J. Noncommut. Geom. 18 (2024), 123–142
DOI 10.4171/JNCG/524

A short proof of an index theorem, II

Yavar Abdolmaleki and Dan Kucerovsky

Abstract. We introduce a slight modification of the usual equivariant KK-theory. We use this to give a KK-theoretical proof of an equivariant index theorem for Dirac–Schrödinger operators on a non-compact manifold of nowhere positive curvature. We incidentally show that the boundary of Dirac is Dirac; generalizing earlier work of Baum and coworkers, and a result of Higson and Roe.

1. Introduction

In this paper, we define a form of approximate equivariance in KK_Γ-theory (Definition 3.4 and Theorem 3.2), and apply it to make the proof of Anghel's theorem in [25] equivariant. Some exposition has been added, in order to enhance the readability of this paper and to motivate and explain the techniques used. The main technical tools are presented in Sections 3 and 4. The purpose of these tools is to assist in modifying non-equivariant proofs in KK-theory to the equivariant case, and the non-equivariant proof that we use as a case study is that of Anghel's theorem in [25]. The tools are demonstrated in two slightly different settings: hyperbolic space with $SO(n)$ acting by isometries, and a Hadamard manifold with a discrete amenable group acting by isometries (Sections 5 and 6).

We begin with an extended introduction and discussion of Anghel's index theorem. Index theorems, generally speaking, express an analytical index in terms of topological information. An analytical index is usually some generalization of the classical Fredholm index of a Fredholm operator, and the relevant topological information is usually given by (the cohomological image of) a K-theory/K-homology pairing. One example is the Atiyah–Singer index theorem [5, 6], and another is in the cases where the Baum–Connes conjecture [9] holds.

Anghel's index theorem is an index formula for the Fredholm index of a Dirac–Schrödinger operator. These operators are of the form $D + iV$, where D is a (generalized) Dirac operator and iV is a skew-adjoint order zero operator, acting on the complex L^2-sections of some bundle. The reason for being interested in an equivariant form of Anghel's theorem is that an equivariant index is much more sensitive than the usual Fredholm index. The Fredholm index is integer-valued, while the equivariant index takes values in the representation ring of a group.

Mathematics Subject Classification 2020: 19K35 (primary); 58J20, 19K56 (secondary).
Keywords: KK-theory, complete CAT(0)-manifold, Hadamard manifold.

The classic Anghel's theorem was first proven by Callias [13] for the case of Euclidean space, and then in the case of warped cones by Anghel [2].

Theorem 1.1 (Anghel's theorem). *Let $D + iV$ be a Dirac–Schrödinger operator over a warped cone M with compact even-dimensional base N. If V^2 becomes arbitrarily large outside a compact subset of M, and $[D, V]$ is bounded, then $D + iV$ is Fredholm, with index given by*

$$\int_N \hat{A}(TN) \wedge \mathrm{ch} V^+ d(\mathrm{vol}_N),$$

where $\hat{A}$ denotes Atiyah's A-genus, and V^+ is the positive eigenbundle of V over a copy of N contained in a neighborhood of infinity such that V is invertible in that neighborhood.

The above theorem applies to *warped cones.* These are manifolds which are isomorphic outside a compact set to $\mathbb{R}^+ \times N$ with Riemannian metric $dr^2 + f(r)^2\widetilde{g}$, where $\widetilde{g}$ is the Riemannian metric of the compact manifold N, and f is a nondecreasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$. We remark that a hyperbolic space H^{n+1} is naturally a warped cone, and locally a warped product, of the form $[0, \infty) \times S^n$ with metric $dr^2 + \sinh^2(r)\widetilde{g}$. Anghel's original proof of his theorem [2] used differential geometry, and a KK-theoretical proof, which we will generalize in our work here, can be found in [25].

2. Formulation of an Anghel-type theorem

Our aim is to first prove that a certain natural exact sequence maps the class of a Dirac operator in equivariant K-homology to the class of a Dirac operator, and thus to prove a simple version of an equivariant Anghel-type theorem. However, one has to formulate the problem appropriately because otherwise the problem may either lack applications or lack a solution.

Suppose that a manifold M is a warped cone, as defined just above, whose collar is of the form $(1, \infty) \times N$ for some compact manifold N. We can define a C^*-algebra $C_\upsilon(M)$ consisting of those continuous bounded functions on M whose radial limits at infinity exist uniformly. In [25] there is a proof of an Anghel-type theorem, in the non-equivariant case, based on the cyclic exact sequences induced in KK-theory by the semisplit exact sequence

$$0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0. \tag{1}$$

In this short exact sequence the quotient map is the operation of taking the limit as the radial variable r goes to infinity. The middle term can be viewed as continuous functions on a compactification, $\overline{M}$, of M. This particular compactification, $\overline{M}$, is a compact topological manifold with boundary. Since we want to have a discrete group acting by isometries on the manifolds M and N, it is reasonable to first look at the case where M is a space of constant curvature, see [14] for more information on these. We will thus first focus on the case of hyperbolic space for simplicity in exposition, but then give a complete proof for the general case of a warped cone of nowhere positive curvature.

We will show the following two facts:

(i) *(The boundary of Dirac is Dirac)* The short exact sequence (1) maps the K-homology cycle associated with a Dirac operator D_M on M to a K-homology cycle associated with a Dirac operator D_N on N, and

(ii) *(Equivariant Anghel)* for suitable classes of approximately equivariant potentials V over M we have $[D_M + iV] = [f] \otimes_{C(N)} [D_N]$, where f is a K-theory element over N and the equality is in the representation ring $KK_\Gamma(\mathbb{C}, \mathbb{C})$.

The proof of the first fact will involve an interesting construction of an equivariant KK-group that is only approximately equivariant but defines the same group as the usual definition. We remark that the term used above, that the boundary of Dirac is Dirac, originates with Baum, Douglas, and Taylor [10, Sect. 4.5]. They construct a geometrical K-homology theory in which it is a fundamental fact that property (i) above holds, and they show this for both the non-equivariant case and then for the case of equivariance under a compact Lie group [10, 11, 17].

The next two sections give some preparatory constructions, and we return to the above topics in Section 5.

3. Approximate equivariance in $KK_\Gamma(A, B)$

Generally, the new issues that may arise when adding a group action to a non-equivariant KK-theoretical proof have to do with the existence of cycles. Thus we begin with some remarks on approximately equivariant cycles.

Following [22, Ch. 8], recall that a locally compact group Γ is said to have an action on a graded C^*-algebra B if there is a homomorphism α from Γ into the degree zero $*$-automorphisms of B. The group acts on a Hilbert B-module $\mathcal{E}$ by a homomorphism, also denoted α, into the invertible bounded linear transformations on $\mathcal{E}$ as a Banach space, with $\alpha_g(eb) = \alpha_g(e)\alpha_g(b)$, $\alpha_g\langle x, y\rangle = \langle \alpha_g x, \alpha_g y\rangle$. The space of Hilbert module operators is denoted $\mathcal{L}(E)$ and the ideal of operators that are compact in the Hilbert module sense is denoted $\mathcal{K}(E)$. Given a representation ϕ of a C^*-algebra A on a Hilbert module E, denote by I_ϕ the algebra of operators on E that commute with ϕ modulo compact operators, and by J_ϕ the algebra of operators $L \in \mathcal{L}(E)$ with $\phi(a)L$ and $L\phi(a)$ compact for all $a \in A$. We recall Kasparov's well-known definition of the set of bounded equivariant Kasparov cycles, $\mathcal{E}_\Gamma(A, B)$, for σ-unital C^*-algebras A and B and a second-countable locally compact group Γ.

Definition 3.1 ([21]). The set $\mathcal{E}_\Gamma(A, B)$ is given by triples $(\mathcal{E}, \phi, F)$ such that:

(i) $\mathcal{E}$ is a countably generated $\mathbb{Z}_2$-graded Hilbert B-module with a continuous degree zero action of Γ;

(ii) the map ϕ is a graded $*$-homomorphism $\phi : A \to \mathcal{L}(E)$;

(iii) there is an action of g on A such that $\alpha_g(\phi(a)) = \phi(\alpha_g(a))$;

(iv) the operator F is such that $g \mapsto \phi(a)\alpha_g(F)$ is norm-continuous;

(v) the degree one operator F is in I_ϕ, and is such that $F^2 - 1$, $F - F^*$, and $\alpha_g(F) - F$ are in J_ϕ.

The following theorem shows that condition (iii) can be replaced by the weaker condition

(iii′) there is an action of g on A such that $\alpha_g(\phi(a)) - \phi(\alpha_g(a)) \in \mathcal{K}(E)$.

Theorem 3.2. *For a discrete amenable group* Γ*, we obtain the same Kasparov group* $KK_\Gamma(A, B)$ *from either of the sets of conditions* (i)*,* (ii)*,* (iii)*,* (iv)*, and* (v) or (i)*,* (ii)*,* (iii′)*,* (iv)*, and* (v)*.*

Proof. Recall that there is a group of Γ-extensions, usually denoted $\mathrm{Ext}_\Gamma^{-1}(A, B)$. It consists of equivariant Busby maps, modulo an equivalence relation, see [28] for more information. These extensions are required to be equivariantly semisplit, as will be explained. There is a well-known isomorphism of $KK_\Gamma{}^1(A, B)$ onto $\mathrm{Ext}_\Gamma^{-1}(A, B)$, of the form $(E, \phi, F) \mapsto \pi(F\phi(\cdot)F^*)$, where π is the natural quotient map $\pi : \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$. This isomorphism takes a Fredholm module to a Busby map, and we could choose to define the equivalence relations on $KK_\Gamma{}^1(A, B)$ through pulling back the equivalence relations from $\mathrm{Ext}_\Gamma^{-1}(A, B)$. If we take this point of view, then it is clear that the difference between the two conditions on Fredholm modules

(iii′) *there is an action of g on A such that* $\alpha_g(\phi(a)) - \phi(\alpha_g(a)) \in \mathcal{K}(E)$*;* and

(iii) *there is an action of g on A such that* $\alpha_g(\phi(a)) = \phi(\alpha_g(a))$*,*

disappears under the equivalence relations, because in either case we have

$$\pi(F(\phi(\alpha_g(a)))F^*) = \pi(F(\alpha_g(\phi(a)))F^*).$$

It remains to check the often delicate semisplitting property of the extension that we obtain under the above isomorphism, while using the weaker condition (iii′). We must verify that the extension obtained has an equivariant semisplitting. The natural candidate for such a semisplitting is the map $a \mapsto F\phi(a)F^*$, and with condition (iii′), we can immediately only conclude that we have a semisplitting, $a \mapsto F\phi(a)F^*$. Baaj and Skandalis have however shown that in the discrete and amenable case, the existence of a semisplitting implies the existence of an equivariant semisplitting , see [8, Prop. 7.13(2)], and [8, Prop. 7.16]. ∎

For the reader's convenience we recall our earlier definition of equivariant unbounded cycles in [22, Def. 8.7], and [24, Def. 4.7, p. 272].

Definition 3.3. The set of unbounded equivariant Kasparov modules $\Psi_\Gamma(A, B)$ is given by triples (E, ϕ, D) where E is a Hilbert B-module with Γ-action; $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism; and D is an unbounded regular degree one self-adjoint operator on E, such that

(i) for each $e \in E$, the map $g \mapsto \{D - \alpha_g(D)\}e$ is continuous as a map from Γ into E;

(ii) the operator $(i + D)^{-1}$ is in J_ϕ;

(iii) the homomorphism ϕ satisfies $\alpha_g(\phi(a)e) = \phi(\alpha_g(a))e$; and

(iv) for all a in some dense subalgebra of A, the commutator $[D, \phi(a)]$ is bounded on the domain of D.

Pointwise norm Γ-continuity (strong continuity in the Hilbert module sense) is all that is needed in part (i) of the above definition. This definition behaves correctly under the bounded transform, and the unbounded connection conditions for a Kasparov product still hold and have the same form in the equivariant case as in the non-equivariant case [22–24]. Because passing to the bounded transform [7] only affects the operator D, we can replace, in Definition 3.3 above, condition (iii) by condition (iii′) using Theorem 3.2 applied to the bounded transforms.

The following definition and corollary summarize our discussion.

Definition 3.4. The set of unbounded approximately equivariant Fredholm modules, $\Psi_\Gamma(A, B)$, is given by triples (E, ϕ, D) where E is a (graded) Hilbert B-module with Γ-action; $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism; and D is an unbounded regular degree one self-adjoint operator on E, such that

(i) for each $e \in E$, the map $g \mapsto \{D - \alpha_g(D)\}e$ is continuous as a map from Γ into E;

(ii) the operator $(i + D)^{-1}$ is in J_ϕ;

(iii′) the homomorphism ϕ satisfies $\alpha_g(\phi(a)) - \phi(\alpha_g(a)) \in \mathcal{K}(E)$, and

(iv) for all a in some dense subalgebra of A, the commutator $[D, \phi(a)]$ is bounded on the domain of D.

An alternative form of condition (i) above is given in Corollary A.3 in the Appendix. The above definition does not require amenability. In the presence of amenability, Theorem 3.2 implies the following corollary.

Corollary 3.5. *In the discrete and amenable case,* $\Psi_\Gamma(A, B)$ *is isomorphic to the usual* $KK_\Gamma(A, B)$ *group.*

By assuming or omitting invertibility one can avoid the amenability condition.

Remark 3.6. If three cycles satisfy the usual connection conditions when the group action is forgotten, then they also form a Kasparov product in the equivariant case.

4. A delocalized L^2-space and amenability

In this section, we construct some convenient Hilbert spaces that will be needed in the proofs of our theorems (specifically, they will be used to define K-homology cycles). Recall the unital exact sequence of C^*-algebras from the introduction, namely

$$0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0.$$

This extension is semisplit by the Choi–Effros theorem. If we now equip this extension with a group action that is compatible with the inclusion map, quotient map, and semisplitting map, it will induce [8, Thm. 7.17] cyclic exact sequences in equivariant KK-theory. Fortunately, it is not necessary to explicitly verify compatibility of the group action with the semisplitting: by [8, Sect. 7], see also [27], we can average an non-equivariant semisplitting, thus obtaining an equivariant semisplitting. Due to the expander-based counterexample in [18, Sect. 7], some mild condition on the extension or on the group (for example, amenability) seems to be needed for such a semisplitting. For an example of an explicit equivariant semisplitting, please see the proof of [26, Thm. 3.4].

Remark 4.1 (On Dirac operators). The topological manifold with boundary $\overline{M}$ naturally inherits a metric from M, and this metric is singular at the boundary. Bismut and Cheeger [12] constructed the natural spinor Dirac operator $D_{\overline{M}}$ on a spin^c manifold with boundary and a radially singular metric at the boundary. Their operator is singular at the boundary. The spin bundle associated to this singular operator $D_{\overline{M}}$ restricts to a spin bundle, say E, over M, and since Dirac operators do not increase support, we obtain a convenient Dirac operator D_M on M. Bismut and Cheeger construct at the same time a related Dirac operator D_N on the boundary. These operators are given by local formulas of the expected form,

$$D_{\overline{M}} = \sum_{i=1}^{n} e_i' \nabla_{e_i'}^{\overline{M}}$$

and

$$D_N = \sum_{i=1}^{n-1} e_i \nabla_{e_i}^{N}.$$

The above are formulas 1.6 and 1.7 of [12]. For more information, please see [12, pp. 319–324], or for general information on Dirac operators see [15].

To define cycles in K-homology we need L^2-spaces. Let us use the notation $L^2(M, E)$ for an L^2 space with coefficients in a vector bundle E. The following proposition will be applied to the spinor bundle associated with a Dirac operator, but can as well be stated in a more general form.

Proposition 4.2. *Consider a semisplit short exact sequence*

$$0 \longrightarrow C_0(M) \longrightarrow C(\overline{M}) \underset{i}{\overset{s}{\rightleftarrows}} C(N) \longrightarrow 0$$

equipped with the action of a discrete group Γ*. Suppose that either the group is amenable* or *that there exists a* Γ*-invariant measure on the space* $\overline{M}$*. Given any vector bundle* E *over the finite-dimensional manifold with boundary* $\overline{M}$*, there exist equivariant* L^2*-spaces over* $\overline{M}$ *and over* N *that are compatible with the equivariant quotient map* i *and its equivariant semisplitting map* s *with*

$$L^2(\overline{M}, E) \underset{i^*}{\overset{s}{\rightleftarrows}} L^2(N, i^*(E)).$$

The L^2*-space map* s *is injective, the* L^2*-space map* i^* *is surjective, and* $i^* \circ s =$ Id*. There exists also a natural* L^2*-subspace* $L^2(M, E) \subset L^2(\overline{M}, E)$*.*

Proof. Suppose we are in the amenable case. Since the middle term $C(\overline{M})$ of the given short exact sequence

$$0 \to C_0(M) \to C(\overline{M}) \to C(N) \to 0$$

is a unital C^*-algebra, it has a weak-* compact state space. Amenability then implies the existence of a Γ-invariant state, ϕ, in that state space. In terms of measures rather than states, this implies the existence of a nontrivial invariant measure on the Gelfand spectrum $\overline{M}$ of this C^*-algebra. Letting E be the given bundle on $\overline{M}$, let $L^2(\overline{M}, E)$ denote the corresponding L^2-space constructed from the bundle E and the above invariant measure coming from ϕ. If we compose the invariant state ϕ with the linear, unital, and completely positive map provided by the previously discussed equivariant semisplitting $s : C(N) \to C_\upsilon(M)$, we obtain an invariant state $\phi \circ s$ on $C(N)$. Let i denote the map of Gelfand spaces induced by the quotient map in our exact sequence. Let $i^*(E)$ denote the pullback of the Dirac bundle on $\overline{M}$ to N. Let $L^2(N, i^*(E))$ denote the L^2-space constructed from the bundle $i^*(E)$ and the invariant state $\phi \circ s$ on $C(N)$. (This is a special case of the KGNS construction [20].)

The semisplitting map $s : C(N) \to C_\upsilon(M)$ is not a homomorphism, but it *is* a linear positive map of ordered Banach spaces. Being completely positive, it induces a linear positive map of the space of sections of trivial bundles over N. By Swan's theorem and the compactness of the manifold N, we can regard the space of sections of $i^*(E)$ as a closed subspace of some trivial bundle over N, and then the map s provides a linear positive isomorphism with a closed subspace of the space of sections of the bundle E over $\overline{M}$. The map s therefore splits the given extension at the level of equivariant L^2-spaces, because the maps s and i^* are at this level respectively an injective map and a surjective map, and $i^* \circ s = \mathrm{Id}$. If we are instead given a Γ-invariant measure on $\overline{M}$, then we replace states by integrals in the above argument. ∎

By the above proposition, the map s embeds $L^2(N, i^*(E))$ as an L^2-subspace in $L^2(\overline{M}, E)$, and even provides a splitting map at the level of L^2-spaces. We will lighten the notation by writing, for example, $L^2(N) \subset L^2(\overline{M})$ instead of $s(L^2(N, i^*(E))) \subset L^2(\overline{M}, E)$.

The space $L^2(N)$ can be said to be *delocalized*, or *"quantum"*, in the sense that it is not obviously induced by a restriction of $L^2(\overline{M})$ to a submanifold of $\overline{M}$.

We now turn to applications.

5. The case of hyperbolic space $\mathbb{H}^{n+1}$

The manifold M will, for the moment, be a finite-dimensional hyperbolic space $\mathbb{H}^{n+1}$, as in [1], and the manifold N will be the corresponding boundary S^n. We refer to the boundary points that are in the bounding hyperplane $\mathbb{R}^n$ with respect to the half space model as the real hyperplane, and we denote the remaining boundary point of $\mathbb{H}^{n+1}$ by $\{\infty\}$. In classic hyperbolic geometry, these boundary points are known as ideal points, and can also be called limit points. Since the space $C(\overline{M})$ is equivalent to $C_\upsilon(M)$, in other words, continuous functions on M with a mild smoothness condition imposed at infinity, isometries of M that preserve this condition extend to uniform homeomorphisms of $\overline{M}$. As is well-known—by Brouwer's theorem—such a homeomorphism will have a fixed point. In order to insure smoothness, consider the class of homeomorphisms given by the isometries of M whose extensions are isometries of the real hyperplane into itself, and have a common fixed point on the boundary; geometrically speaking, these are the elements of the (amenable) real orthogonal group $O(n)$ with their natural action on the half-space model. The fixed point is then exactly the special point defined at the start of this section and denoted $\{\infty\}$. We denote by Γ some given discrete amenable subgroup of this group: we regard it as acting by isometries on S^n and on $\mathbb{H}^{n+1}$. We regard the exact sequence (1) at the beginning of Section 2 as being equipped with an action of the discrete amenable subgroup Γ acting by isometries, and moreover we suppose that the isometries preserve orientation so that they will be compatible with Dirac operators. We will then generalize to the case of warped cones of nowhere positive curvature. It is interesting to note that in the case that such a space has constant curvature, it is isomorphic to a quotient of a hyperbolic or a Euclidean space modulo the action of a discrete subgroup of isometries G [14, Ch. 8]. The positive scalar curvature case has obstructions [16] and will not be considered further. In [11, Sect. B.4] it is shown that the boundary of Dirac is Dirac, in the case of equivariance with respect to the action of a compact Lie group. (See [10, 17] for proofs in the non-equivariant case.) The proof in [11] uses the compactness of the Lie group in several places, and thus does not generalize in any obvious way to the case of equivariance under an action by a discrete group. We aim to prove a similar result in the case of the action of a discrete amenable group. As already mentioned, we treat first the case of hyperbolic space $M = \mathbb{H}^{n+1}$, with $N = S^n$, and a discrete amenable subgroup of $SO(n)$ acting. Recall that a hyperbolic space has a natural warped cone structure: namely $[0,\infty) \times S^n$ with metric $dr^2 + \sinh^2(r)\tilde{g}$. In the following theorem, the cycle $[r] \in KK_\Gamma{}^1(C(N), C_0(M))$ has an operator given by multiplication by the radial variable of this warped cone M. The Dirac operators below are as in Remark 4.1, and the homomorphism ϕ is defined next in Proposition 5.2.

Theorem 5.1. *Let M be a hyperbolic space $M = \mathbb{H}^{n+1}$, and let N be its boundary. Let Γ be a discrete amenable subgroup of $SO(n)$ acting on $0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0$ by oriented isometries. The Dirac operators over N and over M give KK_Γ-classes $[D_N] \in KK_\Gamma{}^0(C(N), \mathbb{C})$ and $[D_M] \in KK_\Gamma{}^1(C_0(M), \mathbb{C})$, and $[D_N] = [r] \otimes [D_M]$. The cycle $[r] \in KK_\Gamma{}^1(C(N), C_0(M))$ is given by $(C_0(M), \phi, r)$.*

The proof will take several lemmas, and concludes in Corollary 5.6. We begin by using the geometry of a hyperbolic space to construct the Γ-invariant homomorphism ϕ that is used in the cycle $[r]$. The exact property that is used is the visibility space property.

Proposition 5.2. *There exists a $*$-homomorphism $\phi : C(S^n) \to \mathcal{M}(C_0(\mathbb{H}^{n+1}))$, and*

$$\alpha_g(\phi(f)) - \phi(\alpha_g(f)) \in C_0(\mathbb{H}^{n+1}).$$

Proof. In the half-space model for hyperbolic space, the given function f is defined on the boundary points (i.e., limit points) of the hyperbolic space. Let n denote a limit point in the real hyperplane. Consider the geodesic through hyperbolic space from the limit point n to the limit point $\{\infty\}$. This limit point is defined at the beginning of this section. Extend the domain of the given function f to all of $\overline{M} \setminus \{\infty\}$ by defining it to be constant on all such geodesics. In other words, extend f to a larger domain by making it constant with respect to the y coordinate of the half-space model, so that $f(n, y) := f(n)$. The function we obtain is bounded and continuous at all ordinary points of the hyperbolic space, and thus we can use the function as a multiplier of elements of $C_0(\mathbb{H}^{n+1})$. Let us, for convenience, define $\phi(f)$ to be zero at the limit point $\{\infty\}$, so that we may regard $\phi(f)$ as a bounded function on $\overline{M}$ that is continuous except at the limit point $\{\infty\}$. Now we notice that $\alpha_g(\phi(f))$ and $\phi(\alpha_g(f))$ are equal when restricted to points of the boundary, S^n. But then $\alpha_g(\phi(f)) - \phi(\alpha_g(f))$ is a function that is continuous at all the points of hyperbolic space and its boundary, except possibly the limit point $\{\infty\}$, and moreover it is zero at all other limit points. This means that this function is in $C_0(\mathbb{H}^{n+1})$ as claimed. Finally, we remark that the mapping $\phi : C(S^n) \to \mathcal{M}(C_0(\mathbb{H}^{n+1}))$ that we have defined is evidently an algebraic $*$-homomorphism, and algebraic $*$-homomorphisms of C^*-algebras are automatically bounded (i.e., are continuous $*$-homomorphisms). ∎

5.1. A Kasparov product

The next lemma is an version of [25, Lemma 3.1] for an approximately equivariant cycle. When defining explicit KK-elements, it will be useful to note that the complex Clifford algebra C_1 is isomorphic to $\mathbb{C} \oplus \mathbb{C}$, with elements of the form (e, e) having even degree, and elements of the form $(e, -e)$ having odd degree. This is sometimes referred to as the odd grading. The other common grading is the diagonal/off-diagonal grading (for operators), where operators represented by diagonal 2-by-2 matrices have the even grading, and antidiagonal ones the odd grading. The space of Hilbert module operators is denoted $\mathcal{L}(E)$ and the ideal of operators that are compact in the Hilbert module sense is denoted $\mathcal{K}(E)$.

Lemma 5.3. *Let S be a Dirac bundle over the nowhere positive curvature warped cone M. Let H be the Hilbert space $L^2(M, S) \oplus L^2(M, S)$ with diagonal/off-diagonal grading. Let D_M be the Dirac operator on S. Let ϕ denote the approximately invariant homomorphism of Proposition 5.2, and let r denote multiplication by r, the radial coordinate of the warped cone. Then the Kasparov product $[r] \otimes [D_M]$ equals $[D_M + ir]$, where*

$$[D_M] := \left(H, m, \begin{pmatrix} 0 & D_M \\ D_M & 0 \end{pmatrix}\right) \in KK(C_0(M) \otimes C_1, \mathbb{C}),$$

$$[r] := \left(C_0(M) \otimes C_1, \phi, \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right) \in KK(C(N), C_0(M) \otimes C_1),$$

and

$$[D_M + ir] := \left(H, \phi, \begin{pmatrix} 0 & D_M - ir \\ D_M + ir & 0 \end{pmatrix}\right) \in KK(C(N), \mathbb{C}).$$

Proof. We follow the proof of [25, Lemma 3.1]. The action of $m : C_0(M) \otimes C_1 \to \mathcal{L}(H)$ is given by

$$m : b \oplus b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \qquad m : b \oplus -b \mapsto \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix},$$

where elements of $C_0(M)$ act on $L^2(M, S)$ as multiplication by functions on M. A calculation shows that there is a Hilbert module isomorphism that identifies $H = L^2(M, S) \oplus L^2(M, S)$ with the inner tensor product of $C_0(M) \otimes C_1$ and H over m.

In order to apply the criterion for an unbounded cycle to be the Kasparov product of two given cycles [23], we have to verify a semiboundedness condition and a connection condition. Since these conditions do not explicitly involve the group action, the proof of the non-equivariant case ([25, Lemma 3.1]) goes through, with $[r]$ in the place of $[A]$ there. We comment that $[r]$ is indeed an approximately equivariant cycle, because $r - \alpha_\gamma(r)$ is bounded for each $\gamma \in \Gamma$. ∎

5.2. A homotopy of virtual spaces

We now make use of these L^2-spaces to construct a Kasparov homotopy of K-homology cycles. This homotopy is an equivariant version of the one used in [25, Lemma 4.1]. The statement of the next lemma uses Proposition 5.2, and this means that the lemma is at present applicable only to the current case of the hyperbolic space M. However, the restriction will be removed once we prove Proposition 6.1. The Dirac bundle S in the next lemma can be taken to be as in Remark 4.1.

Lemma 5.4. *Let S be a Dirac bundle over the nowhere positive curvature warped cone M. Let D_M be the Dirac operator on S. Let ϕ be the approximately invariant homomorphism of Proposition 5.2, and let r denote multiplication by r, the radial coordinate of the warped cone. Then $[r] \otimes [D_M] = [D_N]$, where*

$$[D_M] := \left(L^2(M) \oplus L^2(M), m, \begin{pmatrix} 0 & D_M \\ D_M & 0 \end{pmatrix}\right) \in KK(C_0(M) \otimes C_1, \mathbb{C}),$$

$$[r] := \left(C_0(M) \otimes C_1, \phi, \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right) \in KK(C(N), C_0(M) \otimes C_1),$$

and

$$[D_N] := \left(L^2(N) \oplus L^2(N), m, \begin{pmatrix} 0 & D_N \\ D_N & 0 \end{pmatrix}\right) \in KK(C(N), \mathbb{C}).$$

Proof. Lemma 5.3 shows that $[r] \otimes [D_M] = [D_M + ir]$, so we need only to show that $[D_M + ir]$ is Kasparov homotopic to $[D_N]$. Thus we need to show a suitable homotopy of $L^2(M) \oplus L^2(M)$ and $L^2(N) \oplus L^2(N)$. Proposition 4.2 provides equivariant copies of $L^2(N)$ and $L^2(M)$ embedded in $L^2(\overline{M})$. There is a "dimension drop" Hilbert $C([0, 1])$ module

$$E := \{f \in L^2(\overline{M}) \otimes C([0, 1]): f(0) \in L^2(N), f(1) \in L^2(M)\}.$$

The module consists of the continuous functions from $[0, 1]$ into the space $L^2(\overline{M})$ which take endpoint values in the subspaces $L^2(N)$ at one endpoint and $L^2(M)$ at the other endpoint. Since all spaces used are equivariant, the module is equivariant. The Kasparov triple $(E, \phi, D_M + \lambda ir)$ together with the evaluation map (at $\lambda = 0$) denoted by i^* in Proposition 4.2 provides a homotopy in Kasparov's sense of $(L^2(M), \phi, D_M + ir)$ and $(L^2(N), \phi, D_N)$. Note that the Dirac operator D_M can be regarded as a Dirac operator on $\overline{M}$ by Remark 4.1. We thus obtain a cycle of the form

$$\left(L^2(N) \oplus L^2(N), \phi, \begin{pmatrix} 0 & D_N \\ D_N & 0 \end{pmatrix}\right) \in KK(C(N), \mathbb{C}),$$

but the homomorphism ϕ in fact acts on $L^2(N)$ by ordinary multiplication. Thus we obtain the cycle $[D_N] = \left(L^2(N) \oplus L^2(N), m, \left(\begin{smallmatrix} 0 & D_N \\ D_N & 0 \end{smallmatrix}\right)\right) \in KK(C(N), \mathbb{C})$ as was to be shown. ∎

The above shows that if we regard the operation of taking the Kasparov product with the above cycle $[r]$ as a K-homological boundary map, then the boundary of the Dirac operator on M is a Dirac operator on N. This can be attributed in the non-equivariant case to [10, Sect. 4.5], see also [17, 19], and for the case of a compact Lie group action to [11, Lemma 3.8].

The proof we have given can be viewed as a version of the original non-equivariant proof, which consisted of a Kasparov product calculation followed by noticing that the Busby map associated with their boundary cycle coincides with the Busby map of the given extension. In order to find a natural candidate for the boundary cycle, and to state that cycle in a simple way, we introduced a slight generalization of the usual notion of a Fredholm triple.

Technically speaking, we have yet to relate our boundary cycle $[r]$ to the extension $0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0$. However, the expected equality at the level of Busby maps does remain valid in the equivariant case. This is because the formula for computing the Busby map remains the same with or without equivariance.

Proposition 5.5. *The Busby map associated with the cycle* $[r] \in {KK_\Gamma}^1(C(N), C_0(M))$ *under the isomorphism from the proof of Proposition 3.2 coincides with the Busby map associated with the equivariant extension* $0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0$.

Proof. Same as in the non-equivariant case [17, 19]. ∎

Concluding this part of the proof, we combine Lemma 5.4 and Proposition 5.5 to obtain the following.

Corollary 5.6 (The boundary of Dirac is Dirac). *The equivariant K-homology cycle $[D_M]$ is mapped to $[D_N]$ under the K-homology map induced by the equivariant extension* $0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0$.

The proof of the above result just supposes an amenable discrete group action, acting on finite-dimensional spin^c manifolds. If amenability or finite-dimensionality is not assumed, a proof along our lines runs up against one of the expander-based counterexamples to the Baum–Connes conjecture [18]. These counterexamples are all based on failures of exactness at the level of K-theory. Thus, we may wonder if these counterexamples do in fact explicitly manifest at the level of geometric K-homology. More precisely, we have the following question:

Question 5.7. *Is it true that the boundary of Dirac is Dirac, at the level of equivariant K-homology, for actions of finitely generated groups that may not uniformly embed into Hilbert space as in* [18, Sect. 7]*?*

Furthermore, incidently, the K-theory boundary map associated with the cycle $[r] \in {KK_\Gamma}^1(C(N), C_0(M))$ can be written KK_Γ-theoretically as shown below, where C_1 acts on itself in the natural way. See [26, Thm. 3.4] for an ingenious alternative formulation in the bounded picture, using a Poisson-type transform in the specific case of hyperbolic space.

Proposition 5.8. *Let* $[r] \in KK_\Gamma(C(N) \otimes C_1, C_0(M))$ *denote*

$$\left(C_0(M) \otimes C_1, \phi \otimes \mathrm{Id} : C(N) \otimes C_1 \to \mathcal{L}, \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right).$$

This cycle will map the K*-theory element*

$$[A] := \left(M_n(C(N)) \otimes C_1, 1, \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\right) \in KK_\Gamma(\mathbb{C}, C(N) \otimes C_1)$$

to the K*-theory element*

$$[A + r] := \left(M_n(C_0(M)) \otimes C_1, 1, \begin{pmatrix} \phi(A) + r & 0 \\ 0 & -\phi(A) - r \end{pmatrix}\right) \in KK_\Gamma(\mathbb{C}, C_0(M)).$$

Proof. The (inner) tensor product of Hilbert modules $(M_n(C(N)) \otimes C_1) \otimes_{\phi\otimes\mathrm{Id}} C_0(M) \otimes C_1$, where $\mathrm{Id} : C_1 \to \mathcal{L}(C_1)$ denotes the natural action of the complex Clifford algebra C_1 on itself, is isomorphic to $M_n(C_0(M)) \otimes C_1$ regarded as a bimodule over $C_0(M) \otimes C_1$ and over $C(N)$, where $C(N)$ acts on $C_0(M)$ through the map ϕ. With respect to this isomorphism, the elementary map $T_x : C(M) \otimes C_1 \to M_n(C(N)) \otimes C_1 \otimes_{\phi\otimes\mathrm{Id}} C_0(M) \otimes C_1$ is $\phi(x)$, regarded as a multiplication by a matrix-valued function, $x \in M_n(C(N))$; i.e., a map $T_x : C_0(M) \otimes C_1 \to M_n(C_0(M)) \otimes C_1$. The rest of the proof is a routine verification of the connection conditions. ∎

The case of a degree shift amounts to a factor of C_1 as we see in the following proposition.

Proposition 5.9. *Let* $[r] \in KK_\Gamma(C(N), C_0(M) \otimes C_1)$ *denote*

$$\left(C_0(M) \otimes C_1, \phi : C(N) \to \mathcal{L}, \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right).$$

This cycle will map the K*-theory element*

$$[A] := \left(M_n(C(N)) \otimes C_1, 1, \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}\right) \in KK_\Gamma(\mathbb{C}, C(N))$$

to the K*-theory element*

$$[A + r] := \left(M_n(C_0(M)) \otimes C_1, 1, \begin{pmatrix} \phi(A) + r & 0 \\ 0 & -\phi(A) - r \end{pmatrix}\right) \in KK_\Gamma(\mathbb{C}, C_0(M) \otimes C_1).$$

6. The case of Hadamard manifolds

We started with the case of a hyperbolic space, regarded as a warped cone of constant curvature, with an action of $SO(n)$. In replacing the hyperbolic space by a more general warped cone M of nowhere positive curvature, only one new consideration arises, and this has to do with defining the approximately equivariant $*$-homomorphism $\phi : C(N) \to \mathcal{M}(C_0(M))$. The previous proofs go through if we replace Proposition 5.2 above by the following more general Proposition 6.1, below. We may now assume that Γ is amenable, acts by oriented isometries on $\overline{M}$ and fixes a point in the collar manifold N.

Proposition 6.1. *Let M denote a simply connected warped cone of nowhere positive curvature, with boundary N. Assume that Γ is amenable, acts by oriented isometries on $\overline{M}$, and fixes a point in the boundary, N. There exists a $*$-homomorphism $\phi : C(N) \to \mathcal{M}(C_0(M))$, which is approximately equivariant in the sense that $\alpha_g(\phi(f)) - \phi(\alpha_g(f)) \in C_0(M)$.*

Proof. By the Cartan–Hadamard theorem [14, Thm. 3.1], our space of nonpositive curvature is equipped with a globally defined exponential map, providing well-behaved geodesics radiating outwards from any chosen point. The geodesics extend uniquely to the closure $\overline{M}$.

Consider, thus, the family of geodesics, in M, to points n_i on the boundary N, emanating from the limit point $\{\infty\}$. Two such geodesics with distinct values of n_i intersect only at $\{\infty\}$. Let us now extend the domain of the given function $f \in C(N)$ to all of $\overline{M} \setminus \{\infty\}$ by defining it to be constant on all such geodesics. In terms of Figure 1, below, we propagate the values of f upwards along geodesics, obtaining a function that is well-behaved except possibly at the point $\{\infty\}$. The function we so obtain is bounded and continuous at all regular points of the space $\overline{M}$, and so we can use the function as a multiplier of elements of $C_0(M)$. The remainder of the proof is just as in Proposition 5.2. ∎

Exactly as in the previous section, we deduce the following corollary.

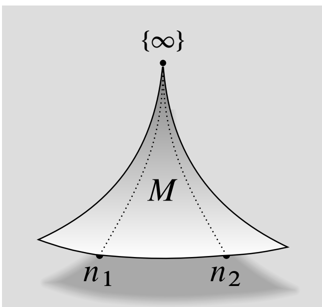

Figure 1. Two geodesics of $\overline{M}$.

Corollary 6.2 (The boundary of Dirac is Dirac). *Let*

$$[D_M] := \left(L^2(M) \oplus L^2(M), m, \begin{pmatrix} 0 & D_M \\ D_M & 0 \end{pmatrix}\right) \in KK(C_0(M) \otimes C_1, \mathbb{C}),$$

and

$$[D_N] := \left(L^2(N) \oplus L^2(N), m, \begin{pmatrix} 0 & D_N \\ D_N & 0 \end{pmatrix}\right) \in KK(C(N), \mathbb{C}),$$

where m is the natural action by multiplication. The equivariant K-homology cycle $[D_M]$ is mapped to $[D_N]$ under the K-homology map induced by the equivariant extension $0 \to C_0(M) \to C_\upsilon(M) \to C(N) \to 0$.

6.1. Application to an Anghel-type theorem

We now turn to the topic of an Anghel-type theorem, on a warped cone of nowhere positive curvature with an amenable discrete group acting quasi-parabolically by oriented isometries. We first recall a classic lemma from the Hilbert space setting. Here, the potential V is a self-adjoint unbounded multiplier acting on (the sections of) the vector bundle E.

Lemma 6.3. *Let D_M be a Dirac-type operator on a spin bundle E over the warped cone M. Let $V : \Gamma(E) \to \Gamma(E)$ go to infinity at infinity in the warped cone. Then $D_M \pm iV$, regarded as an unbounded operator on the Hilbert space $L^2(M, E)$, has compact resolvent.*

For a proof of the above lemma, see any of [2–4, 25]. The above lemma was a key point in showing [25] that $D_M + iV$ defines a (non-equivariant) cycle in $KK(\mathbb{C}, \mathbb{C})$. As shown in the Appendix (Theorem A.2), if $D_m + iV$ does define an equivariant cycle as in Definition 3.4, then it follows that $\alpha_\gamma(D_M - iV) - (D_M - iV)$ is bounded for each $\gamma \in \Gamma$. So the potential V must satisfy the strong but necessary assumption that for each γ, the difference $\alpha_\gamma(V) - V$ is bounded. But, in fact a physically similar condition already appears in the classic Anghel's theorem, see Theorem 1.1, where it was assumed that the gradient of the potential, $[D, V]$, was bounded. Thus the assumption made is plausible. Lemma 6.3 together with the above discussion implies that the necessary conditions for an unbounded equivariant cycle (Definition 3.3) hold, as we see in the following.

Lemma 6.4. *Let D_M be a Dirac-type operator on a spin bundle E over the warped cone M. Let $V : \Gamma(E) \to \Gamma(E)$ go to infinity at infinity in the warped cone, and let $\alpha_\gamma(V) - V$ be bounded for each group element $\gamma \in \Gamma$. Then the cycle*

$$\left(L^2(M, E) \oplus L^2(M, E), 1, \begin{pmatrix} 0 & D_M - iV \\ D_M + iV & 0 \end{pmatrix}\right)$$

is an equivariant $KK_\Gamma(\mathbb{C}, \mathbb{C})$ cycle.

Now, let us recall that:

(i) the Dirac operators over N and over M give KK_Γ-classes

$$[D_N] \in KK_\Gamma{}^0(C(N), \mathbb{C}) \qquad \text{and} \qquad [D_M] \in KK_\Gamma{}^1(C_0(M), \mathbb{C}),$$

(ii) the geometrical properties of M give an approximately equivariant KK_Γ-class $[r] \in KK_\Gamma{}^1(C(N), C_0(M))$, as defined in Lemma 5.3,

(iii) a given endomorphism f that defines a K-theory class $[f] \in KK_\Gamma{}^0(\mathbb{C}, C(N))$, and, by Proposition 5.8/5.9, a KK_Γ class $[V] \in KK_\Gamma{}^1(\mathbb{C}, C_0(M))$, where $V = f + ir$, and

(iv) the index of $D_M + iV$ defines a KK_Γ-class $[D_M + iV] \in KK_\Gamma(\mathbb{C}, \mathbb{C})$, see Lemma 6.4.

Then it follows, from the Kasparov product factorizations in Corollary 5.6, Proposition 5.8/5.9, and associativity of the Kasparov product that

$$\begin{aligned} \operatorname{Ind}(D_M + iV) := [D_M + iV] &= [V] \otimes_{C_0(M)} [D_M] \\ &= [f] \otimes_{C(N)} [r] \otimes_{C_0(M)} [D_M] \\ &= [f] \otimes_{C(N)} [D_N]. \end{aligned}$$

This shows the following.

Proposition 6.5. *Let M be a warped cone of nowhere positive curvature. Let N denote its collar. Let Γ be an amenable discrete group acting by oriented isometries on*

$$0 \to C(N) \to C_v(M) \to C_0(M) \to 0,$$

with a common fixed point in N. Let D_N and D_M denote Dirac operators on N and M respectively, and let f denote a potential on N. Then $[D_M + i(r + \phi(f))] \in KK_\Gamma(\mathbb{C}, \mathbb{C})$ factorizes equivariantly as $[D_N] \otimes_{C(N)} [f]$, an equivariant K-theory and K-homology pairing over the compact manifold N.

The above, then, is the basic form of an Anghel-type theorem that takes into account the restrictions of an equivariant situation. Just as in the non-equivariant case, one can extend it to an apparently larger class of potentials, namely, the case of any $[D_M + iV]$ that is Kasparov homotopic to one of the above form.

We have used hypotheses that are simple to state rather than aiming for maximal generality. With regard to possible generalizations, we should point out that the only nontrivial information from Riemannian geometry that was used is:

- existence of Dirac operators, and
- the visibility space property of Hadamard manifolds (i.e., the Cartan–Hadamard theorem).

The hypothesis of amenability gives a pleasant proof of Proposition 4.2 but can there be replaced by a more lattice-theoretical hypothesis: the existence of a Γ-invariant measure; as already pointed out in (the non-amenable case) of Proposition 4.2. As in Definition 3.3, one could in the remainder of the work proceed by using cycles at the level of $\Psi_\Gamma(A, B)$.

A. Appendix: On strongly Γ-continuous unbounded regular operators

As a convenience to the reader, we recall a known result in this appendix.

Proposition A.1 ([22, Prop. 8.2]). *Let T be a regular operator on a countably generated Hilbert module E and let Γ be a locally compact Hausdorff group acting on E by automorphisms. The following are equivalent:*

(i) *for each $e \in E$, the map $g \mapsto (T - \alpha_g(T))e$ is continuous as a map from Γ into E; and*

(ii) *the function $g \mapsto (T - \alpha_g(T))$ is in $\mathcal{L}(C(K, E))$ for every compact subset $K \subseteq \Gamma$.*

Proof. It is clear that (ii) implies (i). For the other direction, we need to prove uniform boundedness over the compact set K, which we do by using a primæval form of the Banach–Steinhaus theorem.

Theorem A.2 (Banach–Steinhaus). *Suppose that P is a collection of continuous functions $\{p_\lambda : X \to \mathbb{R}\}_{\lambda \in \Lambda}$, and that X is of second category. Then if each cross-section $\lambda \to p_\lambda(x)$ is bounded, there is a nonempty open set $B_0 \subset X$ with $p_\lambda(x)$ uniformly bounded for all $\lambda \in \Lambda$ and $x \in B_0$.*

The above theorem has a standard proof, in consequence of the definition of second category. Now we return to the proof of the proposition. Let L_g denote the given operator $T - \alpha_g(T)$. We apply the Banach–Steinhaus theorem to the collection of functions $\{p_y : \Gamma \to \mathbb{R}\}_{y \in E \setminus \{0\}}$ defined by $p_y(g) = \|L_g y\| / \|y\|$. These functions are continuous because of hypothesis (i), and the cross-sections over $y \in E \setminus \{0\}$, with $g \in \Gamma$ fixed, are bounded by $\|L_g\|$. A locally compact Hausdorff space is a Baire space, so we conclude that there is an open neighborhood $B_0 \subset \Gamma$ with $\|L_g\|$ uniformly bounded on B_0.

We show that every point in Γ will have a neighborhood upon which $\|L_g\|$ is uniformly bounded. Every point in Γ has a neighborhood given by a left translate kB_0 of B_0, and

$$\|L_{kb}\| = \|\alpha_{kb}(T) - T\| = \|\alpha_k(L_b - L_{k^{-1}})\|.$$

Hence L_{kb} is uniformly bounded for all $b \in B_0$, and we complete the proof by covering a given compact set K with finitely many sets of the form kB_0. ∎

Corollary A.3. *The set of unbounded approximately equivariant Fredholm modules $\Psi_\Gamma(A, B)$ is given by triples (E, ϕ, D) where E is a (graded) Hilbert B-module with Γ-action; $\phi : A \to \mathcal{L}(E)$ is a $*$-homomorphism; and D is an unbounded regular degree one self-adjoint operator on E, such that*

(i) *the function $T - \alpha_g(T)$ is in $\mathcal{L}(C(K, E))$ for every compact subset $K \subseteq \Gamma$;*

(ii) *the operator $(i + D)^{-1}$ is in J_ϕ;*

(iii) *the homomorphism ϕ satisfies $\alpha_g(\phi(a)) - \phi(\alpha_g(a)) \in \mathcal{K}(E)$, and*

(iv) *for all a in some dense subalgebra of A, the commutator $[D, \phi(a)]$ is bounded on the domain of D.*

The notation J_ϕ is defined at the start of Section 3.

Acknowledgments. We thank the referee for drawing our attention to the interesting preprint [27], and for sharing his insights with us.

References

[1] Y. Abdolmaleki, *Equivariant KK-theory and its applications in Index theory*. Ph.D. thesis, University of New Brunswick, Fredericton, 2022

[2] N. Anghel, L^2-index formulae for perturbed Dirac operators. *Comm. Math. Phys.* **128** (1990), no. 1, 77–97 Zbl 0697.58048 MR 1042444

[3] N. Anghel, An abstract index theorem on noncompact Riemannian manifolds. *Houston J. Math.* **19** (1993), no. 2, 223–237 Zbl 0790.58040 MR 1225459

[4] N. Anghel, On the index of Callias-type operators. *Geom. Funct. Anal.* **3** (1993), no. 5, 431–438 Zbl 0843.58116 MR 1233861

[5] M. F. Atiyah and I. M. Singer, The index of elliptic operators. I. *Ann. of Math. (2)* **87** (1968), 484–530 Zbl 0164.24001 MR 236950

[6] M. F. Atiyah and G. B. Segal, The index of elliptic operators. II. *Ann. of Math. (2)* **87** (1968), 531–545 Zbl 0164.24201 MR 236951

[7] S. Baaj and P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), no. 21, 875–878 Zbl 0551.46041 MR 715325

[8] S. Baaj and G. Skandalis, C^*-algèbres de Hopf et théorie de Kasparov équivariante. *K-Theory* **2** (1989), no. 6, 683–721 Zbl 0683.46048 MR 1010978

[9] P. Baum, A. Connes, and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-algebras: 1943–1993 (San Antonio, TX, 1993)*, pp. 240–291, Contemp. Math. 167, American Mathematical Society, Providence, RI, 1994 Zbl 0830.46061 MR 1292018

[10] P. Baum, R. G. Douglas, and M. E. Taylor, Cycles and relative cycles in analytic K-homology. *J. Differential Geom.* **30** (1989), no. 3, 761–804 Zbl 0697.58050 MR 1021372

[11] P. Baum, H. Oyono-Oyono, T. Schick, and M. Walter, Equivariant geometric K-homology for compact Lie group actions. *Abh. Math. Semin. Univ. Hambg.* **80** (2010), no. 2, 149–173 Zbl 1242.19006 MR 2734682

[12] J.-M. Bismut and J. Cheeger, Families index for manifolds with boundary, superconnections, and cones. I. Families of manifolds with boundary and Dirac operators. *J. Funct. Anal.* **89** (1990), no. 2, 313–363 Zbl 0696.53021 MR 1042214

[13] C. Callias, Axial anomalies and index theorems on open spaces. *Comm. Math. Phys.* **62** (1978), no. 3, 213–234 Zbl 0416.58024 MR 507780

[14] M. P. do Carmo, *Riemannian geometry*. Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992 Zbl 0752.53001 MR 1138207

[15] J. E. Gilbert and M. A. M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*. Cambridge Stud. Adv. Math. 26, Cambridge University Press, Cambridge, 1991 Zbl 0733.43001 MR 1130821

[16] H. Guo, P. Hochs, and V. Mathai, Positive scalar curvature and an equivariant Callias-type index theorem for proper actions. *Ann. K-Theory* **6** (2021), no. 2, 319–356 Zbl 1484.53087 MR 4301906

[17] N. Higson, On the relative K-homology theory of Baum and Douglas. unpublished manuscript, 1994

[18] N. Higson, V. Lafforgue, and G. Skandalis, Counterexamples to the Baum–Connes conjecture. *Geom. Funct. Anal.* **12** (2002), no. 2, 330–354 Zbl 1014.46043 MR 1911663

[19] N. Higson and J. Roe, *Analytic K-homology*. Oxford Math. Monogr., Oxford University Press, Oxford, 2000 MR 1817560

[20] G. G. Kasparov, The operator K-functor and extensions of C^*-algebras. *Math. USSR, Izv.* **16** (1981), 513–572 Zbl 0464.46054

[21] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture. *Invent. Math.* **91** (1988), no. 1, 147–201 Zbl 0647.46053 MR 918241

[22] D. Kucerovsky, *Kasparov products in KK-theory and unbounded operators with applications to index theory*. Ph.D. thesis, University of Oxford (Magd.), 1995

[23] D. Kucerovsky, The KK-product of unbounded modules. *K-Theory* **11** (1997), no. 1, 17–34 Zbl 0871.19004 MR 1435704

[24] D. Kucerovsky, A lifting theorem giving an isomorphism of KK-products in bounded and unbounded KK-theory. *J. Operator Theory* **44** (2000), no. 2, 255–275 Zbl 0996.46030 MR 1794819

[25] D. Kucerovsky, A short proof of an index theorem. *Proc. Amer. Math. Soc.* **129** (2001), no. 12, 3729–3736 Zbl 0982.58015 MR 1860509

[26] B. Mesland and M. H. Şengün, Hecke operators in KK-theory and the K-homology of Bianchi groups. *J. Noncommut. Geom.* **14** (2020), no. 1, 125–189 Zbl 1473.11112 MR 4107513

[27] K. Thomsen, Excision in equivariant KK-theory. 2001, preprint, Univ. of Århus

[28] K. Thomsen, Equivariant KK-theory and C^*-extensions. *K-Theory* **19** (2000), no. 3, 219–249 Zbl 0945.19002 MR 1756259

Received 26 December 2021

Yavar Abdolmaleki
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, Canada;
yavar.abdolmaleki@unb.ca

Dan Kucerovsky
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, Canada;
dkucerov@unb.ca