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Higher Kazhdan projections, ℓ_2-Betti numbers and Baum–Connes conjectures

Kang Li, Piotr W. Nowak, and Sanaz Pooya

Abstract. We introduce higher-dimensional analogs of Kazhdan projections in matrix algebras over group C^*-algebras and Roe algebras. These projections are constructed in the framework of cohomology with coefficients in unitary representations and in certain cases give rise to non-trivial K-theory classes. We apply the higher Kazhdan projections to establish a relation between ℓ_2-Betti numbers of a group and surjectivity of different Baum–Connes type assembly maps.

Kazhdan projections are certain idempotents whose existence in the maximal group C^*-algebra $C^*_{\max}(G)$ of a group G characterizes Kazhdan's property (T) for G. They have found an important application in higher index theory, as the non-zero K-theory class represented by a Kazhdan projection in $K_0(C^*_{\max}(G))$ obstructs the Dirac-dual Dirac method of proving the Baum–Connes conjecture. This idea has been used effectively in the coarse setting [15,16], where Kazhdan projections were used to construct counterexamples to the coarse Baum–Connes conjecture and to the Baum–Connes conjecture with coefficients. However, Kazhdan projections related to property (T) are not applicable to the classical Baum–Connes conjecture (with trivial coefficients), as the image of a Kazhdan projection in the reduced group C^*-algebra vanishes for any infinite group.

The goal of this work is to introduce higher-dimensional analogs of Kazhdan projections with the motivation of applying them to the K-theory of group C^*-algebra and to higher index theory. These higher Kazhdan projections are defined in the context of higher cohomology with coefficients in unitary representations and they are elements of matrix algebras over group C^*-algebras. Their defining feature is that their image in certain unitary representations is the orthogonal projection onto the harmonic n-cochains. We show that the higher Kazhdan projections exist in many natural cases and that they are non-zero.

We then investigate the classes represented by the higher Kazhdan projection in the K-theory of group C^*-algebras. This is done by establishing a connection between the properties of the introduced projections and ℓ_2-Betti numbers of the group. In particular, we can conclude that the higher Kazhdan projections over the reduced group C^*-algebra

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can be non-zero and can give rise to non-zero K-theory classes in $K_0(C^*_r(G))$. One of the first cases in which this happens is the group $\mathrm{SL}(2,\mathbb{Z})$, discussed in Section 1.5.1. In contrast, the classical Kazhdan projection always vanishes in $C^*_r(G)$ for every infinite group G.

To present the connections between the existence of higher Kazhdan projections and ℓ_2-invariants, we recall that, given a finitely generated group G, the subring $\Lambda^G \subseteq \mathbb{Q}$ is generated from $\mathbb{Z}$ by adjoining inverses of all orders of finite subgroups, as in [23, Theorem 0.3]. The cohomological Laplacian in degree n can be represented by a matrix with coefficients in the group ring $\mathbb{R}G$, denoted by Δ_n. In the case of the regular representation λ and the reduced group C^*-algebra $C^*_r(G)$, the operator $\Delta_n \in \mathbb{M}_k(C^*_r(G))$, for a certain $k \in \mathbb{N}$, is the cohomological Laplacian in degree n in the ℓ_2-cohomology of G. We will say that Δ_n has a spectral gap in a chosen completion of $\mathbb{M}_k(\mathbb{C}G)$ if its spectrum is contained in $\{0\} \cup [\varepsilon, \infty)$ for some $\varepsilon > 0$. Recall that a group G is of type F_n if it has an Eilenberg–MacLane space with a finite n-skeleton. Denote by k_n the number of n-simplices (or n-cells) in the chosen model of the Eilenberg–MacLane space.

Let $\beta^n_{(2)}(G)$ denote the n-th ℓ_2-Betti number of G. The following proposition is a consequence of the existence of the spectral gap for the Laplacian, which then ensures the existence of a higher Kazhdan projection $p_n \in \mathbb{M}_{k_n}(C^*_r(G))$, and the relation

$$\beta^n_{(2)}(G) = \tau_*([p_n]),$$

where τ_* is the canonical trace on $K_0(C^*_r(G))$, as in [23].

Proposition 1. *Let G be of type F_{n+1}. Assume that $\Delta_n \in \mathbb{M}_{k_n}(C^*_r(G))$ has a spectral gap. If the Baum–Connes assembly map $K^G_0(\underline{E}G) \to K_0(C^*_r(G))$ is surjective, then*

$$\beta^n_{(2)}(G) \in \Lambda^G.$$

In particular, if G is torsion-free, then $\beta^n_{(2)}(G) \in \mathbb{Z}$.

The above theorem illustrates a possible strategy for finding counterexamples to the Baum–Connes conjecture: a group G of type F_{n+1} such that $\beta^n_{(2)}(G) \notin \Lambda_G$ (in particular, if $\beta^n_{(2)}(G)$ is irrational) with 0 isolated in the spectrum of Δ_n will not satisfy the Baum–Connes conjecture. More precisely, the Baum–Connes assembly map for G will not be surjective. We refer to [12, 28] for an overview of the Baum–Connes conjecture. It is worth noting that such a counterexample, with bounded orders of finite subgroups, would also not satisfy the Atiyah conjecture, see, e.g., [22].

Our primary application is the construction of classes in the K-theory of the Roe algebra of a box space of a residually finite group G and their relation to the Lück approximation theorem. Let $\{N_i\}$ be a decreasing sequence of finite index normal subgroups of a residually finite group G, satisfying

$$\bigcap N_i = \{e\}.$$

Let λ_i denote the associated quasi-regular representation of G on $\ell_2(G/N_i)$ and consider the family $\mathcal{N} = \{\lambda, \lambda_1, \lambda_2, \ldots\}$ of unitary representations of G. The C^*-algebra $C^*_{\mathcal{N}}(G)$

is the completion of the group ring $\mathbb{C}G$ with respect to the norm induced by the family $\mathcal{N}$ (see Section 1.1 for a precise definition). We will denote by $\beta^n(G) = \dim_{\mathbb{C}} H^n(G, \mathbb{C})$ the standard n-th Betti number of G.

Theorem 2. *Let G be an exact, residually finite group of type F_{n+1} and let k_n, $\{N_i\}$ and $\mathcal{N}$ be as above. Assume that $\Delta_n \in \mathbb{M}_{k_n}(C^*_{\mathcal{N}}(G))$ has a spectral gap and that the coarse Baum–Connes assembly map $KX_0(Y) \to K_0(C^*(Y))$ for the box space $Y = \coprod G/N_i$ of G is surjective, then*

$$\beta^n_{(2)}(N_i) = \beta^n(N_i)$$

for all but finitely many i.

Note that the conclusion of the above theorem in fact is a strengthening of Lück's approximation theorem [20]. Indeed, the expression in the formula in the above theorem can be rewritten as

$$[G : N_i]\left(\beta^n_{(2)}(G) - \frac{\beta^n(N_i)}{[G : N_i]}\right) = 0.$$

Theorem 2 thus forces a much stronger equality of the involved Betti numbers. On the other hand, as shown in [24, Theorem 5.1] there are examples where the speed of convergence of $\beta^n(N_i)/[G : N_i]$ to $\beta^n_{(2)}(G)$ can be as slow as needed.

Theorem 2 in particular provides new strategies for contradicting the coarse Baum–Connes conjecture. It is an important open question whether there exist such counterexamples that do not contain expander graphs. In Section 3 we present several examples of spaces to which our techniques apply, in particular ones for which higher cohomology can be used to deduce that the coarse Baum–Connes conjecture fails for them. These examples are constructed from spaces which contain expanders and thus do not provide new counterexamples to the conjecture. We do expect however, as expressed in Conjecture 21, that there are examples of higher-dimensional expanders to which our techniques apply and which additionally would not contain expander graphs, thus providing essentially new counterexamples to the coarse Baum–Connes conjecture.

On the other hand, it is also natural to conjecture that Theorem 2 could also lead to new counterexamples to the Baum–Connes conjecture with coefficients by embedding a counterexample to the coarse Baum–Connes conjecture, obtained through Theorem 2, isometrically into a finitely generated group, as in [25]. Such constructions could lead to new counterexample to the Baum–Connes conjecture with coefficients, by applying arguments similar to the ones in [16].

Finally, we remark that algebraic conditions implying the existence of gaps in the spectrum of the operators arising from the cochain complexes with coefficients in unitary representations were recently provided in [4]. Those conditions involve writing the elements

$$(\Delta_n^+ - \varepsilon\Delta_n^+)\Delta_n^+ \quad \text{and} \quad (\Delta_n^- - \varepsilon\Delta_n^-)\Delta_n^-,$$

where

$$\Delta_n^+ = d_n^* d_n \quad \text{and} \quad \Delta_n^- = d_{n-1} d_{n-1}^*$$

are matrices over $\mathbb{R}G$ and the two summands of the Laplacian, as sums of squares in $\mathbb{M}_k(\mathbb{R}G)$. Such conditions can be verified using computational methods.

1. Higher Kazhdan projections

Recall that a group has type F_n if it admits an Eilenberg–MacLane space $K(G,1)$ with a finite n-skeleton. The condition F_1 is equivalent to G being finitely generated, and F_2 is equivalent to G having a finite presentation.

1.1. Kazhdan projections related to property (T)

We begin with revisiting the classical notion of Kazhdan projections in the context of property (T). Let G be a finitely generated group with $S = S^{-1}$ a fixed generating set. The real (respectively, complex) group ring of G will be denoted by $\mathbb{R}G$ (respectively, $\mathbb{C}G$). Consider a family $\mathcal{F}$ of unitary representations and let the associated group C^*-algebra be the completion

$$C^*_{\mathcal{F}}(G) = \overline{\mathbb{C}G}^{\|\cdot\|_{\mathcal{F}}},$$

where

$$\|f\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \|\pi(f)\|.$$

We will always assume that $\mathcal{F}$ is a faithful family (i.e., for every $0 \neq f \in \mathbb{C}G$ we have $\pi(f) \neq 0$ for some $\pi \in \mathcal{F}$) to ensure that $\|\cdot\|_{\mathcal{F}}$ is a norm. In particular, we obtain the maximal group C^*-algebra $C^*_{\max}(G)$ if $\mathcal{F}$ is the family of all unitary representations, and the reduced group C^*-algebra $C^*_r(G)$ if $\mathcal{F} = \{\lambda\}$, where λ is the left regular representation.

The following is a classical characterization of property (T) for G due to Akemann and Walter [1]. We will use this characterization as a definition and refer to [5] for a comprehensive overview of property (T).

Theorem 3 (Akemann–Walter [1]). *The group G has property (T) if and only if there exists a projection*

$$p_0 = p_0^* = p_0^2 \in C^*_{\max}(G)$$

with the property that for every unitary representation π of G the image $\pi(p_0)$ is the orthogonal projection onto the subspace of invariant vectors of π.

See, e.g., [17, Section 3.7] and [10] for a broader discussion of Kazhdan projections. We will now interpret Kazhdan projections in the setting of group cohomology. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Denote by d_0 the $\#S \times 1$ matrix

$$\begin{bmatrix} 1 - s_1 \\ \vdots \\ 1 - s_n \end{bmatrix},$$

where s_i runs through the elements of S, with coefficients in $\mathbb{R}G$ and the Laplacian $\Delta_0 \in \mathbb{R}G$ is

$$\Delta_0 = d_0^* d_0 = 2\Big(\#S - \sum_{s\in S} s\Big) \in \mathbb{R}G.$$

For any unitary representation π of G on a Hilbert space $\mathcal{H}$, we have

$$\ker \pi(\Delta_0) = \ker \pi(d_0) = \mathcal{H}^\pi,$$

where $\mathcal{H}^\pi \subseteq \mathcal{H}$ denotes the closed subspace of invariant vectors of π. Note also that the same space can be interpreted as reduced[1] cohomology in degree 0:

$$\ker \pi(\Delta_0) = \mathcal{H}^\pi \simeq \overline{H}^0(G,\pi).$$

The image $\pi(p_0)$ of the Kazhdan projection is the orthogonal projection

$$C^0(G,\pi) \xrightarrow[\pi(p_0)]{} \ker \pi(\Delta_0).$$

The Kazhdan projection p_0 exists in $C^*_{\mathcal{F}}(G)$ if and only if $\pi(\Delta_0)$ has a uniform spectral gap for all $\pi \in \mathcal{F}$. The projection p_0 is non-zero if at least one $\pi \in \mathcal{F}$ has a non-zero invariant vector.

1.2. Kazhdan projections in higher degree

We will now discuss a generalization of Kazhdan projections in the setting of higher group cohomology. We will work in the setting of simplicial complexes however, all of our considerations can be carried out equally in the setting of CW-complexes. Let G be a group of type F_{n+1} with a chosen model X of $K(G,1)$ with finite $n+1$-skeleton.

We can consider the cochain complex for cohomology of G with coefficients in π, where

$$C^n(G,\pi) = \{f : X^{(n)} \to \mathcal{H}\}.$$

By $\mathbb{M}_{m\times m'}(\mathbb{R}G) = \mathbb{M}_{m\times m'} \otimes \mathbb{R}G$ we denote the space of $m \times m'$ matrices with coefficients in the group ring $\mathbb{R}G$. When $m = m'$ the resulting algebra is denoted by $\mathbb{M}_m(\mathbb{R}G)$. The codifferentials can be represented by elements

$$d_n \in \mathbb{M}_{k_{n+1}\times k_n}(\mathbb{R}G),$$

where k_i denotes the number of i-simplices in X, in the sense that for every unitary representation π the codifferential is given by the operator

$$\pi(d_n) : C^n(G,\pi) \to C^{n+1}(G,\pi),$$

where π is applied to an element of $\mathbb{M}_{m\times m'}(C^*_{\mathcal{F}}(G))$ entry-wise.

[1] Recall that in a setting where the cochain spaces in a cochain complex $C^n \xrightarrow[d_n]{} C^{n+1}$ are equipped with a Hilbert space structure coming from a given unitary representation (G,π), the corresponding reduced cohomology is defined as the quotient $\overline{H}^n(G,\pi) = \ker d_n / \overline{\operatorname{im} d_{n-1}}$, where $\overline{\operatorname{im} d_{n-1}}$ is the closure of the image of the codifferential d_{n-1}. Cohomology is said to be reduced in degree n if $\overline{H}^n = H^n$.

The Laplacian element in degree n is defined to be

$$\Delta_n = d_n^* d_n + d_{n-1} d_{n-1}^* \in \mathbb{M}_{k_n}(\mathbb{R}G),$$

and we denote the two summands by

$$\Delta_n^+ = d_n^* d_n, \quad \Delta_n^- = d_{n-1} d_{n-1}^*.$$

In particular, Δ_n, Δ_n^+, and Δ_n^- are elements of any completion of $\mathbb{M}_{k_n}(\mathbb{R}G)$ such as $\mathbb{M}_{k_n}(C^*_{\max}(G))$ or $\mathbb{M}_{k_n}(C^*_r(G))$ (see [4, Section 3]).

More generally, for unitary representation π of G we can now consider the operator $\pi(\Delta_n) \in \overline{\pi(\mathbb{R}G)}$ (we will usually skip the reference to π if it will be clear from the context, as for instance in the previous paragraph). The kernel of $\pi(\Delta_n)$ is the space of harmonic n-cochains for π, and we have the standard Hodge–de Rham isomorphism

$$\ker \pi(\Delta_n) \simeq \overline{H}^n(G, \pi),$$

between that kernel and the reduced cohomology of G with coefficients in π (see, e.g., [4, Section 3]).

Similarly, the kernel of the projection $\pi(\Delta_n^+)$ is the n-cocycles for π, and the kernel of $\pi(\Delta_n^-)$ is the n-cycles.

Definition 4 (Higher Kazhdan projections). Let $\mathcal{F}$ be a family of unitary representations of a group G. A Kazhdan projection in degree n is a projection $p_n = p_n^* = p_n^2 \in \mathbb{M}_{k_n}(C^*_{\mathcal{F}}(G))$ such that for every unitary representation $\pi \in \mathcal{F}$ the projection $\pi(p_n)$ is the orthogonal projection $C^n(G, \pi) \to \ker \pi(\Delta_n)$.

A partial Kazhdan projection in degree n is a projection $p_n^+, p_n^- \in \mathbb{M}_{k_n}(C^*_{\mathcal{F}}(G))$, such that for every unitary representation $\pi \in \mathcal{F}$ the projection $\pi(p_n^+)$ (respectively, $\pi(p_n^-)$) is the orthogonal projection onto the kernel of $\pi(d_n)$ (respectively, onto the kernel of $\pi(d_{n-1}^*)$).

In degree 0 in the case when $\mathcal{F}$ is the family of all unitary representations, the projection $p_0 \in C^*_{\max}(G)$ is the classical Kazhdan projection. Clearly, p_n is non-zero if and only if for at least one $\pi \in \mathcal{F}$ we have $\pi(p_n) \neq 0$.

From now on we will shorten the subscript k_n, denoting the number of n-simplices in the chosen $K(G, 1)$, to k, as the dimension will be clear from the context. Similarly to the case of property (T), higher Kazhdan projections exist in the presence of a spectral gap.

Proposition 5. *Assume that $\Delta_n \in \mathbb{M}_k(C^*_{\mathcal{F}}(G))$ (respectively, Δ_n^+, Δ_n^-) has a spectral gap. Then the Kazdhan projection p_n (respectively, the partial Kazdhan projections p_n^+, p_n^-) exists in the C^*-algebra $\mathbb{M}_{k_n}(C^*_{\mathcal{F}}(G))$.*

Proof. By assumption, the spectrum of Δ_n satisfies

$$\sigma(\pi(\Delta_n)) \subseteq \{0\} \cup [\varepsilon, \infty)$$

for every unitary representation $\pi \in \mathcal{F}$ and some $\varepsilon > 0$. In particular, $\sigma(\Delta_n) \subseteq \{0\} \cup [\varepsilon, \infty)$ when Δ_n is viewed as an element of $\mathbb{M}_k(C^*_{\mathcal{F}}(G))$.

Due to the presence of the spectral gap, we can apply continuous functional calculus and conclude that the limit of the heat semigroup

$$\lim_{t\to\infty} e^{-t\Delta_n}$$

is the spectral projection $p_n \in \mathbb{M}_k(C^*_{\mathcal{F}}(G))$. By construction, p_n has the required properties.

The proof for the partial projections is analogous. ■

Note that the fact that the spectrum of $\Delta_n \in \mathbb{M}_k(C^*_{\mathcal{F}}(G))$ has a gap implies that both Δ_n^+ and Δ_n^- also have spectral gaps and in particular all three higher Kazhdan projections exist in $\mathbb{M}_k(C^*_{\mathcal{F}}(G))$. We have the following decomposition for higher Kazhdan projections.

Lemma 6. *Assume that $\Delta_n \in \mathbb{M}_k(C^*_{\mathcal{F}}(G))$ has a spectral gap. Then*

$$p_n = p_n^+ p_n^-.$$

Proof. It suffices to notice that Δ_n^+ and Δ_n^- commute in $\mathbb{M}_k(C^*_{\mathcal{F}}(G))$ and satisfy

$$\Delta_n^+ \Delta_n^- = \Delta_n^- \Delta_n^+ = 0,$$

using the property that $d_i d_{i-1} = 0$ for every $i \in \mathbb{N}$. Consequently, the corresponding spectral projections commute and their product is the projection onto the intersections of their kernels, which is precisely the kernel of Δ_n. ■

1.3. *K*-theory classes – a special case

In this section we consider the case when $\mathcal{F} = \{\lambda\}$, where λ is the left regular representation of G on $\ell_2(G)$ and $C^*_{\mathcal{F}}(G) = C^*_r(G)$ is the reduced group C^*-algebra of G. We refer to [17, 28] for background on K-theory and assembly maps.

The projection $p_n \in \mathbb{M}_k(C^*_r(G))$ exists if $\lambda(\Delta_n)$ has a spectral gap at 0. In this case the kernel of the Laplacian $\lambda(\Delta_n)$ is isomorphic to the reduced ℓ_2-cohomology of G [22] (see, e.g., [18, Proposition 3.23]):

$$\ker \lambda(\Delta_n) \simeq \ell_2\overline{H}^n(G) \simeq \ker \lambda(d_n) \big/ \overline{\operatorname{im} \lambda(d_{n-1})},$$

and it is non-trivial if and only if the n-th ℓ_2-Betti number $\beta^n_{(2)}(G)$ is non-zero.

The projection $p_n \in \mathbb{M}_k(C^*_r(G))$ gives rise to K-theory class $[p_n] \in K_0(C^*_r(G))$. We are interested in the properties of the classes defined by the higher Kazhdan projections $[p_n] \in K_0(C^*_r(G))$.

Recall that the canonical trace τ_G on $C^*_r(G)$ is given by

$$\tau_G(\alpha) = \langle \alpha\delta_e, \delta_e \rangle,$$

where $\delta_e \in \ell_2(G)$ is the Dirac delta at the identity element $e \in G$. For an element $\alpha \in \mathbb{C}G$ we have $\tau(\alpha) = \alpha(e)$. The induced trace on $\mathbb{M}_k(C_r^*(G))$ is defined as

$$\tau_{k,G}((a_{ij})) = \sum_{i=1}^{k} \tau_G(a_{ii}),$$

and there is an induced map

$$\tau_* : K_0(C_r^*(G)) \to \mathbb{R}$$

defined by $\tau_*([p]) = \tau_{k,G}(p)$, for a class represented by a projection $p \in \mathbb{M}_k(C_r^*(G))$.

Proposition 7. *Let G be a group of type F_{n+1}. Assume that Δ_n has a spectral gap in $\mathbb{M}_k(C_r^*(G))$, then*

$$\tau_*([p_n]) = \beta_{(2)}^n(G).$$

In particular, if $\beta_{(2)}^n(G) \neq 0$, then

$$[p_n] \neq 0 \quad \text{in } K_0(C_r^*(G)).$$

Proof. By definition, the n-th ℓ_2 Betti number of G is the von Neumann dimension (over the group von Neumann algebra $L(G)$) of the orthogonal projection onto the kernel of the Laplacian. Under our assumptions the projection is now an element of the smaller algebra $\mathbb{M}_k(C_r^*(G)) \subseteq \mathbb{M}_k(L(G))$ and

$$\beta_{(2)}^n(G) = \tau_{k,G}(p_n).$$

Therefore, the induced trace on the group $K_0(C_r^*(G))$ satisfies $\tau_*([p_n]) = \beta_{(2)}^n(G)$, as claimed. ∎

A similar argument shows non-vanishing of the classes represented by the partial projections.

Proposition 8. *Assume that 0 is isolated in the spectrum of Δ_n^+ (respectively, Δ_n^-) in $\mathbb{M}_k(C_r^*(G))$. If $\beta_{(2)}^n(G) \neq 0$, then $[p_n^+] \neq 0$ (respectively, $[p_n^-] \neq 0$) in $K_0(C_r^*(G))$.*

Proof. By assumption p_n^+ exists in $\mathbb{M}_k(C_r^*(G))$ and p_n exists in $\mathbb{M}_k(L(G))$ and we will compare the values of the trace over the latter algebra. Since $\tau_{k,G}(p_n) \neq 0$ and

$$\operatorname{im}(p_n) \subseteq \operatorname{im}(p_n^+)$$

by monotonicity of the von Neumann dimension over $L(G)$, we have

$$\tau_{k,G}(p_n^+) \geq \tau_{k,G}(p_n) > 0.$$

Consequently,

$$\tau_*([p_n^+]) \neq 0 \quad \text{in } K_0(C_r^*(G)).$$ ∎

1.4. The Baum–Connes assembly map

The Baum–Connes assembly map is the map

$$\mu_i : K_i^G(\underline{E}G) \to K_i(C_r^*(G)), \quad i = 0, 1.$$

The trace conjecture of Baum and Connes and the modified trace conjecture of Lück [23] predict the range of the composition $\tau_* \circ \mu_0$. In particular, the modified trace conjecture, formulated and proved by Lück [23], states that $\tau_* \circ \mu_i$ takes values in a subring $\Lambda^G \subseteq \mathbb{Q}$, generated from $\mathbb{Z}$ by adjoining the inverses of cardinalities of finite subgroups. See [12, 28, 30] for more details on this conjecture.

Theorem 9 (Lück [23, Theorem 0.3]). *If the Baum–Connes assembly map is surjective, then the composition $\tau_* \circ \mu_0$ takes values in the ring $\Lambda^G \subseteq \mathbb{Q}$.*

We can now state the relation between the Baum–Connes conjecture and ℓ_2-Betti numbers.

Proposition 1. *Let G be of type F_{n+1}. Assume that 0 is isolated in the spectrum of $\Delta_n \in \mathbb{M}_k(C_r^*(G))$. If the Baum–Connes assembly map $K_0^G(\underline{E}G) \to K_0(C_r^*(G))$ is surjective, then*

$$\beta_{(2)}^n(G) \in \Lambda^G.$$

In particular, if G is torsion-free then $\beta_{(2)}^n(G) \in \mathbb{Z}$.

Remark 10 (The Euler class). Consider an infinite group G such that $K(G, 1)$ can be chosen to be a finite complex. Assume that $\lambda(\Delta_n)$ has a spectral gap in $\mathbb{M}_k(C_r^*(G))$ for every $n \geq 0$ (in particular, G is non-amenable). Then we can define the Euler class

$$\Xi(G) = \sum_{i \geq 0} (-1)^i [p_i]$$

in the K-theory $K_0(C_r^*(G))$ so that

$$\tau_*(\Xi(G)) = \chi_{(2)}(G)$$

is the ℓ_2-Euler characteristic of the chosen $K(G, 1)$.

Atiyah's L_2-index theorem associates the L_2-index, analogous to the Fredholm index but defined via the trace on the von Neumann algebra, to a lift $\widetilde{D}$ of an elliptic operator D on a compact manifold M to the universal cover of M. Loosely speaking, in the presence of the spectral gap the L_2-indices of certain operators appearing in the Atiyah L_2-index theorem are in fact traces of their Baum–Connes indices.

1.5. Examples

In case of an infinite group G, the projection $\lambda(p_0) \in C_r^*(G)$ is always 0 since the kernel of $\lambda(\Delta_0)$ consists precisely of constant functions in $\ell_2(G)$.

We will now discuss certain cases in which higher Kazhdan projections exist. We will use the following fact, which is a reformulation of [4, Proposition 16, (2) and (3)].

Lemma 11. *Let π be a unitary representation of G. The Laplacian $\pi(\Delta_n)$ has a spectral gap around 0 if and only if the cohomology groups $H^n(G,\pi)$ and $H^{n+1}(G,\pi)$ are both reduced.*

Sketch of proof. Using the notation of [4], given a cochain complex

$$\cdots \to C^{n-1} \to C^n \to C^{n+1} \to \cdots ,$$

where the C^n have Hilbert space structures and the codifferentials $d_n : C^n \to C^{n+1}$ are bounded operators, we have $\Delta_n = d_n^* d_n + d_{n-1} d_{n-1}^*$. We have an orthogonal decomposition

$$C^n = C_n^+ \oplus \ker \Delta_n \oplus C_n^-,$$

where $C_n^- = \ker d_n \cap \ker(d_{n-1}^*)^\perp$ and $C_n^+ = \ker d_{n-1}^* \cap (\ker d_n^*)^\perp$. Then the restrictions of $d_n^* d_n$ and of $d_{n-1} d_{n-1}^*$ are invertible on C_n^+ and C_n^-, respectively if and only if the cohomology groups H^n and H^{n+1} are reduced [4]. Clearly, the first condition is equivalent to the fact that 0 is isolated in the spectrum of the Laplacian Δ_n. ∎

1.5.1. Free groups. Consider the free group F_n on $n \geq 2$ generators. The standard Eilenberg–MacLane space $K(F_n, 1)$ is the wedge $\bigvee_n S^1$ of n circles, whose universal cover is the tree, the Cayley graph of F_n. Then for F_n the projection $p_1 \in \mathbb{M}_k(C_r^*(F_n))$ exists and gives rise to a non-zero class in K-theory $C_r^*(F_n)$.

Indeed, since $n \geq 2$, the free group F_n is non-amenable, we have that the ℓ_2-cohomology is reduced in degree 1 (i.e., the range of the codifferential into the 1-cochains is closed). Since $K(F_n, 1)$ is 1-dimensional, the cohomology in degree 2 is reduced as there are no 2-cochains. We thus have $d_1 = 0$, $p_1^+ = \mathrm{I}$, and $p_1 = p_1^-$. These facts together with Lemma 11 imply that the cohomological Laplacian Δ_1 in degree 1 has a spectral gap.

The ℓ_2-Betti number of a free group F_n on n generators is

$$\beta^1_{(2)}(F_n) = n - 1.$$

By the previous discussion together with Proposition 7,

$$[p_1] \neq 0 \quad \text{in } K_0(C_r^*(F_n)).$$

Moreover, we can identify precisely the class of p_1 in the K-theory group. Indeed, $K_0(C_r^*(F_n))$ is isomorphic to $\mathbb{Z}$ and generated by $1 \in C_r^*(F_n)$. Because of this, the value of the trace determines the class of p_1 to be

$$[p_1] = n - 1 \in \mathbb{Z} \simeq K_0(C_r^*(F_n)).$$

The corresponding Euler class (see Remark 10) in $K_0(C_r^*(F_n))$ is given by

$$\Xi(F_n) = [p_0] - [p_1] = -[p_1].$$

More generally, the argument used in the above example together with [2, Corollary 4.8] gives the following.

Corollary 12. *Let G be a finitely presented group with infinitely many ends such that $H^2(G, \ell_2(G))$ is reduced. Then the projection $p_1(G)$ exists in $\mathbb{M}_{k_n}(C_r^*(G))$ and*

$$[p_1] \neq 0 \quad \text{in } K_0(C_r^*(G)).$$

Relation to the identity. Consider the group $\mathrm{SL}(2,\mathbb{Z})$. This group contains the free group F_2 as a subgroup of index 12. The ℓ_2-cohomology of $\mathrm{SL}(2,\mathbb{Z})$ in degree 1 and 2 is reduced. Indeed, in degree 1 this is because the group is non-amenable; in degree 2 it follows by Shapiro's lemma. Therefore by Lemma 11 the cohomological Laplacian Δ_1 has a spectral gap in degree 1 and the projection p_1 exists in $M_k(C_r^*(\mathrm{SL}(2,\mathbb{Z})))$. Moreover, we have

$$\tau([p_1]) = \beta^1_{(2)}(\mathrm{SL}(2,\mathbb{Z})) = 1/12,$$

which implies that the K-theory class $[p_1] \in K_0(C_r^*(\mathrm{SL}(2,\mathbb{Z})))$ is not a multiple of the identity.

This argument applies to other virtually-free groups with non-integral ℓ^2-Betti numbers.

1.5.2. Kähler groups. The property that the Laplacian has a spectral gap in the analytic setting (i.e., L_2-cohomology defined in terms differential forms) is equivalent to the existence of the spectral gap in the combinatorial setting was shown in [11], [14, Remark 2], see also [22, Theorem 2.68] or [21, Section 8]. This is most often phrased in terms of Novikov–Shubin invariants associated to spectral density functions: spectral gap is equivalent to the Novikov–Shubin invariant being infinite, which is the same as the property that the spectral density function is constant in the neighborhood of 0. Then it is shown that the combinatorial and analytic spectral density functions are dilation equivalent, and in particular, equal in a neighborhood of 0.

Consider now a closed Kähler hyperbolic manifold M and denote $G = \pi_1(M)$. The Laplacian acting on L_2-differential forms on the universal cover $\widetilde{M}$ has a spectral gap in every dimension, as shown by Gromov [13], see also [22, Section 11.2.3]. Gromov also showed that the ℓ_2-cohomology of M vanishes except for the middle dimension.

Consequently, if G is a fundamental group of such a Kähler manifold, then the assumptions of Proposition 7 are satisfied and the projection p_n exists in $\mathbb{M}_{k_n}(C_r^*(G))$ in every dimension.

Moreover, the Euler class

$$\Xi(G) = \sum_{i \geq 0} (-1)^i [p_i] \in K_0(C_r^*(G))$$

exists and its trace is the ℓ_2-Euler characteristic of G.

1.5.3. Lattices in $\mathbf{PGL}_{n+1}(\mathbb{Q}_p)$. Given $n \geq 2$, a sufficiently large prime p and a lattice $\Gamma \subseteq \mathrm{PGL}_{n+1}(\mathbb{Q}_p)$, we have that $H^n(\Gamma, \pi)$ is reduced for every unitary representation π, see [4, Proposition 19]. This fact is equivalent to $\pi(\Delta_n^-)$ having a spectral gap for every unitary representation π of Γ and it is easy to check that such a spectral gap must be then uniform; i.e., Δ_n^+ has a spectral gap in $\mathbb{M}_k(C^*_{\max}(\Gamma))$.

At the same time there exists a finite index subgroup $\Gamma' \subseteq \Gamma$ such that $H^n(\Gamma', \mathbb{C}) \simeq H^n(\Gamma, \ell_2(\Gamma/\Gamma')) \neq 0$.

Finally, since Γ acts on the Bruhat–Tits building (see, e.g., [27]), there are no cells of dimension $n+1$ and the codifferential d_n is 0.

As a consequence we obtain the following.

Proposition 13. *Let $\Gamma \subseteq \mathrm{PGL}_{n+1}(\mathbb{Q}_p)$ be a lattice, then the Kazhdan projection p_n exists in $\mathbb{M}_k(C^*_{\max}(\Gamma))$ and is non-zero.*

2. The coarse Baum–Connes conjecture

We will now prove our main results and describe the K-theory classes induced by the higher Kazhdan projections in Roe algebras and their applications to the coarse Baum–Connes conjecture of a box space of a residually finite group. The arguments we will use were introduced in [15, 16] and developed further in [29]. We also refer to [30, Chapter 13] for more details.

2.1. Roe algebras, box spaces, and Laplacians

2.1.1. Roe algebras. The Roe algebra $C^*(X)$ of a discrete, bounded geometry metric space X is the completion in $B(\ell_2(X;\mathcal{H}_0))$ of the $*$-algebra $\mathbb{C}[X]$ of finite propagation operators, which are locally compact; i.e., for a discrete space the matrix coefficients are compact operators $T_{x,y} \in \mathcal{K}(\mathcal{H}_0)$ on a fixed, infinite-dimensional Hilbert space $\mathcal{H}_0$.

If G acts on X, then the equivariant Roe algebra is defined to be the closure of the subalgebra $\mathbb{C}[X]^G \subseteq \mathbb{C}[X]$ of equivariant finite propagation operators, i.e., satisfying $T_{x,y} = T_{gx,gy}$, for any $g \in G$ and $x, y \in X$.

The uniform Roe algebra $C_u^*(X)$ of X is the completion in $B(\ell_2(X))$ of the $*$-algebra $\mathbb{C}_u[X]$ of finite propagation operators. The equivariant uniform Roe algebra $\mathbb{C}_u^*(X)^G$ is defined as before by considering G-invariant operators in $\mathbb{C}_u[X]$ and taking the closure inside $C_u^*(X)$.

See [29, Definitions 3.2 and 3.6] for a detailed description of the Roe algebra and the equivariant Roe algebra.

2.1.2. Laplacians on box spaces. Consider a finitely generated residually finite group G. Let $\{N_i\}$ be a family of finite index normal subgroups of G satisfying $\bigcap N_i = \{e\}$. Consider the space

$$Y = \coprod G/N_i,$$

viewed as a box space with $d(G/N_i, G/N_j) \geq 2^{i+j}$.

Let λ_i be the quasi-regular representation of G on $\ell_2(G/N_i)$ given by pulling back the regular representation of G/N_i on $\ell_2(G/N_i)$ to G via the quotient map $G \to G/N_i$

and denote by $\mathcal{N}$ the family of representations

$$\mathcal{N} = \{\lambda, \lambda_1, \lambda_2, \dots\}.$$

The standing assumption in this section will be that the cohomological n-Laplacian Δ_n has a spectral gap in $\mathbb{M}_k(C^*_{\mathcal{N}}(G))$. The cohomological Laplacian element $\Delta_n \in \mathbb{M}_k(\mathbb{C}G)$ in degree n maps to $\lambda_i(\Delta_n)$ in $\mathbb{M}_k(\lambda_i(C^*_{\mathcal{N}}(G)))$ and we will denote

$$\Delta_n^i = \lambda_i(\Delta_n).$$

Define the Laplace element of the box space to be

$$D_n = \bigoplus_i \Delta_n^i \in \bigoplus \mathbb{M}_k(\mathbb{C}_u[G/N_i]) \subseteq \mathbb{M}_n(\mathbb{C}_u[Y]).$$

Note that D_n has a spectral gap in $C^*(Y)$ if and only if all the Δ_n^i have a spectral gap over $C^*_{\mathcal{N}}(G)$.

2.2. The lifting map

From now on we will assume that the group G satisfies the operator norm localization property. As shown by Sako [26], the operator norm localization property is equivalent to exactness for bounded geometry metric spaces, in particular for finitely generated groups, so the assumptions of Theorem 2 guarantee this property for G.

Note that we will only be using a special case of the setup described in [29], as our box space consists of a family of finite quotients of a single group G, which naturally serves as a covering space for all of the finite quotients. This setting is the same as the one in [15], however we do follow the detailed version of the arguments as in [29].

Willett and Yu in [29, Lemma 3.8] define a lifting map, a $*$-homomorphism

$$\phi : \mathbb{C}[Y] \to \frac{\prod_i \mathbb{C}[G]^{N_i}}{\bigoplus_i \mathbb{C}[G]^{N_i}},$$

into the algebra of sequences of elements of the N_i-invariant subspaces of $\mathbb{C}[G]$ modulo the relation of being equal at all but finitely many entries.

We have

$$\mathbb{M}_k\left(\frac{\prod_i \mathbb{C}[G]^{N_i}}{\bigoplus_i \mathbb{C}[G]^{N_i}}\right) = \frac{\prod_i \mathbb{M}_k(\mathbb{C}[G]^{N_i})}{\bigoplus_i \mathbb{M}_k(\mathbb{C}[G]^{N_i})}$$

and for every $k \in \mathbb{N}$ the map ϕ induces a $*$-homomorphism

$$\phi^{(k)} : \mathbb{M}_k(\mathbb{C}[Y]) \to \mathbb{M}_k\left(\frac{\prod \mathbb{C}[G]^{N_i}}{\bigoplus \mathbb{C}[G]^{N_i}}\right),$$

by applying ϕ entry-wise.

We recall that in [29, Lemma 3.12] it was shown that if G has the operator norm localization property then the map ϕ extends to a map

$$\phi : C^*(Y) \to \frac{\prod_i C^*(G)^{N_i}}{\bigoplus_i C^*(G)^{N_i}},$$

and similarly induces the corresponding map $\phi^{(n)}$ on the $k \times k$ matrices over these algebras. Here,

$$\frac{\prod_i C^*(G)^{N_i}}{\bigoplus_i C^*(G)^{N_i}}$$

is the algebra of sequences of elements of $C^*(G)^{N_i}$ modulo the relation of being asymptotically equal.

The same formula as the one for ϕ in [29, Lemma 3.8] gives a uniform version of the lifting map,

$$\phi_u : \mathbb{C}_u[Y] \to \frac{\prod_i \mathbb{C}_u[G]^{N_i}}{\bigoplus_i \mathbb{C}_u[G]^{N_i}},$$

which is in the same way a $*$-homomorphism.

If G has the operator norm localization property (see, e.g., [7, 26]), then it also has the uniform version of that property, as proved by Sako [26]. This implies that ϕ_u extends to a $*$-homomorphism

$$\phi_u : C_u^*(Y) \to \frac{\prod_i C_u^*(G)^{N_i}}{\bigoplus_i C_u^*(G)^{N_i}}.$$

See [30, Lemma 13.3.11 and Corollary 13.3.12] for the detailed arguments, which apply verbatim here. Similarly, $\phi_u^{(n)}$ in both cases induces a map on matrices over the respective algebras.

2.3. Surjectivity of the coarse Baum–Connes assembly map

We will now focus on the lift of the Laplacian D_n. In order to do this, we will use an argument based on [29, Lemma 5.6].

Lemma 14. *Let G be an exact group and assume that Δ_n has a spectral gap over $C^*_{\mathcal{N}}(G)$. Denote by*

$$P_n = \bigoplus \lambda_i(p_n) \in \mathbb{M}_k(C_u^*(Y))$$

the spectral projection associated to $D_n \in \mathbb{M}_k(C_u^*(Y))$, then

$$\phi_u^{(k)}(P_n) = \prod_{i=1}^{\infty} \lambda(p_n),$$

where $\lambda(p_n)$ is the projection onto the harmonic n-cochains in the ℓ_2-cohomology of G.

Proof. The projection P_n is the spectral projection of D_n. The Laplacian element D_n defined above lifts to an element

$$\phi_u^{(k)}(D_n) \in \mathbb{M}_k\left(\frac{\prod_{i=1}^{\infty} \mathbb{C}_u[G]^{N_i}}{\bigoplus_{i=1}^{\infty} \mathbb{C}_u[G]^{N_i}}\right),$$

whose spectrum is contained in $\{0\} \cup [\varepsilon, \infty)$ for some $\varepsilon > 0$. Indeed, since the same is true for $D_n \in \mathbb{M}_k(C_u^*(Y))$, by assumption, and spectral gaps are preserved by homomorphisms of unital C^*-algebras.

It is easy to see that from the definition of ϕ in the proof of [29, Lemma 3.8] that the lift of D_n is represented by the constant sequence

$$\phi_u^{(k)}(D_n) = \prod_{i=1}^{\infty} \Delta_n,$$

in the algebra

$$\frac{\prod_{i=1}^{\infty} \mathbb{M}_k(\mathbb{C}G)}{\bigoplus_{i=1}^{\infty} \mathbb{M}_k(\mathbb{C}G)} \subseteq \mathbb{M}_k\left(\frac{\prod_{i=1}^{\infty} \mathbb{C}_u[G]^{N_i}}{\bigoplus_{i=1}^{\infty} \mathbb{C}_u[G]^{N_i}}\right).$$

Indeed, this follows from the fact that the size of the support of Δ_n^i and the formula for the lift $\phi_{(u)}$ are both independent of the particular quotient G/N_i if i is sufficiently large.

Consider now the spectral projection $\widetilde{P}_n$ associated to $\phi_u^{(k)}(D_n)$, then

$$\widetilde{P}_n = \lim_{t\to\infty} e^{-t\phi_u^{(k)}(D_n)} = \lim_{t\to\infty} \phi_u^{(k)}\big(e^{-tD_n}\big) = \phi_u^{(k)}(P_n).$$

Thus the associated spectral projection is of the form

$$\prod_{i=1}^{\infty} \lambda(p_n),$$

where

$$\lambda(p_n) \in \mathbb{M}_k(C_r^*(G))$$

is the projection onto the harmonic n-cochains in the ℓ_2-cochains of G, as claimed. ∎

We now define the *higher Kazhdan projection of the box space Y* to be $P_n \otimes q \in \mathbb{M}_n(C^*(Y))$, where q is any rank one projection on $\mathcal{H}_0$. We consider the associated K-theory class $[P_n \otimes q] \in K_0(C^*(Y))$. We will show that under certain conditions this class is non-zero and does not lie in the image of the coarse Baum–Connes conjecture for Y.

The first thing to notice is that the lift of the projection $P_n \otimes q$ can be described explicitly as an element of $C^*(G)^G$.

Lemma 15. *Let $q \in \mathcal{K}(\mathcal{H}_0)$, then*

$$\phi(P_n \otimes q) = \phi_u(P_n) \otimes q.$$

Proof. Let α_i be a sequence of finite propagation operators such that $P_n = \lim_i \alpha_i$, then

$$\phi(\alpha_i \otimes q) = \phi_u(\alpha_i) \otimes q,$$

by the definition of ϕ ([29, Lemma 3.8]). Passing to the limit and using continuity of ϕ we obtain the claim. ∎

Following [15, 29] define the map

$$d_* : K_0(C^*(Y)) \to \frac{\prod \mathbb{Z}}{\bigoplus \mathbb{Z}}$$

by taking the map

$$d : C^*(Y) \to \frac{\prod \mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0))}{\bigoplus \mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0))},$$

defined by

$$A \mapsto \prod_{i=1}^{\infty} Q_i A Q_i,$$

where $Q_i : \ell_2(Y) \to \ell_2(G/N_i)$ is the projection on the i-th box G/N_i, and considering the induced map on K-theory

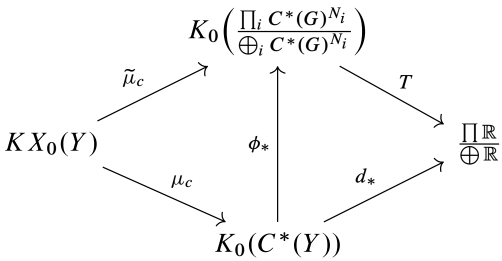
$$d_* : K_0(C^*(Y)) \to K_0\left(\frac{\prod \mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0))}{\bigoplus \mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0))}\right) \simeq \frac{\prod \mathbb{Z}}{\bigoplus \mathbb{Z}}.$$

The above map d_* can recognize when our K-theory class is non-zero. The map d_* applied to the projection P_n gives the sequence of dimensions of images of $\lambda_i(p_n)$.

On the other hand, consider the trace τ_i on $C^*(G)^{N_i} \simeq C^*_r(N_i) \otimes \mathcal{K}$ by considering the tensor product of the standard trace on $C^*_r(N_i)$ with the canonical trace on $\mathcal{K}$. These τ_i induce the map

$$T : K_0\left(\frac{\prod_i C^*(G)^{N_i}}{\bigoplus_i C^*(G)^{N_i}}\right) \to \frac{\prod \mathbb{R}}{\bigoplus \mathbb{R}}.$$

The following diagram was analyzed in [29].



The next lemma was formulated in [29] for projections in $C^*(X)$, however the proof applies equally to projections in $\mathbb{M}_n(C^*(X))$.

Lemma 16 ([29, Lemma 6.5]). *If p is a projection in $\mathbb{M}_n(C^*(Y))$ such that the class $[p] \in K_0(C^*(Y))$ is in the image of the coarse assembly map $\mu_0 : KX_0(Y) \to K_0(C^*(Y))$, then*

$$d_*([p]) = T(\phi_*([p])) \quad \text{in } \frac{\prod \mathbb{R}}{\bigoplus \mathbb{R}}.$$

These maps take values in $\prod \mathbb{R}/\bigoplus \mathbb{R}$ viewed as an object in the category of abelian groups; that is, the equality of the two traces is an equality of coordinates for all but finitely many i (see [30, Section 13.3]). The next statement allows us to show that when it is non-compact, the projection P_n gives rise to a non-zero K-theory class in $K_0(C^*(Y))$.

Proposition 17. *Assume that $H^n(G, \ell_2(G/N_i)) \neq 0$ for infinitely many $i \in \mathbb{N}$. Then $d_*[P_n] \neq 0$.*

Proof. We have

$$d_*(P_n) = \prod_{i=1}^{\infty} \dim \lambda_i(p_n) = \prod_{i=1}^{\infty} \dim H^n(G; \lambda_i).$$

Indeed, since

$$\mathbb{M}_n(C^*(Y)) = C^*(\mathbb{Z}_n \times Y),$$

the map

$$d^{(n)} : \mathbb{M}_n(C^*(X)) \to \frac{\prod \mathbb{M}_n(\mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0)))}{\bigoplus \mathbb{M}_n(\mathcal{K}(\ell_2(G/N_i; \mathcal{H}_0)))}$$

can be rewritten as

$$d : C^*(X \times \mathbb{Z}_n) \to \frac{\prod \mathcal{K}(\ell_2(G/N_i \times \mathbb{Z}_n; \mathcal{H}_0))}{\bigoplus \mathcal{K}(\ell_2(G/N_i \times \mathbb{Z}_n; \mathcal{H}_0))}.$$

By the assumption on the cohomology of G with coefficients in λ_i, the projection p is non-compact in $C^*(Y \times \mathbb{Z}_n)$ and as shown in the proof of [29, Theorem 6.1, p. 1407], we have $d_*[p] \neq 0$. ∎

The next lemma shows that the trace of the lift is naturally related to ℓ_2-Betti numbers.

Lemma 18. *The following holds*

$$T(\phi_*([P_n])) = \prod_{i=1}^{\infty} [G : N_i] \beta^n_{(2)}(G) = \prod_{i=1}^{\infty} \beta^n_{(2)}(N_i).$$

Proof. For a finite index subgroup $N \subseteq G$, the trace τ_N on $C^*(G)^N$ is defined as the tensor product of the canonical trace tr_N on $C^*_r(N)$ with the canonical (unbounded trace) Tr on the compact operators, via the isomorphism

$$\psi_N : C^*(G)^N \underset{\simeq}{\to} C^*_r(N) \otimes \mathcal{K}.$$

The isomorphism ψ_N is defined by considering a fundamental domain $D \subset G$ for $N \subseteq G$ and for $A \in C^*(G)^N$ identifying

$$A \mapsto \sum_{g \in N} u_g \otimes A^{(g)},$$

where $A^{(g)} \in \mathcal{K}(\ell_2(D, \mathcal{H}_0))$ is defined by the formula

$$A^{(g)}_{x,y} = A_{x,gy},$$

for $x, y \in D$.

With this identification we observe that for an element $A \in C^*(G)^G$ of the form $A = \alpha \otimes q$, where $\alpha \in C^*_r(G)$ and q is a rank 1 projection on $\mathcal{H}_0$, we have

$$\begin{aligned}\tau_N(A) &= \tau_N\left(\sum_{g\in N} u_g \otimes A^{(g)}\right)\\ &= \sum_{g\in N} \operatorname{tr}(u_g)\operatorname{Tr}\left(A^{(g)}\right)\\ &= \operatorname{Tr}\left(A^{(e)}\right).\end{aligned}$$

In our case $A^{(e)}$ is a $D \times D$ matrix defined by restricting A to D. Therefore

$$\operatorname{Tr}(A^{(e)}) = \sum_{x\in D} \alpha^{(e)}_{x,x} \cdot \operatorname{rank}(q) = [G : N]\alpha^{(e)}_{e,e}.$$

The same relation passes to traces of matrices over the respective C^*-algebras. In the case of the projection P_n, these formulas yield

$$\tau_N(P_n) = [G : N]\beta^n_{(2)}(G),$$

as claimed. ∎

Before we summarize this discussion, we will observe one more fact that will allow to relate our results to Lück's approximation theorem. Recall that the Betti number of a group G is the number $\beta^n(G) = \dim_{\mathbb{C}} H^n(G, \mathbb{C})$.

Lemma 19. *For a finite index subgroup $N \subseteq G$, we have*

$$\dim_{\mathbb{C}} H^n(G, \ell_2(G/N)) = \beta^n(N).$$

Proof. Since N is of finite index in G, we have

$$\operatorname{CoInd}^G_N \mathbb{C} = \operatorname{Ind}^G_N \mathbb{C} = \ell_2(G/N),$$

see, e.g., [6, Proposition 5.9]. Applying Shapiro's lemma we obtain

$$H^n(G, \ell_2(G/N)) \simeq H^n(G, \operatorname{CoInd}^G_N \mathbb{C}) \simeq H^n(N, \mathbb{C}).$$ ∎

We are now in the position to formulate the main theorem of this section.

Theorem 2. *Let G be an exact, residually finite group of type F_{n+1} and let $\{N_i\}$ and $\mathcal{N}$ be the family of unitary representations defined above. Assume that $\Delta_n \in \mathbb{M}_{k_n}(C^*_{\mathcal{N}}(G))$ has* *a spectral gap and that the coarse Baum–Connes assembly map $KX_0(Y) \to K_0(C^*(Y))$ for the box space $Y = \coprod G/N_i$ of G is surjective, then*

$$\beta^n_{(2)}(N_i) = \beta^n(N_i)$$

for all but finitely many i.

Proof. The claim follows from the explicit computation of the values of both traces in Proposition 17 and Lemmas 18 and 19. ∎

Note that Theorem 2 provides a strengthening of Lück's approximation theorem [20] in the case described by the above theorem. Indeed, we can rewrite the conclusion of Theorem 2 as vanishing of

$$[G : N_i]\left(\beta^n_{(2)}(G) - \frac{\beta^n(N_i)}{[G : N_i]}\right) = 0$$

for all but finitely many i. Compare this with [24, Theorem 5.1], where examples with slow speed of convergence have been constructed. The speed of convergence of Betti numbers of finite quotients to the ℓ_2-Betti number of a residually finite group was also studied in [8], however the techniques used there are different. In our case both the assumptions and the conclusions are stronger.

Remark 20 (Ghost projections). We can extend the notion of a ghost operator to matrix algebras over the Roe algebra by defining an element $T \in \mathbb{M}_n(C^*(Y))$ to be a ghost if the coefficients $T_{i,j} \in C^*(Y)$ are ghost for all $1 \leq i, j \leq n$. It can be shown that the kernel of the lifting map $\phi^{(n)}$ consists precisely of ghost operators. See [30, Corollary 13.3.14]. This observation provides a new cohomological tool to construct ghost projections in the case when ℓ_2-Betti number of G vanishes. The problem of constructing new examples of ghost projection was posed by Willet and Yu [29].

3. Examples

Here we will discuss examples that the above theorem applies to.

3.1. Degree 0

We first recall the counterexample to the coarse Baum–Connes conjecture constructed in [15]. Let G be an infinite, finitely generated, residually finite, and exact group with property (τ) with respect to a family of finite index (normal) subgroup $\{N_i\}$ such that $\bigcap N_i = \{e\}$. Then, as shown in [15] and then in [29], the projection p_0 exists in the Roe algebra of the box space $\square_{i=1}^{\infty} G/N_i$ and

$$d_*([p_0]) = (1, 1, 1, \ldots).$$

At the same time $\beta^0_{(2)}(G) = 0$ as G is infinite. As a consequence, the coarse Baum–Connes assembly map is not surjective.

3.2. Higher degree

Our second example will show that similar phenomena can also appear in higher cohomology and that higher cohomology can also be used to show that the coarse Baum–Connes conjecture fails.

Consider the free group F_2 on 2 generators and let $\{N_i\}_{i=1}^{\infty}$ be a family of finite index normal subgroups such that $\bigcap N_i = \{e\}$ and the family of Cayley graphs of the finite groups $\{F_2/N_i\}$ is a family of expanders.

Now take $F_2 \times F_2$ and the family $\{N_i \times N_i\}$. We will use cohomology in degree 1 to show that the coarse Baum–Connes assembly map is not surjective.

We will work with cellular cohomology. The Eilenberg–McLane space of F_2 is the figure eight space, denoted here by E, with the Cayley graph T of F_2 as its universal cover. Similarly, for $F_2 \times F_2$ the Eilenberg–McLane space is $E \times E$ and $T \times T$ is its universal cover. The cellular structures of these spaces are the obvious ones and are determined by the structure of E given by a 0-cell and two 1-cells attached via identifying the endpoints with the 0-cell.

Denote by λ_i the unitary representation of F_2 on $\ell_2(F_2/N_i)$, for convenience we will also adopt the convention that $\lambda_0 = \lambda$. Observe that for the group F_2, each of the families of operators $\lambda_i(\Delta_0)$ and $\lambda_i(\Delta_1)$ has uniform spectral gaps, since in the cochain complex

$$0 \xrightarrow[\lambda_i(d_{-1})=0]{} C^0(F_2, \lambda_i) \xrightarrow[\lambda_i(d_0)]{} C^1(F_2, \lambda_i) \xrightarrow[\lambda_i(d_1)=0]{} 0,$$

we have $\Delta_0 = d_0^* d_0$ and $\Delta_1 = d_0 d_0^*$. By the assumption that G/N_i give rise to expanders graphs, $\lambda_i(d_0)$ have a uniform spectral gap for $i \geq 1$, while the same hold true for $i \geq 0$ by non-amenability of F_2.

Since we are using cellular cohomology and cells in the product are product of cells of lower dimensions, we have a unitary isomorphism between the chain groups $C^n(F_2 \times F_2, \lambda_j \otimes \lambda_j)$ and the groups $\bigoplus_{p+q=n} C^p(F_2, \lambda_i) \otimes C^q(F_2, \lambda_i)$. The codifferential is given by the formula

$$d^n = \bigoplus_{p+q=n} d^p \otimes I + (-1)^p I \otimes d^q.$$

Consequently, the Laplacian is given by the formula

$$\lambda_i(\Delta_n) = \bigoplus_{p+q=n} \lambda_i(\Delta_p) \otimes I + I \otimes \lambda_i(\Delta_q).$$

It follows that the family $\{\lambda_i(\Delta_n)\}_{i\in\mathbb{N}}$ has a uniform spectral gap for every $n = 0, 1, 2$. Consequently, the projection p_1 exists the Roe algebra of the box space

$$Y = \square(F_2 \times F_2)\big/(N_i \times N_i).$$

We now will show that value $d_*(p_1)$ as a sequence is not eventually zero. Indeed, observe that by the classical Lück approximation theorem we have

$$\left| \frac{\beta^1(N_i)}{[F_2 : N_i]} - \beta^1_{(2)}(F_2) \right| \to 0.$$

Since $\beta^1_{(2)}(F_2) = 1$, we derive that $\beta_1(N_i) \to \infty$. Applying the Künneth theorem to the cohomology group $H^1(F_2, \ell_2(F_2/N_i) \simeq H^1(N_i, \mathbb{C}))$, we obtain that dim $H^1(N_i \times N_i, \mathbb{C}) \neq 0$. Since $H^1(N_i \times N_i, \mathbb{C})$ is isomorphic to $H^1(F_2 \times F_2, \ell_2(F_2 \times F_2/N_i \times N_i))$, we obtain the claim.

Finally, we observe that by the Künneth formula for ℓ_2-Betti numbers we have

$$\beta^1_{(2)}(F_2 \times F_2) = 2\beta^0_{(2)}(F_2)\beta^1_{(2)}(F_2) = 0.$$

By Theorem 2 we conclude that the class $[p_1] \in K_0(C^*(Y))$ does not belong to the image of the coarse Baum–Connes conjecture.

The same arguments apply to k-fold Cartesian products of free groups $F_n \times \cdots \times F_n$ and the family $\{N_i \times \cdots \times N_i\}$ of its finite index subgroups. In that case we obtain that the projection p_{k-1} exists in the Roe algebra of the box space

$$\Box F_n \times \cdots \times F_n / N_i \times \cdots \times N_i,$$

$d_*([p_{k-1}]) \neq 0$ but $T \circ \phi_*([p_{k-1}]) = 0$ since the ℓ_2-Betti number $\beta^{k-1}_{(2)}$ of the k-fold Cartesian product $F_n \times \times \cdots F_n$ vanishes.

We remark that since products of expanders are again expanders, the above counterexample also follows from the method using 0-cohomology, as in the previous example.

3.3. High-dimensional expanders

High-dimensional expanders are simplicial complexes that can be viewed as higher-dimensional analogs of expander graphs. There are several approaches to defining high-dimensional expansion, and one of them is via uniform spectral gaps for the Laplacian in simplicial cohomology. In that sense, an n-dimensional spectral expander is a sequence of finite simplicial complexes $\{S_i\}$ such that a family of operators of type Δ^+_k has a uniform spectral gap, see [19] for more details and background.

Our results allow us to propose the following conjecture.

Conjecture 21. High-dimensional spectral expanders do not satisfy the coarse Baum–Connes conjecture.

A similar conjecture can be made about coboundary expanders, which are defined using a spectral gap-type notion in cohomology with coefficients in $\mathbb{Z}_2$, however at the moment such a conjecture would be more speculative.

4. Final remarks

Question 22. Is there a spectral gap characterization of the existence of higher Kazhdan projections over $C^*_{\mathcal{N}}(G)$?

As mentioned earlier, Kazhdan's property (T) for G is characterized by either the vanishing of first cohomology of G with every unitary coefficients, or equivalently, by the fact that the first cohomology of G with any unitary coefficients is always reduced. Related higher-dimensional generalizations of property (T) were discussed in [3], see also [9]. As pointed out in [4], the generalizations of these two conditions to higher degrees are not equivalent. The existence of higher Kazhdan projections is related to the property that cohomology is reduced and to the existence of gaps in the spectrum of the Laplacian rather than to vanishing of cohomology. Indeed, in our reasoning it is crucial that cohomology does not vanish. It would be interesting to determine if the existence of higher Kazhdan projections can be viewed as a higher-dimensional rigidity property.

Remark 23 (Higher Kazhdan projections and K-amenability). Clearly, if for a particular G we have at the same time that $\beta^n_{(2)}(G) = 0$ and $\pi(\Delta_n)$ has a non-trivial kernel and a spectral gap for some $\pi \neq \lambda$, then G cannot be amenable.

Consider the map

$$K_*(C^*_{\max}(G)) \to K_*(C^*_r(G)).$$

If $[p_n] \neq 0$ in the former, but $[p^r_n] = 0$ in the latter, then the map above cannot be an isomorphism. In other words, if $\beta^n_{(2)}(G) = 0$ and we could ensure that the class of $[p_n]$ is not 0, then the group G would not be K-amenable, as this last condition forces the above map in K-theory to be an isomorphism.

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Kang Li
Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstraße 11, 91058 Erlangen, Germany; kang.li@fau.de

Piotr W. Nowak
Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-656 Warszawa, Poland; pnowak@impan.pl

Sanaz Pooya
Department of Mathematics, Stockholm University, 106 91 Stockholm, Sweden; current address: Institute of Mathematics, University of Potsdam, Campus Golm, Haus 9 Karl-Liebknecht-Str. 24-25, D-14476 Potsdam OT Golm, Germany; sanaz.pooya@uni-potsdam.de