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Chern classes of quantizable coisotropic bundles

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Abstract. Let M be a smooth algebraic variety of dimension $2(p+q)$ with an algebraic symplectic form and a compatible deformation quantization of the structure sheaf. Consider a smooth coisotropic subvariety Y of codimension q and a vector bundle E on Y. We show that if the pushforward of E admits a deformation quantization (as a module), then its "trace density" characteristic class lifts to a cohomology group associated to the null foliation of Y. Moreover, it can only be nonzero in degrees $2q, \dots, 2(p+q)$. For Lagrangian Y, this reduces to a single degree $2q$. Similar results hold in the holomorphic category.

1. Introduction

Let (M, ω) be a smooth variety of dimension $2(p+q)$ over a field k of characteristic zero, with an algebraic symplectic form ω (or corresponding holomorphic objects over $k = \mathbb{C}$). We assume that the structure sheaf $\mathcal{O}_M$ admits a compatible deformation quantization $\mathcal{O}_h$ and fix a choice of such quantization. In other words, $\mathcal{O}_h$ is a sheaf (in the Zariski or analytic topology, respectively) of complete, separated, and flat $k[[h]]$ algebras such that $\mathcal{O}_h / h\mathcal{O}_h \simeq \mathcal{O}_M$ and if $a \mapsto a_0$ is the quotient map, then

$$a * b - b * a = hP(da_0, db_0) \pmod{h^2},$$

where $P \in H^0(M, \Lambda^2 T_M)$ is the Poisson bivector corresponding to the symplectic form ω under the isomorphism $T_M \to \Omega^1_M$, $v \mapsto \iota_v(\omega)$.

Consider a coherent sheaf E_h of $\mathcal{O}_h$-modules which is complete, separated, and flat over $k[[h]]$. See [18] for a general overview of modules over deformation quantization. We view E_h as a quantization of its "principal symbol" $\sigma(E_h) = E_h / hE_h$, a coherent sheaf of $\mathcal{O}_M$-modules. A broad, but difficult, question is to establish necessary and sufficient conditions which would imply the existence of E_h. One way to simplify the situation is to assume that the support $Y \xrightarrow{j} M$ of E_h is smooth, and in fact E_h / hE_h is a direct image $j_* E$ of a locally free sheaf E on Y.

A straightforward observation, which we recall below, is that in this case Y should be coisotropic; i.e., if N is the normal bundle of Y in M, then the projection of P to $H^0(Y, \Lambda^2 N)$ should be zero. Then $p = \frac{1}{2}(\dim M - 2 \operatorname{codim} Y)$ is a non-negative integer.

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When $p = 0$, i.e., Y is Lagrangian, papers [1,3] establish necessary and sufficient conditions for existence of E_h. First, the associated projective bundle $\mathbb{P}(E)$ on Y should admit a flat algebraic connection. In particular, the Chern character of E equals $e \cdot \exp(c_1(E))$ with $e = rk\ E$. To formulate the remaining conditions, recall that by [10] a choice of $\mathcal{O}_h$ induces the Deligne–Fedosov class $c(\mathcal{O}_h) \in \frac{1}{h} H^2_{DR}(X)[[h]]$ of the form

$$c(\mathcal{O}_h) = \frac{1}{h}[\omega] + \omega_0 + h\omega_1 + h^2\omega_2 + \cdots .$$

Note that our present indexing of the coefficients is shifted by 1 as compared to that of [1].

The Lagrangian property of Y implies that $[\omega]$ restricts to zero on Y. In [1,3], it was shown that the existence of quantization also implies that $\omega_i|_Y = 0$, for the cohomology classes ω_i with $i \geq 1$. This may be viewed as a strengthened Lagrangian condition which depends on the choice of $\mathcal{O}_h$. As for the cohomology class ω_0, it admits a canonical lift (depending on $\mathcal{O}_h$) to $H^2_{DR}(Y, \Omega^{\geq 1})$ which is involved in the equation

$$\frac{1}{e}c_1(E) = \omega_0 + \frac{1}{2}c_1(K_Y), \tag{1}$$

where K_Y is the canonical bundle of Y. Square root of the canonical class has appeared in the contact setting in [17]. In the holomorphic setting, quantization of the square root of the canonical class is due to D'Agnolo and Schapira [9]. See also [5,22].

Returning to the case of a non-necessarily Lagrangian smooth coisotropic Y, define the "quantum Chern character" class considered in [21] (see also [6,7] for later results):

$$\tau(E_h) := \widehat{A}(T_M)\exp\big(-c(\mathcal{O}_h)\big)\,\mathrm{ch}\,\big(\sigma(E_h)\big),$$

where the $\widehat{A}$-genus is recalled in Section 2.2. We understand $\tau(E_h)$ as a class with values in the de Rham cohomology $H^*_{DR,Y}(M)((h))$ with support at Y, which can be identified with the de Rham cohomology of Y, due to smoothness. The purpose of this note is the following result

Theorem 1.1. *If the principal symbol sheaf $\sigma(E_h)$ is isomorphic to the direct image j_*E of a locally free sheaf E on a smooth coisotropic subvariety $j : Y \to M$ of codimension q, then the class $\tau(E_h) \in H^*_{DR,Y}(M)((h))$ is zero except in degrees $2q, \ldots, 2(p+q) = \dim_k M$.*

Moreover, if $\Omega_{\mathcal{F}} \subset \Omega^1_Y$ is the sheaf of 1-forms that vanish on the null-foliation (or characteristic foliation) $\mathcal{F} \subset T_Y$ and $F^r\Omega^\bullet_Y$ is the ideal in the de Rham complex generated by the r-th power of $\Omega_{\mathcal{F}}$, then $\tau(E_h)$ is in the image of the map

$$\bigoplus_{r\geq 0}^{p} H^{2r}(Y, F^r\Omega^\bullet_Y)((h)) \to \bigoplus_{r\geq 0}^{p} H^{2r}(Y, \Omega^\bullet_Y)((h)) \simeq \bigoplus_{r\geq 0}^{p} H^{2r+2q}_Y(M, \Omega^\bullet_M)((h)).$$

Our strategy is an application of formal geometry and the Gelfand–Fuks map: first use Riemann–Roch theorem to replace $\tau(E_h)$ by an element $\tau_Y(E) \in H^\bullet_{DR}(Y)((h))$; then show that after completion both the quantized functions and the quantized module are isomorphic to standard objects and construct a Harish-Chandra torsor (foliated over $\mathcal{F}$) and a Lie algebra cohomology class that induces $\tau(E_h)$ via Gelfand–Fuks map. At this point, the vanishing reduces to a vanishing in Lie algebra cohomology for which we use a Lie algebraic version of the index theorem, cf. [6–8, 15, 16, 21, 23], and the fact that in the symplectic situation the trace map can be defined on negative cyclic homology.

Alternatively, one could use an observation due to B. Tsygan that the Chern character of a perfect complex factors through negative cyclic homology of its derived endomorphism algebra, but in the algebraic geometry setting the Lie algebra cohomology route seems a bit shorter.

Remarks. (i) In a forthcoming paper with V. Ginzburg, cf. [2], we prove a similar statement for quantizable sheaves with arbitrary support, including the fact that $\tau(E_h)$ agrees with the general Connes–Chern character and that the algebraic index theorem holds for general algebraic varieties. Since in general a formal completion of a quantized sheaf will not be isomorphic to a "standard formal model", methods of formal geometry do not apply for general sheaves.

(ii) It would be very interesting to relate our main theorem to Bordemann's criterion for the existence of second-order quantization (mod h^3). This might depend on what can be said about the Atiyah–Molino class of the characteristic foliation of Y. See [5] for more details.

The paper is organized as follows. In Section 2, we recall standard constructions related to foliations, characteristic classes and use Riemann–Roch theorem to reduce the main result to a cohomology class on Y. In Section 3, we recall definitions related to Harish-Chandra pairs and torsors and the Gelfand–Fuks map. We further state the Lie cohomology algebraic index theorem and prove a vanishing result for the class involved. The conceptual reason for the vanishing is that the Connes–Chern character with values in periodic cyclic homology lifts to negative cyclic homology. In Section 4, we prove the main result by constructing two Harish-Chandra torsors that induce the class under consideration, and then invoking the vanishing of Section 3.

2. Preliminaries and notation

2.1. Null foliation and a filtration on the de Rham complex

We start by assuming that a pair $(\mathcal{O}_h, E_h)$ is given as in the introduction and that $\sigma(E_h) = E_h/hE_h$ is the direct image j_*E of a locally free sheaf supported on a smooth subvariety Y. We use the same notation E for associated vector bundle on Y.

If $I \subset \mathcal{O}_M$ is the ideal sheaf of functions vanishing on Y and $x \in \mathcal{O}_h$ is a local section projecting to I, then $x \cdot E_h \subset hE_h$. If y is another such section, it follows that the commutator $(x \cdot y - y \cdot x)$ sends E_h to $h^2 E_h$ and hence the image of $\frac{1}{h}(x \cdot y - y \cdot x)$ in $\mathcal{O}_M$ also annihilates j_*E; i.e., it belongs to I. Thus, the ideal sheaf I is closed with respect to the Poisson bracket induced by the Poisson bivector P.

If N is the normal bundle of Y in M, we can restate this by saying that $P|_Y$ projects to the zero section in $H^0(Y, \Lambda^2 N)$, and then the same restriction defines a section in $H^0(Y, T_Y \otimes N)$. We can view the latter as a morphism $N^\vee \to T_Y$ and it is easy to check that it is an embedding of vector bundles. Using $j_* N^\vee \simeq I/I^2$, we can write an explicit local formula for it:

$$N^\vee \ni x \mapsto P|_Y(dx, \cdot) \in T_Y.$$

Denote by $\mathcal{F} \subset T_Y$ the image of this embedding, i.e., the *null-foliation* of Y. By the above, this sub-bundle is *involutive*, i.e., closed with respect to the bracket of vector fields on Y (since the Poisson bracket on I/I^2 is compatible with the bracket on vector fields).

The involutive property can be restated as follows. Let $\Omega_{\mathcal{F}} = (T_Y/\mathcal{F})^\vee \subset \Omega^1_Y$ be the sheaf of 1-forms vanishing along $\mathcal{F}$ and denote by $F^1\Omega^\bullet \subset \Omega^\bullet_Y$ the graded ideal generated by $\Omega_{\mathcal{F}}$ in the sheaf of differential forms on Y, viewed as a sheaf of graded commutative algebras. By a straightforward application of the formula

$$d\omega(v_0, v_1) = v_0\omega(v_1) - v_1\omega(v_0) - \omega\big([v_0, v_1]\big) \tag{2}$$

the involutive property of $\mathcal{F}$ is equivalent to the statement that $F^1\Omega^\bullet$ is a subcomplex of the de Rham complex. It follows that each power of the ideal $F^k\Omega^\bullet := (F^1\Omega^\bullet)^k$ is also a subcomplex.

The main message in this paper is that characteristic classes of interest lift to the cohomology groups $H^{2r}(Y, F^r\Omega^\bullet)$. Note that the rank of $\Omega_{\mathcal{F}}$ is $2p = \dim Y - rk(\mathcal{F}) = \dim M - 2\,\mathrm{codim}\, Y$, hence for $r > 2p$ the relevant cohomology group vanishes as $F^r\Omega^\bullet$ is the zero subcomplex. Moreover, the particular class $\tau(E_h)$ vanishes for $r > p$.

2.2. Riemann–Roch theorem and reduction to a class on Y

Recall that for a power series $G(z)$ with constant term 1 and a vector bundle V of rank v, we can define its multiplicative G-genus as a product $\prod G(z_i)$, where $z_1, \dots z_v$ are the Chern roots of V, i.e., formal variables such that the l-th elementary symmetric function in z_i is equal to the l-th Chern class $c_l(V)$. Recall also that the $\widehat{A}$-genus $\widehat{A}(V)$ and the Todd genus $\mathrm{Td}(V)$ correspond to

$$G_1(z) = \sqrt{\frac{z/2}{\sinh(z/2)}} = \sqrt{\frac{z \cdot \exp(-z/2)}{1 - \exp(-z)}} \quad \text{and} \quad G_2(z) = \frac{z}{1 - \exp(-z)},$$

respectively. The Todd genus is involved in the Grothendieck–Riemann–Roch theorem for a closed embedding $j : Y \to M$ and a coherent sheaf E on Y (see [14, Section 15.2]):

$$\mathrm{ch}(j_*E) = j_*[\mathrm{ch}(E)\,\mathrm{Td}(N)^{-1}],$$

where N is the normal bundle to Y. We use this formula to study the “quantum” class

$$\tau(E_h) = \widehat{A}(T_M)\exp\big(-c(\mathcal{O}_h)\big)\,\mathrm{ch}(j_*E)$$

which appears in the index theorem of [6, 21] and the local index formula of [8]. By Riemann–Roch and the projection formula, we can rewrite this expression as

$$\tau(E_h) = j_*\left[\frac{\widehat{A}(T_M|_Y)}{\mathrm{Td}(N)} \exp\big(-c(\mathcal{O}_h|_Y)\big)\,\mathrm{ch}(E)\right].$$

Note that $G_1(z)$ is an even function of z, hence $\widehat{A}(N) = \widehat{A}(N^\vee)$. Using the multiplicative property of genus and the short exact sequences

$$0 \to T_Y \to T_M|_Y \to N \to 0, \quad 0 \to N^\vee \simeq \mathcal{F} \to T_Y \to Q \to 0$$

(where the second short exact sequence is the definition of Q), we conclude that

$$\frac{\widehat{A}(T_M|_Y)}{\mathrm{Td}(N)} = \frac{\widehat{A}(Q)\cdot(\widehat{A}(N))^2}{\mathrm{Td}(N)} = \widehat{A}(Q)\exp\left(-\frac{c_1(N)}{2}\right).$$

Hence Theorem 1.1 reduces to the statement that the class

$$\tau_Y(E) = \widehat{A}(Q)\exp\left(-\frac{c_1(N)}{2}\right)\exp\big(-c(\mathcal{O}_h)|_Y\big)\,\mathrm{ch}(E)$$

is in the image of $\bigoplus H^{2r}(Y, F^r\Omega^\bullet) \to \bigoplus H^{2r}(Y,\Omega^\bullet)$ and vanishes for $r > p$. Note that when $M = Y$ and M is projective, the cohomology groups will be nonzero up to degree $4p = 2\dim M$.

3. Lie algebra cohomology and Gelfand–Fuks map

The purpose of this section is to review some known results involving Lie algebra cohomology and characteristic classes, as they apply to deformation quantization; and also to fix the notation.

3.1. Lie algebra cohomology and a version of the Chern–Weil map

We follow [19, Section 10.1] and [13, Chapter 1.3]. For a Lie algebra $\mathfrak{g}$ over a field k and a $\mathfrak{g}$-module V, the Lie algebra cohomology groups are defined as $H^\bullet(\mathfrak{g};V) = \mathrm{Ext}^\bullet_{U(\mathfrak{g})}(k,V)$, where $U(\mathfrak{g})$ is the universal enveloping algebra. Using the standard Koszul resolution $\Lambda^\bullet(\mathfrak{g})\otimes U(\mathfrak{g}) \to k$ over $U(\mathfrak{g})$, these can be computed using Chevalley–Eilenberg cochain complex $C^\bullet(\mathfrak{g};V)$:

$$\cdots \to \mathrm{Hom}_k(\Lambda^n\mathfrak{g}, V) \to \mathrm{Hom}_k\big(\Lambda^{n+1}(\mathfrak{g}), V\big) \to \cdots$$

with a differential d_{Lie} (cf. [19, Section 10.1.6]):

$$\begin{aligned} d_{\mathrm{Lie}}\alpha(g_0,\dots,g_n) &= \sum_{i=0}^{n}(-1)^i g_i\alpha(g_0,\dots,\hat{g}_i,\dots,g_n) \\ &\quad + \sum_{i<j}(-1)^{i+j}\alpha\big([g_i,g_j],g_0,\dots,\hat{g}_i,\dots,\hat{g}_j,\dots,g_n\big). \end{aligned} \tag{3}$$

We view the elements $\alpha \in \mathrm{Hom}_k(\Lambda^\bullet(g), V)$ as skew-symmetric functions on $\mathfrak{g}$ with values in V.

For a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the subcomplex of *relative* Lie cochains, cf. [13, Chapter 1.3], $C^\bullet(\mathfrak{g}, \mathfrak{h}; V)$ is given by the condition that both α and $d_{\mathrm{Lie}}(\alpha)$ vanish when one of their arguments is in $\mathfrak{h}$. Its cohomology groups are denoted by $H^\bullet(\mathfrak{g}, \mathfrak{h}; V)$.

When $V^\bullet$ is a dg-Module over $\mathfrak{g}$, the formula (3) is adjusted by a term involving the internal differential of $V^\bullet$.

As a preparation for later proof, we state a homotopy lemma for complexes of Lie cochains.

Lemma 3.1. *Let $M^\bullet$ and $N^\bullet$ be two complexes of modules over a Lie algebra $\mathfrak{g}$ and let $f : M^\bullet \to N^\bullet$, $g : N^\bullet \to M^\bullet$ be chain maps compatible with an action of a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Suppose that $\varphi : M^\bullet \to M^{\bullet-1}$ is a homotopy satisfying $d\varphi + \varphi d = gf - 1_M$. Denote by*

$$\delta : C^\bullet(\mathfrak{g}, \mathfrak{h}; \cdot) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; \cdot) \quad \text{and} \quad d_{\mathrm{Hom}} : C^\bullet(\mathfrak{g}, \mathfrak{h}; \cdot) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; \cdot)$$

the first terms in the formula (3) and the combination of the second term of (3) with the internal differential on $M^\bullet$, $N^\bullet$, respectively (so that d_{Hom} can be identified with the Lie cochain differential for the trivial $\mathfrak{g}$-module structure). Let

$$f_{\mathrm{Hom}} : C^\bullet(\mathfrak{g}, \mathfrak{h}; M^\bullet) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; N^\bullet), \quad g_{\mathrm{Hom}} : C^\bullet(\mathfrak{g}, \mathfrak{h}; N^\bullet) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; M^\bullet)$$

be the morphisms of complexes with d_{Hom} differentials, induced by f, g, respectively, and $\varphi_{\mathrm{Hom}} : C^\bullet(\mathfrak{g}, \mathfrak{h}; M^\bullet) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; M^{\bullet-1})$ a homotopy induced by φ.

(i) *If g is compatible with $\mathfrak{g}$ action and the side conditions $\varphi\varphi = 0$, $\varphi g = 0$, $f\varphi = 0$ hold, then*

$$\tilde{f} = f_{\mathrm{Hom}}\big(1 + (\delta\varphi_{\mathrm{Hom}}) + (\delta\varphi_{\mathrm{Hom}})^2 + (\delta\varphi_{\mathrm{Hom}})^3 + \cdots\big)$$

is compatible with Lie algebra cohomology differentials $d_{\mathrm{Lie}} = d_{\mathrm{Hom}} + \delta$.

(ii) *If $M^\bullet = N^\bullet$, $g = 1_M$, and f is compatible with the $\mathfrak{g}$-action, then f_{Hom} is compatible with $d_{\mathrm{Lie}} = d_{\mathrm{Hom}} + \delta$ and*

$$\widetilde{\varphi} = \varphi_{\mathrm{Hom}}\big(1 + (\delta\varphi_{\mathrm{Hom}}) + (\delta\varphi_{\mathrm{Hom}})^2 + (\delta\varphi_{\mathrm{Hom}})^3 + \cdots\big)$$

is a homotopy between 1 and f_{Hom}.

Proof. Part (i) is a consequence of the basic perturbation lemma; cf. [20]. Part (ii) is easier to establish by direct computation although it is also a very degenerate case of the ideal perturbation lemma; cf. [20]. ∎

One source of relative Lie cocycles (see [21, Section 2.2]) arises from an $ad(\mathfrak{h})$-invariant projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$ and its curvature

$$C(u \wedge v) = \big[\mathrm{pr}(u), \mathrm{pr}(v)\big] - \mathrm{pr}\,\big([u, v]\big) : \Lambda^2\mathfrak{g} \to \mathfrak{h}. \tag{4}$$

Assume for simplicity that the $\mathfrak{g}$ action on V is trivial. Then for any $\mathfrak{h}$-invariant polynomial $S \in \mathrm{Sym}^l(\mathfrak{h}^*)^{\mathfrak{h}} \otimes V$, the cochain $\rho(S) \in C^{2l}(g; V)$ defined by

$$\rho(S)(v_1 \wedge \cdots \wedge v_{2l}) \\ = \frac{1}{l!} \sum_{\sigma \in S_{2l}, \sigma(2i-1)<\sigma(2i)} (-1)^{\sigma} S\big(C(v_{\sigma(1)}, v_{\sigma(2)}), \ldots, C(v_{\sigma(2l-1)}, v_{\sigma(2l)})\big)$$

is relative with respect to $\mathfrak{h}$, closed, and its relative cohomology class is independent on the choice of the projection $\mathrm{pr} : \mathfrak{g} \to \mathfrak{h}$. This defines the Chern–Weil homomorphism

$$\rho : \mathrm{Sym}^{\bullet}(\mathfrak{h}^*)^{\mathfrak{h}} \to H^{2\bullet}(\mathfrak{g}, \mathfrak{h}; k).$$

We will need the following examples of relative cocycles:

(1) when $\mathfrak{h} \simeq \mathfrak{gl}_e(k)$, set $\mathrm{ch}_{\mathrm{Lie}} = \rho(\mathrm{tr}(\exp(x)))$ and $c_{1,\mathrm{Lie}} = \rho(\mathrm{tr}(x))$ for $x \in \mathfrak{h}$;

(2) when $h \simeq \mathfrak{sp}_{2p}(k)$, set $\widehat{A}_{\mathrm{Lie}} = \rho\big(\det\big(\frac{y/2}{\sinh(y/2)}\big)^{1/2}\big)$ for $y \in \mathfrak{h}$;

(3) for a central extension of Lie algebras

$$0 \to \mathfrak{a} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

and a k-vector space splitting $\tilde{\mathfrak{g}} \simeq \mathfrak{a} \oplus \mathfrak{g}$, the 2-cocycle $C : \Lambda^2\mathfrak{g} \to \mathfrak{a}$ is the curvature as above. In the cases, we consider that the cocycle may be chosen in $C^2(\mathfrak{g}, \mathfrak{h}; \mathfrak{a})$.

3.2. Torsors over Harish-Chandra pairs and characteristic classes

Definition. A *Harish-Chandra pair* $(\mathfrak{g}, F)$ consists of a Lie algebra $\mathfrak{g}$, a (pro)algebraic group F over k, an embedding of Lie algebras $\mathfrak{f} = \mathrm{Lie}(F) \subset \mathfrak{g}$ and an action of F on $\mathfrak{g}$ which extends the adjoint action of F on $\mathfrak{f}$. A *module* over a Harish-Chandra pair (g, F) is an F-module V with an F-equivariant Lie morphism $\mathfrak{g} \to \mathrm{End}_k(V)$ extending the tangent Lie morphism on $\mathfrak{f}$.

In this paper, $\mathfrak{f}$ will have finite codimension in $\mathfrak{g}$ and $F \simeq L \ltimes U$ with L a finite dimensional reductive group and U a pro-unipotent infinite dimensional algebraic group.

Definition. A *Harish-Chandra torsor* or a flat $(\mathfrak{g}, F)$-torsor over a scheme Y is an F-torsor $\pi : P \to Y$ with an F-equivariant $\mathfrak{g}$-valued 1-form $\gamma : T_P \to \mathfrak{g} \otimes_k \mathcal{O}_P$ which restricts to the canonical Maurer–Cartan form (with values in $\mathfrak{f} \subset \mathfrak{g}$) on the vector fields tangent to the fibers of $\pi : P \to Y$ and satisfies the Maurer–Cartan equation

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0,$$

where d is the de Rham differential on P and the bracket is computed in $\mathfrak{g}$. In the infinite dimensional case, some care must be taken to define such torsors. One possible approach is to follow the pattern in [24, 25] and work with representable functors. In our case, eventually we will only need direct images of differential forms from P to Y, and since

F is a limit of affine groups, all geometric objects on Y can be defined using shaves on Y with a coaction of functions on F, and so on. See [11, Sections 2 and 3] for a closely related case.

Assume that $\gamma : T_P \to \mathfrak{g} \otimes_k \mathcal{O}_P$ is onto and has kernel T_γ of finite constant rank q. Note that since γ is injective on vertical vector fields, the differential of $\pi : P \to X$ is injective on T_γ. Moreover, since γ is F-equivariant, for points x, y in the same orbit of F, the images $(T_\gamma)_x$ and $(T_\gamma)_y$ in $(T_Y)_{\pi(x)=\pi(y)}$ agree. Let $\mathcal{F} \subset T_Y$ be the resulting rank q sub-bundle on Y.

Lemma 3.2. *In the situation described, let $T_\pi = \mathrm{Ker}(d\pi) \subset T_P$ be the vector fields tangent to the fibers. Then the sub-bundles $\mathcal{F} \subset T_Y$ and $\mathcal{G} = T_\pi \oplus ker(\gamma) \subset T_P$ are integrable (i.e., stable under the Lie bracket of vector fields). Let $\Omega_{\mathcal{F}} \subset \Omega^1_Y$ be the annihilator of $\mathcal{F}$ and $\Omega_{\mathcal{G}} \subset \Omega^1_P$ the annihilator of $\mathcal{G}$. Denote by $F^r\Omega^\bullet_Y$, resp. $F^r\Omega^\bullet_P$, the graded ideal generated by the r-th power of $\Omega_{\mathcal{F}}$, resp. the r-th power of $\Omega_{\mathcal{G}}$. Then both $F^r\Omega^\bullet_Y$ and $F^r\Omega^\bullet_P$ are preserved by the corresponding de Rham differentials and there is a morphism of complexes of sheaves $F^r\Omega^\bullet_Y \to \pi_* F^r\Omega^\bullet_P$.*

Proof. We start with $\mathcal{G}$. If v_1, v_2 are two vector fields in $\mathrm{Ker}(\gamma)$, then formula (2) for $d\gamma$ and the Maurer–Cartan equation for γ imply that the bracket $[v_1, v_2]$ is also annihilated by γ. The fact that vector fields tangent to the fibers are closed with respect to the Lie bracket holds for any smooth π. Finally, let's assume that $v_1 \in \mathrm{Ker}(\gamma)$ and $v_2 \in T_\pi$. Then the quadratic term in the Maurer–Cartan equation vanishes on $v_1 \wedge v_2$ (as $\gamma(v_1) = 0$) and we are left with

$$0 = d\gamma(v_1 \wedge v_2) = v_1 \cdot \gamma(v_2) - v_2\gamma(v_1) - \gamma\big([v_1, v_2]\big).$$

The second term is zero by assumption on v_1 and the first is in $\mathfrak{f} \otimes_k \mathcal{O}_P \subset \mathfrak{g} \otimes_k \mathcal{O}_P$. Hence the third term is in $\mathfrak{f} \otimes_k \mathcal{O}_P$ as well, which means $[v_1, v_2] \in \gamma^{-1}(\mathfrak{f} \otimes_k \mathcal{O}_P) = \mathcal{G}$.

In particular, a bracket of two F-equivariant vector fields in $\mathrm{Ker}(\gamma)$ is again an F-equivariant vector field in $\mathrm{Ker}(\gamma)$. Its F-equivariant descent is a rank q subbundle in the Atiyah algebra At_P of P on Y (= the F-equivariant descent of all vector fields on P), which is also closed under Lie bracket. It projects isomorphically to a sub-bundle $\mathcal{F} \subset T_Y$ (as $\mathrm{Ker}(\gamma)$ has trivial intersection with T_π) which is closed with respect to the Lie bracket since $At_P \to T_P$ is compatible with brackets. By construction, 1-forms on Y which vanish on $\mathcal{F}$ pull back to F-equivariant 1-forms on P which vanish on $\mathcal{G}$. Hence the morphism of sheaves of dg algebras $\Omega^\bullet_Y \to \pi_*\Omega^\bullet_P$ is compatible with multiplicative filtrations $F^r\Omega^\bullet$ induced by the two foliations. ∎

Definition. In the situation of the previous lemma, we will say that the Harish-Chandra torsor (P, γ) is *foliated* over $\mathcal{F} \subset T_Y$. We will see that in this case some of its characteristic classes what apriori belong to $H^\bullet_{DR}(Y)$ admit a lift to cohomology of $F^r\Omega^\bullet_Y$.

Let V be a Harish-Chandra module over $(\mathfrak{g}, F)$ with trivial action (as will be in our applications). For a Lie l-cochain $\alpha : \Lambda^l \mathfrak{g} \to V$, the composition $\Lambda^l T_P \to \Lambda^l \mathfrak{g} \otimes_k \mathcal{O}_P \to V \otimes_k \mathcal{O}_P$ may be viewed as a V-valued l-form on P and the Maurer–Cartan equation ensures that the resulting *Gelfand–Fuks morphism* (cf. [13, Chapter 3.1 (C), (D)]) also agrees with differentials:

$$\mathrm{GF} : C^\bullet(\mathfrak{g}; V) \to \Gamma(P, \Omega^\bullet_P \otimes_k V). \tag{5}$$

We want to use this observation to study characteristic classes of P in the cohomology of Y. Note that to obtain classes on Y, we need to work with objects which are invariant with respect to the reductive subgroup of F.

For a subalgebra $\mathfrak{h} \subset \mathfrak{f} = \mathrm{Lie}(F) \subset \mathfrak{g}$, relative cochains in $C^\bullet(\mathfrak{g}, \mathfrak{h}; V)$ map to the subcomplex $(\Omega^\bullet \otimes_k V)_{\mathfrak{h}\text{-basic}}$ of $\mathfrak{h}$-basic forms, i.e., forms β such that $L_v\beta = 0$ and $\iota_v\beta = 0$ for any $v \subset \mathfrak{h}$ (we use the same letter v for the vertical vector field on P induced by v via the action of F).

Lemma 3.3. *Assume that $F = U \rtimes H$ is a semi-direct product of a finite dimensional connected reductive group H and a pro-unipotent group U. If $\mathfrak{h} = \mathrm{Lie}(H)$, there exists a quasi-isomorphism of sheaves of dg-algebras*

$$\Omega^\bullet_Y \to \pi_*\big((\Omega^\bullet_P)_{\mathfrak{h}\text{-}basic}\big).$$

If P is foliated over $\mathcal{F} \subset T_Y$, then for any $r \geq 0$ the natural morphism of ideal sheaves

$$F^r\Omega^\bullet_Y \to \pi_*\big((F^r\Omega^\bullet_P)_{\mathfrak{h}\text{-}basic}\big)$$

is a quasi-isomorphism of complexes of sheaves.

Sketch of proof. First assume that U is finite dimensional and look at the first quasi-isomorphism. Since H is connected, the pushforward of $\mathfrak{h}$-basic forms from P to Y may be identified with the pushforward of forms on P/H to Y. But $P/H \to Y$ is a bundle with affine fibers so the assertion follows from the relative Poincaré lemma (triviality of relative de Rham cohomology for fibrations by affine spaces). For infinite dimensional pro-unipotent U, we first consider the finite dimensional unipotent factors and then pass to a limit, as in [24, Theorem 6.7.1].

In view of the unfiltered quasi-isomorphism, its filtered version reduces to showing that the maps induced on associated graded quotients are quasi-isomorphisms. First,

$$F^r\Omega^\bullet_Y / F^{r+1}\Omega^\bullet_Y \simeq \Lambda^r(Q) \otimes \Lambda^\bullet\mathcal{F}^\vee,$$

where $Q = T_Y/\mathcal{F}$ and we consider $\Lambda^\bullet\mathcal{F}^\vee$ as a complex of sheaves with the differential similar to that in formula (3) (in other words, it is the de Rham differential of the Lie algebroid $\mathcal{F} \subset T_Y$). On the other hand, since the pullback of $\mathcal{F}$ is isomorphic to $\mathrm{Ker}(\gamma) \subset \mathcal{G}$ and the pullback of Q is isomorphic to $T_P/\mathcal{G}$, by the projection formula we get

$$\pi_*\big((F^r\Omega^\bullet_P)_{\mathfrak{h}\text{-basic}}\big)/\pi_*\big((F^{r+1}\Omega^\bullet_P)_{\mathfrak{h}\text{-basic}}\big) \simeq \Lambda^r(Q) \otimes \Lambda^\bullet\mathcal{F}^\vee \otimes_{\mathcal{O}_Y} \pi_*\big((\Omega^\bullet_\pi)_{\mathfrak{h}\text{-basic}}\big),$$

where $\Omega^\bullet_\pi$ is the relative de Rham complex of $\pi : P \to Y$. Hence we just need to show that

$$\mathcal{O}_Y \to \pi_*\big((\Omega^\bullet_\pi)_{\mathfrak{h}\text{-basic}}\big)$$

is a quasi-isomorphism, which again follows from the relative Poincaré lemma. ∎

In view of the previous result, passing to cohomology in (5) we obtain an $\mathfrak{h}$-relative version of Gelfand–Fuks map, which we denote also by GF:

$$\mathrm{GF}: H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, \mathfrak{h}; V) \to H^\bullet_{DR}(Y) \otimes_k V.$$

Recall that whenever we use GF we assume that $\mathfrak{g}$-action on V is trivial, otherwise the right-hand side would involve the de Rham cohomology of associate vector bundle V_P with a flat connection induced by the Harish-Chandra module structure.

Proposition 3.4. *For $V = k$ in the setting of Lemma 3.3, the composition of the Lie algebraic Chern–Weil map and the Gelfand–Fuks map*

$$\mathrm{Sym}^\bullet(\mathfrak{h}^*)^{\mathfrak{h}} \to C^{2\bullet}(\mathfrak{g}, \mathfrak{h}; k) \to H^{2\bullet}_{DR}(Y)$$

is the classical Chern–Weil map of the torsor P_H, associated to P via the group homomorphism $F \to H$. If P is foliated over $\mathcal{F} \subset T_Y$, for every $r \geq 0$ the composition admits a canonical lift

$$\mathrm{Sym}^r(\mathfrak{h}^*)^{\mathfrak{h}} \to H^{2r}_{DR}(Y, F^r\Omega^\bullet_Y).$$

Proof. Let $P_U = P/H$, then $P \to P_U$ may be viewed as the H-torsor pulled back from Y via $P_U \to Y$. Since $P_U \to Y$ induces isomorphism on de Rham cohomology (relative Poincaré lemma), we can replace Y by P_U and assume that U is trivial. Then the composition

$$\nabla = \mathrm{pr} \circ \gamma : T_P \to \mathfrak{g} \otimes_k \mathcal{O}_P \to \mathfrak{h} \otimes_k \mathcal{O}_P$$

is a connection on $P \to P_U$ and the Lie theoretic curvature $C : \Lambda^2\mathfrak{g} \to \mathfrak{h}$ gives the classical curvature $R_\nabla = \Lambda^2\gamma \circ C : \Lambda^2 T_P \to \mathfrak{h} \otimes_k \mathcal{O}_P$. The assertion follows since the classical Chern–Weil map may be computed by evaluating invariant polynomials on R_∇.

For the second assertion, we observe that relative Lie cochains define a global section of $\pi_*((F^r\Omega^\bullet_P)_{\mathfrak{h}\text{-basic}})$ and the result follows by application of Lemma 3.3. ∎

3.3. Algebraic version of the index theorem

Consider the formal Weyl algebra $\mathcal{D}_p$, the completion (at the augmentation ideal) of the universal enveloping of the Heisenberg Lie algebra with generators $x_1, \dots, x_p, y_1, \dots, y_p$, h, and the only nontrivial commutators given by $[y_j, x_i] = \delta_{ij} h$. In other words, as a vector space $\mathcal{D}_p$ is isomorphic to $k[[x_1, \dots, x_p y_1, \dots, y_p, h]]$ but has nontrivial commutation relations $y_i x_i = x_i y_i + h$.

We consider the associative algebra $\mathcal{E} = \mathfrak{gl}_e(\mathcal{D}_p)$ and the Lie algebra $\mathrm{Der}(\mathcal{E})$ of its continuous $k[[h]]$-linear derivations. Of course, some of the derivations are inner and hence there is a Lie morphism $\mathcal{E} \to \mathrm{Der}(\mathcal{E})$. In addition, any commutator in $\mathcal{D}_p$ is divisible by h, so commuting with a scalar $\frac{1}{h}\mathcal{D}_p$-valued matrix also gives a derivation of $\mathcal{D}_p$. We claim a short exact sequence

$$0 \to \frac{1}{h}k[[h]] \to \left(\frac{1}{h}\mathcal{D}_p + \mathcal{E}\right) \to \mathrm{Der}(\mathcal{E}) \to 0. \tag{6}$$

For $e = 1$, this is well known; see e.g. [6, Section 2.3] and [4, equation (3.2)]. For general e, it follows from the fact that the quotient of all derivations by inner derivations (i.e., first Hochschild cohomology group) is a Morita invariant; cf. [19, Chapter 1.2].

The Lie algebra $\mathfrak{g} = (\frac{1}{h}\mathcal{D}_p + \mathcal{E})$ has a reductive subalgebra $\mathfrak{gl}_e \oplus \mathfrak{sp}_{2p}$ (matrices with values in $k \subset \mathcal{D}_p$ plus an isomorphic copy of $\mathfrak{sp}_{2p}$ in $\frac{1}{h}\mathcal{D}_p$ spanned by commutators of 1, $\frac{1}{h}x_i x_j$, $\frac{1}{h}y_i x_j$, $\frac{1}{h}y_i y_j$). We also consider the abelian subalgebra $\mathfrak{a} = \frac{1}{h}k[[h]]$. To take into account $\mathfrak{a} \cap \mathfrak{gl}_e = k$, introduce $\mathfrak{a}' \subset \mathfrak{a}$ with topological basis given by h^i, $i = -1, 1, 2, \ldots$. Then

$$\mathfrak{h} = \mathfrak{gl}_e \oplus \mathfrak{sp}_{2p} \oplus \mathfrak{a}'$$

is a Lie subalgebra of $\mathfrak{g}$, by Hochschild–Serre spectral sequence

$$H^\bullet(\mathfrak{g}, \mathfrak{h}; V) \simeq H^\bullet\big(\mathrm{Der}(\mathcal{E}), (\mathfrak{gl}_e/k) \oplus \mathfrak{sp}_{2p}; V\big),$$

where V is a module over $\mathrm{Der}(\mathcal{E})$ (and thus also a module over $\mathfrak{g}$). Below, we are interested in the cohomology of $\mathrm{Der}(\mathcal{E})$ but we find it easier to do computations in $\mathfrak{g}$.

Now we would like to state a vanishing lemma for homogeneous components of a particular class in $H^\bullet(\mathfrak{g}, \mathfrak{h}; k((h)))$. Its proof will take the rest of the section. A reader willing to treat it as a black box may wish skip to Section 4. We follow the notation and exposition in [16] which deals with a version of algebraic index theorem that is most convenient for our setting.

This class can be defined via the Chern–Weil construction. To fix a projection

$$\mathrm{pr} : \mathfrak{g} \to \mathfrak{h},$$

we introduce a filtration on $\mathfrak{g}$ by giving h degree 2 and x_i, y_j degree 1. Since elements of $\mathfrak{g}$ involve infinite sums, $\mathfrak{g}$ splits into a direct product $\prod_{i \geq -2} \mathfrak{g}_i$. For the first two factors in $\mathfrak{h}$, we project $\mathfrak{g}$ onto $\mathfrak{g}_0 \simeq \mathfrak{gl}_e \oplus \mathfrak{sp}_{2p}$ (recall that $\mathfrak{sp}_{2p}$ is spanned by the commutators in the degree zero part $(\frac{1}{h}\mathcal{D}_p)_0$) and set $k = \frac{h}{h}k$ to be in the kernel on the projection onto $\mathfrak{sp}_{2p}$. For $\mathfrak{a}'$ we choose any projection $\frac{1}{h}\mathcal{D}_p \to \mathfrak{a}'$ and extend it by 0 to trace zero matrices in $\mathcal{E}$.

Lemma 3.5. *For $\mathfrak{h} = \mathfrak{gl}_e \oplus \mathfrak{sp}_{2p} \oplus \mathfrak{a}'$, let $\mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e)$, $\widehat{A}_{\mathrm{Lie}}(\mathfrak{sp}_{2p})$, and $C(\mathfrak{a}')$ be the classes induced by the Chern–Weil construction at the end of Section 3.1, from the respective factors. Let*

$$\tau_{\mathcal{D}_p} = \mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e)\widehat{A}_{\mathrm{Lie}}(\mathfrak{sp}_{\mathfrak{n}}) \exp\big(-C(\mathfrak{a}')\big) \in H^{\mathrm{even}}_{\mathrm{Lie}}\big(\mathfrak{g}, \mathfrak{h}; k((h))\big).$$

Then the components of $\tau_{\mathcal{D}_p}$ of degree $> 2p$ are equal to zero.

Proof. Our proof is based on the fact that the class of the theorem arises from the study of periodic cyclic homology of the associative algebra $A = gl_e(\mathcal{D}_p) \otimes_{k[[h]]} k((h))$. The vanishing will follow from the fact that this class lifts to the negative cyclic homology of A. We briefly recall the relevant definitions here, omitting details that do not contribute to the proof.

If A is a unital k-algebra, set $\overline{A} = A/k \cdot 1$ and $C_{-l}(A) = A \otimes \overline{A}^{\otimes l}$. This is the cohomological grading, rather than homological grading used in some sources, although the indices are written as subscripts to avoid confusion with the Hochschild cohomology complex. The standard formulas e.g. in [19, Sections 1.1.1 and 2.1.8] define the Hochschild and cyclic differentials,

$$b : C_\bullet(A) \to C_{\bullet+1}(A), \quad B : C_\bullet(A) \to C_{\bullet-1}(A)$$

that satisfy $b^2 = 0$, $B^2 = 0$, $Bb + bB = 0$. Introducing a formal variable u of cohomological degree 2, consider two complexes

$$CC_\bullet^-(A) = \big(C_\bullet[[u]], b + uB\big), \quad CC_\bullet^{\mathrm{per}} = \big(C_\bullet((u)), b + uB\big)$$

with cohomology defining the negative cyclic homology $HC_\bullet^-(A)$ and the periodic cyclic homology $HC_\bullet^{\mathrm{per}}(A)$, respectively. In both cases, $1 \in A = C_0(A)$ satisfies $(b + uB)(1) = 0$ and thus gives a cohomology class.

In the case $A = gl_e(\mathcal{D}_p) \otimes_{k[[h]]} k((h))$, there is a quasi-isomorphism

$$\big(C_\bullet(A)((u)), b + uB\big) \simeq \big(\Omega^{-\bullet}((h))((u)), hL_\pi + ud_{DR}\big), \tag{7}$$

where $\Omega^{-\bullet}$ stands for formal differential forms in $x_1, \ldots, x_p, y_1, \ldots, y_p$ and

$$\pi = \sum (\partial/\partial x_i) \wedge (\partial/\partial y_i)$$

is the standard Poisson bivector. For $e = 1$ and on the level of Hochschild complexes, this is essentially the FFS cocycle (cf. [12]) since $(\Omega^{-\bullet}((h)), hL_\pi)$ has cohomology only in the top degree. The cyclic extension of the FFS cocycle was constructed by Pflaum–Posthuma–Tang (e.g. in [23, Section 2.2]) where the data was interpreted as a cyclic cohomology class, but Definition 3.5 in *loc. cit* allows to re-package it as the morphism from the cyclic chain complex as above.

The Lie algebra $\mathrm{Der}(\mathcal{E})$ of derivations of $\mathcal{E} = \mathfrak{gl}_e(\mathcal{D}_p)$ acts on periodic and negative cyclic complexes. The quasi-isomorphism is compatible with the action of $\mathfrak{pgl}_e(k) \oplus \mathfrak{sp}_{2p} \subset \mathrm{Der}(\mathcal{E})$ (see below). Although it is not compatible with the action of the full algebra of derivations, it can be upgraded to a cocycle in the relative Lie algebra cohomology of $\mathrm{Der}(\mathcal{E})$:

$$\tau_{\mathrm{Lie}} \in C^\bullet\big(\mathfrak{g}, \mathfrak{h}; \mathrm{Hom}_{k((h))}\big(\big(CC_\bullet^-(A), b + uB\big), \big(\widehat{\Omega}^{-\bullet}((h))[[u]], hL_\pi + ud_{DR}\big)\big)\big). \tag{8}$$

The case of general e can be obtained by rephrasing [23, Sections 4.2, 4.3 and Proposition 4.2], while explicit formulas can be found in [16], where the Lie derivative $L_\pi = d\iota_\pi + \iota_\pi d$ is denoted by Δ. The morphism (7) is given in Definition 3.4 of *loc. cit.* and its compatibility with b and B differentials is proved in Lemmas 3.5 and 3.6 of the same paper. The Lie cochain version can be found in Definition 3.7 and Theorem 3.8 of *loc. cit.* (the fact that the Lie cocycle is relative over $\mathfrak{sp}_{2p}$ follows from the argument in Theorem 3.13 of *loc. cit.* or [23, Proposition 2.10] and invariance with respect to the other two factors in the definition of $\mathfrak{h}$ is immediate from definitions).

We shift the grading used in [16] assigning τ_{Lie} cohomological degree 0. We note here that the action of $\mathfrak{g}$ on $\widehat{\Omega}^{-\bullet}((h))$ is not h linear (indeed, the element $h \in \mathfrak{g}$ acts by zero, so the action factors through the quotient by h), and all cochains are only k-linear maps. Although τ_{Lie} is constructed in [16] for the periodic cyclic complex (it is denoted by $\langle -,-\rangle_{int}^{S1}$ in the definition introduced before Theorem 3.13 of *loc. cit.*), we emphasize that at this step no inversion of u is necessary and exactly the same definition works on the level of negative cyclic complexes. Next, one can construct two $\mathfrak{g}$-invariant homomorphisms

$$g_u \circ e^{-\frac{h}{u}\iota_\pi}, g_h \circ e^{-\frac{u}{h}\omega} \in \mathrm{Hom}_{k((h))}\left(\left(\widehat{\Omega}^{-\bullet}((h))[[u]], hL_\pi + ud_{DR}\right), \left(\widehat{\Omega}^{\bullet}((h))((u)), d_{DR}\right)\right)$$

which are homotopic in the full Lie algebra cohomology complex.

The first homomorphism needs inversion of u. We first use the identity

$$e^{\frac{h}{u}\iota_\pi}(hL_\pi + ud_{DR}) = (ud_{DR})e^{\frac{h}{u}\iota_\pi}$$

to land in the complex $(\widehat{\Omega}^{-\bullet}((h))((u)), ud_{DR})$. Then we apply the re-grading operator g_u which is an isomorphism of complexes

$$\left(\widehat{\Omega}^{-\bullet}((h))((u)), ud_{DR}\right) \to \left(\widehat{\Omega}^{\bullet}((h))((u)), d_{DR}\right)$$

sending an i-form α to $u^{-i}\alpha$. Thus, on the left-hand side α has cohomological degree $(-i)$, on the right-hand side degree i and adjustment by u^{-i} makes the re-grading a degree 0 operator.

The second homomorphism uses h^{-1} and the formal symplectic form $\omega = \sum dx_i \wedge dy_i$. In this case, we start with the identity $e^{-\frac{u}{h}\omega}(hL_\pi + ud_{DR}) = hL_\pi e^{-\frac{u}{h}\omega}$ to land in the complex $(\widehat{\Omega}^{-\bullet}((h))[[u^{-1}, u]], hL_\pi)$ and then apply a re-grading isomorphism g_h

$$\left(\widehat{\Omega}^{-\bullet}((h))((u)), hL_\pi\right) \to \left(\widehat{\Omega}^{\bullet}((h))((u)), d_{DR}\right)$$

which sends an i-form α to $\frac{h^{i-n}}{u^n} *_\omega (\alpha)$. Here, the symplectic Hodge operator

$$*_\omega : \Omega^i \to \Omega^{2p-i}$$

is defined by $\beta \wedge *_\omega(\alpha) = \omega^n \langle \beta, \alpha \rangle_\pi$ and $\langle \cdot, \cdot \rangle_\pi$ is the pairing on i-forms induced by π.

With these preliminaries, we now prove the vanishing claimed in Lemma 3.5 in several steps

Step 1. First consider that the class

$$\tau_1 = \sum \tau_1^{m,l} u^l \in \bigoplus C^{2m}\left(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^{-2(m+l)}((h))u^l\right)$$

is obtained by pairing $1 \in C^0_{\text{Lie}}(\mathfrak{g}, \mathfrak{h}; (CC_0^-(A), b + uB))$ with τ_{Lie} defined in (8) (we recall that the grading on differential forms has been inverted at this step). The range of indices is $m \geq 0$ and $0 \leq m + l \leq n$. Then move on to the class $\tau_2 = e^{-\frac{h}{u}\iota_\pi}\tau_1$ which lives in the same complex but with the differential ud_{DR} instead of $hL_\pi + ud_{DR}$. The components $\tau_2^{m,l}u^l$ of the class τ_2 can be nonzero in the same range $0 \leq m, 0 \leq m + l \leq p$.

Step 2. Now consider the regraded class

$$\tau_3 = g_u \tau_2 = \sum \tau_2^{m,l} u^{-2m-2l} u^l \in \bigoplus C^{2m}\big(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^{2(m+l)}((h)) u^{-2m-l}\big)$$

(where the differential forms now have the usual grading) and observe that the exponents of u are in the range $(-2p, \dots, 0)$. The next step is to replace the de Rham complex $(\widehat{\Omega}^\bullet, d_{DR})$ by a quasi-isomorphic complex $(k, 0)$. Note that the projection $\widehat{\Omega}^\bullet \to k$ (which vanishes on forms of positive degrees and sends a power series in degree zero to its constant term) is not $\mathfrak{g}$-equivariant. However, it can be extended to a quasi-isomorphism of complexes $\tilde{f} : C^\bullet(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^\bullet) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; k)$ using Lemma 3.1 (i) above. This leads to a class

$$\tau_4 = \sum \tau_4^{4m+2l} u^{-(2m+l)} = \tilde{f}(\tau_3) \in \bigoplus C^{4m+2l}\big(\mathfrak{g}, \mathfrak{h}; k((h)) u^{-2m-l}\big).$$

Note that $0 \le m, 0 \le m + l \le p$ imply that $0 \le 4m + 2l \le 4p$. Now we use the result of [16] to justify the following.

Claim. If $\tau_{\mathfrak{D}_p} = a_0 + a_2 + a_4 + \cdots$ is an expansion into a sum of cochains of even degrees, $\deg(a_{2i}) = 2i$, set $(\tau_{\mathfrak{D}_p})_u = a_0 + a_2 u^{-1} + a_4 u^{-2} + \cdots$. Then on the level of Lie algebra cohomology classes, one has $\tau_4 = (\tau_{\mathfrak{D}_p})_u$ in $H^\bullet(\mathfrak{g}, \mathfrak{h}; k((h))[[u^{-1}, u]])$.

Indeed, [16, Theorem 3.21] asserts equality of two cohomology classes in the same cohomology group. The right-hand side of *loc. cit.* is the same as $(\tau_{\mathfrak{D}_p})_u$ by definition, except for the normalizing power of u which is due to the fact that we shift the classes to cohomological degree zero, as mentioned above. The left-hand side agrees with our definition too: Definition 3.10 of *loc. cit.* matches our transition from τ_1 to τ_2 and then on to τ_4 (although evaluation at 1 involved in the definition of τ_1 is introduced in that paper a bit later).

Step 3. We are finally in position to prove that on the level of cohomology the coefficients τ_4^{4k+2l} vanish when $4k + 2l > 2n$. Indeed, instead of

$$\tau_3 = g_u e^{-\frac{h}{u}\iota_\pi} \tau_1$$

we could consider

$$\tau_3' = g_h e^{-\frac{u}{h}\omega} \tau_1$$

which has trivial coefficients of $u^{-(2k+l)}$, $2k + l \ge n$ since the only negative power of u created is the factor u^{-n} in the definition of g_h.

Step 4. To show that τ_3 and τ_3' have the same cohomology class, we use the identity

$$e^{\frac{\omega}{uh}} \circ \big(g_h \circ e^{-\frac{u}{h}\omega}\big) = g_u \circ e^{\frac{h}{u}\iota_\pi};$$

which can either be established by direct computation, or by using a basic observation of Hodge Theory that operators $\wedge\omega$ and ι_π generate an $\mathfrak{sl}_2$ action on the de Rham complex, hence the above identity can be obtained as the image of the group level identity in SL_2

$$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & h^{-1} \\ -h & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} = \begin{pmatrix} 1 & h^{-1} \\ 0 & 1 \end{pmatrix}$$

under the action homomorphism.

Finally, we note that $e^{\frac{\omega}{uh}}$ is homotopic to identity. This follows from the fact that $\omega = d_{DR}(\alpha)$, where $\alpha = \frac{1}{2}\sum(x_i dy_i - y_i dx_i)$ is the Euler vector field converted to 1-forms using ω. Therefore, we have $d_{DR}\varphi + \varphi d_{DR} = e^{\frac{\omega}{uh}}$, where

$$\varphi = \sum_{k \geq 0} \frac{\alpha}{uh} \frac{\omega^k}{(uh)^k (k+1)!}.$$

Again, the homotopy does not agree with the $\mathfrak{g}$-action (only with the $\mathfrak{h}$-action) hence we use Lemma 3.1 (ii) to obtain a homotopy between identity operator and $e^{\frac{\omega}{uh}}$:

$$\widetilde{\varphi} : C^{\bullet}\big(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^{\bullet}((h))((u))\big) \to C^{\bullet-1}\big(\mathfrak{g}, \mathfrak{h}; \widehat{\Omega}^{\bullet}((h))((u))\big).$$

Hence we can use the above class τ_3' instead of τ_3. Since τ_3' by construction has at worst poles of order $\leq n$ in the u variable, the assertion follows. ∎

4. Proof of the main result

In this section, we prove Theorem 1.1 by studying the characteristic class

$$\tau_Y(E) = \exp\left(-\frac{c_1(N)}{2}\right) \mathrm{ch}(E)\widehat{A}(B) \exp\big(-c(\mathcal{O}_h)|_Y\big) \in H^{\bullet}_{DR}(Y)((h)).$$

For $y \in Y$, the preimage of the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_{Y,y}$ in the stalk at y, with respect to the reduction mod h map $\mathcal{O}_{h,y} \to \mathcal{O}_{Y,y}$, is a maximal ideal $\mathfrak{m}_{h,y} \subset \mathcal{O}_{h,y}$. Adapting the classical proof of the Darboux theorem, we show that after completion at this maximal ideal the triple $(\mathcal{O}_h, E_h, \mathrm{End}_{\mathcal{O}_h}(E_h))$ is isomorphic – non-canonically! – to a similar triple independent of y or Y. Different choices of isomorphisms will give the Harish-Chandra torsor $P_{\mathcal{D},\mathcal{M}}$ inducing $\tau_Y(E)$ via the Gelfand–Fuks map. Further, it is actually lifted from a quotient torsor $P_{\mathcal{E}}$ and Theorem 1.1 will be reduced to the study of the class $\tau_{\mathcal{D}_p}$, where one uses Lemmas 3.3 and 3.5.

4.1. Standard formal models: $\mathcal{D}$, $\mathcal{M}$, and $\mathcal{E}$

Below for $n = p + q$, we will assume that $\mathcal{D}_q$ is the Weyl algebra built on the variables x_i, y_i, h, with $i = 1, \ldots, q$, that $\mathcal{D}_p$ corresponds to the values $i = q+1, \ldots, q+p$ while $\mathcal{D}$ is the Weyl algebra on the full set of variables with $i = 1, \ldots, p+q = n$. Fixing decomposition $n = p + q$ and an integer $e \geq 1$, define a left $\mathcal{D}$-module

$$\mathcal{M} := \big[\mathcal{D}/\mathcal{D}\langle y_1, \ldots, y_q\rangle\big]^{\oplus e} \simeq \big(\mathcal{D}_q/\mathcal{D}_q\langle y_1, \ldots, y_q\rangle\big) \widehat{\otimes}_{k[[h]]} \mathcal{D}_p^{\oplus e}$$

(on the right-hand side, we use completed tensor product). The second presentation implies the following isomorphism for endomorphisms of $\mathcal{M}$ (which we assume to be acting *on the right*):

$$\mathcal{E} := \mathrm{End}_{\mathcal{D}}(\mathcal{M}) \simeq gl_e(\mathcal{D}_p)$$

Lemma 4.1. *For any Y, $\mathcal{O}_h$, E_h as before and $y \in Y$, denote by $\widehat{(\cdots)}$ the completion of a stalk at y with respect to the maximal ideal $\mathfrak{m}_{h,y}$.*

(1) *There exist compatible isomorphisms*

$$\sigma_{\mathcal{D}} : \widehat{\mathcal{O}_{h,y}} \to \mathcal{D}, \quad \sigma_{\mathcal{E}} : \widehat{\mathrm{End}}_{\mathcal{O}_{h,y}}(E_{h,y}) \to \mathcal{E} \quad \sigma_{\mathcal{M}} : \widehat{E_{h,y}} \to \mathcal{M}$$

as filtered algebras and modules, respectively. Moreover, if an algebra isomorphism $\sigma_{\mathcal{D}}$ admits a compatible module isomorphism $\sigma_{\mathcal{M}}$, then $\sigma_{\mathcal{D}}$ sends the completion $\widehat{I_h}$ of $I_h = \mathrm{Ker}(\mathcal{O}_h \to \mathcal{O}_Y)$ to the double sided ideal $\mathcal{J} \subset \mathcal{D}$ generated by $y_1, \dots, y_q, h$.

(2) *If $\sigma_{\mathcal{D}}(\widehat{I_h}) = \mathcal{J}$, then $\sigma_{\mathcal{D}}$ extends to a pair of compatible isomorphisms $(\sigma_{\mathcal{D}}, \sigma_{\mathcal{M}})$.*

Proof. Modulo h, we can construct an isomorphism of $\widehat{\mathcal{O}}_{M,y}$ and $k[[x_1, \dots, x_n, y_1, \dots y_n]]$ since y is a smooth point, k has characteristic zero, and X has dimension $2n$. Since Y is smooth, its ideal $I_Y \subset \mathcal{O}_{M,y}$ is generated by a regular sequence and we can assume that the isomorphism sends the completion on I_Y to the ideal generated by $y_1, \dots, y_q$.

We can also adjust the isomorphism to be compatible with the symplectic forms. Let α be a 2-form on $\mathrm{Spec}(k[[x_1, \dots, x_n, y_1, \dots, y_n]])$ induced from ω via the initial isomorphism. Decompose it into homogeneous components: $\alpha = \alpha_0 + \alpha_1 + \alpha_2 + \cdots$, where each α_i is a 2-form with coefficients of homogeneous degree i. Since Y is coisotropic, after a linear change of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, we can assume that $\alpha_0 = \sum dx_i \wedge dy_i$.

Note that the ideal $\mathcal{J}$ generated by $(y_1, \dots, y_q)$ is Poisson with respect to the bracket induced by α. This means that the coefficients of $dx_r \wedge dx_s$ are in $\mathcal{J}$ for each α_j and $r, s \le q$. We now want to find a formal vector field μ such that the formal diffeomorphism $\exp(\mu)$ takes α to α_0 and preserves $\mathcal{J}$. In fact, we will construct $\exp(\mu)$ inductively as the composition of $\exp(\mu_1)$, $\exp(\mu_2), \dots,$ where μ_i is a polynomial vector field with coefficients of homogeneous degree i and

$$\exp(\mu_{i-1}) \cdots \exp(\mu_1)(\alpha) = \alpha_0 + \beta_{\ge i}$$

with $\beta_{\ge i}$ a 2-form with coefficients of degree $\ge i$. To ensure that $\mathcal{J}$ is preserved, we need to have $\mu_i(\mathcal{J}) \subset \mathcal{J}$, which is to say, the coefficient of $\partial/\partial y_r$ in μ_i is an element of $\mathcal{J}$ for $r \le q$. Considering $\gamma_i = \alpha_0(\mu_i, \cdot)$, we see that γ_i needs to be a polynomial differential form with coefficients of degree i, such that the coefficient of dx_r is in $\mathcal{J}$ for $r \le q$ and $d\gamma_i = \beta_i$, where β_i is the degree $= i$ component of $\beta_{\ge i}$.

Note that β_i is at least closed since this property holds for α, is preserved after the action of $\exp(\mu_j)$, and is just a homogeneous component of the resulting form $\alpha_0 + \beta_{\ge i}$. Since the formal de Rham complex is exact in degrees ≥ 0, $\beta_i = d\gamma_i$ with $\gamma_i = \iota_{Eu}\beta_i$, the contraction with the Euler vector field $Eu = \sum (x_r \partial/\partial x_r + y_r \partial/\partial y_r)$. Note that by induction, after each formal diffeomorphism $\mathcal{J}$ remains a Poisson ideal hence the coefficient of $dx_r \wedge dx_s$ in β_i is an element of $\mathcal{J}$ for $r, s \leq q$. After the Euler field contraction, every coefficient of dx_r in γ_i is also in $\mathcal{J}$, as required. Thus we have a formal diffeomorphism $\exp(\mu_i)$ that will eliminate β_i.

Passing to the formal limit $i \to \infty$, we get an isomorphism of

$$\widehat{\mathcal{O}}_{M,y} \simeq k[[x_1, \ldots, x_n, y_1, \ldots, y_n]]$$

which takes the completion of I_Y to the ideal $\mathcal{J}$, and is compatible with the symplectic forms. This proves the "quasiclassical" part of the statement.

Both $\widehat{\mathcal{O}}_h$ and $\mathcal{D}$ are deformation quantizations of the same algebra

$$k[[x_1, \ldots, x_n, y_1, \ldots, y_n]],$$

corresponding to two formal Poisson bivectors $h(\sum \partial/\partial x_i \wedge \partial/\partial y_i) + h^2\pi_2 + \cdots$ with the same h-linear part $\pi_1 = \sum \partial/\partial x_i \wedge \partial/\partial y_i$. By the general Maurer–Cartan formalism, Poisson bivectors with fixed linear part correspond to Maurer–Cartan solutions of the algebra of polyvector fields with the nonzero differential $[\pi_1, \cdot]$. Using α_0 to convert polyvector fields to differential forms, we get the complex in which the bracket with π_1 becomes the de Rham differential. Since the formal de Rham complex is exact in degree two, there is a unique quantization with the h-linear part π_1. Hence the above isomorphism modulo h extends to an isomorphism $\sigma_{\mathcal{D}}$. If it can be extended to pair $(\sigma_{\mathcal{D}}, \sigma_{\mathcal{M}})$ compatible with the module action, then $\widehat{I}_h$, resp. $\mathcal{J}$, is the annihilator of $\widehat{E}_{h,y}/h\widehat{E}_{h,y}$, resp. $\mathcal{M}/h\mathcal{M}$, which implies compatibility with ideals stated in (1).

For existence of $\sigma_{\mathcal{M}}$ in part (2), assume that compatibility with ideals does hold, and first construct the isomorphism modulo h and then lift it inductively modulo higher powers of h. Indeed, using $\sigma_{\mathcal{M}}$ we can view $\widehat{E}_{h,y}/h\widehat{E}_{h,y}$ and $\mathcal{M}/h\mathcal{M}$ as projective (hence free) modules of the same rank over the local ring $\widehat{\mathcal{O}}_{Y,y}$. Hence there is an isomorphism $\rho_0 : \mathcal{M}/h\mathcal{M} \to \widehat{E}_{h,y}/h\widehat{E}_{h,y}$

To lift it to $\widehat{E}_{h,y}$ and $\mathcal{M}$, take the standard space of generators $k^{\oplus e} \subset \mathcal{M}$, and choose any lift $\rho : k^{\oplus e} \to \widehat{E}_{h,y}$ of $\rho_0|_{k^{\oplus e}}$. Consider the subalgebra $\mathcal{D}' \subset \mathcal{D}$ with (topological) generators $x_1, \ldots, x_n, y_{q+1}, \ldots, y_n, h$ and the map $\sigma' : \mathcal{D}' \otimes_k k^{\oplus e} \to \widehat{E}_{h,y}$, $f \otimes v \mapsto f \cdot \rho(v)$. By a version of Nakayama's lemma, it is an isomorphism of $k[[h]]$-modules. It induces an isomorphism with $(\mathcal{D}/\mathcal{D}\langle y_1, \ldots, y_q\rangle)^{\oplus e} = \mathcal{M}$ (as $\mathcal{D}$-modules) precisely when $y_s \cdot \rho(v) = 0$ for any $v \in k^{\oplus e}$ and $s \leq q$. Our goal is to adjust ρ to achieve this condition inductively, ensuring that the vanishing holds modulo h^l for $l \geq 1$. This obviously works for $l = 1$ as y_s acts by zero on $\widehat{E}_{h,y}/h\widehat{E}_{h,y}$. To make an inductive step, suppose that we have $\rho_{l+1} : k^{\oplus e} \to \widehat{E}_{h,y}/h^{l+1}\widehat{E}_{h,y}$ and that by inductive assumption $y_s \cdot \rho_{l+1}(v)$ is divisible by h^l for all v and $s \leq q$. Let $U = Im(\rho_{l+1})$ and let $u_1, \ldots, u_e \in U$ be the images of the standard basis vectors. By assumption,

$$y_s \cdot u_i = h^l \sum_j f_{si}^j u_j, \quad f_{si}^j \in \mathcal{D}'.$$

We are looking for elements

$$u_i' = u_i + h^{l-1} \sum_j g_i^j u_j, \quad g_i^j \in \mathcal{D}'$$

which satisfy $y_s \cdot u_i' = 0$ in $\widehat{E}_{h,y}/h^{l+1}\widehat{E}_{h,y}$. For $l \geq 2$, this gives

$$hf_{si}^j + [y_s, g_i^j] = 0 \pmod{h^2}.$$

Introducing matrices $F_s = (f_{si}^j)$, $G = (g_i^j)$ with entries in

$$\mathcal{D}'/h\mathcal{D}' \simeq k[[x_1, \ldots, x_n, y_{q+1}, \ldots, y_n]]$$

and using the fact that $[y_s, \cdot]$ for $s \leq q$ acts as $h(\partial/\partial x_s)$, we get a system of equations $F_s = \partial G/\partial x_s$ on matrices with coefficients in $k[[x_1, \ldots, x_n, y_{q+1}, \ldots, y_n]]$. This has a solution precisely when $\partial F_s/\partial x_t = \partial F_t/\partial x_s$ for $s, t \leq q$ (by vanishing of formal de Rham cohomology in degree 1). But the latter equation is a consequence of $y_t y_s \cdot u_i = y_s y_t \cdot u_i$. For $l = 1$, the equations read

$$hF_s + (y_s G - G y_s) + hGF_s = 0 \pmod{h^2}$$

which is equivalent to $\partial(\log(1 + G))/\partial x_s = F_s$. As before, we can find a matrix H such that $\partial H/\partial x_s = F_s$ and assume that its entries have zero constant terms. Then the matrix entries of $G = \exp(H) - 1$ are formal power series with zero constant term. To achieve $y_s \cdot u_i = 0 \pmod{h^{l+1}}$, we are adjusting u_i by adding vectors that vanish $\pmod{h^{i-1}}$, for all $l \geq 1$. Hence the limit as $l \to \infty$ is well defined, and that gives a system of $\mathcal{D}'$-generators $u_1, \ldots, u_e$ which also satisfy $y_s \cdot u_i = 0$. This gives an isomorphism $\sigma_{\mathcal{M}}$, as required. ■

4.2. Harish-Chandra pairs and torsors associated to $(\mathcal{O}_h, E_h)$

It is clear from the proofs that isomorphisms of Lemma 4.1 are not unique. Two different choices of the pair $(\sigma_{\mathcal{D}}, \sigma_{\mathcal{M}})$ are related by automorphisms

$$\Phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}, \quad \Phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$$

compatible with the module action and filtrations. The group $\mathrm{Aut}(\mathcal{D}, \mathcal{M})$ formed by all such $(\Phi_{\mathcal{D}}, \Phi_{\mathcal{M}})$ has a natural structure of a proalgebraic group, given by reducing the automorphism modulo the image of $\mathfrak{m}^k \subset \mathcal{D}$, where $\mathfrak{m}$ is the kernel of the algebra homomorphism $\mathcal{D} \to k$ sending h, x_i, y_j to zero. The tangent Lie algebra is formed by pairs $(\phi_{\mathcal{D}}, \phi_{\mathcal{M}})$, where $\phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a continuous derivation preserving $\mathfrak{m}$ and $\phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ satisfies

$$\phi_{\mathcal{M}}(fm) = \phi_{\mathcal{D}}(f)m + f\phi_{\mathcal{M}}(m).$$

If we drop the condition that $\phi_{\mathcal{D}}$ should preserve $\mathfrak{m}$, we obtain a somewhat larger Lie algebra $\mathrm{Der}(\mathcal{D}, \mathcal{M})$. The following lemma, established by direct computation, clarifies its structure.

Lemma 4.2. *The following short exact sequences hold.*

(1) *The map* $(\phi_{\mathcal{D}}, \phi_{\mathcal{M}}) \mapsto \phi_{\mathcal{D}}$ *induces a short exact sequence*

$$0 \to \mathcal{E} = gl_e(\mathcal{D}_p) \to \mathrm{Der}(\mathcal{D}, \mathcal{M}) \to \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \to 0, \tag{9}$$

where $\mathrm{Der}(\mathcal{D})_{\mathcal{J}}$ *is the algebra of continuous derivations of* $\mathcal{D}$ *which sent* $\mathcal{J}$ *to itself.*

(2) *Commutator of left action of* $\phi_{\mathcal{M}}$ *with right action of* $\mathcal{E}$ *induces the right arrow in the short exact sequence*

$$0 \to \frac{1}{h}\mathcal{J} \to \mathrm{Der}(\mathcal{D}, \mathcal{M}) \to \mathrm{Der}(\mathcal{E}) \to 0. \tag{10}$$

(3) *The reduction of derivations in* $\mathrm{Der}(\mathcal{E})$*, resp. in* $\mathrm{Der}(\mathcal{D})_{\mathcal{J}}$*, modulo h, resp. modulo* $\mathcal{J}$*, induces short exact sequences:*

$$0 \to \mathcal{E}/\frac{1}{h}k[[h]] \to \mathrm{Der}(\mathcal{E}) \to \mathrm{Ham}_q \to 0,$$
$$0 \to \frac{1}{h}\mathcal{J}/k[[h]] \to \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \to \mathrm{Ham}_q \to 0,$$

where Ham_q *is the algebra of Hamiltonian derivations of* $k[[x_1, \dots, x_q, y_1, \dots, y_q]]$*. The two compositions* $\mathrm{Der}(\mathcal{D}, \mathcal{M}) \to \mathrm{Ham}_q$ *agree and this gives rise to a short exact sequence:*

$$0 \to k[[h]] \to \mathrm{Der}(\mathcal{D}, \mathcal{M}) \to \mathrm{Der}(\mathcal{D})_{\mathcal{J}} \times_{\mathrm{Ham}_q} \mathrm{Der}(\mathcal{E}) \to 0. \tag{11}$$

■

Following the pattern of [25, Section 5], [24, Section 6] or [11, Sections 2 and 3], we see that all pairs $(\sigma_{\mathcal{D}}, \sigma_{\mathcal{M}})$ are parameterized by a Harish-Chandra torsor $P_{\mathcal{D},\mathcal{M}}$ over the pair $(\mathrm{Der}(\mathcal{D}, \mathcal{M}), \mathrm{Aut}(\mathcal{D}, \mathcal{M}))$. Similarly, all isomorphisms $\sigma_{\mathcal{E}}$ are parameterized by a Harish-Chandra torsor $P_{\mathcal{E}}$ over the pair $(\mathrm{Der}(\mathcal{E}), \mathrm{Aut}(\mathcal{E}))$. We note here that for $P_{\mathcal{D},\mathcal{M}}$ the connection form γ of Section 3.2 is an isomorphism (such torsors are called transitive) while for $P_{\mathcal{E}}$ the short exact sequence (10) implies that this torsor is foliated over $\mathcal{F}$.

4.3. The class τ_Y is the image of τ_{Lie}

By Proposition 3.4 and [6, Section 4.0.3], the characteristic class

$$\tau_Y = \hat{A}(Q) \exp\left(-\frac{c_1(N)}{2}\right) e^{-c(\mathcal{O}_h)} \mathrm{ch}(E) \in H^{\bullet}_{DR}(Y)((h))$$

is equal to the image, with respect to the Gelfand–Fuks map of the torsor $P_{\mathcal{D},\mathcal{M}}$, of the class

$$\tau_{\mathrm{Lie}} = \hat{A}_{\mathrm{Lie}}(\mathfrak{sp}_{2p}) \exp\left(-\frac{c_{1,\mathrm{Lie}}(\mathfrak{gl}_q)}{2}\right) e^{-c} \mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e),$$

where the factors other than e^{-c} are defined at the end of Section 3.1 and c is obtained from the extension class of

$$0 \to \frac{1}{h}k[[h]] \to \frac{1}{h}\mathcal{D} \to \mathrm{Der}(\mathcal{D}) \to 0$$

by restricting to the subalgebra $\mathrm{Der}(\mathcal{D})_{\mathcal{J}} \subset \mathrm{Der}(\mathcal{D})$ and then pulling back under the surjection of (9). For the factors other than e^{-c}, we use the fact that the components of τ_Y can be defined by using the Chern–Weil construction on the torsor of symplectic frames in Q and the torsor of usual frames in N, E. Since these torsors can be induced from $P_{\mathcal{D},\mathcal{M}}$, we can apply compatibility of Gelfand–Fuks map with induced torsors to reinterpret the classes via Lie algebra cohomology of $\mathrm{Der}(\mathcal{D}, \mathcal{M})$. We record for future reference that

$$\tau_{\mathrm{Lie}} \in H^*\big(\mathrm{Der}(\mathcal{D}, \mathcal{M}), \mathfrak{gl}_q \oplus \mathfrak{gl}_r \oplus \mathfrak{sp}_{2p} \oplus \mathfrak{a}'; k((h))\big).$$

4.4. Reduction to class $\tau_{\mathcal{D}_p}$ and end of proof

Lemma 4.3. *The cohomology class τ_{Lie} is represented by a cocycle which vanishes if one of its arguments is in $\frac{1}{h}\mathcal{J}$. Hence τ_{Lie} is a pullback of a cohomology class of* $\mathrm{Der}(\mathcal{E})$ *via the surjection in* (10) *and that class is further equal to the class of Lemma* 3.5*:*

$$\tau_{\mathcal{D}_p} \in H^\bullet_{\mathrm{Lie}}\big(\mathrm{Der}(\mathcal{E}), \mathfrak{pgl}_r \oplus \mathfrak{sp}_{2p}; k((h))\big) = H^\bullet_{\mathrm{Lie}}\big(\mathfrak{g}, \mathfrak{gl}_r \oplus \mathfrak{sp}_{2p} \oplus \mathfrak{a}'; k((h))\big).$$

Proof. Step 1. We first recall the definitions. Assign the elements in $\mathfrak{gl}_e(k) \subset \mathrm{Der}(\mathcal{D}, \mathcal{M})$ degree 0 and keep assuming that $\deg h = 2$, $\deg x_i = \deg y_j = 1$. Then any element of $\mathrm{Der}(\mathcal{D}, \mathcal{M})$ is a possibly infinite sum of homogeneous elements of degree ≥ -1 and the Lie bracket is homogeneous.

Then $\mathfrak{gl}_e$ and $\mathfrak{sp}_{2p}$ are spanned by elements $\frac{1}{h}x_i y_j$, $1 \leq i, j \leq q$ and $\frac{1}{h}x_s x_t$, $\frac{1}{h}y_s y_t$, $\frac{1}{h}(x_s y_t + y_t x_s)$, $q+1 \leq s, t \leq p+q$, respectively, and the degree zero part of $\mathrm{Der}(\mathcal{D}, \mathcal{M})$ splits as

$$\mathfrak{gl}_e \oplus \mathfrak{gl}_q \oplus \mathfrak{sp}_{2p} \oplus W,$$

where W is spanned by $\frac{1}{h}y_j y_t$, $\frac{1}{h}x_s y_j$ with $1 \leq j \leq q$, $1 \leq t \leq (p+q)$, $(q+1) \leq s \leq (p+q)$ (we note here that in the specified ranges the variables commute). This gives a projection

$$\mathrm{Der}(\mathcal{D}, \mathcal{M}) \to \mathfrak{gl}_e \oplus \mathfrak{gl}_q \oplus \mathfrak{sp}_{2p}$$

sending the elements of nonzero degree to zero, and vanishing on W. We can combine it with a natural projection to any of the three factors on the right-hand side, to be used for calculation of classes $\mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e)$, $\widehat{A}_{\mathrm{Lie}}(\mathfrak{sp}_{2p})$, $\exp(-\frac{c_{1,\mathrm{Lie}}(\mathfrak{gl}_q)}{2})$ in the definition of τ_{Lie}.

The curvature defined in (4) is not zero only when its arguments have degrees $-1, 0$ or 1. The same applies to the degree zero component c_0 of c. We recall here that c_0 is computed with respect to the projection onto k which vanishes on elements of non-zero degrees, trace zero matrices in $\mathfrak{gl}_e$ and on the subspaces $\mathfrak{sp}_{2p}$, W, and on the elements of the type $\frac{1}{h}(x_i y_j + y_j x_i)$.

Step 2. Let us show that the $\widehat{A}_{\mathrm{Lie}}$ class is pulled back from the quotient by $\frac{1}{h}\mathcal{J}$. In fact, consider the curvature $C(u \wedge v) \in \mathfrak{h}$ for the projection onto $\mathfrak{h} = \mathfrak{sp}_{2p}$ and $u \in \frac{1}{h}\mathcal{J}$ of degree $-1, 0$ or 1. It follows from (10) that $\frac{1}{h}\mathcal{J}$ is a Lie ideal which has zero projection onto $\mathfrak{sp}_{2p}$, so all positive components of $\widehat{A}_{\mathrm{Lie}}(\mathfrak{sp}_{2p})$ vanish if one of the arguments is in $\frac{1}{h}\mathcal{J}$.

Step 3. Let us prove that the cochain representing the class

$$\mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e)\exp\left(-\frac{1}{2}c_{1,\mathrm{Lie}}(\mathfrak{gl}_q)-c_0\right)$$

is zero if one of its arguments is in $\frac{1}{h}\mathcal{J}$. Since c_0 vanishes on elements $\frac{1}{h}(x_i y_j + y_j x_i)$, we have

$$\begin{aligned} c_0\Big(\frac{1}{h}\sum b_{ij}x_i y_j\Big) &= c_0\Big(\frac{1}{2h}\sum b_{ij}(x_i y_j - y_j x_i) + \frac{1}{2h}\sum b_{ij}(x_i y_j + y_j x_i)\Big) \\ &= -\frac{1}{2}\sum b_{ii}. \end{aligned}$$

Since $\sum b_{ii}$ is the invariant polynomial corresponding to $c_{1,\mathrm{Lie}}(\mathfrak{gl}_q)$, we conclude that $c_0 + c_{1,\mathrm{Lie}}(\mathfrak{gl}_q)$ corresponds to the linear function on $\mathfrak{h} = \mathfrak{gl}_e \oplus \mathfrak{gl}_q$ which sends (X_1, X_2) to $\frac{1}{e}\,\mathrm{tr}(X_1)$. Moreover, since $\mathrm{ch}_{\mathrm{Lie}}(\mathfrak{gl}_e)$ comes from $\mathrm{tr}(\exp(x))$ and $\mathrm{tr}(\exp(x-\alpha\cdot I)) = \mathrm{tr}(\exp(x))\exp(-\alpha)$, we can rewrite the above class as the image of the invariant series

$$S(X_1\oplus X_2) = \sum_{n\geq 0}\frac{1}{n!}\,\mathrm{tr}\left(X_1 - \frac{1}{e}\,\mathrm{tr}(X_1)\right)^n$$

under the Chern–Weil map. We denote by $\bar{X} = X_1 - \frac{1}{e}\,\mathrm{tr}(X_1)$ the trace zero part of X_1 and by $S_l(X) = \frac{1}{l!}\,\mathrm{tr}(\bar{X}_1)^l$ the degree l component of the invariant power series. Recall that the Chern–Weil class corresponding to $S_l(X)$ is obtained by polarization of $S_l(X)$:

$$\begin{aligned} &\rho(S_l)(v_1\wedge\cdots\wedge v_{2l}) \\ &\quad= \frac{1}{l!}\sum_{\sigma}(-1)^{\sigma}\,\mathrm{tr}\left(\bar{C}(v_{\sigma(1)}\wedge v_{\sigma(2)})\bar{C}\big((v_{\sigma(3)}\wedge v_{\sigma(4)})\cdots\bar{C}(v_{\sigma(2l-1)}\wedge v_{\sigma(2l)})\big)\right), \end{aligned}$$

where the sum is over all permutations $\sigma\in S_{2n}$ that satisfy $\sigma(2i-1)<\sigma(2i)$. So it suffices to show that $\bar{C}=0$ in $\mathfrak{gl}_e$ if $C = C(u\wedge v)$ with $u\in\frac{1}{h}\mathcal{J}$. This holds since $\frac{1}{h}\mathcal{J}$ is a Lie ideal and its projection onto $\mathfrak{gl}_q$ lands into the subspace of scalar matrices which have trivial $\bar{X}$ part.

Step 4. It remains to show that the class $\exp(-(c-c_0))$ is in the image of the pullback under the projection $\mathrm{Der}(\mathcal{D},\mathcal{M})\to\mathrm{Der}(\mathcal{E})$. Recall that c was defined in Section 4.3. Let θ be the pullback of a similar class under $\mathrm{Der}(\mathcal{D},\mathcal{M})\to\mathrm{Der}(\mathcal{E})$. Then by the short exact sequence (11), the sum $c-\theta$ is zero (the minus sign in front of θ is due to the fact that $\mathcal{M}$ was considered as a right $\mathcal{E}$-module or, equivalently, a left $\mathcal{E}^{op}$-module). We are using the fact that the sum of two extensions descends to the fiber product over Ham_p, that the pullback of the extension class to the extension algebra $\mathrm{Der}(\mathcal{D},\mathcal{M})$ must be zero, and that existence of $P_{\mathcal{D},\mathcal{M}}$ allows to use the Gelfand–Fuks map associated to this torsor. Hence, on the level of cohomology $c=\theta$ and the same holds for each of the coefficients in the expansion in powers of h, e.g. $c_0=\theta_0$.

The fact that the class pulled back from $\mathrm{Der}(\mathcal{E})$ is exactly $\tau_{\mathcal{D}_p}$ follows from the definition of τ_{Lie} and the vanishing proved. ∎

End of proof of Theorem 1.1. By Section 2.2 (application of Riemann–Roch theorem), the class $\tau(E_h)$ is the image of $\tau_Y(E)$ in the cohomology of Y.

By Proposition 3.4, the class $\tau_Y(E)$ in the de Rham cohomology of Y is the image of a class τ_{Lie} under the Gelfand–Fuks map associated to the Harish-Chandra torsor $P_{\mathfrak{D},\mathcal{M}}$ of all isomorphisms $(\sigma_{\mathfrak{D}}, \sigma_{\mathcal{M}})$.

By Lemma 4.3, we can replace the pair $(\tau_{\mathrm{Lie}}, P_{\mathfrak{D},\mathcal{M}})$ by the pair $(\tau_{\mathfrak{D}_p}, P_{\mathcal{E}})$, where the torsor $P_{\mathcal{E}}$ parameterizes isomorphisms $\sigma_{\mathcal{E}} : \mathrm{End}_{\widehat{\mathcal{O}}_h} \widehat{E}_h \to \mathcal{E}$ (we note here that a choice of $(\sigma_{\mathfrak{D}}, \sigma_{\mathcal{M}})$ also induces a choice of $\sigma_{\mathcal{E}}$).

But $P_{\mathcal{E}}$ is a Harish-Chandra torsor foliated over $\mathcal{F} \subset T_Y$, hence the characteristic class τ_Y admits a lift to $\bigoplus_{r\geq 0}^{p} H^{2r}(Y, F^r\Omega_Y^\bullet)((h))$ by Proposition 3.4.

Finally, the components of $\tau_{\mathfrak{D}_p}$ vanish in cohomology groups of degrees $> 2p$ by Lemma 3.5. This finishes the proof of Theorem 1.1 ■

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