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Limits of relatively hyperbolic groups and Lyndon's completions

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Abstract. We describe finitely generated groups H universally equivalent (with constants from G in the language) to a given torsion-free relatively hyperbolic group G with free abelian parabolics. It turns out that, as in the free group case, the group H embeds into Lyndon's completion $G^{\mathbb{Z}[t]}$ of the group G, or, equivalently, H embeds into a group obtained from G by finitely many extensions of centralizers. Conversely, every subgroup of $G^{\mathbb{Z}[t]}$ containing G is universally equivalent to G. Since finitely generated groups universally equivalent to G are precisely the finitely generated groups discriminated by G, the result above gives a description of finitely generated groups discriminated by G. Moreover, these groups are exactly the coordinate groups of irreducible algebraic sets over G.

1. Introduction

Denote by $\mathcal{G}$ the class of all non-abelian torsion-free relatively hyperbolic groups with free abelian parabolics. In this paper we describe finitely generated groups that have the same universal theory as a given group $G \in \mathcal{G}$ (with constants from G in the language). We say that they are *universally equivalent* to G. These groups are central to the study of logic and algebraic geometry of G. It turns out that, as in the case when G is a non-abelian free group [11], a finitely generated group H universally equivalent to G embeds into Lyndon's completion $G^{\mathbb{Z}[t]}$ of the group G, or equivalently, H embeds into a group obtained from G by finitely many extensions of centralizers. Conversely, every subgroup of $G^{\mathbb{Z}[t]}$ containing G is universally equivalent to G [2]. Let H and K be G-*groups* (contain G as a subgroup). We say that a family of G-*homomorphisms* (homomorphisms identical on G) $\mathcal{F} \subset \mathrm{Hom}_G(H, K)$ *separates* [*discriminates*] H into K if for every non-trivial element $h \in H$ [every finite set $H_0 \subset H$ of non-trivial elements] there exists $\phi \in \mathcal{F}$ such that $h^\phi \neq 1$ [$h^\phi \neq 1$ for every $h \in H_0$]. In this case we say that H is G-*separated* [G-*discriminated*] by K. Sometimes we do not mention G and simply say that H is separated [discriminated] by K. When K is a free group we say that H is *freely separated* [*freely discriminated*]. Since finitely generated groups universally equivalent to G are precisely the finitely generated groups discriminated by G ([1], [15]), the result above gives a description of finitely generated groups discriminated by G or *fully residually* G-*groups*. These groups are exactly the coordinate groups of irreducible algebraic sets

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over G. Therefore we obtain a complete description of irreducible algebraic sets over G. Our proof uses the results of [7] and [17], [19].

1.1. Algebraic sets

Let G be a group generated by A, and $F(X)$ the free group on $X = \{x_1, \dots, x_n\}$. A system of equations $S(X, A) = 1$ in variables X and with coefficients from G can be viewed as a subset of $G * F(X)$. A solution of $S(X, A) = 1$ in G is a tuple $(g_1, \dots, g_n) \in G^n$ such that $S(g_1, \dots, g_n) = 1$ in G. The set $V_G(S)$ of all solutions of $S = 1$ in G is called the *algebraic set* defined by S.

The maximal subset $R(S) \subseteq G * F(X)$ with

$$V_G(R(S)) = V_G(S)$$

is the *radical* of $S = 1$ in G. The quotient group

$$G_{R(S)} = G[X]/R(S)$$

is the *coordinate group* of $S = 1$.

The following conditions are equivalent:

- G is *equationally Noetherian*, i.e., every system $S(X) = 1$ over G is equivalent to some finite part of itself;
- the Zariski topology (formed by algebraic sets as a subbasis of closed sets) over G^n is *Noetherian* for every n, i.e., every proper descending chain of closed sets in G^n is finite.
- every chain of proper epimorphisms of coordinate groups over G is finite.

If the Zariski topology is Noetherian, then every algebraic set can be uniquely presented as a finite union of its *irreducible components*:

$$V = V_1 \cup \cdots \cup V_k.$$

Recall that a closed subset V is *irreducible* if it is not the union of two proper closed subsets (in the induced topology).

1.2. Fully residually G-groups

A direct limit of a direct system of finite partial n-generated subgroups of G such that all products of generators and their inverses eventually appear in these partial subgroups, is called a *limit group over* G. The same definition can be given using the notion of "marked group".

A *marked group* (G, S) is a group G with a prescribed family of generators $S = (s_1, \dots, s_n)$. Two marked groups $(G, (s_1, \dots, s_n))$ and $(G', (s_1', \dots, s_n'))$ are *isomorphic as marked groups* if the bijection $s_i \leftrightarrow s_i'$ extends to an isomorphism. For example, $(\langle a \rangle, (1, a))$ and $(\langle a \rangle, (a, 1))$ are not isomorphic as marked groups. Denote by $\mathcal{G}_n$ the set of groups marked by n elements up to isomorphism of marked groups. One can define a metric on $\mathcal{G}_n$ by setting the distance between two marked groups (G, S) and (G', S') to be e^{-N} if they have exactly the same relations of length at most N. (This metric was used in [8], [5], [3].) Finally, a *limit group over* G is a limit (with respect to the metric above) of marked groups (H_i, S_i), where $H_i \leq G$, $i \in \mathbb{N}$, in $\mathcal{G}_n$.

The following two theorems summarize properties that are equivalent for a group H to the property of being discriminated by G (being G-discriminated by G).

Theorem A. [No coefficients] *Let G be an equationally Noetherian group. Then for a finitely generated group H the following conditions are equivalent:*

1. $\mathrm{Th}_\forall(G) \subseteq \mathrm{Th}_\forall(H)$*, i.e.,* $H \in \mathbf{Ucl}(G)$*;*
2. $\mathrm{Th}_\exists(G) \supseteq \mathrm{Th}_\exists(H)$*;*
3. *H embeds into an ultrapower of G;*
4. *H is discriminated by G;*
5. *H is a limit group over G;*
6. *H is defined by a complete atomic type in the theory* $\mathrm{Th}_\forall(G)$*;*
7. *H is the coordinate group of an irreducible algebraic set over G defined by a system of coefficient-free equations.*

For a group A we denote by $\mathcal{L}_A$ the language of groups with constants from A.

Theorem B. [With coefficients] *Let A be a group and G an A-equationally Noetherian A-group. Then for a finitely generated A-group H the following conditions are equivalent:*

1. $\mathrm{Th}_{\forall,A}(G) = \mathrm{Th}_{\forall,A}(H)$*;*
2. $\mathrm{Th}_{\exists,A}(G) = \mathrm{Th}_{\exists,A}(H)$*;*
3. *H A-embeds into an ultrapower of G;*
4. *H is A-discriminated by G;*
5. *H is a limit group over G;*
6. *H is a group defined by a complete atomic type in the theory* $\mathrm{Th}_{\forall,A}(G)$ *in the language $\mathcal{L}_A$;*
7. *H is the coordinate group of an irreducible algebraic set over G defined by a system of equations with coefficients in A.*

Equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ are standard results in mathematical logic. We refer the reader to [20] for the proof of $2 \Leftrightarrow 4$, to [9], [1] for the proof of $4 \Leftrightarrow 7$. Obviously, $2 \Rightarrow 5 \Rightarrow 3$. The above two theorems are proved in [4] for arbitrary equationally Noetherian algebras. Notice that in the case when G is a free group and H is finitely generated, H is a limit group if and only if it is a limit group in the terminology of [21], [3] or [6], [7].

1.3. Lyndon's completions of CSA-groups

The paper [15], following Lyndon [14], introduced a $\mathbb{Z}[t]$-completion $G^{\mathbb{Z}[t]}$ of a given CSA-group G. In [2] it was shown that if G is a CSA-group satisfying the Big Powers condition, then finitely generated subgroups of $G^{\mathbb{Z}[t]}$ are G-universally equivalent to G.

We refer to finitely generated G-subgroups of $G^{\mathbb{Z}[t]}$ as *exponential extensions* of G (they are obtained from G by iteratively adding $\mathbb{Z}[t]$-powers of group elements). The group $G^{\mathbb{Z}[t]}$ is the union of an ascending chain of extensions of centralizers of the group G (see [15]).

A group obtained as the union of a chain of extensions of centralizers

$$\Gamma = \Gamma_0 < \Gamma_1 < \cdots < \bigcup \Gamma_k$$

where

$$\Gamma_{i+1} = \langle \Gamma_i, t_i \mid [C_{\Gamma_i}(u_i), t_i] = 1 \rangle$$

(extension of the centralizer $C_{\Gamma_i}(u_i)$) is called an *iterated extension of centralizers* and is denoted $\Gamma(U, T)$, where $U = \{u_1, \ldots, u_k\}$ and $T = \{t_1, \ldots, t_k\}$.

Every exponential extension H of G is also a subgroup of an iterated extension of centralizers of G.

1.4. Relatively hyperbolic groups

A group G is *hyperbolic* relative to a collection $\{H_\lambda\}_{\lambda \in \Lambda}$ of subgroups (parabolic subgroups) if G is finitely presented relative to $\{H_\lambda\}_{\lambda \in \Lambda}$,

$$G = \left\langle X \cup \left(\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda \right) \;\middle|\; \mathcal{R} \right\rangle,$$

and there is a constant $L > 0$ such that for any word $W \in X \cup \mathcal{H}$ representing the identity in G we have $\mathrm{Area}^{\mathrm{rel}}(W) \le L\|W\|$, where $\mathrm{Area}^{\mathrm{rel}}(W)$ is the minimal number k such that $W = \prod_{i=1}^{k} g_i R_i g_i^{-1}$, $r_i \in \mathcal{R}$, in the free product of the free group with basis X and groups $\{H_\lambda\}_{\lambda \in \Lambda}$.

In [7, Theorem 5.16] Groves showed that groups from $\mathcal{G}$ are equationally Noetherian. By Theorem 1.14 of [17] the centralizer of every hyperbolic element from a group $G \in \mathcal{G}$ is cyclic. Therefore any non-cyclic abelian subgroup is contained in a finitely generated parabolic subgroup. It follows that finitely generated groups from $\mathcal{G}$ are CSA, that is, have malnormal maximal abelian subgroups (see also [6, Lemma 6.7]).

1.5. Big Powers condition

We say that an element $g \in G$ is *hyperbolic* if it is not conjugate to an element of one of the subgroups H_λ, $\lambda \in \Lambda$.

Proposition 1.1. *Groups from $\mathcal{G}$ satisfy the Big Powers condition for hyperbolic elements: if U is a set of hyperbolic elements, $g = g_1 u_1^{n_1} g_2 \ldots u_k^{n_k} g_{k+1}$, $u_1, \ldots, u_k \in U$, and $g_{i+1}^{-1} u_i g_{i+1}$ do not commute with u_{i+1}, then there exists a positive number N such that if $|n_i| \ge N$, $i = 1, \ldots, k$, then $g \ne 1$.*

The proof of this proposition is similar to that of [19, Lemma 4.4] and was suggested by D. Osin.

The Cayley graph of G with respect to the generating set $X \cup \mathcal{H}$ is denoted by $\Gamma(G, X \cup \mathcal{H})$. For a path p in $\Gamma(G, X \cup \mathcal{H})$, $l(p)$ denotes its length, and p_- and p_+ denote the origin and the terminus of p, respectively.

Definition 1.2 ([17]). Let q be a path in the Cayley graph $\Gamma(G, X \cup \mathcal{H})$. A (non-trivial) subpath p of q is called an H_λ*-component* for some $\lambda \in \Lambda$ (or simply a *component*) if

(a) the label of p is a word in the alphabet $H_\lambda \setminus \{1\}$;
(b) p is not contained in a bigger subpath of q satisfying (a).

Two H_λ-components p_1, p_2 of a path q in $\Gamma(G, X \cup \mathcal{H})$ are called *connected* if there exists a path c in $\Gamma(G, X \cup \mathcal{H})$ that connects some vertex of p_1 to some vertex of p_2 and the label of the path, denoted $\phi(c)$, is a word consisting of letters from $H_\lambda \setminus \{1\}$. In algebraic terms this means that all vertices of p_1 and p_2 belong to the same coset gH_λ for a certain $g \in G$. Note that we can always assume that c has length at most 1, as every non-trivial element of $H_\lambda \setminus \{1\}$ is included in the set of generators. An H_λ-component p of a path q is called *isolated* (in q) if no distinct H_λ-component of q is connected to p.

The following lemma can be found in [18, Lemma 2.7].

Lemma 1.3. *Suppose that G is a group hyperbolic relative to a collection $\{H_\lambda \mid \lambda \in \Lambda\}$ of subgroups. Then there exists a constant $K > 0$ and finite subset $\Omega \subseteq G$ such that the following condition holds. Let q be a cycle in $\Gamma(G, X \cup \mathcal{H})$, $p_1, \ldots, p_k$ a set of isolated components of q for some $\lambda \in \Lambda$, and $g_1, \ldots, g_k$ the elements of G represented by the labels of $p_1, \ldots, p_k$, respectively. Then for any $i = 1, \ldots, k$, g_i belongs to the subgroup $\langle \Omega \rangle \leq G$ and the word lengths of g_i with respect to Ω satisfy the inequality*

$$\sum_{i=1}^{k} |g_i|_\Omega \leq K l(q).$$

Recall also that a subgroup is *elementary* if it contains a cyclic subgroup of finite index. The lemma below is proved in [19].

Lemma 1.4. *Let g be a hyperbolic element of infinite order in G. Then*

1. *The element g is contained in a unique maximal elementary subgroup $E_G(g)$ of G.*
2. *The group G is hyperbolic relative to the collection $\{H_\lambda \mid \lambda \in \Lambda\} \cup \{E_G(g)\}$.*

Proof of Proposition 1.1. It suffices to prove the proposition under the following additional assumption: if u_i and u_j are conjugate, then $u_i = u_j$, and if $u_i = u_{i+1}$, then $g_{i+1} \notin E(u_i)$. Indeed, if $u_j = h^{-1}u_i h$, we replace u_j by $\bar{u}_j = u_i = hu_jh^{-1}$, g_j by $\bar{g}_j = g_jh^{-1}$ and g_{j+1} by $\bar{g}_{j+1} = hg_{j+1}$. If $[g_j^{-1}u_{j-1}g_j, u_j] \neq 1$, then $h[g_j^{-1}u_{j-1}g_j, u_j]h^{-1} = [\bar{g}_j^{-1}u_{j-1}\bar{g}_j, \bar{u}_j] \neq 1$. Similarly, if $[g_{j+1}^{-1}u_jg_{j+1}, u_{j+1}] \neq 1$, then $[\bar{g}_{j+1}^{-1}\bar{u}_j\bar{g}_{j+1}, u_{j+1}] \neq 1$. The CSA condition implies that $[g_{i+1}^{-1}u_ig_{i+1}, u_i] = 1$ is equivalent to $g_{i+1} \in E(u_i)$.

Joining $g_1, \ldots, g_{k+1}$ to the finite relative generating set X if necessary, we may assume that $g_1, \ldots, g_{k+1} \in X$. Set

$$\mathcal{F} = \{f \in \langle \Omega \rangle \mid |f|_\Omega \leq 4K\},$$

where K and Ω are given by Lemma 1.3. Suppose that $g_1u_1^{n_1} \ldots g_ku_k^{n_k}g_{k+1} = 1$. We consider a loop $p = q_1r_1q_2r_2 \ldots q_kr_kq_{k+1}$ in $\Gamma(G, X \cup \mathcal{H})$, where q_i (respectively, r_i) is labeled by g_i (respectively by $u_i^{n_i}$).

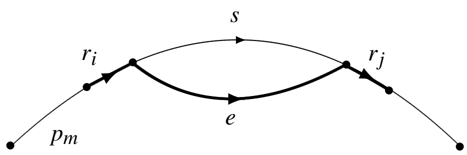

Fig. 1

Note that $r_1, \ldots, r_k$ are components of p. First assume that not all of these components are isolated in p. Suppose that r_i is connected to r_j for some $j > i$ and $j - i$ is minimal possible. Let s denote the segment $[(r_i)_+, (r_j)_-]$ of p, and let e be a path of length at most 1 in $\Gamma(G, X \cup \mathcal{H})$ labeled by an element of H_λ such that $e_- = (r_i)_+$, $e_+ = (r_j)_-$ (see Fig. 1). If $j = i + 1$, then the label $\mathrm{Lab}(s)$ is g_{i+1}. This contradicts the assumption $g_{i+1} \notin E(u_i)$ since $\mathrm{Lab}(s)$ and $\mathrm{Lab}(e)$ represent the same element in G. Therefore, $j = i + 1 + l$ for some $l \geq 1$. Note that the components $r_{i+1}, \ldots, r_{i+1+l}$ are isolated in the cycle se^{-1}. (Indeed otherwise we can pass to another pair of connected components with smaller value of $j - i$.) By Lemma 1.3 we have $u_q^{n_q} \in \langle \Omega \rangle$ for all $i + 1 \leq q \leq i + 1 + l$ and

$$\sum_{q=i+1}^{i+l+1} |u_q^{n_q}|_\Omega \leq K l(se^{-1}) = K(2k + 2).$$

Hence $|u_p^{n_p}|_\Omega \leq K(2 + 2/k) \leq 4K$ for at least one p, which is impossible for large n_p. Thus all components $r_1, \ldots, r_k$ are isolated in p. Applying now Lemma 1.3 again, we obtain

$$\sum_{q=1}^{m} |u_q^{n_q}|_\Omega \leq K l(p) = K(2k + 2).$$

This is again impossible for large $n_1, \ldots, n_k$. □

1.6. Main results and the scheme of the proof

Our main result is the following theorem.

Theorem C. [With constants] *Let* $\Gamma \in \mathcal{G}$. *A finitely generated* Γ*-group* H *is* Γ*-universally equivalent to* Γ *if and only if* H *is embeddable into* $\Gamma^{\mathbb{Z}[t]}$.

The group $\Gamma^{\mathbb{Z}[t]}$ is discriminated by Γ. Indeed, it is enough to prove that any group H obtained from Γ by a finite series of extensions of centralizers is Γ-discriminated. We can obtain H from Γ in two steps. Let K be a subgroup of H that is obtained from Γ by only extending centralizers of elements from parabolic subgroups. Then $K \in \mathcal{G}$ and H is obtained from K by a series of extensions of centralizers of hyperbolic elements. By Proposition 1.1 applied to each centralizer extension, H is discriminated by K. Since K is discriminated by Γ by Lemma 1.3, H is also discriminated by Γ.

The proof of the converse follows the argument in [10], [11] with necessary modifications. It splits into steps. In Section 3 we will prove

Theorem D. *Let $\Gamma \in \mathcal{G}$ and H a finitely generated group discriminated by Γ. Then H embeds into an NTQ extension of Γ.*

In Section 4 we will prove

Theorem E. *Let $\Gamma \in \mathcal{G}$ and Γ^* an NTQ extension of Γ. Then Γ^* embeds into a group $\Gamma(U, T)$ obtained from Γ by finitely many extensions of centralizers.*

2. Quadratic equations and NTQ systems and groups

Definition 2.1. A *standard quadratic equation* over the group G is an equation of one of the following forms (below d, c_i are non-trivial elements from G):

$$\prod_{i=1}^{n}[x_i, y_i] = 1, \qquad n > 0; \tag{1}$$

$$\prod_{i=1}^{n}[x_i, y_i]\prod_{i=1}^{m} z_i^{-1}c_i z_i d = 1, \qquad n, m \geq 0, m + n \geq 1; \tag{2}$$

$$\prod_{i=1}^{n} x_i^2 = 1, \qquad n > 0; \tag{3}$$

$$\prod_{i=1}^{n} x_i^2 \prod_{i=1}^{m} z_i^{-1}c_i z_i d = 1, \qquad n, m \geq 0, n + m \geq 1. \tag{4}$$

Equations (1), (2) are called *orientable* of genus n, while equations (3), (4) are *non-orientable* of genus n.

Let W be a strictly quadratic word over a group G. Then there is a G-automorphism $f \in \mathrm{Aut}_G(G[X])$ such that W^f is a standard quadratic word over G.

To each quadratic equation one can associate a punctured surface. For example, the orientable surface associated to (2) will have genus n and $m + 1$ punctures.

Definition 2.2. Strictly quadratic words of the type $[x, y]$, x^2, $z^{-1}cz$, where $c \in G$, are called *atomic quadratic words* or simply *atoms*.

By definition a standard quadratic equation $S = 1$ over G has the form

$$r_1 \ldots r_k d = 1,$$

where the r_i are atoms and $d \in G$. The number k is called the *atomic rank* of this equation; we denote it by $r(S)$.

Definition 2.3. Let $S = 1$ be a standard quadratic equation written in the atomic form $r_1 \ldots r_k d = 1$ with $k \geq 2$. A solution $\phi : G_{R(S)} \to G$ of $S = 1$ is called:

1. *degenerate* if $r_i^{\phi} = 1$ for some i, and *non-degenerate* otherwise;

2. *commutative* if $[r_i^\phi, r_{i+1}^\phi] = 1$ for all $i = 1, \ldots, k-1$, and *non-commutative* otherwise;
3. *in general position* if $[r_i^\phi, r_{i+1}^\phi] \neq 1$ for all $i = 1, \ldots, k-1$.

Put

$$\kappa(S) = |X| + \varepsilon(S),$$

where $\varepsilon(S) = 1$ if S of the type (2) or (4), and $\varepsilon(S) = 0$ otherwise.

Definition 2.4. Let $S = 1$ be a standard quadratic equation over a group G which has a solution in G. The equation $S(X) = 1$ is *regular* if $\kappa(S) \geq 4$ (equivalently, the Euler characteristic of the corresponding punctured surface is at most -2) and there is a non-commutative solution of $S(X) = 1$ in G, or it is an equation of the type $[x, y]d = 1$ or $[x_1, y_1][x_2, y_2] = 1$.

Let G be a group with a generating set A. A system of equations $S = 1$ is called *triangular quasi-quadratic* (briefly, TQ) over G if it can be partitioned into the following subsystems:

$$\begin{aligned} S_1(X_1, X_2, \ldots, X_n, A) &= 1, \\ S_2(X_2, \ldots, X_n, A) &= 1, \\ &\ldots \\ S_n(X_n, A) &= 1, \end{aligned}$$

where for each i one of the following holds:

1) S_i is quadratic in the variables X_i;
2) $S_i = \{[y, z] = 1, [y, u] = 1 \mid y, z \in X_i\}$ where u is a group word in $X_{i+1} \cup \cdots \cup X_n \cup A$; in this case we say that $S_i = 1$ corresponds to an extension of a centralizer;
3) $S_i = \{[y, z] = 1 \mid y, z \in X_i\}$;
4) S_i is the empty equation.

Sometimes, we join several consecutive subsystems $S_i = 1, S_{i+1} = 1, \ldots, S_{i+j} = 1$ of a TQ system $S = 1$ into one block, thus partitioning the system $S = 1$ into new blocks. It is convenient to call a new system also a triangular quasi-quadratic system.

In the notation above define $G_i = G_{R(S_i, \ldots, S_n)}$ for $i = 1, \ldots, n$ and put $G_{n+1} = G$. The TQ system $S = 1$ is called *non-degenerate* (briefly, NTQ) if the following conditions hold:

5) each system $S_i = 1$, where $X_{i+1}, \ldots, X_n$ are viewed as the corresponding constants from G_{i+1} (under the canonical maps $X_j \to G_{i+1}$, $j = i+1, \ldots, n$), has a solution in G_{i+1};
6) the element in G_{i+1} represented by the word u from 2) is not a proper power in G_{i+1}.

An NTQ system $S = 1$ is called *regular* if each non-empty quadratic equation in S_i is regular (see Definition 2.4). The coordinate group of an NTQ system (resp., a regular NTQ system) is called an *NTQ group* (resp., a *regular NTQ group*).

3. Embeddings into NTQ extensions

Let $\Gamma \in \mathcal{G}$. In this section we will prove Theorem D. Namely, we will show how to embed a finitely generated fully residually Γ-group into an NTQ extension of Γ.

Theorem 3.1 ([7, Theorem 1.1]). *Let $\Gamma \in \mathcal{G}$ and G a finitely generated freely indecomposable group with abelian JSJ decomposition $\mathcal{D}$. Then there exists a finite collection $\{\eta_i : G \to L_i\}_{i=1}^n$ of proper quotients of G such that, for any homomorphism $h : G \to \Gamma$ which is not equivalent to an injective homomorphism, there exists $h' : G \to \Gamma$ with $h \sim h'$ (the relation $\sim$ uses conjugation, canonical automorphisms corresponding to $\mathcal{D}$ and "bending moves"), $i \in \{1, \dots, n\}$ and $h_i : L_i \to \Gamma$ so that $h' = \eta_i h_i$. The quotient groups L_i are fully residually Γ.*

This theorem reduces the description of $\mathrm{Hom}(G, \Gamma)$ to a description of $\mathrm{Hom}(L_i, \Gamma)_{i=1}^n$. We then apply it again to each L_i in turn and so on with successive proper quotients. Such a sequence terminates by equationally Noetherian property. Using this theorem one can construct a Hom-diagram which is the same as the so-called Makanin–Razborov diagram constructed in Section 6 of [7].

The statement of the above theorem is still true if we replace the set of all homomorphisms $h : G \to \Gamma$ by the set of all Γ-homomorphisms. The proof is the same. Therefore, a similar diagram can be constructed for Γ-homomorphisms $G \to \Gamma$.

Proof of Theorem D. Let G be a finitely generated freely indecomposable group discriminated by Γ. According to the construction of the Makanin-Razborov diagram the set $\mathrm{Hom}(G, \Gamma)$ is divided into a finite number of families. Therefore one of these families contains a discriminating set of homomorphisms. Each family corresponds to a sequence of fully residually Γ-groups (see [13])

$$G = G_0, G_1, \dots, G_n,$$

where G_{i+1} is a proper quotient of G_i and $\pi_i : G_i \to G_{i+1}$ is an epimorphism. Similarly to Lemma 16 from [13], for a discriminating family, π_i is a monomorphism for the following subgroups H in the JSJ decomposition $\mathcal{D}_i$ of G_i:

1. H is a rigid subgroup in $\mathcal{D}_i$;
2. H is an edge subgroup in $\mathcal{D}_i$;
3. H is the subgroup of an abelian vertex group A in $\mathcal{D}_i$ generated by the canonical images in A of the edge groups of the edges of $\mathcal{D}_i$ adjacent to A.

We need the following result.

Lemma 3.2 ([13, Lemma 22]).

(1) *Let $H = A *_D B$, D be an abelian subgroup that is maximal abelian in A or B, and $\pi : H \to \bar{H}$ be a homomorphism such that the restrictions of π to A and B are injective. Put*

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(D)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}((\pi(D)) \smallsetminus C_{\bar{H}}(\pi(D))$*, the map*

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ \pi(x)^u, & x \in B, \end{cases}$$

gives rise to a monomorphism $\psi : H \to H^*$.

(2) *Let* $H = \langle A, t \mid d^t = c, d \in D \rangle$*, where D is abelian and either D or its image is maximal abelian in A, and let* $\pi : H \to \bar{H}$ *be a homomorphism such that the restriction of* π *to A is injective. Put*

$$H^* = \langle \bar{H}, y \mid [C_{\bar{H}}(\pi(D)), y] = 1 \rangle.$$

Then for every $u \in C_{H^*}((\pi(D)) \smallsetminus C_{\bar{H}}(\pi(D))$*, the map*

$$\psi(x) = \begin{cases} \pi(x), & x \in A, \\ u\pi(x), & x = t, \end{cases}$$

gives rise to a monomorphism $\psi : H \to H^*$.

Let now $\mathcal{D}$ be an abelian JSJ decomposition of G. Combining foldings and slidings, we can transform $\mathcal{D}$ into an abelian decomposition in which each vertex with non-cyclic abelian subgroup that is connected to some rigid vertex, is connected to only one vertex which is rigid. We suppose from the beginning that $\mathcal{D}$ has this property. Let G_1 be the fully residually Γ proper quotient of G on the next level of the Makanin–Razborov diagram, and π be the canonical epimorphism $\pi : G \to G_1$. Let $G_1 = P_1 * \cdots * P_\alpha * F$ be the Grushko decomposition of G_1 relative to the set of all rigid subgroups and edge subgroups of $\mathcal{D}$. Here F is the free factor and each P_i is freely indecomposable modulo rigid subgroups and edge subgroups of $\mathcal{D}$.

We will construct a canonical extension G^* of $\bar{G} = P_1 * \cdots * P_\alpha$ which is the fundamental group of the graph of groups Λ obtained from a single vertex v with the associated vertex group $G_v = \bar{G}$ by adding finitely many edges corresponding to extensions of centralizers (viewed as amalgamated products) and finitely many QH-vertices connected only to v. By construction of $\bar{G}$, each factor in this decomposition contains a conjugate of the image of some rigid subgroup or an edge group in $\mathcal{D}$. Indeed, the Grushko decomposition of $\bar{G}$ is non-trivial only if the fundamental groups of some separating simple closed curves on the surfaces corresponding to QH subgroups of $\mathcal{D}$ are mapped by π to the identity element. Such curves cut the surface into pieces, and the fundamental groups of all the pieces that are not attached to rigid subgroups are mapped into F.

Let $g_1, \ldots, g_l$ be a fixed finite generating set of $\bar{G}$. For an edge $e \in \mathcal{D}$ we fix a tuple of generators d_e of the abelian edge group G_e. The required extension G^* of $\bar{G}$ is constructed in three steps. On each step we extend the centralizers $C_{\bar{G}}(\pi(d_e))$ of some edges e in $\mathcal{D}$ or add a QH subgroup. Simultaneously, to every edge $e \in \mathcal{D}$ we associate an element $s_e \in C_{G^*}(\pi(d_e))$.

Step 1. Let E_{rig} be the set of all edges between rigid subgroups in $\mathcal{D}$. One can define an equivalence relation $\sim$ on E_{rig} by declaring for $e, f \in E_{\mathrm{rig}}$ that

$$e \sim f \Leftrightarrow \exists g_{ef} \in \bar{G} \left(g_{ef}^{-1} C_{\bar{G}}(\pi(e)) g_{ef} = C_{\bar{G}}(\pi(f)) \right).$$

Let E be a set of representatives of equivalence classes of E_{rig} modulo $\sim$. We construct a group $G^{(1)}$ by extending every centralizer $C_{\bar{G}}(\pi(d_e))$ of $\bar{G}$, $e \in E$, as follows. Let

$$[e] = \{e = e_1, \dots, e_{q_e}\}$$

and $y_e^{(1)}, \dots, y_e^{(q_e)}$ be new letters corresponding to the elements in $[e]$. Then put

$$G^{(1)} = \langle \bar{G}, y_e^{(1)}, \dots, y_e^{(q_e)} (e \in E) \mid [C(\pi(d_e)), y_e^{(j)}] = 1, [y_e^{(i)}, y_e^{(j)}] = 1 \ (i, j = 1, \dots, q_e) \rangle.$$

One can associate with $G^{(1)}$ the system of equations over $\bar{G}$:

$$[\bar{g}_{es}, y_e^{(j)}] = 1, \quad [y_e^{(i)}, y_e^{(j)}] = 1, \quad i, j = 1, \dots, q_e, \ s = 1, \dots, p_e, \ e \in E, \tag{5}$$

where $y_e^{(j)}$ are new variables and the elements $\bar{g}_{e1}, \dots, \bar{g}_{ep_e}$ are constants from $\bar{G}$ which generate the centralizer $C(\pi(d_e))$. We assume that the constants $\bar{g}_{ej}$ are given as words in the generators $g_1, \dots, g_l$ of $\bar{G}$. We associate with the edge $e_i \in [e]$ an element s_{e_i} that is the conjugate of $y_e^{(i)}$ from $C_{G^{(1)}}(\pi(d_{e_i}))$.

Step 2. Let A be a non-cyclic abelian vertex group in $\mathcal{D}$ and A_e the subgroup of A generated by the images in A of the edge groups of edges adjacent to A. Then $A = \text{Is}(A_e) \times A_0$ where $\text{Is}(A_e)$ is the isolator of A_e in A (the minimal direct factor containing A_e) and A_0 a direct complement of $\text{Is}(A_e)$ in A. Notice that the restriction of π_1 to $\text{Is}(A_e)$ is a monomorphism (since π_1 is injective on A_e and A_e is of finite index in $\text{Is}(A_e)$). For each non-cyclic abelian vertex group A in $\mathcal{D}$ we extend the centralizer of $\pi_1(\text{Is}(A_e))$ in $G^{(1)}$ by the abelian group A_0 and denote the resulting group by $G^{(2)}$. Observe that since $\pi_1(\text{Is}(A_e)) \leq \bar{G}$ the group $G^{(2)}$ is obtained from $\bar{G}$ by extending finitely many centralizers of elements from $\bar{G}$.

If the abelian group A_0 has rank r then the system of equations associated with the abelian vertex group A has the form

$$[y_p, y_q] = 1, \quad [y_p, \bar{d}_{ej}] = 1, \quad p, q = 1, \dots, r, \ j = 1, \dots, p_e, \tag{6}$$

where y_p, y_q are new variables and the elements $\bar{d}_{e1}, \dots, \bar{d}_{ep_e}$ are constants from $\bar{G}$ which generate the subgroup $\pi(\text{Is}(A_e))$. We assume that the constants $\bar{d}_{ej}$ are given as words in the generators $g_1, \dots, g_l$ of $\bar{G}$.

Step 3. Let Q be a non-stable QH subgroup in $\mathcal{D}$ (not mapped by π into the same QH subgroup). Suppose Q is given by a presentation

$$\prod_{i=1}^{n} [x_i, y_i] p_1 \cdots p_m = 1$$

where there are exactly m outgoing edges $e_1, \dots, e_m$ from Q and $\sigma(G_{e_i}) = \langle p_i \rangle$, $\tau(G_{e_i}) = \langle c_i \rangle$ for each edge e_i. We add a QH vertex Q to $G^{(2)}$ by introducing new generators and the quadratic relation

$$\prod_{i=1}^{n} [x_i, y_i] (c_1^{\pi_1})^{z_1} \cdots (c_{m-1}^{\pi_1})^{z_{m-1}} c_m^{\pi_1} = 1 \tag{7}$$

to the presentation of $G^{(2)}$. Observe that in the relations (7) the coefficients in the original quadratic relations for Q in $\mathcal{D}$ are replaced by their images in $\bar{G}$.

Similarly, one introduces QH vertices for non-orientable QH subgroups in $\mathcal{D}$.

The resulting group is denoted by $G^* = G^{(3)}$.

We define a (Γ)-homomorphism $\psi : G \to G^*$ with respect to the splitting $\mathcal{D}$ of G and will prove that it is a monomorphism. Let T be the maximal subtree of $\mathcal{D}$. First, we define ψ on the fundamental group of the graph of groups induced from $\mathcal{D}$ on T. Notice that if we consider only Γ-homomorphisms, then the subgroup Γ is elliptic in $\mathcal{D}$, so there is a rigid vertex $v_0 \in T$ such that $\Gamma \leq G_{v_0}$. The mapping π embeds G_{v_0} into $\bar{G}$, hence into G^*.

Let P be a path $v_0 \to v_1 \to \cdots \to v_n$ in T that starts at v_0. With each edge $e_i = (v_{i-1} \to v_i)$ between two rigid vertex groups we have already associated the element s_{e_i}. Let us associate elements to other edges of P:

a) if v_{i-1} is a rigid vertex, and v_i is either abelian or QH, then $s_{e_i} = 1$;
b) if v_{i-1} is a QH vertex, v_i is rigid or abelian, and the image of e_i in the decomposition $\mathcal{D}^*$ of G^* does not belong to T^*, then s_{e_i} is the stable letter corresponding to the image of e_i;
c) if v_{i-1} is a QH vertex and v_i is rigid or abelian, and the image of e_i in the decomposition of G^* belongs to T^*, then $s_{e_i} = 1$;
d) if v_{i-1} is an abelian vertex with $G_{v_{i-1}} = A$ and v_i is a QH vertex, then s_{e_i} is an element from A that belongs to A_0.

Since two abelian vertices cannot be connected by an edge in Γ, and we can suppose that two QH vertices are not connected by an edge, these are all possible cases.

We now define the embedding ψ on the fundamental group corresponding to the path P as follows:

$$\psi(x) = \pi(x)^{s_{e_i} \cdots s_{e_1}} \qquad \text{for } x \in G_{v_i}.$$

This map is a monomorphism by Lemma 3.2. Similarly we define ψ on the fundamental group of the graph of groups induced from $\mathcal{D}$ on T. We extend it to G using the second statement of Lemma 3.2.

Recursively applying this procedure to G_1 and so on, we will construct the NTQ group N such that G is embedded into N. Theorem D is proved.

4. Embedding of NTQ groups into $G(U, T)$

An NTQ group H over Γ is obtained from Γ by a series of extensions:

$$\Gamma = H_0 < H_1 < \cdots < H_n = H,$$

where for each $i = 1, \ldots, n$, H_i is either an extension of a centralizer in H_{i-1} or the coordinate group of a regular quadratic equation over H_{i-1}. In the second case, equivalently, H_i is the fundamental group of the graph of groups with two vertices, v and w, such that v is a QH vertex with QH subgroup Q, and H_{i-1} is the vertex group of w. Moreover, there is a retraction from H_i onto H_{i-1}. In this section we will prove the following theorem which, by induction, implies Theorem E.

Theorem 4.1. *Let H be the fundamental group of the graph of groups with two vertices, v and w, such that v is a QH vertex with QH subgroup Q, $H_w = \Gamma \in \mathcal{G}$, and there is a retraction from H onto Γ such that Q corresponds to a regular quadratic equation. Then H can be embedded into a group obtained from Γ by a series of extensions of centralizers.*

The idea of the proof is as follows. Let S_Q be a punctured surface corresponding to the QH vertex group in the decomposition (say $\mathcal{D}$) of H as the graph of groups. We will find in Proposition 4.9 a finite collection of simple closed curves (s.c.c.) on S_Q and a homomorphism $\delta : H \to K$, where K is an iterated centralizer extension of $\Gamma * F$, with the following properties:

1) δ is a retraction on Γ,
2) each of the simple closed curves in the collection and all boundary elements of S_Q are mapped by δ into non-trivial elements of K,
3) each connected component of the surface obtained by cutting S_Q along this family of s.c.c. has Euler characteristic -1,
4) the fundamental group of each of these connected components is mapped monomorphically into a 2-generated free subgroup of K.

Given this collection of s.c.c. on the surface associated with the QH vertex group in the decomposition $\mathcal{D}$, one can extend $\mathcal{D}$ by further splitting the QH vertex groups along the family of simple closed curves described above. Now the statement of Theorem 4.1 will follow from Lemma 3.2.

Proposition 4.2 ([10, Prop. 3]). *Let $S = 1$ be a non-degenerate standard quadratic equation over a CSA-group G. Then either $S = 1$ has a solution in general position, or every non-degenerate solution of $S = 1$ is commutative.*

Proving the theorem we will consider the following three cases for the equation corresponding to the QH subgroup Q: orientable of genus ≥ 1, genus = 0, and non-orientable of genus ≥ 1. For an orientable equation of genus ≥ 1 we have the following proposition.

Proposition 4.3 (cf. [10, Prop. 4]). *Let $S : \prod_{i=1}^{m}[x_i, y_i]\prod_{j=1}^{n} c_j^{z_j} g^{-1} = 1$ ($m \geq 1$, $n \geq 0$) be a non-degenerate standard quadratic equation over a group $G \in \mathcal{G}$. Then $S = 1$ has a solution in general position in some group H which is an iterated extension of centralizers of $G * F$ (where F is a free group) unless $S = 1$ is the equation $[x_1, y_1][x_2, y_2] = 1$ or $[x, y]c^z = 1$. This solution can be chosen so that the images of x_i and y_i generate a free subgroup (for each $i = 1, \ldots, m$).*

Proof. Let $n = 0$. Then we have a standard quadratic equation of the type

$$[x_1, y_1]\ldots[x_k, y_k] = g,$$

which we will sometimes write as $r_1 \ldots r_k = g$, where, as before, $r_i = [x_i, y_i]$.

Lemma 4.4. *Let $S : [x_1, y_1][x_2, y_2] = g$ be a non-degenerate equation over a group $G \in \mathcal{G}$. Then $S = g$ has a solution in general position in some group H which is an iterated extension of centralizers of $G * F$ unless $S = 1$ is the equation $[x_1, y_1][x_2, y_2] = 1$. Moreover, for each i, x_i, y_i generate a free subgroup.*

Proof. Suppose $S = g$ has a solution ϕ such that $r_1^\phi = 1$ and $r_2^\phi = 1$. Then $g = 1$ and our equation takes the form

$$[x, y][x_2, y_2] = 1. \tag{8}$$

From now on we assume that for all solutions ϕ either $r_1^\phi \neq 1$ or $r_2^\phi \neq 1$.

Suppose now that just one of the equalities $r_i^\phi = 1$ $(i = 1, 2)$ holds, say $r_1^\phi = 1$. Write $x_2^\phi = a$ and $y_2^\phi = b$. Then the equation is

$$[x, y][x_2, y_2] = [a, b] \neq 1.$$

This equation has other solutions, for example, for a new letter c and $p > 2$,

$$\psi: \;\; x \to (ca^{-1})^{-p}c, \;\; y \to c^{(ca^{-1})^p}, \;\; x_2 \to a^{(ca^{-1})^p}, \;\; y_2 \to (ca^{-1})^{-p}b \tag{9}$$

for which

$$r_1^\psi = [c, (ca^{-1})^p] \neq 1 \quad \text{and} \quad r_2^\psi = [(ca^{-1})^p, a][a, b] \neq 1.$$

We claim that $[r_1^\psi, r_2^\psi] \neq 1$. Indeed, $[r_1^\psi, r_2^\psi] = 1$ if and only if

$$[[c, (ca^{-1})^p], [(ca^{-1})^p, a][a, b]] = 1,$$

but this is not true in $G * \langle c \rangle$.

Thus, just one case is left to consider. Suppose that $[r_1^\phi, r_2^\phi] = 1$ and $r_i^\phi \neq 1$ $(i = 1, 2)$ for all solutions ϕ. Suppose $x^\phi = a$, $y^\phi = b$, $x_2^\phi = c$ and $y_2^\phi = d$. We will use ideas from [12] to change the solution. Let

$$H = \langle G, t_1, t_2, t_3, t_4, t_5 \mid 1 = [t_1, b] = [t_2, t_1 a] = [t_3, d] = [t_4, t_3 c] = [t_5, t_2 b c^{-1} t_3^{-1}] \rangle.$$

Let $x^\psi = t_5^{-1} t_1 a$, $y^\psi = (t_2 b)^{t_5}$, $x_2 = (t_3 c)^{t_5}$, $y_2^\psi = t_5^{-1} t_4 d$. This ψ is also a solution of the same equation. But now x^ψ and y^ψ generate a free subgroup of H. If we have a word $w(x, y)$ then $w(x^\psi, y^\psi) = 1$ in H if all occurrences of t_5 disappear. This can only happen if $w(x, y)$ is made up of the blocks $x^{-1}yx$. But these blocks commute, hence $w = x^{-1}y^n x$. But now $w^\psi = a^{-1}t_1^{-1}(t_2 b)^n t_1 a$, therefore w^ψ contains t_2 that does not disappear. Therefore $w^\psi \neq 1$. Similarly, x_2^ψ and y_2^ψ generate a free subgroup of H.

We will show now that $[r_1^\psi, r_2^\psi] \neq 1$. Indeed,

$$r_1^\psi r_2^\psi = [x^\psi, y^\psi][x_2^\psi, y_2^\psi] = [a, b][c, d],$$

but

$$r_2^\psi r_1^\psi = [x_2^\psi, y_2^\psi][x^\psi, y^\psi] = t_5^{-1} c^{-1} t_3^{-1} t_5 d^{-1} t_3 c d a^{-1} t_1^{-1} b^{-1} t_2^{-1} t_1 a t_5^{-1} t_2 b t_5.$$

And there is no way to make a pinch and cancel t_5 in the second expression. Therefore $[r_1^\psi, r_2^\psi] \neq 1$ and the proposition is proved. □

Similarly, one can prove the following lemma.

Lemma 4.5 (cf. [10, Lemma 13]). *Let $S : [x_1, y_1] \dots [x_k, y_k] = g$ be a non-degenerate equation over a group $G \in \mathcal{G}$ and assume that $k \geq 3$. Then $S = g$ has a solution in general position over some group H which is an iterated extension of centralizers of $G * F$. Moreover, for each i, x_i, y_i generate a free subgroup.*

Proof. The proof is by induction on k.

Let $k = 3$. Assume that $g = 1$. This means we have the equation

$$[x_1, y_1][x_2, y_2][x_3, y_3] = 1,$$

which has a solution

$$x_1^\phi = a, \quad y_1^\phi = b, \quad x_2^\phi = b, \quad y_2^\phi = a, \quad x_3^\phi = 1, \quad y_3^\phi = 1,$$

where a, b are arbitrary generators of F. Then the conclusion follows from Proposition 4 of [10]. But for convenience of the reader we will give a proof here. The equation

$$[x_2, y_2][x_3, y_3] = [b, a]$$

is non-degenerate of atomic rank 2; hence, by the lemma above, it has a solution θ such that $[r_2^\theta, r_3^\theta] \neq 1$, and the images x_2^θ, y_2^θ (resp., the images x_3^θ, y_3^θ) generate a free non-abelian subgroup. We got a solution ψ such that

$$x_1^\psi = a, \quad y_1^\psi = b, \quad x_i^\psi = x_i^\theta, \quad y_i^\psi = y_i^\theta \quad \text{for } i = 2, 3.$$

Now we are in a position to apply the previous lemma to the equation

$$[x_1, y_1][x_2, y_2] = [y_3^\psi, x_3^\psi].$$

It follows that there exists a solution to $S = g$ in general position and such that the subgroups generated by the images of x_i, y_i are free non-abelian for $i = 1, 2, 3$.

Assume now that $g \neq 1$. Then there exists a solution ϕ such that for at least one i we have $r_i^\phi \neq 1$. Renaming variables one can assume that exactly $r_3^\phi = [a, b] \neq 1,\ a, b \in G$. Then the equation

$$r_1 r_2 = g[b, a]$$

has a solution in G. Again, we have two cases. If $g[b, a] \neq 1$, then we can argue as in Lemma 4.4. We obtain first a solution ϕ such that $x_i^\phi = c_i, y_i^\phi = d_i, i = 1, 2, x_3^\phi = a, y_3^\phi = b, [r_1^\phi, r_2^\phi] \neq 1, [c_1, d_1] \neq g$, and c_i, d_i generate a free subgroup for $i = 1, 2$. Then we consider the equation $[x_2, y_2][x_3, y_3] = [d_1, c_1]g$ and apply Lemma 4.4 once more.

If $g[b, a] = 1$ then $g = [a, b]$ and the initial equation $S = g$ actually has the form

$$r_1 r_2 r_3 = [a, b].$$

In this event consider a solution θ such that

$$x_1^\theta = c, \quad y_1^\theta = d, \quad x_2^\theta = (ca^{-1})^{-1} d, \quad y_2^\theta = c^{ca^{-1}}, \quad x_3^\theta = a^{ca^{-1}}, \quad y_3^\theta = (ca^{-1})^{-1} b,$$

where c, d are non-commuting elements from F. Then $[r_i^\theta, r_j^\theta] \neq 1$, $i, j = 1, 2, 3$, and obviously x_i^θ, y_i^θ generate a free group.

Let $k > 3$. The equation

$$r_1 \dots r_k = g$$

has a solution ϕ such that for at least one i, say $i = k$ (by renaming variables we can always assume this), we have $r_k^\phi = [a, b] \neq 1$. Then the equation

$$r_1 \dots r_{k-1} = g[b, a]$$

is non-degenerate and by induction there is a solution θ such that $[r_i^\theta, r_{i+1}^\theta] \neq 1$ for all $i = 1, \dots, k-2$, and x_i, y_i generate a free subgroup for $i = 1, \dots, k-1$. Define now a solution θ_1 of the initial equation $S = g$ as follows:

$$x_i^\theta = x_i^{\theta_1}, \quad y_i^\theta = y_i^{\theta_1} \quad \text{for } i = 1, \dots, k-2,$$
$$x_{k-1}^{\theta_1} = t_5^{-1} t_1 x_{k-1}^\theta, \quad y_{k-1}^{\theta_1} = (t_2 y_{k-1}^\theta)^{t_5}, \quad x_k^{\theta_1} = (t_3 a)^{t_5}, \quad y_k^{\theta_1} = t_5^{-1} t_4 b,$$

where

$$[t_1, y_{k-1}^\theta] = [t_2, t_1 x_{k-1}^\theta] = [t_3, b] = [t_4, t_3 a] = [t_5, t_2 y_{k-1}^\theta a^{-1} t_3^{-1}] = 1.$$

This solution satisfies the requirements of the lemma. □

Thus, Proposition 4.3 is proved for the case $n = 0$. Consider now the case $n > 0$.

Lemma 4.6 (cf. [10, Lemma 14]). *The equation $S : [x, y]c^z = g$, where $g \neq 1$, which is consistent over a group $G \in \mathcal{G}$ always has a solution in general position in some iterated centralizer extension H of G such that the images of x and y generate a free subgroup.*

Proof. Let $x \to a$, $y \to b$, $z \to d$ be an arbitrary solution of $[x, y]c^z = g$, where $g \neq 1$. Then $g = [a, b]c^d$ and the equation takes the form

$$[x, y]c^z = [a, b]c^d.$$

We can assume that $[a, b] \neq 1$. Indeed, suppose $[a, b] = 1$. If $[c, d] \neq 1$, then we can write the equation as

$$[x, y]c^z = c^d = [d, c^{-1}]c,$$

which has the solution $x \to d$, $y \to c^{-1}$, $z \to 1$ such that $[x, y] \to [d, c^{-1}] \neq 1$. So we can assume now that $[c, d] = 1$, in which case we have the equation

$$[x, y]c^z = c \quad \text{or equivalently} \quad [x, y] = [c^{-1}, z].$$

The group G is a non-abelian CSA-group; hence the center of G is trivial. In particular, there exists an element $h \in G$ such that $[c, h] \neq 1$. We see that $x \to c^{-1}$, $y \to h$, $z \to h$ is a solution ϕ for which $[x, y]^\phi \neq 1$.

Thus we have the equation $[x, y]c^z = [a, b]c^d$, where $[a, b] \neq 1$. Let $H = \langle G, t \mid [t, bc^d] = 1\rangle$. Consider the map ψ defined as follows:

$$x^\psi = t^{-1}a, \quad y^\psi = t^{-1}bt, \quad z^\psi = dt.$$

Straightforward computations show that

$$[x, y]^{\psi} = [a, b][b, t], \qquad (c^z)^{\psi} = c^{dt};$$

hence

$$[x^{\psi}, y^{\psi}]c^{z^{\psi}} = [a, b]c^d,$$

and consequently ψ is a solution.

We claim that $[r_1^{\psi}, r_2^{\psi}] \neq 1$. Indeed, suppose $[r_1^{\psi}, r_2^{\psi}] = 1$; then we have

$$[[x, y]^{\psi}, c^{z^{\psi}}] = 1, \quad [[a, b][b, t], c^{dt}] = 1,$$
$$t^{-1}b^{-1}tb[b, a]t^{-1}d^{-1}c^{-1}dt[a, b]b^{-1}t^{-1}bd^{-1}cdt = 1,$$

which implies

$$t^{-1}b^{-1}tb[b, a]t^{-1}d^{-1}c^{-1}dt[a, b]b^{-1}bd^{-1}cd = 1.$$

The letter t disappears only if c^d commutes with b or b^a commutes with bc^d. In both cases the last equality implies that $[a, b]$ commutes with c^d and b commutes with b^a. Therefore $[a, b] = 1$, which contradicts the choice of a, b, c, d. □

Now suppose that $m = 1, n > 1$. Let $\phi : G_S \to G$ be an arbitrary solution of $S = g$. Write

$$h = g\Big(\prod_{j=3}^{n} c_j^{z_j}\Big)^{-\phi}$$

and consider the equation

$$[x, y]c_1^{z_1}c_2^{z_2} = h. \tag{10}$$

If this equation satisfies the conclusion of Proposition 4.3, then by induction the equation $S = g$ will satisfy the conclusion. So we need to prove the proposition just for equation (10). There are now two possible cases.

Case (a): There exists a solution ξ of the equation (10) such that $(c_2^{z_2})^{\xi} \neq h$. In this event by Lemma 4.6 the equation

$$[x, y]c_1^{z_1} = h(c_2^{z_2})^{-\xi} \neq 1$$

has a solution θ in general position. Hence we can extend this θ to a solution of (10) in such a way that $r_i^{\theta} \neq 1$ for $i = 1, 2$ and $[r_1^{\theta}, r_2^{\theta}] \neq 1$. Consequently, by Proposition 4.2 we can construct a solution ψ in general position. It will automatically satisfy the conclusion of Proposition 4.3.

Case (b): Assume now that $(c_2^{z_2})^{\phi} = h$ for all solutions ϕ of (10). Then we actually have

$$[x, y]c_1^{z_1} = 1, \qquad c_2^{z_2} = h,$$

and this system of equations has a solution in G. It follows that $c_1 = [a, b] \neq 1$ for some $a, b \in G$. Therefore equation (10) is

$$[x, y][a, b]^{z_1}c_2^{z_2} = h,$$

and has a solution ψ of the type

$$x^{\psi} = b^{f}, \quad y^{\psi} = a^{f}, \quad z_1^{\psi} = f, \quad z_2^{\psi} = z_2^{\phi}$$

where f is an arbitrary element in G and ϕ is an arbitrary solution of (10). The two elements $[a, b]$ and h are non-trivial in the CSA-group G, hence there exists $f^* \in G$ such that $[[a, b]^{f^*}, h] \neq 1$. But this implies that if we take $f = f^*$ then the solution ψ will satisfy $[r_2^{\psi}, r_3^{\psi}] \neq 1$. Now it is sufficient to apply Proposition 4.2.

Now we suppose that $m = 2, n > 1$. Then we have the equation

$$[x_1, y_1][x_2, y_2] \prod_{j=1}^{n} c_j^{z_j} = g.$$

Again, if there exists a solution ϕ of this equation such that

$$\Big(\prod_{j=1}^{n} c_j^{z_j}\Big)^{\phi} \neq g,$$

then we can write

$$h = g\Big(\prod_{j=1}^{n} c_j^{z_j}\Big)^{-\phi},$$

and consider the equation

$$[x_1, y_1][x_2, y_2] = h$$

which according to Lemma 4.5 has a solution ξ in general position such that the images of x_i, y_i generate a free subgroup. We can extend it to a solution of $S = g$, and by Proposition 4.3 applied to the equation

$$[x_1^{\xi}, y_1^{\xi}][x_2, y_2] \prod_{j=1}^{n} c_j^{z_j} = g$$

we can construct a solution ψ in general position with the required properties.

Assume now that

$$\Big(\prod_{j=1}^{n} c_j^{z_j}\Big)^{\phi} = g$$

for all solutions ϕ of the equation $S = g$. This implies that an arbitrary map of the type

$$x_1 \to a, \quad y_1 \to b, \quad x_2 \to b, \quad y_2 \to a$$

extends by means of any ϕ above to a solution ψ of the equation $S = g$. Choose $a, b \in F$; then $[[b, a], r_3^{\phi}] \neq 1$ for the given solution ϕ. And we again just need to appeal to Proposition 4.3 for the equation

$$[a, b][x_2, y_2] \prod_{j=1}^{n} c_j^{z_j} = g.$$

The case $m > 2$ is easy since if ϕ is a solution of the equation

$$\prod_{i=1}^{m}[x_i, y_i]\prod_{j=1}^{n} c_j^{z_j} g^{-1} = 1,$$

then we can consider the equation

$$\prod_{i=1}^{m}[x_i, y_i] = g\Big(\prod_{j=1}^{n} c_j^{z_j}\Big)^{-\phi},$$

which by Lemma 4.5 has a solution in general position such that the images of x_i, y_i generate a free subgroup; after that to finish the proof we need only apply Proposition 4.2.

Proposition 4.3 is proved. □

The following proposition settles the genus 0 case.

Proposition 4.7. *Let $S : c_1^{z_1} \dots c_k^{z_k} = g$ be a non-degenerate standard quadratic equation over a group $G \in \mathcal{G}$. Then either $S = g$ has a solution in general position in some iterated centralizer extension of $G * F$, or every solution of $S = g$ is commutative.*

Proof. By the definition of a standard quadratic equation, $c_i \neq 1$ for all $i = 1, \dots, k$. Hence every solution of $S = g$ is non-degenerate. Now the result follows from Proposition 4.2. □

The following proposition can be proved similarly to Proposition 8 in [10].

Proposition 4.8. *Let $S : x_1^2 \dots x_p^2 c_1^{z_1} \dots c_k^{z_k} g = 1$ be a non-degenerate regular standard quadratic equation over a group $G \in \mathcal{G}$. Then there is a solution in general position in some iterated centralizer extension of $G * F$. If $p > 2$ and $p + k > 3$, then the equation is regular.*

We now introduce some notation. For $S : \prod_{i=1}^{m}[x_i, y_i]\prod_{j=1}^{n} c_j^{z_j} = g$, denote $p_j = c_j^{z_j}$, $p_{n+1} = g^{-1}$, $q_k = \prod_{i=1}^{k}[x_i, y_i]$ for $k \leq m$ and $q_{m+k} = \prod_{i=1}^{m}[x_i, y_i]\prod_{j=1}^{k} p_k$.

For $S : \prod_{i=1}^{m} x_i^2 \prod_{j=1}^{n} c_j^{z_j} = g$, denote $p_j = c_j^{z_j}$, $p_{n+1} = g^{-1}$, $q_k = \prod_{i=1}^{k} x_i^2$ for $k \leq m$ and $q_{m+k} = \prod_{i=1}^{m} x_i^2 \prod_{j=1}^{k} p_k$.

Proposition 4.9. *Let $S = g$ be a regular quadratic equation over a group $G \in \mathcal{G}$. Then there exists a solution δ in $G * F$ such that for any $j = 1, \dots, m + n - 1$:*

1. $[q_j^{\delta}, r_{j+1}^{\delta}] \neq 1$;
2. $[q_j^{\delta}, (r_{j+1} \dots r_{n+m})^{\delta}] \neq 1$;
3. *there exists a solution δ in an iterated centralizer extension of $G * F$ such that the following subgroups are free non-abelian: $\langle q_j^{\delta}, r_{j+1}^{\delta}\rangle$ for any $j = 1, \dots, m + n - 1$; $\langle q_j^{\delta}, x_{j+1}^{\delta}\rangle$ for any $j = 1, \dots, m - 1$; $\langle q_{j+1}^{\delta}, x_{j+1}^{\delta}\rangle$ for any $j = 1, \dots, m - 1$.*

Proof. Let $S = g$ be an orientable equation. We begin with the first statement. Let ϕ be a solution in general position constructed in Proposition 4.3. Let $q_{j-1} = \prod_{i=1}^{j-1}[x_i, y_i]$, $A = q_{j-1}^{\phi}, x_j^{\phi} = a, y_j^{\phi} = b, x_{j+1}^{\phi} = c, y_{j+1}^{\phi} = d$. If $[A[a,b],[c,d]] \neq 1$, then the statement is proved for j. Suppose that $[A[a,b],[c,d]] = 1$. We can assume that $[b,c] \neq 1$ (taking ab instead of b if necessary). Let $t = bc^{-1}$. Take another solution ψ such that $q_{j-1}^{\psi} = q_{j-1}^{\phi}$, $x_j^{\psi} = t^{-s}a, y_j^{\psi} = b^{t^s}, x_{j+1}^{\psi} = c^{t^s}, y_{j+1}^{\psi} = t^{-s}d$ for a large $s \in \mathbb{N}$.

If $[q_{j-1}^{\psi}[x_j^{\psi}, y_j^{\psi}], [x_{j+1}^{\psi}, y_{j+1}^{\psi}]] = 1$, then

$$A[a,b][b,t^s][t^s,c][c,d] = [t^s,c][c,d]A[a,b][b,t^s],$$

and therefore

$$A[a,b][c,d] = [t^s,c]A[a,b][c,d][b,t^s].$$

If we denote $B = A[a,b][c,d]$, this is equivalent to $B = [t^s,c]B[b,t^s]$, which is equivalent, by commutation transitivity, to $[t, cBb^{-1}] = 1$ or $[t, B^{c^{-1}}] = 1$, or $[B, c^{-1}b] = 1$.

We take instead of c, d respectively $(d^p)c, ((d^p)c)^k d$ and denote the new solution by $\delta_{s,p,k}$. If $[q_j^{\delta_{s,p,k}}, [x_{j+1}^{\delta_{s,p,k}}, y_{j+1}^{\delta_{s,p,k}}]] = 1$ for all s, p, k, then by the CSA property $[b(d^p c)^{-1}, (d^p c)^k d] = 1$ for all p, k, contrary to c, d freely generating a free subgroup.

The proof for $j \geq m$ is similar.

The same solution $\delta_{s,p,k}$ can be used to prove the second statement.

We will now prove the third statement by induction on j. Let δ be a solution with properties 1 and 2. Let $j = 1$ and

$$H_1 = \langle G * F, t_1 \mid [t_1, (r_2 \dots r_{m+n})^{\delta}] = 1 \rangle.$$

We transform δ into a solution δ_1 in the following way. If $m \neq 0$, then

$$x_1^{\delta_1} = x_1^{\delta}, \quad y_1^{\delta_1} = y_1^{\delta},$$

and

$$x_i^{\delta_1} = x_i^{\delta t_1}, \quad y_i^{\delta_1} = y_i^{\delta t_1}, \quad z_k^{\delta_1} = z_k^{\delta} t_1$$

for $i = 2, \dots, m, k = 1, \dots, n$. The subgroup generated by $q_1^{\delta_1}, r_2^{\delta_1}$ is free. Using Proposition 4.3 one can see that the subgroups generated by $q_1^{\delta_1}, x_2^{\delta_1}$ (if $m \geq 2$), and by $q_2^{\delta_1}, x_2^{\delta_1}$ are also free. In the case $m = 0$ we define

$$z_1^{\delta_1} = z_1^{\delta}, \quad z_k^{\delta_1} = z_k^{\delta} t_1$$

for $i = 2, \dots, m, k = 1, \dots, n$.

Suppose by induction that a solution δ_{i-1} in a group H_{j-1} which is an iterated centralizer extension of $G * F$ and satisfying the third statement of the proposition for indices from 1 to $j-1$ has been constructed. Let

$$H_j = \langle H_{j-1}, t_j \mid [t_j, (r_{j+1} \dots r_{m+n})^{\delta}] = 1 \rangle.$$

We begin with the solution δ_{j-1} and transform it into a solution δ_j in the following way:

$$\begin{aligned} x_i^{\delta_j} &= x_i^{\delta_{j-1}}, \quad y_i^{\delta_j} = y_i^{\delta_{j-1}}, \quad i = 1, \dots, j, \\ x_i^{\delta_j} &= x_i^{\delta_{j-1} t_j}, \quad y_i^{\delta_j} = y_i^{\delta_{j-1} t_j}, \quad i = j+1, \dots, m, \\ z_i^{\delta_j} &= z_i^{\delta_{j-1}} t_j. \end{aligned}$$

The subgroups generated by $q_j^{\delta_j}, r_{j+1}^{\delta_j}$, by $q_j^{\delta_j}, x_{j+1}^{\delta_j}$ and by $q_{j+1}^{\delta_j}, x_{j+1}^{\delta_j}$ are free.

The proof for a non-orientable equation is very similar and we skip it. □

We can now prove Theorem 4.1. Let H be the fundamental group of the graph of groups with two vertices, v and w, such that v is a QH vertex, $H_w = \Gamma \in \mathcal{G}$, and there is a retraction from H onto Γ. Let S_Q be a punctured surface corresponding to a QH vertex group in this decomposition of H. Elements q_j, x_j correspond to simple closed curves on the surface S_Q. By Proposition 4.9, we found a collection of simple closed curves on S_Q and solution δ with a properties 1)–4) from the beginning of Section 4.

Theorem E now follows from Theorem 4.1 by induction.

Notice that Proposition 4.9 also implies the following

Corollary 4.10 (cf. [21, Lemma 1.32]). *Let Q be the fundamental group of a punctured surface S_Q of Euler characteristic at most -2. Let $\mu : Q \to \Gamma$ be a homomorphism that maps Q into a non-abelian subgroup of Γ and the image of every boundary component of Q is non-trivial. Then either:*

1. *there exists a separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into Γ, and the image in Γ of the fundamental group of each connected component obtained by cutting S_Q along γ is non-abelian, or*
2. *there exists a non-separating s.c.c. $\gamma \subset S_Q$ such that γ is mapped non-trivially into Γ, and the image of the fundamental group of the connected component obtained by cutting S_Q along γ is non-abelian.*

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