \ldots, o_K, S_K, o_1, S_1$ in $\mathcal{GCC}(T_{v\sim w})$ where $S_k$ is a colored component of $T_{v\sim w}$ attached to the connectors $o_k$ and $o_{k+1}$, with indices $k$ modulo $K$ (the $o_k$'s and $S_k$'s are pairwise distinct).

Thanks to the enumeration below, all colored components of $\mathcal{CC}(T_{v\sim w})$ on this cycle are original components of $\mathcal{GCC}(T)$, except at most two consecutive ones that contain both $v$ and $w$. For $v$ (and $w$), we distinguish if it is an output vertex, associated to a connector, or if it is an internal vertex (an vertex which is not an output). We decompose five alternatives, illustrated in Figure 6:

(1) If $v$ and $w$ are internal vertices of components of the same color, then $\mathcal{GCC}(T_{v\sim w})$ is obtained by identifying these components in $\mathcal{GCC}(T)$.

(2) If $v$ and $w$ are internal vertices of components of different colors, then $\mathcal{GCC}(T_{v\sim w})$ is obtained by creating a new connector between them in $\mathcal{GCC}(T)$.

(3) If $v$ and $w$ are not internal vertices, then $\mathcal{GCC}(T_{v\sim w})$ is obtained by identifying them in $\mathcal{GCC}(T)$, then identifying the possible components of same colors attached to $v$ and $w$, and then reducing the number of edges attaching them to the connectors from two to one. In general for a partition $\pi$ of $\mathcal{P}(V)$, there may exist a component attached both to $v$ and $w$ (which results in other operations), but this is not possible if $v$ and $w$ do not belong to a same component.

(4) If $v$ is an internal vertex and $w$ a connector, that is, not attached to a component of the same color as the one containing $v$, then $\mathcal{GCC}(T_{v\sim w})$ is obtained by putting an edge between $w$ and the component of $v$ in $\mathcal{GCC}(T)$.

(4') If $v$ is an internal vertex and $w$ a connector attached to a component of the same color as the one containing $v$, then $\mathcal{GCC}(T_{v\sim w})$ is obtained by identifying these components in $\mathcal{GCC}(T)$.

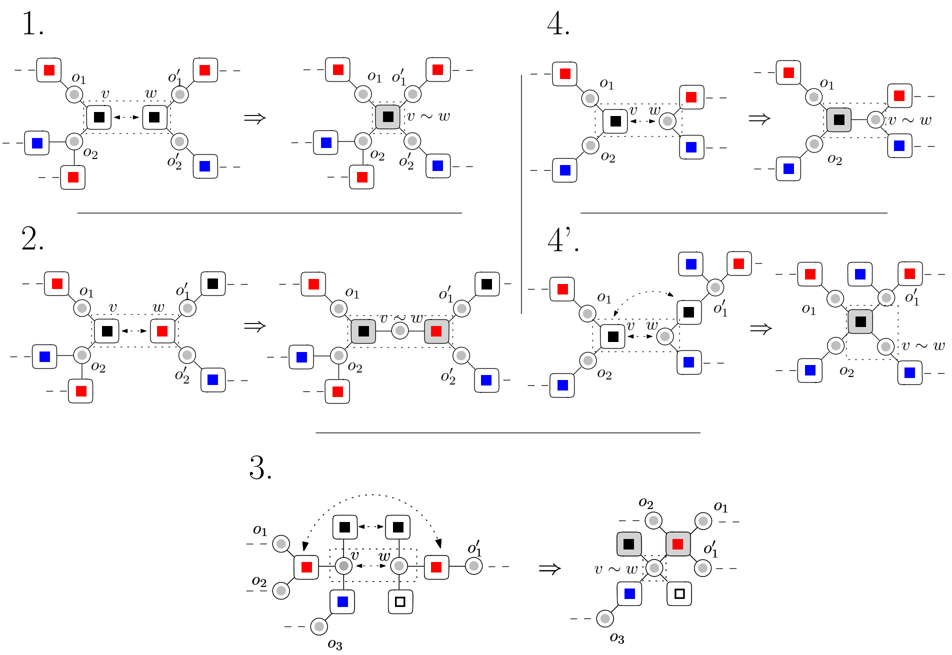


**Figure 6.** For each of the five items of the figure, the left-most picture is a local detail of the graph $\mathcal{GCC}(T)$. Square vertices represent inputs, circle vertices represent connectors, and the different colors for inputs represent different labels $\mathcal{A}_j$. There are two different parts of $\mathcal{GCC}(T)$ (on the left and on the right) that contain respectively the vertex $v$ and $w$ (inside the dotted rectangle) identified to give the right-most picture for each item. The right-most picture of each item is a detail of $\mathcal{GCC}(T_{v\sim w})$. They are different cases, depending if $v$ and $w$ are input or output vertices and on the colors of the input vertices. An input vertex is in grey when it is involved in the identification (it is not a colored component of the original graph $T$).

Hence the components $S_1, \ldots, S_K$ of $\mathcal{GCC}(T_{v\sim w})$ are the original components of $\mathcal{GCC}(T)$ except at most for two new components attached to a same connector (in grey in Figure 6). In particular, the components $S_1, \ldots, S_K$ of $\mathcal{GCC}(T_{v\sim w})$ are solid for $\pi$, except at most for two consecutive components.

Let us prove that $T^\pi$ cannot be a tree by contradiction. We assume that $T^\pi$ is a tree, and we find a contradiction by applying Lemma 2.19 to the graph $T_{v\sim w}$, the induced partition $\pi_{v\sim w}$ such that

$$T^\pi = (T_{v\sim w})^{\pi_{v\sim w}},$$

and the simple cycle $\mathcal{C} : o_1, S_1, o_2, S_2, \ldots, o_K, S_K$ of $\mathcal{GCC}(T_{v\sim w})$. We conclude that there exist at least two indices $k, k' \in \{1, \ldots, K\}$ such that $o_k \sim_\pi o_{k+1}$ and $o_{k'} \sim_\pi o_{k'+1}$ (with indices modulo $K$). As in Lemma 2.19, we divide the proof into two cases: $K \geq 4$ and $K \in \{2, 3\}$.

In the case $K \geq 4$, Lemma 2.19 implies that we can find two colored components $S_k$ and $S_{k'}$ which are not consecutive, and such that $o_k \sim_\pi o_{k+1}$ and $o_{k'} \sim_\pi o_{k'+1}$. However, thanks to the discussion above, one of them is solid for $\pi$, and $\pi$ cannot identify its outputs, which is a contradiction.

When $K = 2$, Lemma 2.19 says that the two connectors of the cycle are identified, and so are the two outputs of the colored components $S_1$ and $S_2$. Except if $v$ and $w$ are in the second situation in the above list, at least one of the components is an original one, which is a contradiction as all original components are solid for $\pi$. Let us examine the remaining case, where $v$ and $w$ are internal vertices of colored components of different colors (case 2 in Figure 6), and a new connector appears in $T_{v\sim w}$. In that case, the fact that $\mathcal{GCC}(T^\pi)$ is a tree implies that $v \sim w$ is identified with the connector between the original components of $v$ and $w$, which is a contradiction as all original components are solid for $\pi$. The case $K = 3$ is similar, as Lemma 2.19 says that the three connectors need to be identified by $\pi$. ∎

**2.2.3. End of the proof of Theorem 2.8.** We can now complete the proof of Theorem 2.8. To start with, we prove that the first two properties are equivalent. Assume first that the $\mathcal{A}_j$, $j \in J$, are independent and let us prove that every alternated 0-bigraph polynomial in reduced elements is centered using the preliminary results of the section. By Lemma 2.14 and Corollary 2.15, it is sufficient to consider

$$h = T_g(t_1 \otimes \cdots \otimes t_L) \in \mathbb{C}\mathcal{T}\Big\langle \bigsqcup_j \mathcal{A}_j \Big\rangle$$

where $g \in \mathcal{B}_{\text{alt}}^{(0)}$ and $t_1 \otimes \cdots \otimes t_L$ $g$-colored such that $t_\ell = p(m_\ell)$ for a graph monomial $m_\ell$ for each $\ell = 1, \dots, L$. Without loss of generality, assume that the indices $\ell$ for which $d_\ell = 1$ are $1, \dots, K$ for $K \leq L$. For any $\mathbf{i} = (i_1, \dots, i_K)$ in $\{0, 1\}^K$, let $h_{\mathbf{i}}$ be the graph polynomial $T_g(\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L)$ where

- $\tilde{t}_\ell = t_\ell$ if $\ell > K$,
- $\tilde{t}_\ell = m_\ell$ if $\ell \leq K$ and $i_\ell = 0$,
- $\tilde{t}_\ell = -\tau[m_\ell] \times (\cdot)$ if $\ell \leq K$ and $i_\ell = 1$, where $(\cdot)$ is the 1-graph monomial with a single vertex and no edges,

in such a way one has $h = \sum_{\mathbf{i}\in\{0,1\}^K} h_{\mathbf{i}}$. We apply Corollary 2.20 to each $h_i$: one has

$$\tau[h] = \mathbb{1}(g \text{ is a tree}) \times \prod_{\ell=1}^{L} \tau\big[p(T_\ell)\big],$$

where for $d_\ell = 1$ we denote $p(T_\ell) = T_\ell - \tau[T_\ell] \times (\cdot)$. When $g$ is a tree, there is at least one $\ell$ with $d_\ell = 1$; for the latter $\tau[p(T_\ell)] = 0$, we find $\tau[h] = 0$ as expected. Note that we need to reason on the leaves because the centering of $p(T_\ell)$ implies $\tau[p(T_\ell)] = 0$ only for leaves.

Conversely, let $\tau$ be an unital linear form on $\mathbb{C}\langle \bigsqcup_j \mathcal{A}_j \rangle$. Assume it satisfies $\tau[h] = 0$ for any $h$ given by an alternated bigraph operation in alternated and reduced elements. Then by Lemma 2.14 and the previous paragraph, it coincides with the free product of the traffic distribution of the $\mathcal{A}_\ell$'s on $\mathbb{C}\mathcal{T}\langle \bigsqcup_j \mathcal{A}_j \rangle$. Hence the $\mathcal{G}$-subalgebras $\mathcal{A}_\ell$ are independent.

The second and fourth items (the same property for 2-bigraph polynomials and w.r.t. the anti-trace $\Psi$) are equivalent since an element $h = T_g(t_1 \otimes \cdots \otimes t_L)$ of $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ is an alternated bigraph operation in reduced elements if and only if the element $\tilde{h} = T_{\tilde{g}}(t_1 \otimes \cdots \otimes t_L)$ of $\mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$ is as well, where in $\tilde{g}$ we forget the position of the input and output. We recall that $\Psi(h) = \tau(\tilde{h})$ by definition of $\Psi$.

The third item (the property for 2-bigraph polynomials and w.r.t. the trace $\Phi$) implies the second one since if an element $h = T_g(t_1 \otimes \cdots \otimes t_L)$ of $\mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$ is an alternated bigraph operation in reduced elements, then so is the element $\tilde{h} = T_{\tilde{g}}(t_1 \otimes \cdots \otimes t_L)$ of $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ obtained by declaring that a vertex is both the input and the outputs.

Assume now that the second item is satisfied and let us prove the third one. There we use again an argument of the previous section. Let $h = T_g(\mathbf{t})$ in $\mathbb{C}\mathcal{G}^{(2)}\langle\mathcal{A}\rangle$ where $\mathbf{t}$ is $g$-alternated reduced and given by monomials $t_\ell = p(m_\ell)$ as usual. If the two outputs $v$ and $w$ of $g$ are equal, then $g = \Delta(g)$ so $\Phi(h) = 0$. Assume the outputs are distinct, so that $\Delta(g)$ is possibly not alternated at the position where $w$ and $v$ are identified (Figure 6). As in the proof of Lemma 2.21, we apply Lemma 2.19 to the graph $T_{v\sim w}$, any partition, and a cycle given by a path between $v$ and $w$ in $T$, and it yields that two output vertices of one of the colored components have to be identified. Since the colored components are centered, this identification is not possible and we get $\Phi(h) = 0$.

## 3. Products of traffic spaces

This section is mainly devoted to the construction of the free product of traffic spaces, in particular under the context where we assume a positivity condition for the combinatorial trace. In the last subsection we also consider the tensor product of traffic spaces which will be used a couple of times in Part II.

### 3.1. The free product of algebraic traffic spaces

Let us first consider an arbitrary ensemble $X$. The free $\mathcal{G}$-algebra generated by $X$ is the space $\mathbb{C}\mathcal{G}\langle X\rangle$ generated by graph monomials whose edges are labeled by elements of $X$. It is endowed with the natural structure of $\mathcal{G}$-algebra given by the composition maps of the operad $\mathcal{G}$ (Section 1.2.1): for any graph operation $g \in \mathcal{G}_K$ and any graph polynomials $g_1, \dots, g_K \in \mathcal{G}\langle X\rangle$ labeled in $X$,

$$Z_g(g_1 \otimes \cdots \otimes g_K) = g(g_1, \dots, g_K),$$

where in the right hand side we identify the graph operation $g \in \mathcal{G}_K$ with the associated graph monomial in $K$ variables. Hence $\mathcal{G}$ is a well-defined $\mathcal{G}$-algebra.

Let $\tau : \mathbb{C}\mathcal{T}\langle X\rangle \to \mathbb{C}$ be an arbitrary linear map, unital in the sense that $\tau[(\cdot)] = 1$. Then it always induces the structure of an algebraic traffic space on $\mathbb{C}\mathcal{G}\langle X\rangle$. To explain this fact, we first define a combinatorial trace $\tilde{\tau} : \mathbb{C}\mathcal{T}\big\langle \mathbb{C}\mathcal{G}\langle X\rangle\big\rangle \to \mathbb{C}$ as follows. For any test graph $T$ labeled in $\mathcal{G}\langle X\rangle$ with $K \geq 1$ edges denoted $e_1, \dots, e_K$ and labeled respectively by monomials $g_1, \ldots, g_K$ in $\mathcal{G}\langle X \rangle$, we set $\tilde{\tau}[T] = \tau[T_{\mathbf{g}}]$, where $T_{\mathbf{g}}$ is the graph labeled in $X$ obtained from $T$ by replacing the edge $e_k$ by the graph $g_k$ for any $k = 1, \ldots, K$. Then we extend $\tilde{\tau}$ by multi-linearity with respect to the edges of $T$ and set $\tilde{\tau}[(\cdot)] = 1$.

**Lemma 3.1.** *The map* $\tilde{\tau} : \mathbb{C}\mathcal{T}\langle \mathbb{C}\mathcal{G}\langle X \rangle\rangle \to \mathbb{C}$ *satisfies the associativity property, and so endows* $\mathbb{C}\mathcal{G}\langle X \rangle$ *with the structure of an algebraic traffic space.*

*Proof.* Let $T \in \mathcal{T}\langle X \rangle$ whose edges are denoted $e_1, \ldots, e_n$, where $e_1$ has label

$$Z_h(g_1 \otimes \cdots \otimes g_K) = h(g_1, \ldots, g_K)$$

and $e_i$, $i \geq 2$, has label $g_{K+i-1}$ for graph monomials $g_1, \ldots, g_{K+n-1}$ labeled in $X$. We have by definition $\tilde{\tau}[T] = \tau[\tilde{T}]$ where $\tilde{T}$ is the graph labeled in $X$ obtained by replacing $e_1$ by $h(g_1, \ldots, g_K)$ and $e_i$, $i \geq 2$, by $g_{K+i-1}$. But we have $\tau[\tilde{T}] = \tilde{\tau}[T_h]$, where $T_h$ is the graph labeled in $\mathcal{G}\langle X \rangle$ obtained by replacing in $T$ the edge $e_1$ by $h$. This implies the associativity property $\tilde{\tau}[T] = \tilde{\tau}[T_h]$. ∎

Let now $J$ be a labeling set and for each $j \in J$ let $X_j$ be an ensemble. Recall that we denote by $\bigsqcup_{j \in J} X_j$ the set of couples $(j, x)$ where $j \in J$ and $x \in X_j$. Assume that for each $j \in J$ we are given a unital linear map $\tau_j : \mathbb{C}\mathcal{T}\langle X_j \rangle \to \mathbb{C}$, and denote by

$$\tau : \mathbb{C}\mathcal{T}\Big\langle \bigsqcup_{j \in J} X_j \Big\rangle \to \mathbb{C}$$

the free product of the $\tau_j$, $j \in J$. Denote by $\tilde{\tau}$ the combinatorial trace on

$$\mathbb{C}\mathcal{T}\Big\langle \mathbb{C}\mathcal{G}\Big\langle \bigsqcup_{j \in J} X_j \Big\rangle\Big\rangle \to \mathbb{C}$$

induced by $\tau$ and by $\tilde{\tau}_j$ the restrictions of $\tilde{\tau}$ to the subspaces $\mathbb{C}\mathcal{T}\big\langle \mathbb{C}\mathcal{G}\langle X_j \rangle\big\rangle$ generated by test graphs whose labels are graphs labeled in $X_j$, $j \in J$.

**Lemma 3.2.** *The map* $\tilde{\tau}$ *is the free product of the* $\tilde{\tau}_j$*'s,* $j \in J$. *Hence the* $\mathcal{G}$*-subalgebras* $\mathbb{C}\mathcal{G}\langle X_j \rangle$, $j \in J$ *are traffic independent in* $(\mathbb{C}\mathcal{G}\langle \bigsqcup_j X_j \rangle, \star_j \tau_j)$.

This fact is proved in [17, Proposition 2.14], based only on the definition of traffic independence in terms of the injective trace. The proof of Theorem 2.8 is somehow a strengthening of this proof, and now the lemma is actually a direct consequence of the new characterization of traffic independence.

*Proof.* Let $h = T_g(t_1 \otimes \cdots \otimes t_L)$ be an alternated bigraph operation in reduced elements labeled in $\bigsqcup_j \mathbb{C}\mathcal{G}\langle X_j \rangle$ and let us prove that $\tilde{\tau}[h] = 0$. Let $\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L$ be the tensor product of elements labeled in $\bigsqcup_j X_j$ obtained as follows: for each graph $t_\ell$, we replace each edge by the linear combination of the graphs that appear on their labels. By definition of $\tilde{\tau}$, we have $\tilde{\tau}[h] = \tau[\tilde{h}]$ where $\tilde{h} = T_g(\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L)$. Moreover, thanks Lemma 2.11 and the multi-linearity of the function $p$ in this Lemma, $\tilde{h}$ is still an alternated bigraph operation in reduced elements. By Corollary 2.20, we hence get $\tilde{\tau}[h] = 0$. ∎

We can now define the free product of $\mathcal{G}$-algebras. The map $g \mapsto Z_g$ is extended for $g$ by linearity for linear combinations of graph operations.

**Definition 3.3.** For any family of $\mathcal{G}$-algebras $(\mathcal{A}_j)_{j\in J}$, we denote by $*_{j\in J}\mathcal{A}_j$ the vector space $\mathbb{C}\mathcal{G}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle$, quotiented by the following relations: for any $i\in J$, any $a_1,\dots,a_k \in \mathcal{A}_i$, $a_{k+1},\dots,a_n \in \bigcup_{j\in J}\mathcal{A}_j$, any $g$ in $\mathcal{G}_{n-k+1}$ and any linear combination of graph operations $h$ in $\mathcal{G}_k$,

$$Z_g\big(\cdot \xleftarrow{Z_h(a_1\otimes\cdots\otimes a_k)} \cdot \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_n} \cdot\big)$$
$$\sim Z_g\big(Z_h(\cdot \xleftarrow{a_1} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_k} \cdot) \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_n} \cdot\big).$$

For instance, an edge labeled by the unit $(\cdot \xleftarrow{1_{\mathcal{A}}} \cdot)$ is equal to the graph with no edge $(\cdot)$. Similarly to the proof of Lemma 3.1, the above relations allow us to endow the quotient $*_{j\in J}\mathcal{A}_j$ with a $\mathcal{G}$-algebra structure. The relations

$$Z_h(\cdot \xleftarrow{a_1} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_K} \cdot) \sim (\cdot \xleftarrow{Z_h(a_1\otimes\cdots\otimes a_K)} \cdot),$$

when $a_1,\dots,a_K$ are in a same algebra $\mathcal{A}_i$, allow to consider the $\mathcal{G}$-algebra homomorphisms $V_j : \mathcal{A}_j \to *_{j\in J}\mathcal{A}_j$ given by the image of $a \mapsto (\cdot \xleftarrow{a} \cdot)$ by the quotient map. The $\mathcal{G}$-algebra $*_{j\in J}\mathcal{A}_j$ is the free product of the $\mathcal{G}$-algebras in the following sense.

**Proposition 3.4.** *Let $\mathcal{B}$ be a $\mathcal{G}$-algebra, and $f_j : \mathcal{A}_j \to \mathcal{B}$ a family of $\mathcal{G}$-morphisms. There exists a unique $\mathcal{G}$-morphism $*_{j\in J} f_j : *_{j\in J}\mathcal{A}_j \to \mathcal{B}$ such that $f_j = (*_{j\in J} f_j)\circ V_j$ for all $j \in J$. As a consequence, the maps $V_j$ are injective.*

*Proof.* The existence is given by the following definition of $*_{j\in J} f_j$ on $\mathbb{C}\mathcal{G}^{(2)}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle$:

$$*_{j\in J} f_j\big(g(\cdot \xleftarrow{a_1} \cdot, \dots, \cdot \xleftarrow{a_n} \cdot)\big) = g\big(f_{j(1)}(a_1),\dots,f_{j(n)}(a_n)\big)$$

whenever $a_1 \in \mathcal{A}_{j(1)},\dots,a_n \in \mathcal{A}_{j(n)}$. It obviously respects the relation defining $*_{j\in J}\mathcal{A}_j$.

The uniqueness follows from the fact that $*_{j\in J} f_j$ is uniquely determined on $\bigcup_j V_j(\mathcal{A}_j)$ (indeed, $*_{j\in J} f_j(a)$ must be equal to $f_j(b)$ whenever $a = V_j(b)$) and that $\bigcup_j V_j(\mathcal{A}_j)$ generates $*_{j\in J}\mathcal{A}_j$ as a $\mathcal{G}$-algebra. ∎

We now construct the free product of algebraic traffic spaces.

**Proposition 3.5.** *Let $(\mathcal{A}_j,\tau_j)_{j\in J}$ be a family of algebraic traffic spaces. Let*

$$\tilde{\tau} : \mathbb{C}\mathcal{T}\Big\langle\mathbb{C}\mathcal{G}\Big\langle\bigsqcup_{j\in J}\mathcal{A}_j\Big\rangle\Big\rangle \to \mathbb{C}$$

*be the unital linear map induced by $*_{j\in J}\tau_j : \mathbb{C}\mathcal{T}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle \to \mathbb{C}$ as in the first paragraph of the section. Then $\tilde{\tau}$ respects the quotient structure of $*_{j\in J}\mathcal{A}_j$. Still denoting the quotient map $\star_{j\in J}\tau_j : \mathbb{C}\mathcal{T}\langle *_{j\in J}\mathcal{A}_j\rangle \to \mathbb{C}$, we then get an algebraic traffic space*

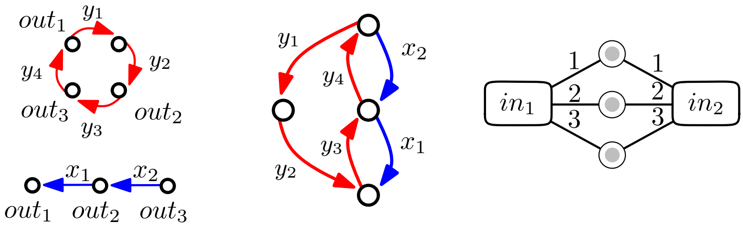


**Figure 7.** Left: two 3-graph monomials $g$ and $g'$. Middle: the test graph $g|g'$. Right: the bigraph operation $\mathfrak{g}$ such that $g|g' = T_{\mathfrak{g}}(g \otimes g')$.

$(*_{j\in J}\mathcal{A}_j, \star_{j\in J}\tau_j)$ *called the free product of the algebraic traffic spaces. Furthermore, we have* $\tau_i = (\star_{j\in J}\tau_j)\circ V_i$*, where* $V_i$ *is the canonical injective algebra homomorphism from* $\mathcal{A}_i$ *to* $*_{j\in J}\mathcal{A}_j$*, and the* $\mathcal{A}_i$*,* $i \in J$*, are traffic independent in* $(*_{j\in J}\mathcal{A}_j, \star_{j\in J}\tau_j)$.

*Proof.* Let $T \in \mathbb{C}\mathcal{T}\langle \mathbb{C}\mathcal{G}\langle \bigsqcup_{j\in J}\mathcal{A}_j\rangle\rangle$ such that an edge $e$ has label $(\cdot \xleftarrow{Z_h(a_1\otimes\cdots\otimes a_k)} \cdot)$, where $h = \sum_i a_i h_i$ is linear combination of graph operations labeled in the same $\mathcal{A}_i$. It suffices to prove that

$$\tilde{\tau}[T] = \tilde{\tau}[T_h],$$

where $T_h = \sum_i a_i T_{h_i}$ with $T_{h_i}$ the graph obtained by replacing $e_1$ by the graph $h_i$ evaluated in $(\cdot \xleftarrow{a_1} \cdot, \dots, \cdot \xleftarrow{a_k} \cdot)$. But when decomposing $T$ and $T_h$ on $\mathbb{C}\mathcal{T}\langle \mathbb{C}\mathcal{G}\langle \bigsqcup_{j\in J}\mathcal{A}_j\rangle\rangle$ according to the direct sum of Lemma 2.14, we get the same coefficient on $\mathbb{C}(\cdot)$. Since the $\mathcal{G}$-subalgebras are independent by Lemma 3.2, $\tilde{\tau}[T]$ and $\tilde{\tau}[T_h]$ are equal to these constants and so they are equal. Hence $\tilde{\tau}$ respects the quotient structure defining $*_{j\in J}\mathcal{A}_j$. ∎

### 3.2. Definition of positivity and traffic spaces

We first define an analogue of $^*$-algebras. On the set of graph operations $\mathcal{G}$, we define an *involution* $t : g \to g^t$ , where $g^t$ is obtained from $g$ by reversing the orientation of its edges and interchanging the input and the output.

**Definition 3.6.** A $\mathcal{G}^*$-algebra is a $\mathcal{G}$-algebra $\mathcal{A}$ endowed with an anti-linear involution $* : \mathcal{A} \to \mathcal{A}$ which is compatible with the action of $\mathcal{G}$, in the following sense: for all $K$-graph operation $g$ and $a_1, \dots, a_K \in \mathcal{A}$, $(Z_g(a_1 \otimes \cdots \otimes a_K))^* = Z_{g^t}(a_1^* \otimes \cdots \otimes a_K^*)$. A $\mathcal{G}^*$-subalgebra is a $\mathcal{G}$-subalgebra closed under adjoint. A $\mathcal{G}^*$-morphism between $\mathcal{A}$ and $\mathcal{B}$ is a $\mathcal{G}$-morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f(a^*) = f(a)^*$ for any $a \in \mathcal{A}$.

Recall that for any $n \geq 1$, a $n$-graph monomial is a test graph with the data of a $n$-tuple of vertices (see Definition 2.1). Let $g, g'$ be two $n$-graph monomials labeled in some set $\mathcal{A}$. We set $g|g'$ the test graph obtained by merging the $i$-th output of $g$ and $g'$ for any $i = 1, \dots, n$. We extend the map $(g, g') \mapsto g|g'$ to a bilinear application $\mathbb{C}\mathcal{G}^{(n)}\langle\mathcal{A}\rangle^2 \to \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$. Note that one can also realize $g|g'$ as a bigraph operation evaluated in $g \otimes g'$, see Figure 7.

Assume moreover that $\mathcal{A}$ is endowed with an anti-linear involution $* : \mathcal{A} \to \mathcal{A}$. Given an $n$-graph monomial $g = (V, E, \gamma, \mathbf{v})$ we set $g^\dagger = (V, E^\dagger, \gamma^\dagger, \mathbf{v})$, where $E^\dagger$ is obtained by reversing the orientation of the edges in $E$ and with $\gamma^\dagger$ given by $e \mapsto \gamma(e)^*$. Note that for $n = 2$, $g^\dagger(a_1 \otimes \cdots \otimes a_n) \neq g^t(a_1^* \otimes \cdots \otimes a_n^*)$ since there is no inversion of the two outputs in the definition of $g^\dagger$ as in Definition 3.6. We extend the map $g \mapsto g^\dagger$ to an anti-linear map on $\mathbb{C}\mathcal{G}^{(n)}\langle \mathcal{A} \rangle$.

**Definition 3.7.** A *traffic space* is an algebraic traffic space $(\mathcal{A}, \tau)$ such that:

- $\mathcal{A}$ is a $\mathcal{G}^*$-algebra,
- the combinatorial trace on $\mathcal{A}$ satisfies the following *positivity condition*: for any $n \geq 1$ and any $n$-graph polynomials $g$ labeled in $\mathcal{A}$,

$$\tau[g|g^\dagger] \geq 0. \tag{3.1}$$

  We call $\tau$ a combinatorial state.

A homomorphism between two traffic spaces is a $\mathcal{G}^*$-morphism which is a homomorphism of algebraic traffic space.

Note that for any $n$-graph polynomial $g|g^\dagger = g^\dagger|g$.

For $n = 2$, (3.1) is equivalent to the positivity of the trace $\Phi$ induced by $\tau$ on the $*$-algebra $\mathcal{A}$. Moreover, (3.1) for $n = 1$ implies the positivity of the anti-trace $\Psi$ (Definition 1.11): indeed we have $\Psi[aa^*] = \tau[g|g^\dagger]$ where $g$ is the 1-graph monomial with one simple edge whose source is the output.

As a consequence, every traffic space $(\mathcal{A}, \tau)$ has two $^*$-probability space structures $(\mathcal{A}, \Phi)$ and $(\mathcal{A}, \Psi)$ (endowed with the product $Z_{\cdot \xleftarrow{1} \cdot \xleftarrow{2} \cdot}$). Positivity of $\tau$ implies the Cauchy Schwarz inequality

$$\big|\tau[g_1|g_2]\big| \leq \sqrt{\tau[g_1|g_1^\dagger]\tau[g_2|g_2^\dagger]}.$$

**Example 3.8** (Example 1.13 continued). The algebraic traffic space of random matrices is actually a traffic space since $\tau_N$ is positive. Indeed, recall that in Remark 2.3 for any $n$-graph monomial $g$ labeled in $J$ and a family $\mathbf{A}_N = (A_j)_{j \in J}$ we have defined a random tensor matrix $g(\mathbf{A}_N) \in (\mathbb{C}^N)^{\otimes n}$. The positivity is clear since one has

$$\tau_N\big[(g|g^\dagger)(\mathbf{A}_N)\big] := \mathbb{E}\bigg[\frac{1}{N} \sum_{\mathbf{i} \in [N]^n} g(\mathbf{A}_N)_{\mathbf{i}} \overline{g(\mathbf{A}_N)_{\mathbf{i}}}\bigg] \geq 0$$

We see now a consequence of the positivity, which will be an additional motivation for Part II. Let $(\mathcal{A}, \tau)$ be a traffic space and let $a_1, \ldots, a_n \in \mathcal{A}$ be such that $\Phi(a_1 \ldots a_n) \neq 0$. Denote by $T_1$ the oriented simple cycle with $n$ edges labeled $\cdots \xleftarrow{a_i} \cdot \xleftarrow{a_{i+1}} \cdots$ along the cycle. Let $t_1$ be a 1-graph monomial with test graph $T_1$ and whose output is an arbitrary vertex. With $(\cdot)$ denoting the 1-graph monomial with no edge, we have

$$\Phi(a_1 \ldots a_n) = \tau[T_1] = \tau\big[t_1|(\cdot)\big] \neq 0.$$

Then, since $\tau$ is positive, the Cauchy-Schwarz inequality gives

$$\big|\tau\big[t_1|(\cdot)\big]\big|^2 \leq \tau[t_1|t_1^\dagger] \times \tau\big[(\cdot)|(\cdot)\big] = \tau[t_1|t_1^\dagger].$$

Hence the test graph $T_2 = t_1|t_1^\dagger$ satisfies $\tau[T_2] \neq 0$. It consists in two simple cycles that share exactly one vertex. We iterate, assuming we have a test graph $T_n$ such that $\tau[T_n] \neq 0$. Let $t_n$ be a 1-graph monomial with test graph $T_n$ and output an arbitrary vertex. Then $T_{n+1} = t_n|t_n^\dagger$ satisfies $\tau[T_{n+1}] \neq 0$. We have proved the following.

**Lemma 3.9.** *Let* $(\mathcal{A}, \tau)$ *be a traffic space. Then* $\tau$ *is nonzero on an infinite number of* cacti, *that is, test graphs such that each edge belongs to a unique cycle (see Part II).*

In the second part of the article, given a non-commutative probability space $(\mathcal{A}, \Phi)$ we construct a traffic space $(\mathcal{B}, \tau)$ such that $\mathcal{B}$ contains $\mathcal{A}$ and the trace associated to $\tau$ and restricted on $\mathcal{A}$ is $\Phi$. The lemma shows that the naive answer for this question,

- $\tau[T] = \Phi(a_1 \ldots a_n)$ if $T$ is an oriented simple cycle with consecutive edges $a_1, \ldots, a_n$,
- $\tau[T] = 1$ for the test graph with no edge,
- and $\tau[T] = 0$ otherwise,

does not yield a positive combinatorial trace. There are no matrices converging to a traffic with such a simple distribution.

### 3.3. Positivity of the free product

For each $j \in J$, let $(\mathcal{A}_j, \tau_j)$ be a traffic space. By Section 3.1, we can consider the algebraic traffic space $(*_{j\in J}\mathcal{A}_j, \star_{j\in J}\tau_j)$, the free product of the $(\mathcal{A}_j, \tau_j)$'s. We shall now prove that $\tau := \star_{j\in J}\tau_j$ satisfies the positivity condition (3.1). Therefore, we give in Lemma 3.11 a structural result for the canonical space $\mathbb{C}\mathcal{G}^{(n)}\langle\bigsqcup_{j\in J}\mathcal{A}_j\rangle$, introduced in Definition 1.9. The ideas of the current section are inspired by the counterpart of this construction for the free product of unital algebras with identification of units (see [26, Chapter 6] and [26, Formula (6.2)]). The proofs build on the preliminary material presented in Section 2.2.

**Definition 3.10.** Let us consider for $n \geq 1$ a colored bigraph operation $g \in \mathcal{B}_{\mathrm{col}}^{(n)}$ (Definition 2.13). A bijection of the vertex set of $g$ is called an automorphism of $g$ if it preserves the adjacency, the bipartition, the ordered set of outputs and the coloring of $g$. Their set forms a group denoted $\mathrm{Aut}_g$ that acts on $\mathcal{B}_{\mathrm{col}}^{(n)}$ and on the subspace $\mathcal{B}_{\mathrm{alt}}^{(n)}$ of alternated colored bigraph operations with $n$ outputs. The quotient space is denoted by $\widetilde{\mathcal{B}}_{\mathrm{col}}^{(n)}$ (resp. $\widetilde{\mathcal{B}}_{\mathrm{alt}}^{(n)}$) and the equivalent class of a colored bigraph operations $g \in \mathcal{B}_{\mathrm{col}}^{(n)}$ is denoted by $\bar{g}$.

See Figure 8 for an example. Note that an automorphism does not necessarily respect the ordering of the inputs nor the ordering of the neighbor connectors.

Every $\sigma \in \mathrm{Aut}_g$ and every $g$-alternated tensor product $\mathbf{m} = (m_1 \otimes \cdots \otimes m_L)$ of graph monomials induces a new $g$-alternated tensor product $\mathbf{m}_\sigma = (m_{1,\sigma} \otimes \cdots \otimes m_{L,\sigma})$, such that $T_g(\mathbf{m}) = T_{\sigma(g)}(\mathbf{m}_\sigma)$ by reordering the labels of the inputs and of neighbor connectors as follow (see Figures 8 and 9):

- if $\ell_v$ denotes the order of the input vertex $v$ of $g$, then $m_{\ell_v,\sigma} = m_{\ell_{\sigma^{-1}(v)}}$,
- the order of neighbor connectors of an input of $m_{\ell,\sigma}$ is the order of its pre-image by $\sigma$.

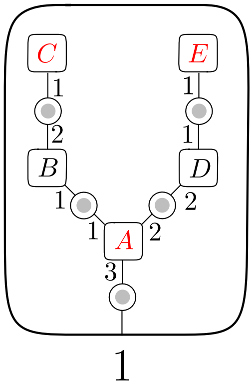


**Figure 8.** The above colored bigraph operation $(g, \gamma)$ has a single nontrivial automorphism $\sigma$, corresponding to vertical mirror symmetry.

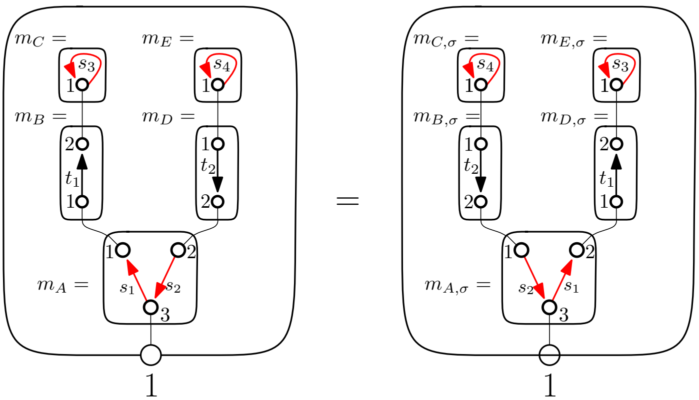

**Figure 9.** Illustration of the equality $T_g(\mathbf{m}) = T_{\sigma(g)}(\mathbf{m}_\sigma)$. With $g$ the colored bigraph operation of Figure 8 and $(m_A \otimes \cdots \otimes m_E)$ a $g$-colored tensor product of graph monomials, then $(m_A \otimes \cdots \otimes m_E)_\sigma$ is obtained by exchanging $m_B$ and $m_D$, $m_C$ and $m_E$, and by permuting the outputs 1 and 2 of $m_A, m_B$ and $m_D$.

We extend this definition by linearity for graph polynomials. Note that we have the property $(\mathbf{t}_{\sigma_1})_{\sigma_2} = \mathbf{t}_{\sigma_2\sigma_1}$ for all $\sigma_1, \sigma_2 \in \mathrm{Aut}_g$. For every alternated bigraph operation $g$, the space $\mathcal{W}_{\bar{g}}$ spanned by $T_g(\mathbf{t})$ for $\mathbf{t}$ reduced and $g$-colored does not depends on $g$ but only on the class $\bar{g} \in \widetilde{\mathcal{B}}^{(n)}_{\mathrm{alt}}$.

**Lemma 3.11.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and $\mathcal{A}_j$, $j \in J$, be independent $\mathcal{G}$-subalgebras.*

(1) *When considering the bilinear form $\tau[\,\cdot \mid \cdot^{\dagger}]$ defined in (3.1), the space of graph-polynomials admits the orthogonal decomposition*

$$\mathbb{C}\mathcal{G}^{(n)}\Big\langle \bigsqcup_{j \in J} \mathcal{A}_j \Big\rangle = \mathbb{C}(\cdot) \oplus^{\perp} \overset{\perp}{\bigoplus_{\bar{g} \in \widetilde{\mathcal{B}}^{(n)}_{\mathrm{alt}}}} \mathcal{W}_{\bar{g}}.$$

(2) *If $g$ is not a tree, then $\mathcal{W}_{\bar{g}}$ is included in the kernel of $\tau[\,\cdot\,|\,\cdot\,^\dagger]$, that is, for any $h \in \mathcal{W}_{\bar{g}}$ and $h' \in \mathbb{C}\mathcal{G}^{(n)}\langle \bigsqcup_{j\in J} \mathcal{A}_j\rangle$, $\tau[h|h'] = 0$.*

(3) *If $g$ is a tree, then for any $h = T_g(t_1 \otimes \cdots \otimes t_L)$, $h' = T_g(t'_1 \otimes \cdots \otimes t'_L)$ in $\mathcal{W}_{\bar{g}}$, we have*

$$\tau[h|h'] = \sum_{\sigma\in \mathrm{Aut}_g} \tau[t_1|t'_{1,\sigma}]\cdots\tau[t_L|t'_{L,\sigma}].$$

**Example 3.12.** With $g$ consisting in a single path between two outputs, the only automorphism of $g$ is the identity, and we then get the following formula: for any $L, L' \geq 2$, any $j_1 \neq j_2 \neq \cdots \neq j_L$ and $j'_1 \neq j'_2 \neq \cdots \neq j'_{L'}$ in $J$, and any $a_{j_\ell} \in \mathcal{A}_{j_\ell}$, $a'_{j'_\ell} \in \mathcal{A}_{j'_{\ell'}}$, $\ell = 1,\dots,L$, $\ell' = 1,\dots,L'$, one has

$$\begin{aligned}&\Phi\big((a_{j_1} - \Delta(a_{j_1}))\dots(a_{j_L} - \Delta(a_{j_L})) \times (a'_{j'_{L'}} - \Delta(a'_{j'_{L'}}))\dots(a'_{j'_1} - \Delta(a'_{j'_1}))\big)\\ &\quad = \mathbb{1}(L = L',\ j_\ell = j'_\ell\ \forall \ell = 1,\dots,L)\prod_{\ell=1}^{L}\Phi\big((a_{j_\ell} - \Delta(a_{j_\ell}))\times(a'_{j_\ell} - \Delta(a'_{j_\ell}))\big).\end{aligned}$$

With $g$ the colored bigraph operation of Figure 8, the automorphisms of $g$ are the identity and the vertical mirror symmetry: hence for any $h = g(t_A \otimes\cdots\otimes t_E)$ where $(t_1\otimes\cdots\otimes t_E)$ reduced and $g$-colored, one has

$$\tau[h|h] = \tau[t_A|t_A]\dots\tau[t_E|t_E] + \tau[t_A|\tilde{t}_A]\tau[t_B|\tilde{t}_D]\tau[t_D|\tilde{t}_B]\tau[t_C|t_E]^2,$$

where $\tilde{t}_X$ is obtained from $t_X$ by permuting outputs 1 and 2 for $X \in \{A, B, D\}$.

*Proof of Theorem 1.2.* Assuming Lemma 3.11 for now, let us deduce Theorem 1.2 (positivity of the trace on the free product of traffic spaces). By Corollary 2.15, it suffices to prove that $\tau[h|h^\dagger] \geq 0$ for each finite combination $h = \sum_i \beta_i T_{g_i}(\mathbf{t}^i)$ for bigraph operations $g_i$ and tensor products of reduced polynomials $\mathbf{t}^i = t^i_1 \otimes\cdots\otimes t^i_{L_i}$, where $t^i_\ell = p(m^i_\ell)$ with a graph monomial $m^i_\ell$. Moreover the previous lemma allows to restrict our consideration to the case where all $g_i$ are in the equivalence class of one particular colored tree $g$ and the color of $m^i_\ell$ depends only on $\ell$, not on $i$.

In particular, the automorphism group of colored graph $\mathrm{Aut}_{g_i}$ is equal to $\mathrm{Aut}_g$ for any $i$. With this notation at hand, we can write

$$\begin{aligned}\tau[h|h^\dagger] &= \sum_{ij}\beta_i\bar{\beta}_j\tau\big[T_g(\mathbf{t}^i)\big|T_g(\mathbf{t}^{j\,\dagger})\big]\\ &= \frac{1}{\#\mathrm{Aut}_g}\sum_{\substack{i,j\\ \sigma\in\mathrm{Aut}_g}}\beta_i\bar{\beta}_j\tau\big[T_g(\mathbf{t}^i_\sigma)\big|T_g(\mathbf{t}^{j\,\dagger}_\sigma)\big],\\ &= \frac{1}{\#\mathrm{Aut}_g}\sum_{\substack{i,j\\ \sigma,\sigma'\in\mathrm{Aut}_g}}\beta_i\bar{\beta}_j\tau\big[t^i_{1,\sigma}\big|{t^j_{1,\sigma'\circ\sigma}}^\dagger\big]\cdots\tau\big[t^i_{L,\sigma}\big|{t^j_{L,\sigma'\circ\sigma}}^\dagger\big],\\ &= \frac{1}{\#\mathrm{Aut}_g}\sum_{\substack{i,j\\ \sigma,\sigma'\in\mathrm{Aut}_g}}\beta_i\bar{\beta}_j\tau\big[t^i_{1,\sigma}\big|{t^j_{1,\sigma'}}^\dagger\big]\cdots\tau\big[t^i_{L,\sigma}\big|{t^j_{L,\sigma'}}^\dagger\big].\end{aligned}$$

We shall now see that the r.h.s. is non-negative. First, for any $\ell = 1, \dots, L$, the matrices $(\tau[t^i_{\ell,\sigma} | t^{j}_{\ell,\sigma'}{}^\dagger])_{(i,\sigma),(j,\sigma')}$ are non-negative since $\tau$ is non-negative on each $\mathcal{G}$-subalgebra $\mathcal{A}_j$. Moreover, their entrywise product $(\tau[t^i_{1,\sigma}|t^{j}_{1,\sigma'}{}^\dagger]\cdots\tau[t^i_{L,\sigma}|t^{j}_{L,\sigma'}{}^\dagger])_{(i,\sigma),(j,\sigma')}$ is also non-negative [26, Lemma 6.11]. This yields the non-negativity of the above right-hand side. ■

*Proof of Lemma* 3.11. According to Lemmas 2.14 and Corollary 2.15, in order to prove any of these three statements, it is enough to consider $\tau[h|h']$, where $h = T_g(\mathbf{t})$ and $h' = T_{g'}(\mathbf{t}')$, with $g, g' \in \mathcal{B}^{(n)}_{\text{alt}}$, $\mathbf{t} = t_1 \otimes \cdots \otimes t_L$ a $g$-colored tensor product, $\mathbf{t}' = t'_1 \otimes \cdots \otimes t'_{L'}$ a $g'$-colored tensor product, such that for each $\ell = 1, \dots, L$, $\ell' = 1, \dots, L'$, $t_\ell = p(m_\ell)$, $t'_{\ell'} = p(m'_{\ell'})$, where $m_\ell$ (respectively $m'_{\ell'}$) is $n_\ell$-graph monomial (respectively a $n'_{\ell'}$-graph monomial) whose outputs are pairwise distinct. It suffices to prove that $\tau[h|h'] = 0$ if $g$ or $g'$ is not a tree and if $g$ and $g'$ do not belong to the same class of alternated colored bigraph operations, and to prove the formula of the third statement.

Assume that the integers $\ell, \ell'$ such that $n_\ell, n_{\ell'} = 1$ are $\{1, \dots, K\}$ and $\{1, \dots, K'\}$ respectively. For any multi-index $(\mathbf{i}, \mathbf{i}') = (i_1, \dots, i_K, i'_1, \dots, i'_{K'})$ in $\{0,1\}^{K+K'}$, let $h_{\mathbf{i},\mathbf{i}'}$ be the graph polynomial $T_g(\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L) | T_{g'}(\tilde{t}'_1 \otimes \cdots \otimes \tilde{t}'_{L'})$ where

- $\tilde{t}_\ell = t_\ell$ if $\ell > K$,
- $\tilde{t}_\ell = m_\ell$ if $\ell \leq K$ and $i_\ell = 0$,
- $\tilde{t}_\ell = (\cdot)$ if $\ell \leq K$ and $i_\ell = 1$,

and $\tilde{t}'_{\ell'}$ is defined similarly, so that

$$h|h' = \sum_{(\mathbf{i},\mathbf{i}') \in \{0,1\}^{K+K'}} \prod_{\substack{\ell=l,\dots,K \\ \text{s.t. } i_k=1}} \big(-\tau[t_k]\big) \times \prod_{\substack{k'=1,\dots,K' \\ \text{s.t. } i_{k'}=1}} \big(-\tau[t_{k'}]\big) \times h_{\mathbf{i},\mathbf{i}'}.$$

We can apply Lemma 2.17 to each graph polynomial $h_{\mathbf{i},\mathbf{i}'}$. Denote by $g_{\mathbf{i}}$ and $g'_{\mathbf{i}'}$ the colored bigraph operations obtained by erasing 1-graph monomials such that $i_\ell = 1$ and $i'_{\ell'} = 1$ respectively, and $g_{\mathbf{i}}|g'_{\mathbf{i}'}$ the bigraph operation obtained by identifying the $i$-th outputs of $g_{\mathbf{i}}$ and $g'_{\mathbf{i}'}$ for any $i = 1, \dots, n$. Denote by $(\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L \otimes \tilde{t}'_1 \otimes \cdots \otimes \tilde{t}'_{L'})_{\mathbf{i},\mathbf{i}'}$ the tensor product where 1-graph monomials are discarded when $\ell \leq K$ and $i_\ell = 1$ and $\ell' \leq K'$ and $i'_{\ell'} = 1$. Then we have the identity

$$h_{\mathbf{i},\mathbf{i}'} = T_{g_{\mathbf{i}}|g'_{\mathbf{i}'}}(\tilde{t}_1 \otimes \cdots \otimes \tilde{t}_L \otimes \tilde{t}'_1 \otimes \cdots \otimes \tilde{t}'_{L'})_{\mathbf{i},\mathbf{i}'},$$

and this graph polynomial satisfies the assumptions of Lemma 2.17. Hence, we get

$$\tau[h_{\mathbf{i},\mathbf{i}'}] = \sum_{\substack{\pi \in \mathcal{P}(V_{\mathbf{i},\mathbf{i}'}) \\ \text{solid}}} \tau^0[T^\pi_{\mathbf{i},\mathbf{i}'}],$$

where we denote

- the graph monomial $T_{\mathbf{i},\mathbf{i}'} = T_g(\tilde{m}_1 \otimes \cdots \otimes \tilde{m}_L) | T_g(\tilde{m}'_1 \otimes \cdots \otimes \tilde{m}'_{L'})$, with $\tilde{m}_\ell$ defined as $\tilde{t}_\ell$ with $m_\ell$ instead of $t_\ell$ in the first case,
- $V_{\mathbf{i},\mathbf{i}'}$ the vertex set of $T_{\mathbf{i},\mathbf{i}'}$,

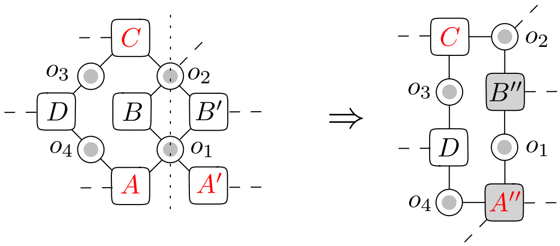


**Figure 10.** Left: a local detail of the bigraph operation $g_{\mathbf{i}}|g'_{\mathbf{i}'}$, with the vertical dotted line separating $g_{\mathbf{i}}$ and $g'_{\mathbf{i}'}$. Right: the bigraph operation $g_{\mathrm{alt},\mathbf{i},\mathbf{i}'}$. The sequence $A, o_1, B, o_2, C, o_3, D, o_4$ forms a simple cycle in $g$. Going from $g_{\mathbf{i}}|g'_{\mathbf{i}'}$ to $g_{\mathrm{alt},\mathbf{i},\mathbf{i}'}$, the inputs $A$ and $A'$ are identified. Yet, if a partition does not identify $o_1$ and $o_4$ in the leftmost picture, then it does not identify $o_1$ and $o_4$ in the rightmost one.

- $\mathcal{O}_\ell$ and $\mathcal{O}'_{\ell'}$ are the sets of outputs of $m_\ell$ and $m'_{\ell'}$ respectively, seen in $V_{\mathbf{i},\mathbf{i}'}$ for $\ell > K$, $\ell' > K'$,
- $\pi_{|\mathcal{O}_\ell}$ and $\pi_{|\mathcal{O}_{\ell'}}$ the restriction of $\pi$ to these sets.

Solidity is with respect to the graph monomials $m_1, \dots, m_L, m'_1, \dots, m'_L$ (excluding the 1-graph monomials such that $i_\ell = i'_{\ell'} = 1$). Note that these graphs are not the colored components of $T$, because of possible identifications between inputs of $g_{\mathbf{i}}$ and $g'_{\mathbf{i}'}$ that are neighbors of the outputs when forming $g_{\mathbf{i}}|g_{\mathbf{i}'}$, as in the third example of Figure 6 of the previous section.

We first assume that $g$ or $g'$ is not a tree and prove that a solid partition is not valid, so we will conclude that $\tau[h|h'] = 0$ for any $h'$, as we expect. Note that for any $\mathbf{i}, \mathbf{i}'$, we have that $g_{\mathbf{i}}$ or $g_{\mathbf{i}'}$ is not a tree: indeed, deleting leaves from a graph does not change whether or not it is a tree. We apply Lemma 2.19 to $T_{\mathbf{i},\mathbf{i}'}$, any partition $\pi$ solid w.r.t. the $m_\ell$'s and $m'_{\ell'}$'s, and a cycle $\mathcal{C}$ on $\mathcal{GCC}(T_{\mathbf{i},\mathbf{i}'})$ coming from a simple cycle of $g_{\mathbf{i}}$. Note that $\mathcal{C}$ is indeed simple since identifications with inputs of $g'$ do not change the cycle, see Figure 10. Solidity of the $m_\ell$'s implies that there are no possible identifications of connectors neighbouring the same input on the cycle $\mathcal{C}$. Hence $\pi$ cannot be valid. From now on, we shall assume that $g$ and $g'$ are trees.

Let us use now the centering of 1-graph polynomials. Let $k = 1, \dots, K$ be an index such that $i_k = 0$ ($m_k$ is in $g_{\mathbf{i}}$) and let $\pi$ be a partition of $V_{\mathbf{i},\mathbf{i}'}$. We say that $m_k$ is *isolated* by $\pi$ whenever no vertex of $m_k$ is identified with a vertex of another colored component except in the trivial way for a vertex of a neighboring component identified with the connector linking them. We say that $\pi$ is not isolating whenever no $m_k$ nor $m'_{k'}$ is isolated, for $k = 1, \dots, K$ and $k' = 1, \dots, K'$. By the multiplicativity property w.r.t. the colored components in the definition of traffic independence, for any valid partition $\pi$

$$\tau^0[T^\pi_{\mathbf{i},\mathbf{i}}] = \left( \prod_{\substack{\ell=1,\dots,L \\ \text{s.t. } m_\ell \text{ isolated}}} \tau^0\big[T_\ell^{\pi|V_\ell}\big] \times \prod_{\substack{\ell'=1,\dots,L' \\ \text{s.t. } m'_{\ell'} \text{ isolated}}} \tau^0\big[T'^{\,\pi|V'_{\ell'}}_{\ell'}\big] \right) \times \tau^0\big[T^{\pi|V_{\mathbf{j},\mathbf{j}'}}_{\mathbf{j},\mathbf{j}'}\big],$$

where $(\mathbf{j},\mathbf{j}') \in \{0,1\}^{K+K'}$ is defined by $j_\ell = 1$ if and only if $i_\ell = 1$ or $m_\ell$ is isolated. In other words, the indices $(\mathbf{j},\mathbf{j}')$ are defined in order to separate the term $\tau^0[T_{\mathbf{j},\mathbf{j}'}^{\pi_{|V_{\mathbf{j},\mathbf{j}'}}}]$, which uses a partition $\pi_{|V_{\mathbf{j},\mathbf{j}'}}$ which is *not isolating*, from the rest of the product. Hence, with the notations

- $\varepsilon(\mathbf{i},\mathbf{i}') := \prod_{k=1}^{K}(-1)^{i_k}\prod_{k'=1}^{K}(-1)^{i'_{k'}}$,
- $\alpha(\mathbf{i},\mathbf{i}') = \prod_{k=1,\dots,K}\tau[t_k]^{i_k}\prod_{k'=1,\dots,K'}\tau[t'_{k'}]^{i'_{k'}}$,

we have

$$\begin{aligned}
\tau[h|h'] &= \sum_{(\mathbf{i},\mathbf{i}')}\varepsilon(\mathbf{i},\mathbf{i}')\times\alpha(\mathbf{i},\mathbf{i}')\times\tau[h_{\mathbf{i},\mathbf{i}'}]\\
&= \sum_{(\mathbf{i},\mathbf{i}')}\varepsilon(\mathbf{i},\mathbf{i}')\times\Big(\sum_{\substack{\pi\in\mathcal{P}(V_{\mathbf{i},\mathbf{i}'})\\ \text{solid}}}\alpha(\mathbf{i},\mathbf{i}')\times\tau^0[T_{\mathbf{i},\mathbf{i}'}^{\pi}]\Big)\\
&= \sum_{(\mathbf{i},\mathbf{i}')}\varepsilon(\mathbf{i},\mathbf{i}')\times\Big(\sum_{\substack{(\mathbf{j},\mathbf{j}')\\ j_k\geq i_k\ \forall k\\ j'_{k'}\geq i'_{k'}\ \forall k'}}\ \sum_{\substack{\pi\in\mathcal{P}(V_{\mathbf{j},\mathbf{j}'})\\ \text{solid}\\ \text{not isolating}}}\alpha(\mathbf{j},\mathbf{j}')\times\tau^0[T_{\mathbf{j},\mathbf{j}'}^{\pi}]\Big)\\
&= \sum_{(\mathbf{j},\mathbf{j}')}\Big(\sum_{\substack{(\mathbf{i},\mathbf{i}')\\ i_k\leq j_k\ \forall k\\ i'_{k'}\leq j'_{k'}\ \forall k'}}\varepsilon(\mathbf{i},\mathbf{i}')\Big)\times\alpha(\mathbf{j},\mathbf{j}')\sum_{\substack{\pi\in\mathcal{P}(V_{\mathbf{j},\mathbf{j}'})\\ \text{solid}\\ \text{not isolating}}}\tau^0[T_{\mathbf{j},\mathbf{j}'}^{\pi}].
\end{aligned}$$

Note that, whenever there exists $k$ such that $j_k = 1$, the factor $(-1)^{i_k}$ in $\varepsilon(\mathbf{i},\mathbf{i}')$ is alternatively equal to $+1$ or $-1$ (whether $i_k = 0$ or $i_k = 1$) in the sum, which yields

$$\sum_{\substack{(\mathbf{i},\mathbf{i}')\\ i_k\leq j_k\ \forall k\\ i'_{k'}\leq j'_{k'}\ \forall k'}}\varepsilon(\mathbf{i},\mathbf{i}') = 0.$$

Consequently, the only non-vanishing term remaining in the sum is the one corresponding to $(\mathbf{j},\mathbf{j}') = (0,\dots,0)$, and we get

$$\tau[h|h'] = \sum_{\substack{\pi\in\mathcal{P}(V)\\ \text{solid}\\ \text{non isolating}}}\tau^0[T^\pi], \tag{3.2}$$

where $T = T_{g|g'}(m_1\otimes\cdots\otimes m_L\otimes m'_1\otimes\cdots\otimes m'_{L'})$ and $V$ its vertex set. In words, the trace of $h$ is the sum of the injective traces of quotients of $T$ by solid and non isolating partitions.

On the other hand, we claim that the valid partitions of $T$ solid w.r.t. the $m_\ell$'s and $m'_{\ell'}$'s satisfy half of the primitivity property of Lemma 2.21: two vertices $v$ and $w$ of $T_{\mathbf{i},\mathbf{i}'}$ that come from $g_{\mathbf{i}}$ (respectively from $g'_{\mathbf{i}'}$) can be identified by a valid partition solid w.r.t. the $m_\ell$'s only in the trivial situation: they belong to a same colored component, or they belong to neighboring components and are identified with the vertex that belong to both

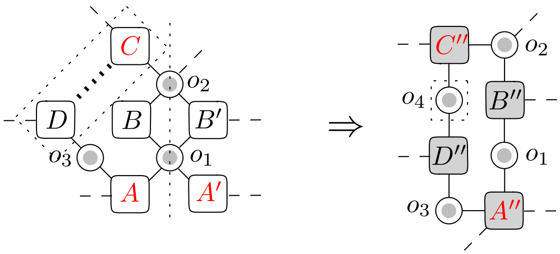


**Figure 11.** Left: the bigraph operation $g_{\mathrm{i}}|g'_{\mathrm{i}'}$ with the dot rectangle representing the identification of two vertices. Right: the graph of colored components of $(S_{\mathrm{i},\mathrm{i}'})_{v\sim w}$. Identifications $o_4 \sim_\pi o_3$ and $o_4 \sim_\pi o_3$ are possible since $o_4$ appeared while identifying $v$ and $w$, but other identifications $o_2 \sim_\pi o_1$ and $o_1 \sim_\pi o_3$ are not possible if they are not allowed in the leftmost graph.

components. Indeed, let us assume conversely that $\pi$ is a solid partition that identifies $v$ and $w$. We apply as usual Lemma 2.19 to the graph $(g_{\mathrm{i}})_{v\sim w}$, the induced partition, and the cycle given by a path between $v$ and $w$ in $g$. Solidity of the $m_\ell$'s implies that there is no possible identifications of connectors which are neighbors of a same input, except possibly around $v \sim w$, see Figure 11. So $\pi$ is not valid except in the trivial case.

We are now ready to prove that if $\tau[h] \neq 0$ then $g$ and $g'$ are isomorphic. Recall that we assume $g$ and $g'$ are trees. Let $\pi$ be a valid and solid partition which does not isolate 1-graph monomials, as in Formula (3.2). Because of the argument of the previous paragraph, each 1-graph monomial of $g$ must be identified with a single 1-graph monomial of $g'$, which defines a bijection $\sigma$ between the leaves of $g$ and $g'$. We now show that

- for any $i_1, i_2 = 1, \dots, n$, the unique path from the $i_1$-th to the $i_2$-th outputs of $g$ is isomorphic to the unique path between the same outputs in $g'$,
- if $S$ and $S'$ are two leaves of $g$ and $g'$ such that $\sigma(S) = S'$, then for any $i = 1, \dots, n$, the unique path from $S$ to the $i$-th output of $g$ is isomorphic to the unique path from $S'$ to the same output in $g'$.

All these isomorphisms of paths are consistent and form together an isomorphism between $g$ and $g'$. The argument for consistency is the same as Buneman's theorem [5] that a tree can be recovered up to isomorphism from the distances between the leaves. Alternatively, one can construct the isomorphism by induction: start with one arbitrary output, and for the inductive step, extend the isomorphism by considering all the vertices in the path linking this output with one of the other outputs or one of the leaves; as all vertices belong to such paths, it leads at the end to an isomorphism between $g$ and $g'$.

In order to show the first point, consider a simple path $\mathcal{D} : o_1, S_1, \dots, S_Q, o_{Q+1}$ in $\mathcal{GCC}(h)$ between two outputs in $g$ and the simple path $\mathcal{D}' : o'_1, S'_1, \dots, S'_{Q'}, o'_{Q'+1}$ in $\mathcal{GCC}(h')$ between the same outputs. We apply Lemma 2.19 to $T_{\mathrm{i},\mathrm{i}'}$, a solid and valid partition $\pi$, and a cycle $\mathcal{C}$ in $\mathcal{GCC}(T_{\mathrm{i},\mathrm{i}'})$ formed by the concatenation of $\mathcal{D}$ and $\mathcal{D}'$. Denote by $j_q$ the color of $S_q$ and $j'_{q'}$ the one of $S'_{q'}$. The inputs $S_q, S'_{q'}$ different from $S_1, S'_1, S_Q, S'_{Q'}$ are components of $g$ or $g'$ and they are therefore solid in $\pi$. The inputs $S_1$

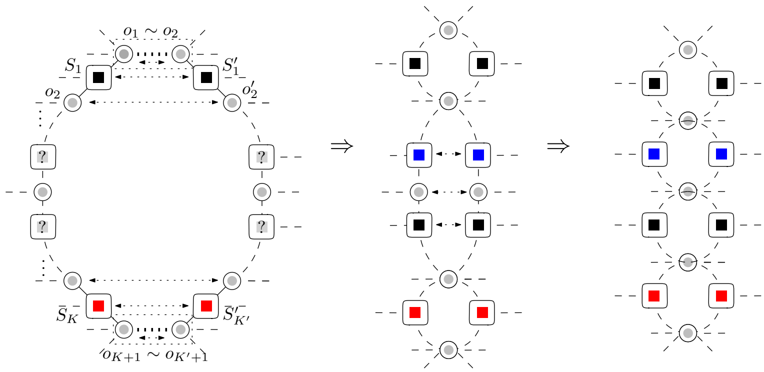


**Figure 12.** Left: two paths from the same outputs that form a simple cycle in $g|g'$. The pair of extremal inputs must be of a same color if a quotient whose $\mathcal{GCC}$ is a tree exists. Such a quotient must be a quotient of the graph with identifications $o_2 \sim o'_2$ and $o_Q \sim o_{Q'}$, so we can iterate the reasoning.

and $S'_1$ are identified in $\mathcal{GCC}(T_{\mathbf{i},\mathbf{i}'})$ if and only if $j_1 = j'_1$, and the same holds for the last inputs $S_Q$ and $S'_{Q'}$. Hence necessarily they are the only pairs of identification. If $Q \geq 2$ and $Q' \geq 2$, we can iterate this reasoning on the graph $\tilde{T}_{\mathbf{i},\mathbf{i}'}$ obtained from $T_{\mathbf{i},\mathbf{i}'}$ with these two identifications (see Figure 12) as $\pi$ is also solid and valid for this new graph (in fact, $\mathcal{GCC}(T^{\pi}_{\mathbf{i},\mathbf{i}'}) = \mathcal{GCC}(\tilde{T}^{\pi}_{\mathbf{i},\mathbf{i}'})$). Hence the two colored paths $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic: one has $Q = Q'$ and $j_q = j'_q$ for any $q = 1, \ldots, Q$, and moreover the partition $\pi$ identifies pairwise $o_k \sim_\pi o'_k$ for any $k = 1, \ldots, K$. For the second point, the proof is the same with paths from the colored components that are leaves in $\mathcal{GCC}$ to the outputs.

Valid partitions identifying pairwise a connectors of $g$ with connectors of $g'$, the multiplicativity property in the definition of traffic independence yields the expected formula.

■

### 3.4. The tensor product of traffic spaces

Let $J$ be an integer and for each $j = 1, \ldots, J$, let $(\mathcal{A}_j, \tau_j)$ be an algebraic traffic space. We construct a traffic space $(\bigotimes_j \mathcal{A}_j, \bigotimes_j \tau_j)$, that contain each traffic space $\mathcal{A}_j$ and such that the $\mathcal{A}_j$'s commute. Their algebraic tensor product $\bigotimes_j \mathcal{A}_j$ is indeed a $\mathcal{G}$-algebra with action of $K$-graph operations

$$\begin{aligned} &Z_g\big((a_{1,1} \otimes \cdots \otimes a_{1,J}) \otimes \cdots \otimes (a_{K,1} \otimes \cdots \otimes a_{K,J})\big) \\ &\quad = Z^{(1)}_g(a_{1,1} \otimes \cdots \otimes a_{K,1}) \otimes \cdots \otimes Z^{(J)}_g(a_{1,J} \otimes \cdots \otimes a_{K,K}), \end{aligned}$$

for any $g \in \mathcal{G}_K$ and any $a_{k,j} \in \mathcal{A}_j$, where $Z^{(j)}$ denotes the action of graph operations on $\mathcal{A}_j$, $j = 1, \ldots, J$. The tensor product of the combinatorial traces is defined, for any $T \in \mathcal{T}\langle \bigotimes_j \mathcal{A}_j \rangle$ whose edges are labeled by pure tensor products, by $\bigotimes_j \tau_j[T] = \tau_1[T_1] \cdots \tau_J[T_J]$, where $T_j$ is obtained from $T$ by replacing a label $a_1 \otimes \cdots \otimes \cdots a_J$ by $a_j$.

We will need later the following lemma.

**Lemma 3.13.** *The injective version of $\bigotimes_j \tau_j$ is given as follow. For any test graph $T \in \mathcal{T}\langle \bigotimes_j \mathcal{A}_j \rangle$, denote by $\Lambda_T$ the set of $J$-tuples $(\pi_1, \ldots, \pi_J) \in \mathcal{P}(V)^J$ such that if two elements belong to a same block of $\pi_i$ then they belong to different blocks of $\pi_j$ for some $j \neq i$. Then*

$$\Big(\bigotimes_j \tau_j\Big)^0[T] = \sum_{(\pi_1,\ldots,\pi_J)\in\Lambda_T} \tau_1^0[T_1^{\pi_1}] \cdots \tau_J^0[T_J^{\pi_J}].$$

*Proof.* We clearly have

$$\sum_{\pi\in\mathcal{P}(V)} \Big( \sum_{(\pi_1,\ldots,\pi_J)\in\Lambda_{T^\pi}} \prod_{j=1}^{J} \tau_j^0[T_j^{\pi_j}] \Big) = \prod_{j=1}^{J} \sum_{\pi_j\in\mathcal{P}(V)} \tau_j^0[T_j^{\pi_j}] = \bigotimes_j \tau_j[T].$$

This implies the expected result by uniqueness of Möbius transform. ∎

If the spaces are traffic spaces, i.e., if the maps $\tau_j$'s are positive, then their tensor product is also a traffic space by the usual argument of positivity of the Hadamard product [26, Lemma 6.11].

## Part II
# On the three types of traffics associated to non-commutative independences

## Presentation

In [17], three types of traffics were identified, one for each notion of the three non-commutative notions of independence. It uses the limiting traffic distribution of $(U_N, U_N^*)$ for a Haar unitary random matrix $U_N$ (this convergence is shown in [17, Proposition 3.7]).

**Definition 3.14.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathbf{a} = (a_j)_{j\in J}$ be a family of elements of $\mathcal{A}$. We say that $\mathbf{a}$ is of

- *free type* if it is *unitarily invariant*, in the sense that $\mathbf{a}$ has the same traffic distribution as $u\mathbf{a}u^* = (ua_ju^*)_{j\in J}$, where $(u, u^*)$ is traffic independent of $\mathbf{a}$ and limit of $(U_N, U_N^*)$ for a Haar unitary random matrix $U_N$;
- *Boolean type* if, for any $T \in \mathcal{T}\langle J\rangle$, one has $\tau[T] = 0$ if $T$ is not a tree;
- *tensor type* if the traffics are diagonals, in the sense that $a_j = \Delta(a_j)$ for all $j \in J$.

The precise link with the usual notions of independences is given by [17, Theorem 5.5]:

- the traffic independence of traffics of *free type* implies the *free independence* with respect to the trace $\Phi$ of $(\mathcal{A}, \tau)$;
- the traffic independence of traffics of *Boolean type* implies the *Boolean independence* with respect to the anti-trace $\Psi$ of $(\mathcal{A}, \tau)$;

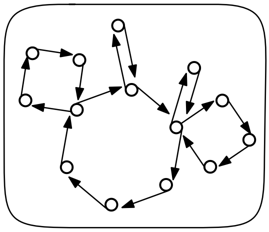

**Figure 13.** A well oriented cactus.

- the traffic independence of traffics of *tensor type* implies the *tensor independence* with respect to the trace $\Phi$ of $(\mathcal{A}, \tau)$;

This section is mostly devoted to the study of traffics of free types. We give an explicit description of the injective distribution of traffics of this type, and we prove Theorems 1.1 and 4.7 about unitarily invariant random matrices.

More generally, we give two characterizations of the above three types of traffics: a characterization as a particular symmetry of the traffic distribution, and a characterization with respect to the injective distribution. Besides, for any non-commutative probability space and any type of traffics among the above three, under mild assumptions, we construct a canonical traffic space of the given type, which includes the initial space as a subalgebra. The whole picture is contained in Section 8.

## 4. Generalities on unitarily invariant traffics

The canonical construction of free type consists in proving that any tracial non-commutative probability space $(\mathcal{A}, \Phi)$ can be realized as an algebra of unitarily invariant traffics. A crucial step is an explicit description of the distribution of traffics of this type, which is given in the two next sections.

### 4.1. Cacti and non-crossing partitions

Recall that we call simple cycle of a graph a closed path visiting pairwise distinct vertices (orientation of the edges is ignored).

**Definition 4.1.** A cactus is a finite connected graph such that each edge belongs exactly to one simple cycle. A well oriented cactus is a cactus such that the simple cycles of the graph are oriented, see Figure 13.

Well oriented cacti are related to non-crossing partitions in the following way. Let $T$ be a test graph consisting in a simple cycle with consecutive edges $(\cdot \overset{1}{\leftarrow} \cdots\cdots \overset{n}{\leftarrow} \cdot)$. Let $\sigma$ be a non-crossing partition of the set $E := \{1, \dots, n\}$ of edges of $T$. Let us denote by $V = \{1', \dots, n'\}$ the set of vertices of $T$, so that $i'$ is neighbor of $i$ and $i + 1$ with notation modulo $n$. The *Kreweras* complement $\widehat{\sigma}$ of $\sigma$ is the largest partition of $V$ such that the partition $\sigma \sqcup \widehat{\sigma}$ of $E \sqcup V$ is non-crossing (with the convention $1 < 1' < 2 < \cdots < n < n'$).

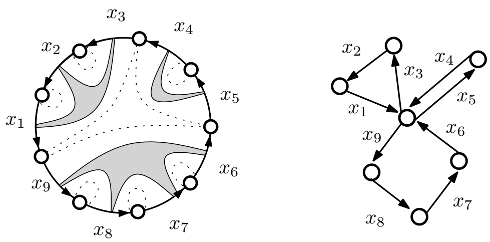


**Figure 14.** Left: A cycle of length nine, a non-crossing partition $\nu$ of its edges (grey) and the Kreweras complement $\pi$ (dotted) of $\nu$. Right: the quotient of the cycle by $\pi$.

**Lemma 4.2.** *For any partition $\pi$ of $V$, the quotient $T^\pi$ is a well oriented cactus if and only if $\pi$ is non-crossing. Moreover, in this case, there exists a non-crossing partition $\sigma$ of $E$ such that $\pi = \hat{\sigma}$.*

The correspondence between non-crossing partitions and cacti is illustrated in Figure 14.

*Proof.* The Kreweras operation $\sigma \mapsto \hat{\sigma}$ is a bijection $\mathrm{NC}(n) \to \mathrm{NC}(n')$. Hence the content of the lemma is unchanged if we replace the sentence "$\pi \in \mathrm{NC}(n')$" by "$\exists \sigma \in \mathrm{NC}(n)$ such that $\pi = \hat{\sigma}$". Let us prove the equivalence between $T^\pi$ is a well oriented cactus and there exists a non-crossing partition $\sigma$ of $E$ such that $\pi = \hat{\sigma}$.

For any partition $\pi$ of $V$, denote $\sigma(\pi)$ the partition of the edges of $T$ such that $i \sim_\sigma i'$ if and only if $i$ and $i'$ belong to the same simple cycle of $T$. Note first that $T^\pi$ is a cactus if and only if there exists at least one *isolated* simple cycle, that is, a subgraph attached to the rest of the graph by a single vertex, and the graph without this simple graph is a cactus. Indeed, let $\mathcal{G}$ be the (undirected) graph whose vertices are the simple cycles of $T$ with an edge between two cycles for each vertex they have in common. Then $T^\pi$ is a cactus if and only if $G$ is a tree. A leaf of this tree is a simple cycle with the expected property. On the other hand, $\sigma$ is a non-crossing partition if and only if there one block of $\sigma$ is an interval $I$ of $[n]$ such that the restriction of $\sigma$ to $[n] \setminus I$ is non-crossing (the proof is similar, see for instance [15, Property 17.9]). Since isolated simple cycles of $T$ correspond to intervals of $\sigma$, we get the desired property by induction. ∎

**Corollary 4.3.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space with trace $\Phi$ and let $\mathbf{a}$ be a family of elements of $\mathcal{A}$. Assume that the injective distribution of $\mathbf{a}$ is supported on well oriented cacti and is multiplicative w.r.t. their cycles, that is, for any test graph $T \in \mathcal{T}\langle \mathbf{a} \rangle$,*

$$\tau^0[T] = \mathbb{1}(T \text{ is a well oriented cactus}) \times \prod_C \tau^0[C],$$

*where the product is over the simple cycles of $T$. Then for any simple cycle $C$ with consecutive edges $(\cdot \overset{a_1}{\longleftarrow} \cdots\cdots \overset{a_n}{\longleftarrow} \cdot)$ we have $\tau^0[C] = \kappa_n(a_1, \ldots, a_n)$ where $\kappa_n$ is the $n$-th free cumulant function relative to the trace $\Phi$.*

*Proof.* Let $T$ denotes a simple cycle with consecutive edges $(\cdot \xleftarrow{a_1} \cdots\cdots \xleftarrow{a_n} \cdot)$. Then the definition of $\Phi$, the formula for $\tau^0$, and the lemma yield

$$\begin{aligned}\Phi(a_1 \dots a_n) = \tau[T] &= \sum_{\pi \in \mathcal{P}(V)} \mathbb{1}(T^\pi \text{ well oriented cactus}) \prod_C \tau^0[C] \\ &= \sum_{\sigma \in \mathrm{NC}(n)} \prod_{C \in \sigma} \tau^0[C],\end{aligned}$$

with in the second line the abuse of notation that a cycle $C$ with consecutive edges $(\cdot \xleftarrow{a_{i_1}} \cdots\cdots \xleftarrow{a_{i_\ell}} \cdot)$ of a cactus $T^{\hat{\sigma}}$ is identified with the corresponding block $\{i_1, \dots, i_\ell\}$ of $\sigma$. Since $\tau^0$ is multi-linear when seen as function of the labels of its edges, this property characterizes the free cumulant functions by the Möbius inversion formula stated in Section 1.2.3. ∎

The motivation to introduce this notion is that, under the assumptions of Corollary 4.3, the traffic distribution of $\mathbf{a}$ is completely determined by its non-commutative distribution in $(\mathcal{A}, \Phi)$ since free cumulants are determined by $\Phi$. This is the starting point of the canonical construction which is developed in Section 7. Before stating this, we first present properties and example of such traffics.

### 4.2. Unitarily invariant traffics

Let us temporarily say that a family of traffics is *of cactus type* when its injective distribution is supported on well oriented cacti and multiplicative w.r.t. their cycles, as in Corollary 4.3. We characterize this ensemble of traffics in terms on the following distributional symmetry.

**Definition 4.4.** Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and $\mathbf{a} = (a_j)_{j \in J}$ be a family of elements of $\mathcal{A}$. We say that $\mathbf{a}$ is *unitarily invariant* if and only if it has the same traffic distribution as $u\mathbf{a}u^* = (ua_ju^*)_{j \in J}$, where $(u, u^*)$ is traffic independent of $\mathbf{a}$ and distributed as the limit of $(U_N, U_N^*)$ for a Haar unitary random matrix $U_N$.

**Proposition 4.5.** *A family of traffics is unitarily invariant if and only if it is of cactus type.*

The proof of the proposition is given in Section 5 and requires an analysis of the geometry of cacti and graph of colored components.

**Corollary 4.6.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathbf{a}$ be a family of traffics of cactus type. Then the unital algebra generated by $\mathbf{a}$ is of cactus type.*

*Proof.* Let $\mathbf{b} = (P_j(\mathbf{a}))_{j \in J}$ for some non-commutative polynomials $P_j$, $j \in J$. Then $u\mathbf{b}u^* = (P_j(u\mathbf{a}u^*))_{j \in J}$ has the same traffic distribution as $\mathbf{b}$, so it is of cactus type. ∎

For all $N \geq 1$, let $\mathbf{A}_N$ be a family of random matrices in $\mathrm{M}_N(\mathbb{C})$. We recall that under the assumptions of Theorem 1.1 (the convergence in $^*$-distribution and the asymptotic factorization of $^*$-moments), $\mathbf{A}_N$ converges in traffic distribution toward a unitarily invariant family.

**Theorem 4.7.** *Under the above setting, the asymptotic factorization property holds for the traffic distribution: for all test graphs $T_1, \dots, T_k$, we have the following convergence*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(T_1(\mathbf{A}_N)\right)\cdots\frac{1}{N}\operatorname{Tr}\left(T_k(\mathbf{A}_N)\right)\right] = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(T_1(\mathbf{A}_N)\right)\right]\cdots\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(T_k(\mathbf{A}_N)\right)\right]. \quad (4.1)$$

The proof of Theorems 1.1 and 4.7 is given in Section 6 and is based on Weingarten calculus. Factorization property of $^*$-moments is required to get the multiplicativity of the injective distribution with respect to the cycles of cacti as in Corollary 4.3.

**Example 4.8.** The following list shows examples of large random matrices converging to traffics of free types:

(1) A Haar unitary matrix $U_N$ converges to a unitarily invariant traffic $u$ in some traffic space $(\mathcal{A}, \tau)$, and we can assume that $u$ is unitary ($u^*u = uu^* = 1$), see [17]. Denote by $\Phi$ the trace associated to $\tau$. It is known that in the non-commutative probability space $(\mathcal{A}, \Phi)$, $u$ is a *Haar unitary*, characterized by $\Phi(u^k(u^*)^\ell) = \mathbb{1}(k = \ell)$ for any $k, \ell \geq 0$. Recall that the only nonzero free cumulants of $u$ are

$$\kappa_{2n}(u, u^*, \dots, u, u^*) = \kappa_{2n}(u^*, u, \dots, u^*, u) = c_{n-1}(-1)^{n-1}$$

where $c_n = \frac{2n!}{(n+1)!n!}$ are the Catalan numbers. In particular, the injective traffic distribution of $u$ is supported on well oriented cacti whose cycles have even size and whose labels are alternated.

(2) Let $X_N = (\frac{x_{i,j}}{\sqrt{n}})_{i,j=1,\dots,N}$ be a complex Wigner matrix (the $x_{i,j}$ are independent and identically distribution along and out of the diagonal, the distribution of $x_{i,j}$ does not depend on $N$ and admit moments of all orders). Assume the entries are centered, invariant in law by complex conjugation ($x_{i,j} \overset{\text{law}}{=} \overline{x_{i,j}}$) and that $\mathbb{E}[|x_{i,j}|^2] = 1$, $\mathbb{E}[x_{i,j}^2] = 0$. Then $X_N$ converges to a unitarily invariant traffic $x$ in some traffic space $(\mathcal{A}, \tau)$, and we can assume that $x$ is self-adjoint ($x^* = x$), see [17]. It is known that in the non-commutative probability space $(\mathcal{A}, \Phi)$, $x$ is a *semicircular variable*, characterized $\Phi(a^k) = \mathbb{1}(k \text{ even})c_{k/2}$ for any $k$, where $c_n$ are the $n$-th Catalan numbers. The only nonzero free cumulant of $x$ is $\kappa_2(x, x) = 1$. In particular, the injective traffic distribution of $x$ is supported on cacti whose cycles have size two (called the double trees in [17]).

(3) An interest of the notion of unitarily invariant traffics is that it is not restricted to the limit of unitarily invariant matrices, as we have seen in the previous example with Wigner matrices. Matrices which are asymptotically unitarily invariant can even be more structured. For instance, convergence to a unitarily invariant semicircular traffic remains true when Wigner matrix models is generalized to Wigner matrices with intermediated exploding moments (like diluted Erdös–Reńyi graphs) [16], for uniform regular graphs with large degree [18] (when restricting the traffic distribution to *cyclic test graphs*), periodic band Wigner matrices and band Wigner

matrices with slow growth [2]. Hence, the properties of unitarily invariant traffic we state below are asymptotically true for these models.

### 4.3. Relation with freeness and large random matrices

**4.3.1. Abstract statement.** The following proposition motivates that unitarily invariant traffics are referred as traffics of *free type*.

**Proposition 4.9.** *Let* $(\mathcal{A}, \tau)$ *be an algebraic traffic space with trace* $\Phi$*. For each* $j \in J$ *let* $\mathbf{a}_j$ *be a family of traffics in* $\mathcal{A}$ *and set* $\mathbf{a} = \bigcup_j \mathbf{a}_j$*. Let* $\mathbf{b}$ *be an arbitrary family of traffics in* $\mathcal{A}$*.*

1. *If* $\mathbf{a}_j$ *is unitarily invariant for each* $j \in J$ *and the* $\mathbf{a}_j$*'s are traffic independent then* $\mathbf{a}$ *is unitarily invariant and the* $\mathbf{a}_j$*'s are freely independent in* $(\mathcal{A}, \Phi)$*.*
2. *Reciprocally if* $\mathbf{a}$ *is unitarily invariant and the* $\mathbf{a}_j$*'s are freely independent in* $(\mathcal{A}, \Phi)$ *then they are traffic independent in* $(\mathcal{A}, \tau)$*.*
3. *If* $\mathbf{a}$ *is unitarily invariant and is traffic independent from* $\mathbf{b}$ *then* $\mathbf{a}$ *and* $\mathbf{b}$ *are freely independent in* $(\mathcal{A}, \Phi)$*.*

**Remark 4.10.** For the first and third parts of the statement, it is sufficient to assume, instead of the unitary invariance of the $\mathbf{a}_j$'s that for any test graph $T$ with no cutting edge, $\tau^0[T] = 0$ whenever $T$ is not a cactus.

A proof of the proposition is given in [17, Section 5.2] based on the property of unitary invariance (Definition 4.4). For completeness, we give a proof using the cactus property.

*Proof.* (1) Let $T \in \mathcal{T}\langle \bigsqcup_j \mathbf{a}_j \rangle$. Under the assumptions of the proposition, we can write, using *w.o.* as a shortcut for *well oriented*,

$$\tau^0[T] = \mathbb{1}\big(\mathcal{GCC}(T) \text{ is a tree}\big) \prod_{S \in \mathcal{CC}(T)} \mathbb{1}(S \text{ w.o. cactus}) \prod_{C \text{ cycle of } S} \tau^0[C].$$

Let us say that a cactus $T$ in variables $\mathbf{a} = \bigsqcup_j \mathbf{a}_j$ is well colored (in short w.c.) whenever each cycle of $T$ is labeled by variables in a same family $\mathbf{a}_j$. Note that $T$ is well colored if and only if $\sigma$ is a non-mixing non-crossing partition, that is, each of its blocks contains variables in the same family $\mathbf{a}_j$. Note that $T$ is a w.o.w.c. cactus if and only $\mathcal{GCC}(T)$ is a tree and the colored components are cacti. We then get

$$\tau^0[T] = \mathbb{1}(T \text{ w.o.w.c. cactus}) \prod_{C \text{ cycle of } S} \tau^0[C]. \tag{4.2}$$

Moreover, let $C \in \mathcal{T}\langle \bigsqcup_j \mathbf{a}_j \rangle$ be a simple cycle with consecutive edges $(\cdot \xleftarrow{a_{i_1}} \cdot \cdots \cdot \xleftarrow{a_{i_n}} \cdot)$, where the $a_i$ are elements of the $\mathbf{a}_j$'s. The above formula yields

$$\Phi(a_{i_1} \dots a_{i_n}) = \tau[C] = \sum_{\substack{\sigma \in \mathrm{NC}(n) \\ \text{non-mixing}}} \prod_{\{i_1 < \cdots < i_\ell\} \in \sigma} \kappa_\ell(a_{i_1}, \dots, a_{i_\ell}),$$

which characterizes free variables. Moreover, this implies the correspondence between injective traces of well-oriented cycles and free cumulants. Hence, coming back to equation (4.2) for general $T$ we can write

$$\tau^0[T] = \mathbb{1}(T \text{ w.o. cactus}) \prod_{C \text{ cycle of } S} \tau^0[C]. \tag{4.3}$$

since for test graphs $T$ that are not well colored, there are mixed cumulants along some cycles. Hence $\mathbf{a}$ is of cactus type, so it is unitarily invariant.

(2) Reciprocally, let us assume that $\mathbf{a}$ is unitarily invariant and that the $\mathbf{a}_j$'s are free independent. Let us prove that they are traffic independent. Since $\mathbf{a}$ is of cactus type, for any test graph $T \in \mathcal{T}\langle \bigsqcup_j \mathbf{a}_j \rangle$, equation (4.3) is satisfied. Freeness of the $\mathbf{a}_j$'s implies vanishing of mixed cumulants, so that $\tau^0[C] = 0$ for some cycle if $T$ is not a well colored cactus. But $T$ is a w.o.w.c. cactus if and only $\mathcal{GCC}(T)$ is a tree and the colored components are cacti. This yields the formula (4.2) and by the above computation that the $\mathbf{a}_j$'s are traffic independent.

(3) Let now $\mathbf{a}$ be a unitarily invariant family of traffics independent of an arbitrary family $\mathbf{b}$, and let us prove that $\mathbf{a}$ and $\mathbf{b}$ are free independent in $(\mathcal{A}, \Phi)$. Without loss of generality, we can assume that the families of matrices contain the identity. By [26, Theorem 14.4], it suffices to prove that for any $a_1, \ldots, a_n$ in $\mathbf{a}$ and any $b_1, \ldots, b_n$ in $\mathbf{b}$, the following is satisfied

$$\Phi(a_1 b_1 \ldots a_n b_n) = \sum_{\sigma \in \mathrm{NC}(n)} \kappa_\sigma(a_1, \ldots, a_n) \times \Phi_{\widehat{\sigma}}(b_1, \ldots, b_n),$$

where $\widehat{\sigma}$ is the Kreweras complement of $\sigma$ defined in Section 4.1, and

$$\kappa_\sigma(a_1, \ldots, a_n) = \prod_{\{i_1 < \cdots < i_\ell\} \in \sigma} \kappa_\ell(a_{i_1}, \ldots, a_{i_\ell}),$$

with a similar definition for $\Phi_{\widehat{\sigma}}$.

Let $T$ be a simple cycle with consecutive edges $(\cdot \xleftarrow{a_1} \cdot \xleftarrow{b_1} \cdots\cdots \xleftarrow{a_n} \cdot \xleftarrow{b_n} \cdot)$. Then by definition of traffic independence and the cactus property of $\mathbf{a}$, denoting by $V$ the vertex set of $T$ one has

$$\begin{aligned} &\Phi(a_1 b_1 \ldots a_n b_n) \\ &\quad = \tau[T] = \sum_{\pi \in \mathcal{P}(V)} \mathbb{1}\big(\mathcal{GCC}(T^\pi) \text{ is a tree}\big) \\ &\qquad \times \Bigg( \prod_{S \in \mathcal{CC}_{\mathbf{b}}(T^\pi)} \tau^0[S] \times \prod_{S \in \mathcal{CC}_{\mathbf{a}}(T^\pi)} \Big( \mathbb{1}(S \text{ w.o. cactus}) \prod_{C \text{ cycle of } S} \tau^0[C] \Big) \Bigg), \end{aligned}$$

where $\mathcal{CC}_{\mathbf{a}}(T)$ is the set of colored components of $T$ labeled in $\mathbf{a}$, and $\mathcal{CC}_{\mathbf{b}}(T)$ is defined similarly.

The arguments of the proof are those used in [16] (replacing the so-called *fat trees* by the cacti). Given $\pi \in \mathcal{P}(V)$, denote by $S_{\mathbf{a},\pi}$ the graph obtained from $T^\pi$ by identifying

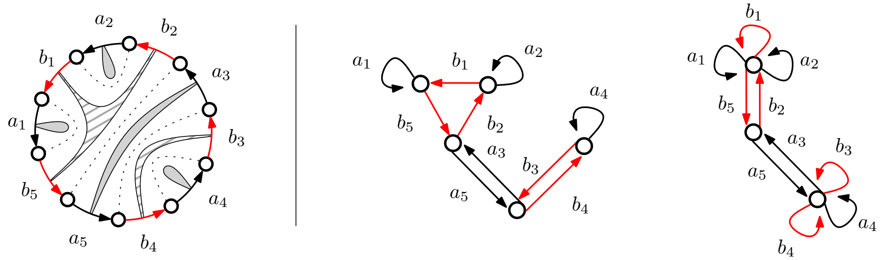


**Figure 15.** Left: the cycle $T$ with a non-crossing partition $\sigma_{\mathbf{a}}$ (full grey blocks), its kreweras complement $\sigma_{\mathbf{b}}$ (striped grey blocks), and the Kreweras complement $\pi_0$ of $\sigma_{\mathbf{a}} \sqcup \sigma_{\mathbf{b}}$ (dotted lines). Center and right: the quotient graph $T^{\pi_0}$, and another quotient graphs $T^{\pi}$ such that $S_\pi$ is the cactus of $\sigma_{\mathbf{a}}$.

the source and target of each edge labeled in $\mathbf{b}$ and suppressing these edges. If $\mathcal{GCC}(T^\pi)$ is a tree and $\mathcal{CC}_{\mathbf{a}}$ is a set of cacti, then $S_{\mathbf{a},\pi}$ is a cactus. By Lemma 4.2, $\pi$ induces a non-crossing partition $\sigma_{\mathbf{a},\pi}$ of the set $E_{\mathbf{a}}$ of edges of $T$ labeled by $\mathbf{a}$, whose blocks are associated to variables labeled $\mathbf{a}$ in the same cycle of $T$ (the cyclic order of $E_{\mathbf{a}}$ is the one around the cycle $T$).

Reciprocally, consider a non-crossing partition $\sigma_{\mathbf{a}}$ of $E_{\mathbf{a}}$ and then a cactus $S(\sigma_{\mathbf{a}})$ labeled by $\mathbf{a}$. Let $\sigma_{\mathbf{b}} = \hat{\sigma}_{\mathbf{a}}$ be the Kreweras complement of $\sigma_{\mathbf{a}}$, which is a partition of the set $E_{\mathbf{b}}$ of edges of $T$ labeled $\mathbf{b}$, and let $S(\sigma_{\mathbf{b}})$ denotes the cactus associated to $\sigma_{\mathbf{b}}$. Once more we consider the Kreweras complement of $\sigma = \sigma_{\mathbf{a}} \sqcup \sigma_{\mathbf{b}}$, which is now a partition $\pi_0 \in \mathcal{P}(V)$ of the vertex set of $T$. By Lemma 4.2, $T^{\pi_0}$ and $S(\sigma_{\mathbf{b}})$ are cacti. Moreover, the partitions $\pi \in \mathcal{P}(V)$ such that $\mathcal{GCC}(T^\pi)$ is a tree, $\mathcal{CC}_{\mathbf{a}}$ is a set of cacti and $S_{\mathbf{a},\pi} = S(\sigma_{\mathbf{a}})$ are those that only identifies vertices in a same cycle of $T^{\pi_0}$ labeled $\mathbf{b}$, which are the cycles of $S(\sigma_{\mathbf{b}})$, see Figure 15. Then we have, using that $\mathbf{a}$ is of cactus type and the definition of $\tau^0$ in the second line,

$$\begin{aligned}\tau[T] &= \sum_{\sigma_{\mathbf{a}} \in \mathrm{NC}(E_{\mathbf{a}})} \prod_{C_{\mathbf{a}} \text{ cycle of } S_{\sigma_{\mathbf{a}}}} \tau^0[C_{\mathbf{a}}] \times \prod_{C_{\mathbf{b}} \text{ cycle of } S_{\hat{\sigma}_{\mathbf{a}}}} \sum_{\pi \in \mathcal{P}\left(V(C_{\mathbf{b}})\right)} \tau^0[C_{\mathbf{b}}^{\pi}] \\ &= \sum_{\sigma_{\mathbf{a}} \in \mathrm{NC}(E_{\mathbf{a}})} \prod_{C_{\mathbf{a}} \text{ cycle of } S(\sigma_{\mathbf{a}})} \kappa(C_{\mathbf{a}}) \times \prod_{C_{\mathbf{b}} \text{ cycle of } S(\hat{\sigma}_{\mathbf{a}})} \Phi(C_{\mathbf{b}}),\end{aligned}$$

where $V(C_{\mathbf{b}})$ denotes the vertex set of $C_{\mathbf{b}}$, $\kappa(C)$ means the free cumulants $\kappa(x_1, \dots, x_\ell)$ for a cycle with consecutive edges $(x_1, \dots, x_\ell)$, and $\Phi(C)$ is defined similarly. With this notation, this is the desired formula. ■

**4.3.2. Asymptotic freeness of random matrices.** The previous proposition implies a universal property of free independence for asymptotically unitarily invariant matrices.

**Corollary 4.11.** *Let* $\mathbf{A}_j^{(N)}$, $j \in J$, *be independent families of random matrices such for each* $j \in J$,

(H0) $U\mathbf{A}_j^{(N)}U^*$ *has the same law as* $\mathbf{A}_j^{(N)}$ *for any permutation matrix* $U$.

(H1) $\mathbf{A}_j^{(N)}$ *converges in traffic distribution to a unitarily invariant family of traffics.*

(H2) $\mathbf{A}_j^{(N)}$ *satisfies the factorization property* (4.1).

*Then the* $\mathbf{A}_j^{(N)}$*'s are asymptotically freely independent with respect to* $\mathbb{E}[\frac{1}{N}\mathrm{Tr}]$ *and* $\bigcup_j \mathbf{A}_j^{(N)}$ *is asymptotically freely independent from any auxiliary independent family of random matrices converging in traffic distribution and satisfying* (H2).

The *universal* aspect of this statement is that it holds for any auxiliary matrices, without assumptions on the form of their limiting traffic distribution.

*Proof.* The first three assumptions imply the asymptotic traffic independence by [17], so the corollary follows directly from Proposition 4.9. ∎

One can work under a slightly weaker assumption than the convergence in traffic distribution of the matrices, since the conclusion is about non-commutative distribution. This is allowed by the modification of the asymptotic traffic independence theorem of [16]. Let us say that a test graph $T$ is *cyclic* if there exists a cycle visiting each oriented edge once (in the right direction). For a family $\mathbf{B}_N$ of matrices, we denote by $\|\mathbf{B}_N\|$ the supremum of the operation norm (square-root of the largest singular value) of the matrices of $\mathbf{B}_N$.

**Corollary 4.12.** *Let* $\mathbf{A}_j^{(N)}$*,* $j \in J$*, be independent families of random matrices such for each* $j \in J$*,* $\mathbf{A}_j^{(N)}$ *satisfies* (H0) *and the following modifications of the previous hypotheses:*

- (H1') $\mathbb{E}[\frac{1}{N}\mathrm{Tr}\, T(\mathbf{A}_N)]$ *converges for any cyclic test graph* $T$ *and the limit satisfies the cactus formula.*
- (H2') $\mathbf{A}_N$ *satisfies the factorization property on cyclic test graphs, and furthermore it satisfies the tightness condition of* [16]*, for instance* $\|\mathbf{A}_N\|$ *is uniformly bounded as* $N$ *goes to infinity.*

*Then the* $\mathbf{A}_j^{(N)}$*'s are asymptotically free independent and* $\bigcup_j \mathbf{A}_j^{(N)}$ *is asymptotically free independent from any independent family of random matrices converging in traffic distribution on cyclic test graphs and satisfying* (H2').

For instance, a normalized adjacency matrix $\mathbf{A}_N$ of a regular large graph with large degree may converge to a unitarily invariant traffics $a$ on cyclic graphs (see [18]). It cannot converges to $a$ on all test graphs since $\deg(\mathbf{A}_N)$ is a constant matrix whereas $\deg(a)$ is a nontrivial random variable for a nonzero unitarily invariant traffic $a$.

*Proof.* The first assumption implies the asymptotic traffic independence when distribution are restricted to cyclic graphs by [16]. Computation of trace of cyclic test graphs involves only computation of injective trace of cyclic graphs and reciprocally. Moreover the trace depends only on combinatorial traces of such graphs. Hence all the computation of the section is valid with this restriction. ∎

## 5. Equivalence between unitary invariance and cactus type

This section is dedicated to the proof of Proposition 4.5.

### 5.1. On the geometry of cacti

**Definition 5.1.** A cutting edge of a finite graph is an edge whose removal increases the number of connected component. A two-edge connected (t.e.c.) graph is a connected graph with no cutting edges. The cut number between two vertices is the minimal number of edges whose removal separate them. Two vertices of a graph form a 3-connection whenever there exist three edge-distinct paths joining them.

We will use Menger's theorem [19]:

**Theorem 5.2.** *Let $v$ and $w$ two distinct vertices of a connected graph. The cut number between $v$ and $w$ is equal to the maximum number of edge-disjoint paths from $v$ to $w$.*

In particular, a 3-connection consists in vertices with cut number at least three. A t.e.c. graph is a graph whose vertices have cutting numbers at least two. We can then deduce the following characterization of cacti.

**Proposition 5.3.** *A finite graph is a cactus if and only if the cut number between any two vertices is constant, equal to two.*

*Proof.* Let $T$ be a finite graph with cut number constant equal to two. It is connected since the cut number is finite. There is no vertices $v$ and $w$ with cut number equal to one, so every edge $e = (v, w)$ is contained in a simple cycle. Moreover, if an edge $e$ of a graph belongs to more than two distinct simple cycles, the union of these cycles with $e$ remove is still t.e.c. so one can find a 3-connection in the graph. Hence $T$ is a cactus.

Let now $T$ be a cactus. The cut number between two vertices is greater than one since the graph is connected and each edge belong to a cycle. Moreover, the cut number between two vertices $v$ and $w$ is always two. Let us consider a simple path between $v$ and $w$. Consider an edge $e$ of this path. Because this path is simple, it does not visit the whole cycle containing $e$: there exists another edge $e'$ of this cycle which does not belong to the path, and removing $e$ and $e'$ separates $v$ and $w$. ■

Let $T$ be a test graph and let $\pi$ be a partition of its vertices. Since edge-disjoint paths on $T$ induce edge-disjoint paths on the quotient graph $T^\pi$, the cut number of two vertices $v$ and $w$ in $T$ cannot decrease if $v$ and $w$ are not identified in $T^\pi$. This implies the following lemma.

**Lemma 5.4.** *Let $T$ be a connected finite graph, let two vertices $v, w$ forming a 3-connection, and let $\pi$ a partition of the vertex set of $T$. If the quotient graph $T^\pi$ is a cactus then $v \sim_\pi w$.*

We now deduce from this lemma three properties characterizing unitarily invariant traffics that we use in next section.

**Corollary 5.5.** *Let $\mathbf{a}$ be a family of traffics of cactus type in an algebraic traffic space $(\mathcal{A}, \tau)$. Let $T$ be a test graph labeled in $\mathbf{a}$ with two vertices $v$ and $w$ forming a 3-connection. Let $T_{v\sim w}$ be the test graph obtained by identifying $v$ and $w$ in $T$. Then one has $\tau[T] = \tau[T_{v\sim w}]$.*

In particular, by iterating this procedure, we get that $\tau[T] = \tau[\widetilde{T}]$ where $\widetilde{T}$ is obtained by identifying all pairs of 3-connections. Note that the order in which the identifications are made does not matter, and gives the same $\widetilde{T}$. The cut number of pairs of vertices of $\widetilde{T}$ is always smaller than or equal to two.

*Proof.* If a quotient graph $T^\pi$ of $T$ is a cactus then $v \sim w$. Since $\tau^0$ is supported on cacti Lemma 5.4 implies

$$\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi] = \sum_{\substack{\pi \in \mathcal{P}(V) \\ v \sim_\pi w}} \tau^0[T^\pi] = \tau[T_{v \sim w}]. \qquad \blacksquare$$

**Corollary 5.6.** *Let* **a** *be a family of traffics of cactus type in an algebraic traffic space* $(\mathcal{A}, \tau)$. *Let $T$ be a test graph that can be obtained by identifying one vertex of $S$ with one vertex of $S'$, where $S$ is t.e.c. Then*

$$\tau[T] = \tau[S] \times \tau[S'].$$

In particular, by iterating this procedure, we get that if $T$ is a cactus then

$$\tau[T] = \prod_{C \text{ cycle of } T} \tau[C].$$

*Proof.* Denote by $o$ the vertex of $T$ that belong both to $S$ and $S'$. Let $v$ (resp. $v'$) be a vertex of $S$ (resp. $S'$), seen in $T$ and different from $o$. Let $\pi$ be a partition of $T$ such that $v \sim_\pi v'$ and $T^\pi$ is a cactus. Then $T^\pi$ is a quotient of $T_{v \sim_\pi v'}$ for which $(v, o)$ forms a 3-connection and so by Lemma 5.4 one has $v \sim_\pi o \sim_\pi v'$. Hence, each partition $\pi$ such that $T^\pi$ is a cactus is the union $\pi = \sigma \cup \sigma'$ of a partition $\sigma$ of the vertices of $S$ and a partition $\sigma'$ of those of $S'$. For such a partition $\pi = \sigma \cup \sigma'$, $T^\pi$ is given by $S^\sigma$ and $(S')^{\sigma'}$ which are glued by the vertex $o$. The definition of cactus type traffics given in Corollary 4.3 gives a factorization on such components: we have

$$\tau^0[T^\pi] = \tau^0[S^\sigma] \times \tau^0\big[(S')^{\sigma'}\big].$$

Hence we get, denoting by $V_S$ and $V_{S'}$ the vertex sets of $S$ and $S'$ respectively,

$$\tau[T] = \sum_{\sigma \in \mathcal{P}(V_S)} \tau^0[S^\sigma] \times \sum_{\sigma' \in \mathcal{P}(V_S)} \tau^0\big[(S')^{\sigma'}\big] = \tau[S] \times \tau[S']. \qquad \blacksquare$$

It remains to show how to handle test graphs with cutting edges.

**Lemma 5.7.** *Let* **a** *be a family of traffics of cactus type in an algebraic traffic space* $(\mathcal{A}, \tau)$. *Let $T$ be a test graph labeled in* **a** *and denote by $\mathcal{O}$ the set of vertices of $T$ with odd degree (the degree is the number of neighbors, here we forget the orientation of the edges). For any partition $\sigma$ of $\mathcal{O}$, let us denote*

$$p_\sigma(T) = \sum_{\sigma' \geq \sigma} \text{Möb}_{\mathcal{P}(\mathcal{O})}(\sigma, \sigma') T^{\sigma'},$$

*where* $\text{Möb}_{\mathcal{P}(\mathcal{O})}$ *denotes the Möbius function of the poset of partitions of* $\mathcal{O}$ *and* $T^{\sigma'}$ *denotes the graph obtained by identifying vertices in the same block of* $T$*. Then one has*

$$\tau[T] = \sum_{\substack{\sigma\in\mathcal{P}(\mathcal{O}) \\ |B| \text{ is even } \forall B\in\sigma}} \tau\big[p_\sigma(T)\big].$$

In particular, we get that $\tau[T]$ can be written as a linear combination of $\tau[S]$ where $S$ has no cutting edges. Since the cutting number can only increase when taking quotients, together with the above lemmas, one gets an expression of $\tau[T]$ in terms of linear combinations of products of $\tau[C]$ where $C$ are simple cycles.

*Proof.* Cacti have only vertices of even degree. Hence, if $\pi$ is a partition such that $T$ is a cactus then it must re-group the vertices in $\mathcal{O}$ in blocks of even size. Hence

$$\tau[T] = \sum_{\substack{\sigma\in\mathcal{P}(\mathcal{O}) \\ |B| \text{ is even } \forall B\in\sigma}} \sum_{\substack{\pi\in\mathcal{P}(V) \\ \text{s.t. } \pi_{\mathcal{O}}=\sigma}} \tau^0[T^\pi].$$

By the same proof as Lemma 2.16 applied to $T^\sigma$, the second sum in nothing else than $\tau[p_\sigma(T)]$. ■

## 5.2. Proof of the equivalence between unitary invariance and cactus type

Let $\mathbf{a} = (a_j)_{j\in J}$ be an arbitrary family of traffics. We consider the limit $(u, u^*)$ of a Haar unitary matrix and its conjugate, independent from $\mathbf{a}$, and we set $\mathbf{b} = (ua_ju^*)_{j\in J}$. Let $\mathbf{c} = (c_j)_{j\in J}$ be a family of traffics of cactus type such that $\mathbf{a}$ and $\mathbf{c}$ has the same distribution with respect to the trace $\Phi$. We shall prove that $\mathbf{b}$ and $\mathbf{c}$ have the same traffic distribution.

For a test graph $T$ labeled in $\mathbf{b}$ we denote by $\tilde{T}$ the graph labeled in $\mathbf{a}, u, u^*$ obtained by replacing each edge $(\cdot \overset{b}{\longleftarrow} \cdot)$ by the sequence of edges $(\cdot \overset{u}{\longleftarrow} \cdot \overset{a}{\longleftarrow} \cdot \overset{u^*}{\longleftarrow} \cdot)$ and by $\tilde{V}$ the vertex set of $\tilde{T}$. In this section, we say that a partition $\pi$ of $\tilde{V}$ is *adapted* whenever $\mathcal{GCC}(\tilde{T}^\pi)$ is a tree and the colored components of $\tilde{T}^\pi$ labeled in $(u, u^*)$ are well oriented cacti whose edges along each cycle alternate between $u$ and $u^*$.

As in Lemma 5.4, each 3-connection needs to be identified under the quotient considered.

**Lemma 5.8.** *If* $\pi$ *is an adapted partition then for any* 3*-connection* $(v, w)$ *of* $\tilde{T}$ *one has* $v \sim_\pi w$.

*Proof.* Let $(v, w)$ be such a pair in $\tilde{T}$ and $\pi$ an adapted partition. Assume moreover $v \not\sim_\pi w$ and let us find a contradiction. Let $S_1, \dots, S_n$ be the path in $\mathcal{GCC}(\tilde{T}^\pi)$ between the colored components $S_1$ and $S_n$ containing $v$ and $w$ respectively. If $n \geq 2$, then one of these components is labeled in $(u, u^*)$ and it is traversed by at least three edge disjoint paths, so it cannot be a cactus. If $n = 1$, the colored component $S$ containing $v$ and $w$ is not a cactus because we have three edge disjoint paths from $v$ to $w$. Consequently, it is not labeled in $(u,u^*)$. Any path from $v$ to $w$ in $T$ can be lifted in $\tilde{T}$, yielding a path which goes through an odd number of edges labeled in $(u,u^*)$ before entering into $S$. As a consequence, there exists some cycle labeled in $(u,u^*)$ which has an odd number of edges, which is not possible if the edges of each cycle alternate between $u$ and $u^*$. ∎

We now prove that the three properties stated in Corollary 5.5, Corollary 5.6 and Lemma 5.7 hold for **b**. By independence of **a** and $(u,u^*)$ and by the formula for the traffic distribution of $(u,u^*)$, we have

$$\tau[T]=\tau[\tilde{T}]=\sum_{\pi\in\mathcal{P}(\tilde{V})}\mathbb{1}\big(\mathcal{GCC}(\tilde{T}^\pi)\text{ is a tree}\big)\prod_{S\in\mathcal{CC}_{\mathbf{a}}(\tilde{T}^\pi)}\tau^0[S]$$
$$\times\prod_{S\in\mathcal{CC}_{(u,u^*)}(\tilde{T}^\pi)}\mathbb{1}(S\text{ w.o. cactus})\prod_{C\text{ cycle of }S}\tau^0[S],$$

where $\mathcal{CC}_{\mathbf{a}}$ and $\mathcal{CC}_{(u,u^*)}$ denote the set of colored components labeled in **a** and $(u,u^*)$ respectively.

The 3-connections of $T$ correspond to those of $\tilde{T}$. Hence for any such pair $(v,w)$ in $T$, we get $\tau[T]=\tau[\tilde{T}]=\tau[\tilde{T}_{v,w}]=\tau[T_{v,w}]$, so the first property (the analog of Corollary 5.5) is clear. The proof of the second property is similar: if $T$ is a t.e.c. graph that can be obtained by identifying two vertices of disjoint test graphs $S$ and $S'$, then $\tilde{T}$ can be obtained by identifying two vertices $\tilde{S}$ and $\tilde{S}'$ and the proof is unchanged, using the above lemma instead of Lemma 5.4.

Let now $T$ be an arbitrary test graph. Denote by $\mathcal{O}$ the set of vertices of odd degree of $T$, and let $\tilde{\mathcal{O}}$ be the corresponding set of vertices in $\tilde{T}$. Consider the set $\tilde{\mathcal{O}}'$ of vertices of $\tilde{T}$ which are endpoints of both an edge labeled in **a** and an edge labeled in $(u,u^*)$. In particular, $\tilde{\mathcal{O}}\cup\tilde{\mathcal{O}}'$ is the set of vertices in $\tilde{T}$ which have an odd number of $(u,u^*)$ edges. If a partition $\pi$ of the vertices of $\tilde{T}$ is adapted, it must regroup the vertices of $\tilde{\mathcal{O}}\cup\tilde{\mathcal{O}}'$ in blocks of even size (since vertices of considered cacti are of even degree). But if $\pi$ identifies a vertex of $\tilde{\mathcal{O}}$ and a vertex of $\tilde{\mathcal{O}}'$, then $T^\pi$ has a cycle with an odd number of edges in $(u,u^*)$, so $\pi$ is not adapted. We hence get

$$\tau[\tilde{T}]=\sum_{\substack{\sigma\in\mathcal{P}(\tilde{\mathcal{O}})\\ |B|\text{ even }\forall B\in\sigma}}\tau\big[p_\sigma[\tilde{T}]\big]=\sum_{\substack{\sigma\in\mathcal{P}(\mathcal{O})\\ |B|\text{ even }\forall B\in\sigma}}\tau\big[p_\sigma[T]\big].$$

Hence, using Lemma 5.7 for **c**, $\tau[T]$ has the same expression as if labels were in **c**. Since for a simple cycle $C$ labeled $b_{j_1},\dots,b_{j_n}$ one has

$$\tau[S]=\Phi(b_{j_1},\dots,b_{j_n})=\Phi(a_{j_1},\dots,a_{j_n})=\Phi(c_{j_1},\dots,c_{j_n}),$$

we get as expected that **b** and **c** have the same traffic distribution.

Let us conclude: if **a** is of cactus type, or equivalently if **a** and **c** have the same traffic distribution, we get that **a** and **b** have the same traffic distribution, which means that **a** is unitarily invariant; on the other hand, if **a** is unitarily invariant, or equivalently if **a** and **b** have the same traffic distribution, we get that **a** and **c** have the same traffic distribution, which means that **a** is of cactus type.

## 6. Asymptotically unitarily invariant random matrices

The purpose of this section is to prove Theorems 1.1 and 4.7. Namely, for any unitarily invariant families of matrices $\mathbf{X}_N$ satisfying the assumptions, for any test graphs $T_1, \dots, T_m$,

$$\tau_{\mathbf{X}_N}(T_1, \dots, T_m) := \frac{1}{N^m} \mathbb{E}\Big[ \prod_{i=1}^m \mathrm{Tr}\, T_i(\mathbf{X}_N) \Big]$$

converges to

$$\prod_{i=1}^m \sum_{\pi \in \mathcal{P}(V_i)} \Big( \mathbb{1}(T_i^\pi \text{ well oriented cactus}) \prod_{C \in \mathrm{Cycle}(T_i^\pi)} \lim_{N\to\infty} \big(\tau^0_{\mathbf{X}_N}[C]\big)\Big), \tag{6.1}$$

where $V_i$ denotes the vertex set of $T_i$.

Before reviewing some results about the free cumulants, some results about the Weingarten function, and the links between those two objects in large dimension, let us mention two applications of this result.

### 6.1. Applications

**Lemma 6.1.** *If $\mathbf{A}_N$ is a family of matrices converging in traffic distribution to a unitarily invariant family, then $\mathbf{A}_N$, $\mathbf{A}_N^t$ and $(\deg(\mathbf{A}_N), \deg(\mathbf{A}_N^t))$ are asymptotically freely independent.*

This generalizes a recent result of Mingo and Popa [20] stating the asymptotic free independence of $\mathbf{A}_N$ and $\mathbf{A}_N^t$ for unitarily invariant matrices. Here we only assume that unitary invariance holds asymptotically.

*Proof.* Let $(\mathcal{A}, \tau)$ be an algebraic traffic space with trace $\Phi$ and let $\mathbf{a} = (a_j)_{j\in J}$ be a unitarily invariant family of traffics. It is sufficient to prove that the families $\mathbf{a}$, $\mathbf{a}^t = (a_j^t)_{j\in J}$ and $(\deg(\mathbf{a}), \deg(\mathbf{a}^t)) = (\deg(a_j), \deg(a_j^t))$ are free independent in $(\mathcal{A}, \Phi)$.

We first prove that $\mathbf{a}$ and $\mathbf{a}^t$ are free. Let us consider $2n$ elements $c_1, \dots, c_{2n}$ alternatively in $\mathbb{C}\langle \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A}\rangle$ and $\mathbb{C}\langle \cdot \overset{a}{\rightarrow} \cdot : a \in \mathcal{A}\rangle$ such that

$$\tau_\Phi(c_1) = \dots = \tau_\Phi(c_{2n}) = 0.$$

We want to prove that $\tau_\Phi(\Delta(c_1 \dots c_{2n})) = 0$. Using the *substitution* property of Definition 1.10 in order to regroup consecutive edges which are oriented in the same direction, we can assume that the $c_i'$s are written as $\cdot \overset{a_i}{\longleftarrow} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, and $c_i$ and $c_{i+1}$ not oriented in the same direction.

Consider now a partition $\pi$ such that $\tau^0_\Phi(\Delta(c_1 \dots c_{2n})^\pi) \neq 0$. Then, take a leaf of the oriented cactus $\Delta(c_1 \dots c_{2n})^\pi$. This leaf is a cycle of only one edge, because if not, the cycle cannot be oriented, since two consecutive edges in $\Delta(c_1 \dots c_{2n})$ are not oriented in the same way. This produces a term $\tau^0_\Phi(\Delta(c_i)) = 0$ in the product $\tau^0_\Phi(\Delta(c_1 \dots c_{2n})^\pi)$, which leads at the end to a vanishing contribution. Finally, $\tau_\Phi(c_1 \dots c_{2n}) = 0$ and we have the freeness wanted.

Now, let us prove that $\mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ is free from $\mathbb{C}\langle \cdot \overset{a}{\rightarrow} \cdot, \cdot \overset{a}{\leftarrow} \cdot : a \in \mathcal{A}\rangle$. By the same argument as above, we can consider that we have a cycle $\Delta(c_1 \dots c_n)$ which consists in an alternating sequence of $c_i's$ written as $\cdot \overset{a_i}{\longleftarrow} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, $\cdot \overset{a_i}{\longrightarrow} \cdot$ with $a_i \in \mathcal{A}$ such that $\Phi(a_i) = 0$, and $c_i \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ such that $\tau_\Phi(c_i) = 0$. We want to prove that $\tau_\Phi(\Delta(c_1 \dots c_n)) = 0$. If there is no term $c_i \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$, we are in the case of the previous paragraph. Let us assume that there exists at least one such term, say $c_1$. By linearity, we can consider that the term $c_1 \in \mathbb{C}\langle \overset{\uparrow a}{\cdot} : a \in \mathcal{A}\rangle$ is written as $\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot} - \tau_\Phi(\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})$, where $\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot}$ is some vertex input/output from which start $k$ edges labeled by $b_1, \dots, b_k \in \mathcal{A}$. Let us prove that $\tau_\Phi(\Delta((\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})c_2 \dots c_n))$ and $\tau_\Phi(\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})\tau_\Phi(\Delta(c_2 \dots c_n))$ are equal, which implies by linearity that $\tau_\Phi(\Delta(c_1 \dots c_n)) = 0$. Decomposing into injective trace, we are left to prove that for every partition $\pi$ of the vertices of $\Delta((\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})c_2 \dots c_n)$ which does not respect the blocks $(\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})$ and $\Delta(c_2 \dots c_n)$, $\tau^0_\Phi(\Delta((\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})c_2 \dots c_n)^\pi) = 0$. The same argument as previous paragraph works again. If one of the vertices of $(\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})$ is identified by $\pi$ with one of the vertex of $\Delta(c_2 \dots c_n)$, and $\Delta((\overset{\uparrow b_1}{\cdot} \cdots \overset{\uparrow b_k}{\cdot})c_2 \dots c_n)^\pi$ is a cactus there exists a cycle not oriented or a leaf labeled by one $a_i$, which leads to a vanishing contribution. ■

### 6.2. The Weingarten function

We need to integrate polynomials in the entries of a unitary matrix against the $\mathrm{U}(N)$-Haar measure. Expressions for these integrals appeared in [32] and were first proven in [8] and given in terms of a function on symmetric group called the Weingarten function. We recall here its definition and some of its properties. For any $n \in \mathbb{N}^*$ and any permutation $\sigma \in \mathcal{S}_n$, let us set

$$\Omega_{n,N}(\sigma) = N^{\#\sigma},$$

where $\#\sigma$ is the number of cycles of $\sigma$. When $n$ is fixed and $N \to \infty$, $N^{-n}\Omega_{n,N} \to \delta_{\mathrm{Id}_n}$. For any pair of functions $f, g : \mathcal{S}_n \to \mathbb{C}$ and $\pi \in \mathcal{S}_n$, let us define the convolution product

$$f \star g(\sigma) = \sum_{\pi \preccurlyeq \sigma} f(\pi) g(\pi^{-1}\sigma).$$

Hence, for $N$ large enough, $\Omega_{n,N}$ is invertible in the algebra of function on $\mathcal{S}_n$ endowed with convolution as a product. We denote by $\mathrm{Wg}_{n,N}$ the unique function on $\mathcal{S}_n$ such that

$$\mathrm{Wg}_{n,N} * \Omega_{n,N} = \Omega_{n,N} * \mathrm{Wg}_{n,N} = \delta_{\mathrm{Id}_n}.$$

Then, [8, Corollary 2.4] says that, for any indices $i_1, i'_1, j_1, j'_1, \dots, i_n, i'_n, j_n, j'_n \in \{1, \dots, N\}$ and $U = (U(i,j))_{i,j=1,\dots,N}$ a Haar distributed random matrix on $\mathrm{U}(N)$,

$$\mathbb{E}\big[U(i_1, j_1) \dots U(i_n, j_n)\bar{U}(i'_1, j'_1) \dots \bar{U}(i'_n, j'_n)\big] = \sum_{\substack{\alpha,\beta \in \mathcal{S}_n \\ i_{\alpha(k)} = i'_k, j_{\beta(k)} = j'_k}} \mathrm{Wg}_{n,N}(\alpha\beta^{-1}). \quad (6.2)$$

### 6.3. Free cumulants and the Möbius function $\mu$

As explained in [4], it is equivalent to consider lattices of non-crossing partitions or sets of permutations endowed with an appropriate distance. For our purposes, it is more suitable to define the free cumulants using sets of permutations. Let us endow $\mathcal{S}_n$ with the metric $d$, by setting for any $\alpha, \beta \in \mathcal{S}_n$,

$$d(\alpha, \beta) = n - \#(\beta\alpha^{-1}),$$

where $\#(\beta\alpha^{-1})$ is the number of cycles of $\beta\alpha^{-1}$. We endow the set $\mathcal{S}_n$ with the partial order given by the relation $\sigma_1 \preceq \sigma_2$ if $d(\mathrm{Id}_n, \sigma_1) + d(\sigma_1, \sigma_2) = d(\mathrm{Id}_n, \sigma_2)$, or similarly if $\sigma_1$ is on a geodesic between $\mathrm{Id}_n$ and $\sigma_2$.

Given a state $\Phi : \mathbb{C}\langle x_j, x_j^*\rangle_{j\in J} \to \mathbb{C}$, we define the free cumulants $(\kappa_n)_{n\in\mathbb{N}}$ recursively on $\mathbb{C}\langle x_j, x_j^*\rangle_{j\in J}$ by the system of equations: $\forall y_1, \dots, y_n \in \mathbb{C}\langle x_j, x_j^*\rangle_{j\in J}$

$$\Phi(y_1 \cdots y_n) = \sum_{\sigma \preccurlyeq (1\cdots n)} \prod_{\substack{(c_1 \dots c_k)\\ \text{cycle of } \sigma}} \kappa(y_{c_1}, \dots, y_{c_k}).$$

Let us fix $y_1, \dots, y_n \in \mathbb{C}\langle x_j, x_j^*\rangle_{j\in J}$ and denote by respectively $\Xi$ and $k$ the functions from $\mathcal{S}_n$ to $\mathbb{C}$ given by

$$\Xi(\sigma) = \prod_{\substack{(c_1 \dots c_k)\\ \text{cycle of } \sigma}} \Phi(y_{c_1} \dots y_{c_k}) \quad \text{and} \quad k(\sigma) = \prod_{\substack{(c_1 \dots c_k)\\ \text{cycle of } \sigma}} \kappa(y_{c_1}, \dots, y_{c_k}),$$

which are such that $\Xi((1\cdots n)) = \sum_{\pi \preccurlyeq (1\cdots n)} k(\pi)$. In fact, we have more generally the relation

$$\Xi(\sigma) = \sum_{\pi \preccurlyeq \sigma} k(\pi).$$

Note that $\Xi = k \star \zeta$, where $\zeta$ is identically equal to one. The identically one function $\zeta$ is invertible for the convolution $\star$ (see [4]), and its inverse $\mu$ is called Möbius function. It allows us to express the free cumulants in terms of the trace:

$$k = \Xi \star \mu. \tag{6.3}$$

### 6.4. Asymptotics of the Weingarten function

One can observe that, for any pair of functions $f, g : \mathcal{S}_n \to \mathbb{C}$ and $\pi \in \mathcal{S}_n$,

$$\sum_{\pi \in \mathcal{S}_n} N^{d(\mathrm{Id}_n, \sigma) - d(\mathrm{Id}_n, \pi) - d(\pi, \sigma)} f(\pi) g(\pi^{-1}\sigma) = f \star g(\sigma) + o(1),$$

where $o(1)$ is a quantity which converges to 0 as $N$ tends to $\infty$. Defining the convolution $\star_N$ as

$$\begin{aligned} f \star_N g &= N^n \Omega_{n,N}^{-1}\big((N^{-n}\Omega_{n,N} f) * (N^{-n}\Omega_{n,N} g)\big) \\ &= \sum_{\pi \in \mathcal{S}_n} N^{d(\mathrm{Id}_n, \sigma) - d(\mathrm{Id}_n, \pi) - d(\pi, \sigma)} f(\pi) g(\pi^{-1}\sigma), \end{aligned}$$

it follows that $\star$ is the limit of $\star_N$. Because $\mathrm{Wg}_{n,N}$ is the inverse of $\Omega_{n,N}$ for the convolution $*$, we have

$$(N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N}) \star_N \zeta = N^{-n}\Omega_{n,N},$$

from which we deduce that $(N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N}) \star \zeta = \delta_{\mathrm{Id}_n} + o(1)$, or similarly that

$$N^{2n}\Omega_{n,N}^{-1}\mathrm{Wg}_{n,N} = \mu + o(1).$$

More generally, if $f, f_N : \mathcal{S}_n \to \mathbb{C}$ are such that $f_N = f + o(1)$, then

$$N^n\Omega_{n,N}^{-1}\big((\Omega_{n,N} f_N) * \mathrm{Wg}_{n,N}\big) = (f_N) \star_N (\mathrm{Wg}_{n,N}) = f \star \mu + o(1). \tag{6.4}$$

*Proof of Theorem* 1.1. Let $\mathbf{X}_N = (X_j)_{j\in J}$ a family of unitarily invariant random matrices which converges in $^*$-distribution, as $N$ goes to infinity, to a family $\mathbf{x} = (x_j)_{j\in J}$ in some non-commutative probability space $(\mathcal{A}, \Phi)$. We fix $m \geq 1$ and test graphs $T_i = (V_i, E_i, j_i) \in \mathbb{C}\mathcal{T}\langle J\rangle$, $i = 1, \dots, m$, and show the convergence stated in (6.1).

By taking the real and the imaginary parts, we can assume that the matrices of $\mathbf{X}_N$ are Hermitian and so we do not consider adjoint of the matrices. We shall denote by $T = (V, E, j)$ the labeled graph obtained from the disjoint unions of $T_1, \dots, T_m$, where the label map is given by restriction: $j_{|E_i} = j_i$ for $i = 1, \dots, m$.

We consider a random unitary matrix $U$, distributed according to the Haar distribution, and independent of $\mathbf{X}_N$. By assumption,

$$\mathbf{Z}_N := U\mathbf{X}_N U^* \in M_N(\mathbb{C})$$

has the same distribution as $\mathbf{X}_N$. We denote respectively by $\underline{e}$ and $\bar{e}$ the origin vertex and the goal vertex of $e$. Then

$$\begin{aligned}
&\tau_{\mathbf{X}_N}[T_1, \dots, T_m] \\
&\quad = \frac{1}{N^m}\sum_{\phi:V\to[N]} \mathbb{E}\Big[\prod_{e\in E} Z_{j(e)}\big(\phi(\underline{e}), \phi(\bar{e})\big)\Big] \\
&\quad = \frac{1}{N^m}\sum_{\substack{\phi:V\to[N]\\ \varphi,\varphi':E\to[N]}} \mathbb{E}\Big[\prod_{e\in E} U\big(\phi(\underline{e}), \varphi(e)\big)\bar{U}\big(\phi(\bar{e}), \varphi'(e)\big)\Big]\mathbb{E}\Big[\prod_{e\in E} X_{j(e)}\big(\varphi(e), \varphi'(e)\big)\Big].
\end{aligned}$$

In the integration formula (6.2), the number $n$ of occurrence of each term $U(i, j)$ is the cardinality of $E$ and the sum over permutations of $\{1, \dots, n\}$ is replaced by a sum over the set $\mathcal{S}_E$ of permutations of the edge set $E$. By identifying $E$ with the set of integers $\{1, \dots, |E|\}$, we consider that $\mathrm{Wg}_{n,N}$ is defined on $\mathcal{S}_E$ instead of $\mathcal{S}_n$. Then, one has

$$\begin{aligned}
&\tau_{\mathbf{X}_N}[T_1, \dots, T_m] \\
&\quad = \frac{1}{N^m}\sum_{\alpha,\beta\in\mathcal{S}_E} \mathrm{Wg}_{n,N}(\alpha\beta^{-1}) \sum_{\substack{\phi:V\to\{1,\dots,N\}\\ \varphi,\varphi':E\to\{1,\dots,N\}\\ \phi(\underline{\alpha(e)})=\phi(\bar{e}),\varphi(\beta(e))=\varphi'(e)}} \mathbb{E}\Big[\prod_{e\in E} X_{j(e)}\big(\varphi(e), \varphi'(e)\big)\Big].
\end{aligned}$$

For any permutation $\alpha \in \mathcal{S}_E$, let $\pi(\alpha)$ be the smallest partition of $V$ such that, for all $e \in E$, $\bar{e}$ is in the same block as $\underline{\alpha(e)}$. Summing over $\phi$ in the previous expression yields

$$\begin{aligned}
&\tau_{\mathbf{X}_N}[T_1, \dots, T_m] \\
&= \sum_{\alpha,\beta \in \mathcal{S}_E} N^{\#\pi(\alpha)-m} \mathrm{Wg}_{n,N}(\alpha\beta^{-1}) \sum_{\substack{\varphi,\varphi' : E \to \{1,\dots,N\} \\ \varphi(\beta(e))=\varphi'(e)}} \mathbb{E}\Big[\prod_{e \in E} X_{j(e)}\big(\varphi(e), \varphi'(e)\big)\Big] \\
&= \sum_{\alpha,\beta \in \mathcal{S}_E} N^{\#\pi(\alpha)-m} \mathrm{Wg}_{n,N}(\alpha\beta^{-1}) \mathbb{E}\Big[\prod_{\substack{(e_1 \dots e_k) \\ \text{cycle of } \beta}} \mathrm{Tr}(X_{j(e_1)} X_{j(e_2)} \dots X_{j(e_k)})\Big]
\end{aligned}$$

To conclude we will need the following.

**Lemma 6.2.** (i) *For any permutation $\alpha \in \mathcal{S}_E$, $\#\pi(\alpha) + \#\alpha \leq \#E + m$ and the equality implies that the graph of $T^{\pi(\alpha)}$ is the disjoint union of $m$ oriented cacti, with resp. set of edges $E_1, \dots, E_m$, and that $\alpha$ fixes the sets $E_1, \dots, E_m$.*

(ii) *The map*

$$\pi : \{\alpha : \#\pi(\alpha) + \#\alpha = \#E + m\} \to \{\sigma : \text{the graph of } T^\sigma \text{ is the disjoint union of } m \text{ oriented cacti with resp. edges set } E_1, \dots, E_m\}$$

*is a bijection whose inverse $\gamma$ is given, for all $\sigma \in \mathcal{P}(V)$ such that $T^\sigma$ is a disjoint union of $m$ oriented cacti with resp. edges $E_1, \dots, E_m$, by the permutation $\gamma(\sigma)$ whose cycles are the simple cycles of $T^\sigma$.*

*Proof Lemma 6.2.* (i) Let $\alpha \in \mathcal{S}_E$. Let us define a connected graph $G_\alpha$ whose vertices are the cycles of $\alpha$ altogether with the blocks of $\pi(\alpha)$, and whose edges are defined as follows. There is an edge between a cycle $c$ of $\alpha$ and a block $b$ of $\pi(\alpha)$ if and only if there is an edge $e$ of $T$ such that $e \in c$ and $\bar{e} \in b$. This way, the edges of $G_\alpha$ are in bijective correspondence with the edges of $T$, and the number of vertices of $G_\alpha$ is $\#\pi(\alpha) + \#\alpha$. Note that, if an edge $e$ belongs to some connected component $S$ of $G_\alpha$, not only the block $b$ containing $\bar{e}$ is in $S$ (as an endpoint of $e$), but also the block $b'$ containing $\underline{e}$ is also in $S$: indeed, the cycle $c$ containing $e$ is also the one containing $\alpha^{-1}(e)$, and is connected to $b' \ni \overline{\alpha^{-1}(e)}$ by the edge $\alpha^{-1}(e)$. Consequently, the number of connected components of $G_\alpha$ is no bigger than $m$.

Now, the number of vertices of $G_\alpha$ is less than or equal to the number of edges plus the number of connected components of $G_\alpha$. Therefore, we have $\#\pi(\alpha) + \#\alpha \leq \#E + m$ with equality if and only $G_\alpha$ is the disjoint unions of $m$ trees.

In fact, each cycle of $\alpha$ yields a cycle in $T^{\pi(\alpha)}$, and in the case where $G_\alpha$ is acyclic, there exists no other cycle in $T^{\pi(\alpha)}$. What is more, since $T_1, \dots, T_m$ are connected, if $T^{\pi(\alpha)}$ has $m$ connected components, the latter cannot use edges of several sets among $E_1, \dots, E_m$. Hence, the simple cycles of $T^{\pi(\alpha)}$ are exactly the cycles of $\alpha$, that cannot use edges from several sets $E_1, \dots, E_m$, and $T^{\pi(\alpha)}$ is therefore the disjoint union of $m$ oriented cacti, with $\alpha$ fixing each set $E_i$, $i = 1, \dots, m$.

(ii) $\pi \circ \gamma$ and $\gamma \circ \pi$ are the identity functions: $\pi$ is one-to-one and its inverse is $\gamma$. ■

For all $\alpha \in \mathcal{S}_E$, set

$$\Xi_N(\alpha) = N^{-\#\alpha}\mathbb{E}\Big[\prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of } \alpha}} \mathrm{Tr}(X_{\gamma(e_1)}X_{\gamma(e_2)}\ldots X_{\gamma(e_k)})\Big]$$

and

$$\Xi(\alpha) = \prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of } \alpha}} \Phi(x_{\gamma(e_1)}x_{\gamma(e_2)}\ldots x_{\gamma(e_k)})$$

in such a way that $\Xi_N = \Xi + o(1)$ as $N$ tends to $\infty$. Let us fix $\alpha \in \mathcal{S}_E$. On the one hand we have

$$N^{\#\pi(\alpha)+\#\alpha-\#E-m} = \mathbb{1}_{\#\pi(\alpha)+\#\alpha=\#E+m} + o(1).$$

On the other hand, according to (6.4), the quantity

$$\sum_{\beta\in\mathcal{S}_E} N^{\#E-\#\alpha}\mathrm{Wg}_{n,N}(\alpha\beta^{-1})\mathbb{E}\Big[\prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of } \beta}} \mathrm{Tr}(X_{\gamma(e_1)}X_{\gamma(e_2)}\ldots X_{\gamma(e_k)})\Big]$$

is equal to $((\Xi_N) \star_N \mathrm{Wg}_{n,N})(\alpha) = (\Xi \star \mu)(\alpha) + o(1)$. Let us write $\alpha_1 \times \cdots \times \alpha_m$ for the permutation whose restriction to $E_1, \ldots, E_m$ is given by $\alpha_i \in \mathcal{S}_{E_i}$, for $i = 1, \ldots, m$. It follows that

$$\tau_{\mathbf{X}_N}(T_1, \ldots, T_m) = \sum_{\substack{\alpha_i\in\mathcal{S}_{E_i}, i=1,\ldots,m\\ \#\pi(\alpha_i\times\cdots\times\alpha_m)+\#\alpha_1\times\cdots\times\alpha_m=\#E+m}} (\Xi \star \mu)(\alpha_1 \times \cdots \times \alpha_m) + o(1).$$

From (6.3), we know that $(\Xi \star \mu)(\alpha) = k(\alpha) = \prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of } \alpha}} \kappa(x_{\gamma(e_1)}, \ldots, x_{\gamma(e_k)})$. Let $\pi_1 \sqcup \cdots \sqcup \pi_m$ be the partition of $E$, that is, finer than $\{E_1, \ldots, E_m\}$ and whose restriction of these $m$ sets is fixed, when $\pi_i \in \mathcal{P}(V_i)$, $i = 1, \ldots, m$. Thanks to Lemma 6.2, we can now write

$$\begin{aligned}\tau_{\mathbf{X}_N}(T) &= \sum_{\substack{\pi_i\in\mathcal{P}(V_i), i=1,\ldots,m\\ T_i^{\pi_i} \text{ is an oriented cactus}}} \prod_{\substack{(e_1\ldots e_k)\\ \text{cycle of } \gamma(\pi_1\sqcup\cdots\sqcup\pi_m)}} \kappa(x_{\gamma(e_1)}, \ldots, x_{\gamma(e_k)}) + o(1)\\ &= \sum_{\substack{\pi_i\in\mathcal{P}(V_i), i=1,\ldots,m\\ T_i^{\pi_i} \text{ is an oriented cactus}}} \prod_{\substack{(e_1\ldots e_k)\\ \text{simple cycle of one graph } T_i^{\pi_i}}} \kappa(x_{\gamma(e_1)}, \ldots, x_{\gamma(e_k)}) + o(1).\end{aligned}$$

In order to pursue the computation, let $t_i$ be the test graph $(V_i, E_i, \lambda_i(e)) \in \mathbb{C}\mathcal{T}\langle\mathcal{G}(\mathcal{A})\rangle$ such that $\lambda_i(e) = x_{j_i(e)}$, for $i = 1, \ldots, m$. By Definition of unitarily invariant traffics, we get

$$\tau_{\mathbf{X}_N}(T_1, \ldots, T_m) = \sum_{\pi_i\in\mathcal{P}(V_i), i=1,\ldots,m} \prod_{i=1}^m \tau^0_\Phi[t_i^{\pi_i}] + o(1) = \prod_{i=1}^m \tau_X[t_i] + o(1)$$

so that $\tau_{\mathbf{X}_N}(T_1, \ldots, T_m)$ converges towards the expected limit. ∎

**Remark 6.3.** From the above proof, it is tempting to believe that expansions of moments of the evaluation of test graphs in powers of $N^{-1}$ should actually be expansions in powers of $N^{-2}$, so that for any $*$-test graph $T = (V, E, j \times \varepsilon) \in \mathbb{C}\mathcal{T}\langle J \times \{1, *\}\rangle$, the fluctuations of $\frac{1}{N}\operatorname{Tr}(T(\mathbf{X}_N)) - \mathbb{E}[\frac{1}{N}\operatorname{Tr}(T(\mathbf{X}_N))]$ should be of order $N^{-1}$. This is nonetheless wrong as shows the following simple example. Consider a random $N \times N$ matrix $A$, whose law is invariant by unitary conjugation and the test graph $T$ with one simple edge labeled by $A$ and one extremity equal both to the input and output. For the associated traffic distribution as in Example 1.8, $\operatorname{Tr}(T(\mathbf{X}_N)) = \sum_{1 \le i,j \le N} A_{i,j}$. In the setting of the central limit theorem where entries of $A$ have variance of order $\frac{1}{N}$, the fluctuations of $\frac{1}{N}\operatorname{Tr}(T(\mathbf{X}_N))$ are of order $O(\frac{1}{\sqrt{N}})$.

## 7. Canonical construction of spaces of free type

The purpose of this section is to prove the Theorem 1.3, which states that any tracial $*$-probability space can be enlarged into a traffic space.

### 7.1. Free $\mathcal{G}$-algebra generated by an algebra

We first describe how an algebra can be canonically extended into a $\mathcal{G}$-algebra.

**Definition 7.1.** Let $\mathcal{A}$ be an algebra. We denote by $\mathcal{G}(\mathcal{A})$ the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}\langle\mathcal{A}\rangle$ of graph polynomials labeled in $\mathcal{A}$, quotiented by the following relations: for all $g \in \mathcal{G}_{n-k+1}$, $a_1, \dots, a_n \in \mathcal{A}$ and $P$ non-commutative polynomial in $n$ variables, we have

$$\begin{aligned} &Z_g(\cdot \xleftarrow{P(a_1,\dots,a_k)} \cdot \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_n} \cdot) \\ &\quad = Z_g\big(P(\cdot \xleftarrow{a_1} \cdot, \dots, \cdot \xleftarrow{a_k} \cdot) \otimes \cdot \xleftarrow{a_{k+1}} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_n} \cdot\big) \end{aligned} \tag{7.1}$$

which allows to consider the algebra homomorphism $V : \mathcal{A} \to \mathcal{G}(\mathcal{A})$ given by $a \mapsto (\cdot \xleftarrow{a} \cdot)$.

Just as for the free product of $\mathcal{G}$-algebras in Section 3.1, the space $\mathcal{G}(\mathcal{A})$ is a $\mathcal{G}$-algebra. Moreover, it is the free $\mathcal{G}$-algebra generated by the algebra $\mathcal{A}$ in the following sense.

**Proposition 7.2.** *Let $\mathcal{B}$ be a $\mathcal{G}$-algebra and $f : \mathcal{A} \to \mathcal{B}$ a algebra homomorphism. There exists a unique $\mathcal{G}$-algebra homomorphism $f' : \mathcal{G}(\mathcal{A}) \to \mathcal{B}$ such that $f = f' \circ V$. As a consequence, the algebra homomorphism $V : \mathcal{A} \to \mathcal{G}(\mathcal{A})$ is injective.*

*Proof.* The existence is given by the following definition of $f'$ on $\mathcal{G}(\mathcal{A})$:

$$f'\big(Z_g(\cdot \xleftarrow{a_1} \cdot \otimes \cdots \otimes \cdot \xleftarrow{a_n} \cdot)\big) = Z_g\big(f(a_1) \otimes \cdots \otimes f(a_n)\big)$$

for all $a_1, \dots, a_n \in \mathcal{A}$; which obviously respects the relation defining $*_{j \in J}\mathcal{A}_j$.

The uniqueness follows from the fact that $f'$ is uniquely determined on $V(\mathcal{A})$ (indeed, $f'(a)$ must be equal to $f(b)$ whenever $a = V(b)$) and that $V(\mathcal{A})$ generates $\mathcal{G}(\mathcal{A})$ as a $\mathcal{G}$-algebra. ∎

For example, the free $\mathcal{G}$-algebra generated by the variables $\mathbf{x} = (x_i)_i \in J$ and $\mathbf{x}^* = (x_i^*)_i \in J$ is the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}\langle \mathbf{x}, \mathbf{x}^* \rangle$ of graphs whose edges are labeled by $\mathbf{x}$ and $\mathbf{x}^*$.

### 7.2. Algebraic construction

Let $(\mathcal{A}, \Phi)$ be a non-commutative probability space such that $\Phi$ is a trace. We want to equip the $\mathcal{G}$-algebra $\mathcal{G}(\mathcal{A})$ with a combinatorial distribution, that is, of cactus type and whose induced distribution on $\mathcal{A} \subset \mathcal{G}(\mathcal{A})$ is $\Phi$. We firstly define $\tau : \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle \to \mathbb{C}$ by the cactus formula, namely for any test graph $T$ labeled in $\mathcal{A}$,

$$\tau^0[T] = \mathbb{1}(T \text{ is a w.o. cactus }) \prod_{C \text{ cycle of } T} \kappa(C),$$

where as usual $\kappa$ is the free cumulant function with respect to $\Phi$ of the variables along the oriented cycle. Then, as in Section 3.1, we consider the map $\tilde{\tau} : \mathbb{C}\mathcal{T}\langle \mathbb{C}\mathcal{G}\langle \mathcal{A} \rangle \rangle \to \mathbb{C}$ defined as follow: for any test graph $T$ with edges $e_1, \dots, e_k$ labeled respectively by graph monomials $g_1, \dots, g_K$, we set $\tilde{\tau}[T] = \tau[T_g]$ where $T_g$ is obtained by replacing the egde $e_k$ by the graph $g_k$ for any $k = 1, \dots, K$. We extend the definition by multi-linearity with respect to the edges and set $\tilde{\tau}[(\cdot)] = 1$. By Lemma 3.1, $\tilde{\tau}$ satisfies the associativity property and then endows $\mathbb{C}\mathcal{G}\langle \mathcal{A} \rangle$ with a structure of algebraic traffic space. It remains to prove that it induces such a structure on $\mathcal{G}(\mathcal{A})$.

**Proposition 7.3.** *The linear form $\tilde{\tau}$ is invariant under the relations (7.1) defining $\mathcal{G}(A)$, and consequently yields to an algebraic traffic space $(\mathcal{G}(A), \tilde{\tau})$. Furthermore, the trace induced by $\tilde{\tau}$ coincides with $\Phi$ on $\mathcal{A}$, seen as a subalgebra of $\mathcal{G}(\mathcal{A})$.*

*Proof.* It is sufficient to prove the following:

(1) For any test graph $T$ having an edge $e$ labeled $a_1 + \alpha a_2$, where $a_1, a_2 \in \mathcal{A}$ and $\alpha \in \mathbb{C}$, one has
$$\tau[T] = \tau[T_1] + \alpha\tau[T_2]$$
where $T_i$ is obtained from $T$ by putting label $a_i$ on $e$.

(2) For any test graph $T$ having an edge $e$ labeled $1_{\mathcal{A}}$, one has
$$\tau[T] = \tau[T_\bullet]$$
where $T_\bullet$ is obtained by identifying source and target of $e$ and suppressing this edge.

(3) For any test graph $T$ having an edge $e$ labeled $a_1 a_2$, where $a_1, a_2 \in \mathcal{A}$, one has
$$\tau[T] = \tau[T_\times]$$
where $T_\times$ is obtained by replacing $e$ by two consecutive edges $(\cdot \xleftarrow{a_1} \cdot \xleftarrow{a_2} \cdot)$.

The first property is an immediate consequence of the linearity of the cumulants. Let us prove the others properties at the level of the injective trace.

**Lemma 7.4.** *With notations as above, we have the following formulas:*

(1) *Whenever $e$ has label $1_{\mathcal{A}}$, one has $\tau^0[T] = \tau^0[T_\bullet]$ if the goal and the source of the edge $e$ are equal in $T$, and $\tau^0[T] = 0$ otherwise.*

(2) *Whenever $e$ has label $a_1a_2$, denote by $V$ the vertex set of $T$ and by $v_0$ the new vertex in $T_\times$. Then for any partition $\pi$ of $V$, one has*

$$\tau^0[T^\pi] = \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \\ \sigma \setminus \{v_0\} = \pi}} \tau^0[T_\times^\sigma].$$

This implies the proposition as we see now. When $e$ has label $1_{\mathcal{A}}$, we get

$$\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi] = \sum_{\pi \in \mathcal{P}(V_\bullet)} \tau^0[T_\bullet^\pi] = \tau[T_\bullet].$$

Moreover, when $e$ has label $a_1a_2$, one has

$$\tau[T] = \sum_{\pi \in \mathcal{P}(V)} \tau^0[T^\pi] = \sum_{\pi \in \mathcal{P}(V)} \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \\ \sigma \setminus \{v_0\} = \pi}} \tau^0[T_\times^\sigma] = \sum_{\sigma \in \mathcal{P}(V \cup \{v_0\})} \tau^0[T_\times^\sigma] = \tau[T_\times].$$

Finally, for any $a \in \mathcal{A}$, seen as an element of $\mathcal{G}(\mathcal{A})$, its trace associated to $\tau$ is given by

$$\tau(\circlearrowleft_a) = \tau^0(\circlearrowleft_a) = \kappa(a) = \Phi(a)$$

as expected. This finishes the proof of the proposition. ∎

*Proof of Lemma 7.4.* The first item follows from the fact that a cumulant involving $1_{\mathcal{A}}$ is equal to 0, except $\kappa(1_{\mathcal{A}}) = 1$ (see [26, Proposition 11.15]). As a consequence, for a cactus $T$ having a loop labeled $1_{\mathcal{A}}$, we can remove the loop without changing the value of the invective trace.

Let us prove the second item, and consider a test graph $T$ with an edge $e$ labeled $a_1a_2$ and $T_\times$ defined as before. Let $\pi$ be a partition of the vertex set of $T$. If $T^\pi$ is not a cactus, then both sides of the equation are equal to zero. Assume that $T^\pi$ is a cactus. We denote by $c$ the cycle of $\cdot \xleftarrow{a_1a_2} \cdot$ in $T^\pi$ and $a_1a_2, b_2, \dots, b_{k-1}$ the elements of the cycle $c$ starting at $a_1a_2$.

Let us consider a partition $\sigma \in \mathcal{P}(V \cup \{v_0\})$ such that $T_\times^\sigma$ is a cactus and $\pi = \sigma \setminus \{v_0\}$. Then, we have two cases:

(1) $v_0$ is of degree 2 in $T_\times^\sigma$ (this occurs for only one partition $\sigma$ given by $\pi \cup \{\{v_0\}\}$). Denoting by $c^+$ the cycle of $T_\times^\sigma$ which contains $v_0$, we have

$$c^+ = (a_2, b_2, \dots, b_{k-1}, a_1).$$

The cycles of $T^\pi$ and $T_\times^\sigma$ different from $c$ and $c^+$ are the same, and by consequence

$$\tau^0[T^\pi]/k(a_1a_2, b_2, \dots, b_{k-1}) = \tau^0[T_\times^\sigma]/k(a_2, b_2, \dots, b_{k-1}, a_1).$$

(2) $v_0$ is of degree $> 2$ in $T^\sigma_\times$. We denote by $c_1$ the cycle of $\cdot \overset{a_1}{\longleftarrow} \cdot$ in $T^\sigma_\times$, $c_2$ the cycle of $\cdot \overset{a_2}{\longleftarrow} \cdot$ in $T^\sigma_\times$ (of course, $c_1$ and $c_2$ are not equal, because if it is the case, $T^\pi$ would be disconnected, which is not possible). The cycles of $T^\pi$ other than $c$ are exactly the cycles of $T^\sigma_\times$ other than $c_1$ and $c_2$. We have $c_1 = (a_2, b_2, \dots, b_l)$ and $c_2 = (b_{l+1}, \dots, b_k, a_1)$ with $l$ the place of the vertex which is identified with $v_0$ in $T^\sigma_\times$. By definition, we have

$$\tau^0[T^\pi]/k(a_1a_2, b_2, \dots, b_{k-1}) = \tau^0[T^\sigma_\times]/\big(k(a_2, b_2, \dots, b_l) \cdot k(b_{l+1}, \dots, b_k, a_1)\big).$$

Conversely, for each vertex $v_1$ in the cycle $c$, we are in the above situation for $\sigma = \pi_{|v_0 \simeq v_1}$.

Finally, using [26, Theorem 11.12] for computing $k(a_1a_2, b_2, \dots, b_{k-1})$, we can compute

$$\begin{aligned}
\tau^0[T^\pi] &= \tau^0[T^\pi]/k(a_1a_2, b_2, \dots, b_{k-1}) \cdot k(a_1a_2, b_2, \dots, b_{k-1}) \\
&= \tau^0[T^\pi]/k(a_1a_2, b_2, \dots, b_{k-1}) \\
&\quad \cdot \Big(k(a_2, b_2, \dots, b_{k-1}, a_1) + \sum_{1 \le l \le k} k(a_2, b_2, \dots, b_l) \cdot k(b_{l+1}, \dots, b_k, a_1)\Big) \\
&= \tau^0[T^{\pi \cup \{\{v_0\}\}}_\times] + \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \setminus \{\pi \cup \{\{v_0\}\}\} \\ \sigma \setminus \{v_0\} = \pi}} \tau^0[T^\sigma_\times] \\
&= \sum_{\substack{\sigma \in \mathcal{P}(V \cup \{v_0\}) \\ \sigma \setminus \{v_0\} = \pi}} \tau^0(T^\sigma_\times). \qquad \blacksquare
\end{aligned}$$

### 7.3. Positivity

Let $(\mathcal{A}, \Phi)$ be a $^*$-probability space. We define $\tau : \mathbb{C}\mathcal{T}\langle \mathcal{A} \rangle \to \mathbb{C}$ by the cactus formula with respect to $\Phi$ and then $(\mathcal{G}(\mathcal{A}), \tilde{\tau})$ as in Proposition 7.3. It remains to prove that $\tilde{\tau}$ satisfies the positivity condition (3.1), and it is actually sufficient to prove that $\tau$ is positive.

In the four steps of the proof, we will prove successively that $\tau[t|t^*] \ge 0$ for $n$-graph polynomials $t = \sum_{i=1}^L \alpha_i t_i$ with an increasing generality:

(1) the $t_i$ are 2-graph monomials without cycles and the leaves are outputs, that is, chains of edges with possibly different orientations;

(2) the $t_i$ are trees whose leaves are the outputs;

(3) the $t_i$ are such that $t_i|t_i^*$ have no cutting edges (see Definition 5.1);

(4) the $t_i$ are $n$-graph monomials.

**Step 1.** By Proposition 7.3, the trace associated to $\tau$ coincides with $\Phi$ on $\mathcal{A} \subset \mathcal{G}(\mathcal{A})$. We still denote it by $\Phi$. Hence we get the positivity if all the $t_i$'s consist in chains of edges all oriented in the same direction. Indeed, we can write $t_i = \cdot \overset{a_i}{\longleftarrow} \cdot$ for all $i$ (or $t_i = \cdot \overset{a_i}{\longrightarrow} \cdot$ for all $i$) and so, we get

$$\tau\big[t|t^*\big] = \tau\left[\sum_{i,j=1}^L \alpha_i \bar{\alpha}_j t_i t_j^*\right] = \Phi\left(\sum_{i,j=1}^L \alpha_i \bar{\alpha}_j a_i a_j^*\right) \ge 0,$$

by positivity of $\Phi$ on $\mathcal{A}$. We deduce that $\Phi$ is positive on the subalgebras $\mathbb{C}\langle \cdot \xleftarrow{a} \cdot : a \in \mathcal{A}\rangle$ and $\mathbb{C}\langle \cdot \xrightarrow{a} \cdot : a \in \mathcal{A}\rangle$ of $\mathcal{G}(\mathcal{A})$. By Lemma 6.1, these subalgebras are freely independent, so $\Phi$ is also positive on the mixed algebra $\mathbb{C}\langle \cdot \xleftarrow{a} \cdot, \cdot \xrightarrow{a} \cdot : a \in \mathcal{A}\rangle$ (the free product of positive traces is positive [26, Lecture 6]). Finally, if the $t_i$'s consist in chains of edges labeled by elements of $\mathcal{A}$, we know that

$$\tau[t|t^*] = \Phi\left[\sum_{i,j=1}^{L} \alpha_i \bar{\alpha}_j t_i t_j^*\right] \geq 0.$$

**Step 2.** Assume that the $t_i$'s are trees whose leaves are the outputs. Let us prove by induction on the number $D$ of all edges of the $t_i$'s that we have $\tau[t|t^*] \geq 0$.

If the number of edges of the $t_i$'s is 0, we have $\tau_\Phi[t|t^*] = \sum_{i,j} \alpha_i \alpha_j^* \geq 0$. We suppose that $D \geq 1$ and that this result is true whenever the number of edges of the $t_i$'s is less than $D - 1$.

We can remove one edge in the following way. Let us choose one leaf $v$ of one of the $t_i's$ which has at least one edge. It is an output and for each tree $t_i$ we denote by $v^{(i)}$ the first node (or distinct leaf if there is no node) of the tree of $t_i$ encountered by starting from this output $v$, and by $t^{(i)}$ the branch of $t_i$ between this output $v$ and $v^{(i)}$. Of course, $v^{(i)}$ can be equal to $v$ and $t^{(i)}$ can be trivial, but there is at least one of the $t^{(i)}$'s which is not trivial. Denote by $\breve{t}_i$ the $n$-graph obtained from $t_i$ after discarding the $t^{(i)}$'s, and whose output $v$ is replaced by $v^{(i)}$. We claim that

$$\tau[t_i|t_j^*] = \tau[t^{(i)}|t^{(j)*}] \times \tau[\breve{t}_i|\breve{t}_j^*].$$

Firstly, we can identify the pairs $v^{(i)}$ and $v^{(j)}$ in the computation of the left hand-side. Indeed, we write $\tau[t_i|t_j^*] = \sum_\pi \tau^0[(t_i|t_j^*)^\pi]$, and consider a term in the sum for which $\pi$ does not identify $v^{(i)}$ and $v^{(j)}$. Because $\breve{t}_i|\breve{t}_j^*$ is t.e.c., there exists two disjoints paths between $v^{(i)}$ and $v^{(j)}$. But because $t^{(i)}|t^{(j)*}$ contains a third distinct path, by Lemma 5.4 $\pi$ cannot be a cactus if it does not identify $v^{(i)}$ and $v^{(j)}$ and so $\tau^0[(t_i|t_j^*)^\pi]$ is zero.

Consider a term in the sum $\sum_\pi \tau^0[(t_i|t_j^*)^\pi]$ for which $\pi$ identifies the pairs $v^{(i)}, v^{(j)}$. Assume that a vertex $v_1$ of $\breve{t}_i|\breve{t}_j^*$ is identified with a vertex $v_2$ which is not in $\breve{t}_i|\breve{t}_j^*$. Assume that $\pi$ does not identify $v^{(i)}$ with $v_1$ and $v_2$. Because $\breve{t}_i|\breve{t}_j^*$ is t.e.c. there exist two edge-disjoint paths between $v_1$ and $v^{(i)}$ outside of $\tau[t^{(i)}|t^{(j)*}]$. But there exists also a path between $v_2$ and $v^{(i)}$ in $t^{(i)}|t^{(j)*}$. By Lemma 5.4, we get that $(t_i|t_j^*)^\pi$ is not a cactus and so $\tau^0[(t_i|t_j^*)^\pi]$ is zero.

Hence, to determine which vertices of $\breve{t}_i|\breve{t}_j^*$ are identified with some vertices of $t^{(i)}|t^{(j)*}$, one can first determine which vertices of $\breve{t}_i|\breve{t}_j^*$ are identified with $v^{(i)} = v^{(j)}$ and which vertices of $t^{(i)}|t^{(j)*}$ are identified with this vertex. Hence the sum over $\pi$ partition of the set of vertices of $t_i|t_j^*$ can be reduced to a sum over partitions $\pi_1$ of the set of vertices of $\breve{t}_i|\breve{t}_j^*$ and partitions $\pi_2$ of the set of vertices of the graph $t^{(i)}|t^{(j)*}$. Moreover, by definition of $\tau$, for two test graphs $T_1$ and $T_2$, if $T$ is obtained by considering the disjoint union of $T_1$ and $T_2$ and merging one of their vertices, one has $\tau^0[T] = \tau^0[T_1] \times \tau^0[T_2]$. Hence, the contribution of $\check{t}_i|\check{t}_j^*$ factorizes in $\tau[\check{t}_i|\check{t}_j^*]$ and the contribution of $t^{(i)}|t^{(j)*}$ factorizes in $\tau[t^{(i)}|t^{(j)*}]$, and we get the expected result.

From Step 1, we know that $A = (\tau[t^{(i)}|t^{(j)*}])_{i,j}$ is nonnegative. By induction hypothesis, we know that $B = (\tau[\check{t}_i|\check{t}_j^*])_{i,j}$ is also nonnegative. We obtain as desired that the Hadamard product of $A$ and $B$ is nonnegative ([26, Lemma 6.11]) and in particular, for all $\alpha_i$, we have

$$\sum_{i,j} \alpha_i \bar{\alpha}_j \tau\big[t_i|t_j^*\big] \geq 0.$$

**Step 3.** Assume that $t_i|t_i^*$ have no cutting edges for all $t_i$. Let us prove that $\tau[t|t^*] \geq 0$.

For a graph $T$, recall that the t.e.c. components are the maximal subgraphs of $T$ with no cutting edges. We define the *tree of t.e.c. components* of $T$ as the graph whose vertices are the t.e.c. components of $T$, and whose edges are the cutting edges of $T$. First of all, our condition is equivalent to the condition that, for each $t_i$, any leaf of the tree of the t.e.c. components of $t_i$ is a component containing an output. Here again, we can proceed by induction. Let $D$ be the total number of t.e.c. components of the $t_i$'s which do not consists in a single vertex.

If $D = 0$, we are in the case of the previous step. Let us assume that $D > 0$ and that the result is true up to the case $D - 1$. We can remove one t.e.c. in the following way. Choose a non-singleton t.e.c. component $t^{(k)}$ of some $n$-graph monomial $t_k$, for some $k$ in $\{1, \dots, L\}$. We consider $t^{(k)}$ as a multi $^*$-graph monomial, where the outputs are the vertices which are attached to cutting edges. Let $\check{t}_k$ be the $n$-graph monomial obtained from $t_k$ by replacing the component $t^{(k)}$ by one single vertex. We define also for $i \neq k$ the $^*$-graph monomial $t^{(i)}$ to be the trivial leaf and set $\check{t}_i = t_i$. We claim that

$$\tau[t_i|t_j^*] = \tau[\check{t}_i|\check{t}_j^*] \times \tau[t^{(i)}] \times \tau[t^{(j)*}]$$

(of course, this equality is nontrivial only if we consider $i = k$ or $j = k$).

Firstly, the outputs of $t^{(i)}$ can be identified. Indeed, consider $v_1, v_2$ two distinct outputs of $t^{(i)}$. Writing $\tau[t_i|t_j^*] = \sum_\pi \tau^0[(t_i|t_j^*)^\pi]$, consider a term in the sum for which $\pi$ does not identify $v_1$ and $v_2$. Since $t^{(i)}$ is t.e.c. there exist two distinct simple paths $\gamma_1$ and $\gamma_2$ between $v_1$ and $v_2$. Consider a path from $v_2$ to $v_1$ that does not visit $t^{(i)}$ in $t_i|t_j^*$. Such a path exists as $v_1$ and $v_2$ belong to two subtrees of $t_i$ that are attached to outputs of $t_i$, themselves being attached to the connected graph $t_j^*$. The quotient by $\pi$ yields three distinct paths $\gamma$ between $v_1$ and $v_2$ in $(t_i|t_j^*)^\pi$ which implies that $(t_i|t_j^*)^\pi$ is not a cactus by Lemma 5.4. Hence, by definition of $\tau$, $\tau^0[(t_i|t_j^*)^\pi]$ is zero. Thus, when we write $\tau[t_i|t_j^*] = \sum_\pi \tau^0[(t_i|t_j^*)^\pi]$ we can restrict the summation to partitions $\pi$ that identify $v_1$ and $v_2$, therefore, we can replace $t_i$ by the graph $\tilde{t}_i$ where we have identify $v_1$ and $v_2$. Hence we have $\tau[t_i|t_j^*] = \tau[\tilde{t}_i|\tilde{t}_j^*]$. Inductively, we can identify all the outputs of $t^{(i)}$ to one single vertex. Let us denote this vertex by $w^{(i)}$.

Let us write $\tau[\tilde{t}_i|\tilde{t}_j^*] = \sum_\pi \tau^0[(\tilde{t}_i|\tilde{t}_j^*)^\pi]$. Let $\pi$ be as in the sum. Assume that a vertex $v_1$ of $t^{(i)}$ is identified by $\pi$ with a vertex $v_2$ which is not in $t^{(i)}$. Assume that $\pi$ does not identify $w^{(i)}$ with $v_1$ and $v_2$. Since $t^{(i)}$ is t.e.c. there exist two distinct paths between $v_1$ and $w^{(i)}$ in $t^{(i)}$. But $\check{t}_i$ is connected and there exists a third path between $v_2$ and $w^{(i)}$. As usual this implies that $(\tilde{t}_i|\tilde{t}_j^*)^\pi$ is not a cactus and so $\tau_\Phi^0[(\tilde{t}_i|\tilde{t}_j^*)^\pi]$ is zero.

Hence, to determine which vertices of $t^{(i)}$ are identified with some vertices outside of $t^{(i)}$, one can first determine which vertices of $t^{(i)}$ are identified with $w^{(i)}$ and which vertices outside of $t^{(i)}$ are identified with this vertex. Thus the sum over partitions $\pi$ of the set of vertices of $\tilde{t}_i|\tilde{t}_j^*$ can be reduced to a sum over partitions $\pi_1$ of the set of vertices of $t^{(i)}$, and partitions $\pi_2$ of the set of vertices of the graph with $t^{(i)}$ removed. Moreover, by definition of $\tau$, for two * test graphs $T_1$ and $T_2$, if $T$ is obtained by considering the disjoint union of $T_1$ and $T_2$ and merging one of their vertices, one has $\tau^0[T] = \tau^0[T_1] \times \tau^0[T_2]$. Hence, the contribution of $T(t_i, t_j^*)$ factorizes in $\tau[T(\check{t}_i, t_j^*)]$ and the contribution of $t^{(i)}$ factorizes in $\tau[t^{(i)}]$. We can do the same factorization for the $n$-graph monomial $t_j^*$, and we get the expected result.

Now, setting $\beta_i = \alpha_i \tau[t^{(i)}]$, we have

$$\tau\big[T(t,t^*)\big] = \sum_{i,j} \beta_i \overline{\beta}_j \tau\big[T(\check{t}_i, \check{t}_j^*)\big]$$

which is nonnegative thanks to the induction hypothesis.

**Step 4.** A direct proof of the positivity in general case requires appropriate tools, and we bypass this difficulty using both the positivity of the free product (Theorem 1.2) and the fact that unitary invariant traffics are of cactus type (Proposition 4.5).

We define an auxiliary distribution of traffic $\tau' : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}$ which is defined to be equal to $\tau$ on the test graphs without cutting edges and equal to 0 on those with cutting edges. This map $\tau'$ induces a combinatorial distribution on the $\mathcal{G}$-algebra $\mathbb{C}\mathcal{G}\langle\mathcal{A}\rangle$ of graph polynomials labeled in $\mathcal{A}$.

On the one hand, the map $\tau'$ does satisfy the positivity property since for any $n$-graph polynomial $t = \sum_i \alpha_i t_i$, we have

$$\begin{aligned}\tau'[t|t^*] &= \sum_{i,j} \alpha_i \bar{\alpha}_j \tau'[t_i|t_j^*] \\ &= \sum_{\substack{i,j \\ t_i|t_j^* \\ \text{without cutting edges}}} \alpha_i \bar{\alpha}_j \tau[t_i|t_j^*] = \sum_{\substack{i,j \\ t_i|t_i^*,\, t_j|t_j^* \\ \text{without cutting edges}}} \alpha_i \bar{\alpha}_j \tau_\Phi[t_i|t_j^*] \geq 0,\end{aligned}$$

using the result of the previous step. By positivity of the free product, the distribution remains positive if we enlarge $\mathbb{C}\mathcal{G}\langle\mathcal{A}\rangle$ into a traffic space $\mathcal{B}$ with a unitary traffic $u$ such that $(u, u^*)$ is the limit of a Haar unitary matrix, traffic independent from the elements of $\mathbb{C}\mathcal{G}\langle\mathcal{A}\rangle$. We consider the function $f : \mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle \to \mathbb{C}\mathcal{T}\langle\mathcal{B}\rangle$ which replaces each edge $e_a := \overset{a}{\leftarrow}$ of a graph in $\mathbb{C}\mathcal{T}\langle\mathcal{A}\rangle$ by the edges $\overset{u}{\leftarrow} \cdot \overset{e_a}{\longleftarrow} \cdot \overset{u^*}{\longleftarrow}$, obtaining a graph whose edges are labeled by elements of $\mathcal{G}\langle\mathcal{A}\rangle \cup \{u, u^*\} \subset \mathcal{B}$. Because conjugating by a unitary variable does not change the distribution on simple cycles, $\tau' \circ f$ and $\tau$ coincide on simple cycles.

| Type | Distributional symmetry | Injective distribution |
| --- | --- | --- |
| **Tensor** | *Diagonality*: $a = \Delta(a)$ where $\Delta = Z_{\Omega}$ is the diagonal projection | Supported on flowers |
| **Boolean** | $\mathbb{J}$*-Invariance*: $a \equiv \mathbb{J} \otimes a$ in distribution, for $\mathbb{J}$ the limit of the matrix whose entries are $\frac{1}{N}$ | Supported on trees |
| **Free** | *Unitary invariance*: $a \equiv uau^*$ in distribution, for $u$ traffic independent and limit of Haar unitary matrix | Supported on cacti and multiplicative on cycles |

**Table 1.** The three types of traffics.

By unitary invariance, the traffic distribution $\tau' \circ f$ is of cactus type. Hence, as $\tau$, it is completely determined by its value on simple cycles. Finally, the traffic distribution $\tau' \circ f$ is exactly $\tau$. The traffic distribution $\tau$ is the restriction of a positive combinatorial distribution, so it is positive.

## 8. Three types of traffics

From Proposition 4.5, we recall the following for traffics of free type. Let $(\mathcal{A}, \tau)$ be a traffic space. A family $\mathbf{a} = (a_j)_{j \in J}$ of elements of $\mathcal{B}$ is of *free type* if one of the following equivalent properties holds:

(1) *Cactus type.* The injective distribution is supported on well oriented cacti that are multiplicative w.r.t. their cycles.

(2) *Unitary invariance.* The family $\mathbf{a}$ has the same traffic distribution as $u\mathbf{a}u^* = (ua_ju^*)_{j\in J}$ where $u$ is traffic independent from $\mathbf{a}$ and is a Haar unitary on $\mathcal{A}$ (i.e. $u$ is unitary and $\Phi(u^k u^{*\ell}) = \delta_{k,\ell}$ for any $k, \ell \geq 0$).

Thus we have two different characterizations of traffic of free type. A distributional symmetry and a property of the injective distribution. In this section, we will state the corresponding characterization for the two other types of traffics (see Table 1).

### 8.1. Boolean type

Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathcal{Y}$ a family of elements of $\mathcal{A}$. Let us remark that $\mathcal{Y}$ is of Boolean type whenever one of the following equivalent conditions is satisfied:

(1) For any $T \in \mathcal{T}\langle \mathcal{Y} \rangle$, one has $\tau[T] = 0$ if $T$ is not a tree, or

(2) for any $T \in \mathcal{T}\langle \mathcal{Y} \rangle$, one has $\tau^0[T] = 0$ if $T$ is not a tree.

In this case, the plain and injective combinatorial distributions coincide, namely

$$\tau[T] = \tau^0[T] \quad \text{for any } T \in \mathcal{T}\langle \mathcal{Y} \rangle.$$

With respect to the trace $\Phi$ associated to $\tau$, $\mathcal{Y}$ has the null distribution since $\Phi(y) = \tau[\Omega(y)] = 0$ for any $y$ in the algebra spanned by $\mathcal{Y}$.

**Lemma 8.1.** *If $\mathcal{Y}$ is of Boolean type, then the non-unital algebra generated by $\mathcal{Y}$ is of Boolean type.*

*Proof.* Let $T$ be a test graph whose edges are labeled by monomials $m_i = y_{i,1} \dots y_{i,n_i}$ with $y_{i,j}$ in $\mathcal{Y}$. Then $\tau[T] = \tau[\tilde{T}]$ where $\tilde{T}$ is obtained by replacing each edge of $T$ by the sequence of edges $(\cdot \xleftarrow{y_1} \dots \xleftarrow{y_n} \cdot)$. The graph $T$ is a tree if and only if $\tilde{T}$ is a tree, hence the result. ■

We now associate a *distributional symmetry* for Boolean type variables. The matrix $\mathbb{J}_N$ whose all entries are $\frac{1}{N}$ converges in traffic distribution to a traffic $\mathbb{J}$ of Boolean type, whose distribution is given by $\tau[T] = \tau^0[T] = \mathbb{1}(T \text{ is a tree})$ for any $T \in \mathcal{T}\langle \mathbb{J} \rangle$.

**Proposition 8.2.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathcal{Y}$ a family of elements of $\mathcal{A}$. A family of traffics $\mathbf{A}$ is of Boolean type whenever one of the following equivalent conditions is satisfied:*

(1) Trees. *For any $T \in \mathcal{T}\langle \mathcal{Y} \rangle$, one has $\tau^0[T] = 0$ if $T$ is not a tree.*

(2) $\mathbb{J}$-invariance. *The family $\mathbf{A}$ has the same distribution as $\mathbb{J} \otimes \mathbf{A}$ in the tensor product of traffic spaces.*

*Proof.* We have for any $T \in \mathcal{T}\langle \mathbb{J} \otimes \mathcal{A} \rangle$,

$$\tau[T] = \tau[T_{\mathbb{J}}] \times \tau[T_{\mathcal{A}}] = \mathbb{1}(T \text{ is a tree})\tau[T_{\mathcal{A}}].$$

Hence the $\mathbb{J}$-invariance is equivalent to the fact that the traffic distribution of $\mathcal{A}$ is supported on trees, or equivalently the fact that the injective combinatorial distribution of $\mathcal{A}$ is supported on tree. ■

**Example 8.3.** Let $\mathbf{A}_N$ be a family of random matrices that converges in traffic distribution (such families can be built from Theorem 1.1). Then for any $M = (M_N)$, sequence of integers that converges to infinity, the family $\mathbb{J}_{M_N} \otimes \mathbf{A}_N$ converges to a family of traffics of Boolean type. Moreover the distribution of $\mathbb{J}_{M_N} \otimes \mathbf{A}_N$ with respect to $\Psi_N$ is the same as for $\mathbf{A}_N$.

Together with the asymptotic traffic independence theorem, this gives a new procedure to produce asymptotically Boolean independent matrices. More precisely, if $\mathbf{A}_N$ and $\mathbf{B}_N$ are independent families of random matrices that converge in traffic distribution, and $S$ is a uniform matrix of permutation of size $(M_N \cdot N) \times (M_N \cdot N)$, then $S(\mathbb{J}_{M_N} \otimes \mathbf{A}_N)S^*$ and $\mathbb{J}_{M_N} \otimes \mathbf{B}_N$ are independent and asymptotically traffic independent, thanks to [17, Theorem 1.8]. Because the limiting traffics are of Boolean type, $S(\mathbb{J}_{M_N} \otimes \mathbf{A}_N)S^*$ and $\mathbb{J}_{M_N} \otimes \mathbf{B}_N$ are asymptotically Boolean independent with respect to the anti-trace $\Psi_N = \frac{1}{N}\sum_{i,j}\langle E_{ij}, \cdot\rangle$

Note that the size of the matrices is $(M_N \cdot N) \times (M_N \cdot N)$. In contrast, in [10, Section 3.1], the author describe a procedure that leads to Boolean independence using tensor product, which produces matrices of size $N^n$, where $n$ is the number of Boolean independent variables.

### 8.2. Tensor type

We say that $\mathcal{Y}$ is of tensor type whenever for any $a \in \mathcal{Y}$, one has $a = \Delta(a)$. A test-graph is a *flower* if it has only one vertex. Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathcal{Y}$ a family of elements of $\mathcal{A}$; [17, Proposition 5.8] says that if $\mathcal{Y}$ is of tensor type, for any $T \in \mathcal{T}\langle\mathcal{Y}\rangle$, one has $\tau^0[T] = 0$ if $T$ is not a flower.

In fact, the converse is also true and we have the following.

**Proposition 8.4.** *Let $(\mathcal{A}, \tau)$ be an algebraic traffic space and let $\mathcal{Y}$ a family of elements of $\mathcal{A}$. $\mathcal{Y}$ is of* tensor type *whenever one of the following equivalent conditions is satisfied:*

(1) Diagonality. *For any $a \in \mathcal{Y}$, one has $a = \Delta(a)$.*

(2) Flowers. *for any $T \in \mathcal{T}\langle\mathcal{Y}\rangle$, one has $\tau^0[T] = 0$ if $T$ is not a flower.*

*Proof.* It remains to prove that if the injective distribution of $\mathcal{Y}$ is supported on flowers, we have $a = \Delta(a)$ for all $a \in \mathcal{Y}$. It suffices to compute $\Phi((a - \Delta(a))(a - \Delta(a))^*) = 0$ and we deduce that $a = \Delta(a)$. ∎

**Lemma 8.5.** *If $\mathcal{Y}$ is of tensor type, then the traffic space generated by $\mathcal{Y}$ is of tensor type.*

*Proof.* For all $K$-graph operation $g$, we have

$$\begin{aligned} Z_g(a_1 \otimes \cdots \otimes a_K) &= Z_g\big(\Delta(a_1) \otimes \cdots \otimes \Delta(a_K)\big) \\ &= Z_{g\circ(\Delta,\ldots,\Delta)}(a_1 \otimes \cdots \otimes a_K) \\ &= Z_{\Delta\circ g\circ(\Delta,\ldots,\Delta)}(a_1 \otimes \cdots \otimes a_K) \\ &= \Delta\big(Z_{g\circ(\Delta,\ldots,\Delta)}(a_1 \otimes \cdots \otimes a_K)\big) \\ &= \Delta\big(Z_g(\Delta(a_1) \otimes \cdots \otimes \Delta(a_K))\big) \\ &= \Delta\big(Z_g(a_1 \otimes \cdots \otimes a_K)\big). \end{aligned}$$

∎

### 8.3. Canonical traffic spaces

Proposition 7.3 and Section 7.3 allow also to conclude the following.

**Proposition 8.6.** *Let $(\mathcal{A}, \Phi)$ be a tracial $^*$-probability space. There exists a traffic space $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$ as $*$-algebras, the trace induced by $\mathcal{B}$ on $\mathcal{A}$ is $\Phi$, and the family of traffics $\mathcal{A}$ is of* free type.

We now deduce from this canonical construction of traffic spaces of free type an analogous construction for traffics of Boolean type.

**Proposition 8.7.** *Let $(\mathcal{A}, \Psi)$ be a non-unital $^*$-probability space. Then, there exists a traffic space $(\mathcal{B}, \tau)$, and an injective morphism of non-commutative probability spaces $\psi : \mathcal{A} \to \mathcal{B}$ such that $\psi(\mathcal{A})$ is a family of traffics of Boolean type.*

*Proof.* One the one hand, let $\psi_1 : (\mathcal{A}, \Psi) \to (\mathcal{B}_1, \tau_1)$ be the universal construction of Part II, namely whose image consists in unitarily invariant traffics. One the other hand, let $(\mathcal{B}_2, \tau_2)$ be a traffic space generated by the limit $\mathbb{J}$ of the matrix $\mathbb{J}_N$. Then $(\mathcal{B}, \tau) := (\mathcal{B}_1 \otimes \mathcal{B}_2, \tau_1 \otimes \tau_2)$ and $\psi : a \mapsto \psi_1(a) \otimes \mathbb{J}$ satisfy the expected properties. ■

Finally, we have the same result for traffics of tensor type.

**Proposition 8.8.** *Let $(\mathcal{A}, \Phi)$ be a commutative $^*$-probability space. There exists a traffic space $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B}$ as $*$-algebras, the trace induced by $\mathcal{B}$ on $\mathcal{A}$ is $\Phi$, and the family of traffics $\mathcal{A}$ is of* tensor type.

*Proof.* It is the first example of [17, Example 4.10.]. One has just to recall that, for a test-graph $T$ whose edges are labeled by $\gamma : E \to \mathcal{A}$, we have

$$\tau(T) = \Phi\Big(\prod_{e \in E} \gamma(e)\Big),$$

which allows to prove the positivity of the traffic space easily from the positivity of $\Phi$. ■

## 8.4. Relations between the traffics of different types, conclusion

We now investigate the independence relations between traffics of tensor, Boolean and free types.

**Proposition 8.9.** *Let $\mathcal{Y}$ be a family of traffics of Boolean type, traffic independent from a unital subalgebra $\mathcal{Z}$ of traffics of free or tensor type. Then, with respect to the anti-trace, $\mathcal{Z}$ is* monotone independent *from $\mathcal{Y}$.*

*More generally, the result holds whenever the unital subalgebra $\mathcal{Z}$ is such that $\Psi(z) = \Phi(z)$ for any $z \in \mathcal{Z}$.*

*Proof.* For any $n \geq 2$, any $z_i$ in $\mathcal{Z}$, $i = 0, \dots, n$ and any $y_i$ in $\mathcal{Y}$, $i = 1, \dots, n$,

$$\Psi[z_0 y_1 z_1 \dots y_n z_n] = \tau[\cdot \overset{z_0}{\longleftarrow} \cdot \overset{y_1}{\longleftarrow} \dots \overset{z_{n-1}}{\longleftarrow} \cdot \overset{y_n}{\longleftarrow} \cdot \overset{z_n}{\longleftarrow} \cdot].$$

Let $\pi$ be a partition of the above test graph $T$ such that the graph of colored components of $T^\pi$ is a tree and the colored components of $T^\pi$ labeled in $\mathcal{Y}$ are trees. Then $\pi$ does not identify vertices that are not extremal vertices of an edge labeled $z_i$, $i = 1, \dots, n$. If $\pi$ does not identify two vertices of an edge labeled $z_i$, then one can factor $\tau^0[\cdot \overset{z_i}{\longleftarrow} \cdot]$ out of the expression for $\tau^0[T^\pi]$. But $\tau^0[\cdot \overset{z_i}{\longleftarrow} \cdot] = \Psi(z_i) - \Phi(z_i) = 0$. Hence we have $\Psi[z_0 y_1 z_1 \dots y_n z_n] = \tau^0[T^\pi]$ where $\pi$ is the partition identifying the source and target of each edge labeled in $\mathcal{Z}$. We then get

$$\begin{aligned}\Psi[z_0 y_1 z_1 \dots y_n z_n] &= \prod_{i=1}^n \tau^0[\overset{z_i}{\circlearrowleft}] \times \tau^0[\overset{y_1}{\longleftarrow} \cdot \overset{y_2}{\longleftarrow} \dots \overset{y_n}{\longleftarrow} \cdot] \\ &= \prod_{i=1}^n \Phi(z_i) \times \Psi[y_1 y_2 \dots y_n].\end{aligned}$$

We use in the last line the fact that $\tau$ and $\tau^0$ coincide for test graphs labeled by traffics of Boolean types. Since $\Phi = \Psi$ for elements of $\mathcal{Z}$, we get the result. ■

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