

Simplicity of twisted C^* -algebras of Deaconu–Renault groupoids

Becky Armstrong, Nathan Brownlowe, and Aidan Sims

Abstract. We consider Deaconu–Renault groupoids associated to actions of finite-rank free abelian monoids by local homeomorphisms of locally compact Hausdorff spaces. We study simplicity of the twisted C^* -algebra of such a groupoid determined by a continuous circle-valued groupoid 2-cocycle. When the groupoid is not minimal, this C^* -algebra is never simple, so we focus on minimal groupoids. We describe an action of the quotient of the groupoid by the interior of its isotropy on the spectrum of the twisted C^* -algebra of the interior of the isotropy. We prove that the twisted groupoid C^* -algebra is simple if and only if this action is minimal. We describe applications to crossed products of topological-graph C^* -algebras by quasi-free actions.

1. Introduction

The purpose of this paper is to characterise simplicity of twisted C^* -algebras arising from continuous 2-cocycles on Deaconu–Renault groupoids of actions of \mathbb{N}^k on second-countable locally compact Hausdorff spaces. The study of twisted C^* -algebras associated to continuous groupoid 2-cocycles dates back to Renault’s seminal work [36]. They serve both as a very flexible C^* -algebraic framework for modelling dynamical systems, and as a source of tractable models for classifiable C^* -algebras [8, 20, 32, 34]. So it is important to be able to determine when a given twisted groupoid C^* -algebra is simple; but this is in general a complicated question.

Deaconu–Renault groupoids encode actions of submonoids of abelian groups by local homeomorphisms of locally compact Hausdorff spaces. In hindsight, the first example of such a groupoid was the one associated to the one-sided full shift on n letters, introduced by Renault in [36] as a model for the Cuntz algebra. However groupoids of this type for generic local homeomorphisms (that is, actions of \mathbb{N}) were first studied by Deaconu [12], and have come to be known as (rank-1) Deaconu–Renault groupoids. Shortly afterwards they were used as models for graph C^* -algebras in [24, 25], and later still, Yeend [43] showed that rank-1 Deaconu–Renault groupoids provide models for the topological-graph C^* -algebras of Katsura [21].

For the dual reasons that most of the key examples studied had been related to 0-dimensional spaces, and that \mathbb{N} embeds in \mathbb{Z} , which has trivial cohomology, no work was done on twisted C^* -algebras associated to Deaconu–Renault groupoids for many years. However, in 2000, Kumjian and Pask [23] introduced higher-rank graphs (or k -graphs) and demonstrated that the associated C^* -algebras can be described as the C^* -algebras of Deaconu–Renault groupoids of actions of \mathbb{N}^k . This led to the development [26–28] of twisted k -graph C^* -algebras. Kumjian, Pask, and Sims showed that from a 2-cocycle on a k -graph, one can construct a 2-cocycle on the associated Deaconu–Renault groupoid so that the twisted C^* -algebras coincide, and they used this model to characterise simplicity of twisted k -graph C^* -algebras [29], as well as to describe applications of this characterisation to the study of crossed products of graph algebras by quasi-free actions.

Here we build substantially on elements of the analysis of [29] to describe precisely when the twisted C^* -algebra of a Deaconu–Renault groupoid for an action of \mathbb{N}^k by local homeomorphisms is simple (Theorem 6.1). To demonstrate the applicability of our main theorem, we use this result to investigate simplicity of crossed products of C^* -algebras associated to rank-1 Deaconu–Renault groupoids by actions of \mathbb{Z} induced by \mathbb{T} -valued 1-cocycles (Theorem 7.4), and we specialise to the Deaconu–Renault groupoids of topological graphs to characterise simplicity of crossed products of topological-graph C^* -algebras by quasi-free automorphisms (Corollaries 7.6 and 7.7).

The paper is organised as follows. In Section 2 we establish background and notation. In Section 3 we describe the periodicity group P_T of a minimal action T of \mathbb{N}^k on a second-countable locally compact Hausdorff space X , and we show that the interior of the isotropy of \mathcal{G}_T is isomorphic to the group bundle $X \times P_T$. In Section 4 we show that every 2-cocycle on \mathcal{G}_T is cohomologous to one whose restriction to $X \times P_T$ is determined by a fixed bicharacter ω of P_T that vanishes on its own centre Z_ω , and we use this to give a concrete description of the spectral action θ of $\mathcal{G}_T/\mathcal{I}_T$ on $X \times \widehat{Z}_\omega$. Then in Section 6 we state and prove our main theorem. We finish in Section 7 by describing an application to crossed products of rank-1 Deaconu–Renault groupoid C^* -algebras by automorphisms induced by continuous 1-cocycles. We provide two appendices—one on group cohomology and one on twisted group C^* -algebras—to provide a handy reference to some key results on these two topics that we need in the body of the paper, and have found difficult to locate explicitly in the literature.

2. Background

2.1. Group 2-cocycles and bicharacters

Here we briefly recall some key facts about second cohomology for discrete groups. For more detail see [9, Chapter IV] and [6, 22] (or [3, Chapter 2] for the key points relevant here collected in one place).

Let G be a discrete group and let A be a multiplicative abelian group. We write $Z^2(G, A)$ for the group of normalised A -valued 2-cocycles on G , $B^2(G, A)$ for the sub-

group of coboundaries, δ^1 for the coboundary map, and $H^2(G, A)$ for the second cohomology group $Z^2(G, A)/B^2(G, A)$. Given $\sigma \in Z^2(G, A)$, we write σ^* for the 2-cocycle $(g, h) \mapsto \sigma(h, g)^{-1}$. We call σ *antisymmetric* if $\sigma = \sigma^*$.

A *bicharacter* of G is a map $\omega: G \times G \rightarrow \mathbb{T}$ such that $\omega(\cdot, g)$ and $\omega(g, \cdot)$ are homomorphisms from $G \rightarrow \mathbb{T}$ for each $g \in G$. Every bicharacter is a \mathbb{T} -valued 2-cocycle. If G is a discrete abelian group, then $\sigma \mapsto \sigma\sigma^*$ is a homomorphism from $Z^2(G, \mathbb{T})$ to the group of antisymmetric bicharacters of G , which descends to an isomorphism of $H^2(G, \mathbb{T})$ onto the same group [33, Proposition 3.2].

In this paper, the *centre* Z_σ of a 2-cocycle σ on G is the joint kernel of the associated antisymmetric bicharacter: $Z_\sigma := \{g \in G : (\sigma\sigma^*)(g, h) = 1 \text{ for all } h \in G\}$. If $\sigma(g, h) = 1 = \sigma(h, g)$ for all $g \in Z_\sigma$ and $h \in G$, then we say that σ *vanishes on its centre*. An adaptation of the argument of [33, Proposition 3.2] (see [3, Theorem 2.2.8] for details) shows that every 2-cocycle on a finitely generated discrete abelian group is cohomologous to a bicharacter that vanishes on its centre.

2.2. Hausdorff étale groupoids

We refer to a topological groupoid \mathcal{G} with a locally compact Hausdorff topology under which multiplication and inversion are continuous as a *Hausdorff groupoid*. We write $\mathcal{G}^{(0)}$ for the unit space of \mathcal{G} , and $\mathcal{G}^{(2)}$ for the set of composable pairs in \mathcal{G} . Given subsets $A, B \subseteq \mathcal{G}$, we write $AB := \{\alpha\beta : (\alpha, \beta) \in (A \times B) \cap \mathcal{G}^{(2)}\}$ and $A^{-1} := \{\alpha^{-1} : \alpha \in A\}$, and for $\gamma \in \mathcal{G}$, we write $\gamma A := \{\gamma\alpha : \alpha \in A\}$ and $A\gamma := \{\alpha\gamma : \alpha \in A\}$. We say that \mathcal{G} is *étale* if the range and source maps $r, s: \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ are local homeomorphisms. We call a subset B of \mathcal{G} a *bisection* if B is contained in an open subset U of \mathcal{G} such that $r|_U$ and $s|_U$ are homeomorphisms onto open subsets of $\mathcal{G}^{(0)}$. Every second-countable Hausdorff étale groupoid has a countable basis of open bisections. For each $x \in \mathcal{G}^{(0)}$, we define $\mathcal{G}_x := s^{-1}(x)$ and $\mathcal{G}^x := r^{-1}(x)$; and $\mathcal{G}_x^x := \mathcal{G}_x \cap \mathcal{G}^x$. We say that \mathcal{G} is *minimal* if $r(\mathcal{G}_x)$ is dense in $\mathcal{G}^{(0)}$ for every $x \in \mathcal{G}^{(0)}$. The *isotropy subgroupoid* of \mathcal{G} is the groupoid $\text{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} \mathcal{G}_x^x = \{\gamma \in \mathcal{G} : r(\gamma) = s(\gamma)\}$. The interior \mathcal{I} of the isotropy of a Hausdorff étale groupoid \mathcal{G} is itself a Hausdorff étale groupoid with unit space $\mathcal{I}^{(0)} = \mathcal{G}^{(0)}$. We say that \mathcal{G} is *effective* if $\mathcal{I} = \mathcal{G}^{(0)}$, and we say that \mathcal{G} is *topologically principal* if $\{x \in \mathcal{G}^{(0)} : \mathcal{G}_x^x = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$. By [7, Lemma 3.1], every topologically principal Hausdorff étale groupoid is effective, and every effective second-countable Hausdorff étale groupoid is topologically principal.

The following definition of a groupoid action comes from [18, Definition 1.60].

Definition 2.1. Suppose that \mathcal{G} is a topological groupoid and X is a topological space. We say that \mathcal{G} *acts continuously on (the left of) X* , and that X is a *continuous (left) \mathcal{G} -space*, if there is a continuous surjective map $R: X \rightarrow \mathcal{G}^{(0)}$ and a continuous map $\theta: (\gamma, x) \mapsto \gamma \cdot x$ from $\mathcal{G} \star X := \{(\gamma, x) \in \mathcal{G} \times X : s(\gamma) = R(x)\}$ to X , satisfying

- (A1) if $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $(\beta, x) \in \mathcal{G} \star X$, then $(\alpha\beta, x), (\alpha, \beta \cdot x) \in \mathcal{G} \star X$, and we have $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$; and
- (A2) for all $x \in X$, we have $(R(x), x) \in \mathcal{G} \star X$, and $R(x) \cdot x = x$.

We refer to the map θ as a *continuous (left) action* of \mathcal{G} on X . For each $x \in X$, the *orbit* of x under θ is the set

$$[x]_\theta := \{\gamma \cdot x : (\gamma, x) \in \mathcal{G} \star X\}.$$

2.3. Cohomology of groupoids

We now recall the relevant cohomology theory for groupoids from [36, Section I.1].

Definition 2.2. Let \mathcal{G} be a topological groupoid, and let A be a topological abelian group with identity e_A .

- (i) A *continuous A -valued 1-cochain* on \mathcal{G} is a continuous map $b: \mathcal{G} \rightarrow A$. We say that b is *normalised* if $b(r(\gamma)) = b(s(\gamma)) = e_A$ for all $\gamma \in \mathcal{G}$.
- (ii) A *continuous A -valued 1-cocycle* on \mathcal{G} is a continuous 1-cochain $c: \mathcal{G} \rightarrow A$ satisfying $c(\alpha\beta) = c(\alpha)c(\beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$.
- (iii) A *continuous A -valued 2-cocycle* on \mathcal{G} is a continuous map $\sigma: \mathcal{G}^{(2)} \rightarrow A$ that satisfies the *2-cocycle identity*: $\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\alpha, \beta\gamma)\sigma(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in \mathcal{G}$ such that $s(\alpha) = r(\beta)$ and $s(\beta) = r(\gamma)$, and is *normalised*, in the sense that $\sigma(r(\gamma), \gamma) = \sigma(\gamma, s(\gamma)) = e_A$ for all $\gamma \in \mathcal{G}$. We write $Z^2(\mathcal{G}, A)$ for the group of continuous A -valued 2-cocycles on \mathcal{G} .
- (iv) The *continuous 2-coboundary* associated to a continuous normalised A -valued 1-cochain $b: \mathcal{G} \rightarrow A$ is the map $\delta^1 b: \mathcal{G}^{(2)} \rightarrow A$ given by

$$\delta^1 b(\alpha, \beta) := b(\alpha)b(\beta)b(\alpha\beta)^{-1}.$$

- (v) We say that two continuous 2-cocycles $\sigma, \tau: \mathcal{G}^{(2)} \rightarrow A$ are *cohomologous* if there exists a continuous normalised 1-cochain such that $\delta^1 b(\alpha, \beta) = \sigma(\alpha, \beta)^{-1}\tau(\alpha, \beta)$ for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$.

We write \mathbb{T} for the multiplicative group of complex numbers of modulus 1. Suppose that \mathcal{G} is a Hausdorff groupoid and take $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Let $\mathcal{G} \times_\sigma \mathbb{T}$ be the set $\mathcal{G} \times \mathbb{T}$ endowed with the product topology, and equipped with the multiplication operation

$$(\alpha, w)(\beta, z) := (\alpha\beta, \sigma(\alpha, \beta)wz), \tag{2.1}$$

defined for all $(\alpha, \beta) \in \mathcal{G}^{(2)}$ and $w, z \in \mathbb{T}$, and the inversion operation

$$(\alpha, w)^{-1} := (\alpha^{-1}, \overline{\sigma(\alpha, \alpha^{-1})w}), \tag{2.2}$$

defined for all $(\alpha, w) \in \mathcal{G} \times \mathbb{T}$. Then $\mathcal{G} \times_\sigma \mathbb{T}$ is a Hausdorff groupoid.

2.4. Twisted groupoid C*-algebras

We now recall Renault’s construction of the full twisted groupoid C*-algebra $C^*(\mathcal{G}, \sigma)$ associated to a Hausdorff étale groupoid \mathcal{G} and a continuous \mathbb{T} -valued 2-cocycle σ on \mathcal{G} . Note that Renault gives this construction for groupoids that are not necessarily étale, but

we specialise to the étale case since we will primarily be dealing with Deaconu–Renault groupoids, which are étale. Renault also defines reduced twisted groupoid C^* -algebras, but we will only be working with amenable groupoids, and in this setting, the full and reduced C^* -algebras coincide. Let $C_c(\mathcal{G}, \sigma)$ denote the complex vector space of continuous compactly supported complex-valued functions on \mathcal{G} , equipped with multiplication given by the *twisted convolution* formula

$$(f * g)(\gamma) := \sum_{\substack{(\alpha, \beta) \in \mathcal{G}^{(2)}, \\ \alpha\beta = \gamma}} \sigma(\alpha, \beta) f(\alpha) g(\beta) = \sum_{\xi \in \mathcal{G}^{r(\gamma)}} \sigma(\xi, \xi^{-1}\gamma) f(\xi) g(\xi^{-1}\gamma),$$

and involution given by

$$f^*(\gamma) := \overline{\sigma(\gamma, \gamma^{-1}) f(\gamma^{-1})}.$$

Then $C_c(\mathcal{G}, \sigma)$ is a $*$ -algebra. We write fg for the twisted convolution product $f * g$ when the intended meaning is clear. The *full twisted groupoid C^* -algebra* $C^*(\mathcal{G}, \sigma)$ is the completion of $C_c(\mathcal{G}, \sigma)$ with respect to the *full C^* -norm*, which is given by

$$\|f\| := \sup \{ \|\pi(f)\| : \pi \text{ is a } *\text{-representation of } C_c(\mathcal{G}, \sigma) \}.$$

Given a locally compact Hausdorff space Y and a function $f \in C_c(Y)$, we define the *open support* of f to be the set $\text{osupp}(f) := f^{-1}(\mathbb{C} \setminus \{0\})$, and the *support* of f to be the set $\text{supp}(f) := \overline{\text{osupp}(f)}$.

2.5. Deaconu–Renault groupoids

We recall the definition of the Deaconu–Renault groupoid associated to an action of \mathbb{N}^k by local homeomorphisms. Details appear in [40, Proposition 3.1].

Fix $k \in \mathbb{N} \setminus \{0\}$. Let $T : n \mapsto T^n$ be an action of \mathbb{N}^k on a locally compact Hausdorff space X by local homeomorphisms. We call the pair (X, T) a *rank- k Deaconu–Renault system*. Define

$$\mathcal{G}_T := \{ (x, m - n, y) \in X \times \mathbb{Z}^k \times X : m, n \in \mathbb{N}^k, T^m(x) = T^n(y) \},$$

and

$$\mathcal{G}_T^{(2)} := \{ ((x, m, y), (w, n, z)) \in \mathcal{G}_T \times \mathcal{G}_T : y = w \}.$$

If $((x, m, y), (y, n, z)) \in \mathcal{G}_T^{(2)}$, then $(x, m + n, z), (y, -m, x) \in \mathcal{G}_T$. We define multiplication from $\mathcal{G}_T^{(2)}$ to \mathcal{G}_T by $(x, m, y)(y, n, z) := (x, m + n, z)$, and inversion on \mathcal{G}_T by $(x, m, y)^{-1} := (y, -m, x)$. Then \mathcal{G}_T is a groupoid, called a *Deaconu–Renault groupoid*. The unit space of \mathcal{G}_T is $\mathcal{G}_T^{(0)} = \{ (x, 0, x) : x \in X \}$, and we identify it with X . The range and source maps of \mathcal{G}_T are given by $r(x, m, y) := x$ and $s(x, m, y) := y$. For open sets $U, V \subseteq X$ and for $m, n \in \mathbb{N}^k$, we define

$$Z(U, m, n, V) := \{ (x, m - n, y) : x \in U, y \in V, \text{ and } T^m(x) = T^n(y) \}.$$

The collection $\{Z(U, m, n, V) : U, V \subseteq X \text{ are open, and } m, n \in \mathbb{N}^k\}$ is a basis for a locally compact Hausdorff topology on \mathcal{G}_T . The sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms onto their ranges and $T^m(U) = T^n(V)$ form a basis for the same topology. Under this topology, \mathcal{G}_T is a locally compact Hausdorff étale groupoid. If X is second-countable, then \mathcal{G}_T is also second-countable.

Remark 2.3. The action of \mathbb{N}^k in the above definition of a Deaconu–Renault system can be replaced with an action of a more general monoid P contained in a group G , and this gives rise to a G -graded Deaconu–Renault groupoid. Such groupoids are studied in [15], but we do not investigate them here.

Lemma 2.4. *Let (X, T) be a rank- k Deaconu–Renault system. The map $c : (x, n, y) \mapsto n$ is a continuous \mathbb{Z}^k -valued 1-cocycle on \mathcal{G}_T , and for each $x \in X$, the restriction of c to $(\mathcal{G}_T)_x^x$ is injective.*

Proof. Fix $\alpha = (x, p, y)$ and $\beta = (y, q, z) \in \mathcal{G}_T$. Then $c(\alpha\beta) = p + q = c(\alpha) + c(\beta)$, and so c is a 1-cocycle. Since each $c|_{Z(U, m, n, V)}$ is constant, c is locally constant and hence continuous. ■

Definition 2.5. Let (X, T) be a rank- k Deaconu–Renault system. The orbit under T of $x \in X$ is

$$[x]_T := \bigcup_{m, n \in \mathbb{N}^k} (T^m)^{-1}(T^n(x)) = \{y \in X : T^m(y) = T^n(x) \text{ for some } m, n \in \mathbb{N}^k\}.$$

We say that (X, T) is *minimal* if $[x]_T$ is dense in X for each $x \in X$. We frequently just write $[x]$ for $[x]_T$.

Remark 2.6. We have $[x] = r(s^{-1}(x)) \subseteq \mathcal{G}_T^{(0)}$, and so \mathcal{G}_T is minimal if and only if (X, T) is minimal.

Remark 2.7. By [40, Lemma 3.5], every Deaconu–Renault groupoid is amenable, and so we can discuss the twisted C*-algebra associated to a Deaconu–Renault groupoid and a continuous 2-cocycle without any ambiguity as to whether we mean the full or reduced C*-algebra.

Remark 2.8. The C*-algebras studied here are related to previous work. Suppose that Λ is a proper, source-free topological k -graph with infinite-path space Λ^∞ (as defined in [5, Section 3]). For each $n \in \mathbb{N}^k$, let $T^n : \Lambda^\infty \rightarrow \Lambda^\infty$ be the shift map. Then (Λ^∞, T) is a rank- k Deaconu–Renault system, and the associated Deaconu–Renault groupoid $\mathcal{G}_\Lambda := \mathcal{G}_T$ is called the *boundary-path groupoid* of the topological k -graph. The twisted C*-algebras $C^*(\mathcal{G}_\Lambda, \sigma)$ associated to continuous 2-cocycles $\sigma \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ on proper, source-free topological k -graphs generalise the twisted C*-algebras of discrete k -graphs studied in [26–29], and are studied in the first-named author’s Ph.D. thesis [3]. In [5], the first- and second-named authors study an alternative notion of a twisted C*-algebra of a topological k -graph associated to a continuous 2-cocycle on the topological k -graph itself, which is

constructed using a product system of Hilbert bimodules. In the case where Λ is a discrete k -graph, it is known (see [2, Theorem 7.2.2]) that these two constructions give the same C*-algebra, but in the more general topological setting, the relationship is unknown.

3. The interior of the isotropy of a Deaconu–Renault groupoid

In this section, we introduce the periodicity group P_T of a minimal Deaconu–Renault groupoid (X, T) , and we show that the interior I_T of the isotropy of \mathcal{G}_T can be identified with $X \times P_T$.

Definition 3.1. Let (X, T) be a rank- k Deaconu–Renault system. For each nonempty precompact open set $U \subseteq X$, we define

$$P_T(U) := \{m - n : m, n \in \mathbb{N}^k, \text{ and } T^m|_U = T^n|_U \text{ is injective}\}.$$

We define

$$P_T := \bigcup_{\emptyset \neq U \subseteq X \text{ precompact open}} P_T(U).$$

Remark 3.2. When $k = 1$, the set $P_T(U)$ is related to the group $\text{Stab}^{\text{ess}}(x)$ from [11, page 29]. Specifically, $\text{Stab}^{\text{ess}}(x)$ contains $P_T(U)$ for any precompact open set U containing x ; but also, since $\text{Stab}^{\text{ess}}(x)$ is a subgroup of \mathbb{Z}^k , and hence finitely generated, it is not too hard to check that there is an open cover of X by sets U such that $P_T(U) = \text{Stab}^{\text{ess}}(x)$ for each x in U .

In addition to being needed for our own arguments, our next result, Proposition 3.5, plugs a gap in the literature—it is mentioned without proof in [11, Page 30].

Proposition 3.3. *Let (X, T) be a minimal rank- k Deaconu–Renault system. Then*

$$P_T = \{p \in \mathbb{Z}^k : (x, p, x) \in \mathcal{G}_T \text{ for all } x \in X\},$$

and P_T is a subgroup of \mathbb{Z}^k .

In order to prove Proposition 3.3, we need the following lemma.

Lemma 3.4. *Let (X, T) be a minimal rank- k Deaconu–Renault system. Suppose that $m, n \in \mathbb{N}^k$ and $y \in X$ satisfy $T^m(y) \neq T^n(y)$. Then there exists an open neighbourhood $W \subseteq X$ of y such that $T^m|_W$ and $T^n|_W$ are injective and $T^m(W) \cap T^n(W) = \emptyset$.*

Proof. Since X is Hausdorff, we can choose open neighbourhoods $U \subseteq X$ of $T^m(y)$ and $V \subseteq X$ of $T^n(y)$ such that $U \cap V = \emptyset$. Define $A := (T^m)^{-1}(U) \cap (T^n)^{-1}(V)$. Then $y \in A$. Since T^m and T^n are local homeomorphisms, there is an open neighbourhood $W \subseteq A$ of y such that $T^m|_W$ and $T^n|_W$ are injective, and we have $T^m(W) \cap T^n(W) \subseteq U \cap V = \emptyset$. ■

Proof of Proposition 3.3. Fix $p \in P_T$. Then there exist $m, n \in \mathbb{N}^k$ and a nonempty open set $U \subseteq X$ such that $p = m - n$, and $T^m|_U = T^n|_U$ is injective. Fix $z \in X$. We claim that $(z, p, z) \in \mathcal{G}_T$. Since $[z]$ is dense in X , we have $U \cap [z] \neq \emptyset$, and so there exist $y \in U$ and $a, b \in \mathbb{N}^k$ such that $T^a(y) = T^b(z)$. Thus,

$$T^{b+m}(z) = T^m(T^b(z)) = T^m(T^a(y)) = T^a(T^m(y)),$$

and

$$T^{b+n}(z) = T^n(T^b(z)) = T^n(T^a(y)) = T^a(T^n(y)).$$

Since $y \in U$, we have $T^m(y) = T^n(y)$, and hence $T^{b+m}(z) = T^{b+n}(z)$. Therefore, $(z, p, z) = (z, (b + m) - (b + n), z) \in \mathcal{G}_T$, and so

$$P_T \subseteq \{p \in \mathbb{Z}^k : (x, p, x) \in \mathcal{G}_T \text{ for all } x \in X\}.$$

We now show that $\mathbb{Z}^k \setminus P_T \subseteq \{p \in \mathbb{Z}^k : (x, p, x) \notin \mathcal{G}_T \text{ for some } x \in X\}$. To see this, fix $p \in \mathbb{Z}^k \setminus P_T$. Let $(m_i, n_i)_{i=1}^\infty$ be an enumeration of $\{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m - n = p\}$. We must find $x \in X$ such that $T^{m_i}(x) \neq T^{n_i}(x)$ for all $i \geq 1$. We claim that there exist nonempty precompact open subsets V_0, V_1, V_2, \dots of X satisfying

- (1) $\overline{V_i} \subseteq V_{i-1}$ for all $i \geq 1$,
- (2) $T^{m_i}|_{\overline{V_{i-1}}}$ and $T^{n_i}|_{\overline{V_{i-1}}}$ are injective for all $i \geq 1$, and
- (3) $T^{m_i}(\overline{V_i}) \cap T^{n_i}(\overline{V_i}) = \emptyset$ for all $i \geq 1$.

To start, let V_0 be a nonempty precompact open subset of X such that $T^{m_1}|_{\overline{V_0}}$ and $T^{n_1}|_{\overline{V_0}}$ are injective. Now fix $i \geq 1$ and suppose that V_0, \dots, V_{i-1} satisfy (1)–(3). Since $m_i - n_i = p \notin P_T$, we have $T^{m_i}|_{V_{i-1}} \neq T^{n_i}|_{V_{i-1}}$, and so there exists $y \in V_{i-1}$ such that $T^{m_i}(y) \neq T^{n_i}(y)$. Thus, by Lemma 3.4, there exists an open neighbourhood $W \subseteq V_{i-1}$ of y such that $T^{m_i}(W) \cap T^{n_i}(W) = \emptyset$. Since X is locally compact and Hausdorff and $T^{m_{i+1}}$ and $T^{n_{i+1}}$ are local homeomorphisms, there is an open neighbourhood V_i of y such that $\overline{V_i} \subseteq W$ and $T^{m_{i+1}}|_{\overline{V_i}}$ and $T^{n_{i+1}}|_{\overline{V_i}}$ are injective. So induction gives the desired sets V_i . Each V_i is contained in the compact set $\overline{V_0}$, and so the descending intersection $\bigcap_{i=1}^\infty \overline{V_i}$ is nonempty. Any $x \in \bigcap_{i=1}^\infty \overline{V_i}$ satisfies $T^{m_i}(x) \neq T^{n_i}(x)$ for all $i \geq 1$.

We conclude by showing that P_T is a subgroup of \mathbb{Z}^k . For all $x \in X$, we have $(x, 0, x) \in \mathcal{G}_T^{(0)} \subseteq \mathcal{G}_T$, and so $0 \in P_T$. Suppose that $p, q \in P_T$. For all $x \in X$, we have $(x, p, x), (x, q, x) \in \mathcal{G}_T$, and hence $(x, p - q, x) = (x, p, x)(x, q, x)^{-1} \in \mathcal{G}_T$. Thus $p - q \in P_T$, and so P_T is a subgroup of \mathbb{Z}^k . ■

Given a rank- k Deaconu–Renault system (X, T) , we write \mathcal{I}_T for the topological interior of $\text{Iso}(\mathcal{G}_T)$. Since \mathcal{G}_T is a locally compact Hausdorff étale groupoid, so is \mathcal{I}_T . From this point forward, we will assume that X is second-countable (and hence so are \mathcal{G}_T and \mathcal{I}_T). We know from [40, Lemma 3.5] that \mathcal{G}_T is amenable, and hence [1, Proposition 5.1.1] implies that \mathcal{I}_T is amenable.

Proposition 3.5. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Let P_T be as in Definition 3.1. Then*

$$\mathcal{I}_T = \{(x, p, x) : x \in X, p \in P_T\} \cong X \times P_T.$$

Proof. For \subseteq , fix $\gamma \in \mathcal{I}_T$. Let $c: \mathcal{G}_T \rightarrow \mathbb{Z}^k$ be the continuous 1-cocycle defined in Lemma 2.4. Let $p := c(\gamma)$ so that $\gamma = (x, p, x)$ for some $x \in X$. We claim that $p \in P_T$. By Remark 2.6, \mathcal{G}_T is minimal, and hence [29, Proposition 2.1] implies that for all $y \in X$,

$$c(\mathcal{I}_T \cap (\mathcal{G}_T)_y^y) = c(\mathcal{I}_T \cap (\mathcal{G}_T)_x^x),$$

and thus

$$p = c(x, p, x) \in c(\mathcal{I}_T \cap (\mathcal{G}_T)_x^x) = c(\mathcal{I}_T \cap (\mathcal{G}_T)_y^y).$$

So Proposition 3.3 gives $p \in P_T$.

For \supseteq , fix $x \in X$ and $p \in P_T$. By the definition of P_T , there exist $m, n \in \mathbb{N}^k$ and a nonempty precompact open set $U \subseteq X$ such that $p = m - n$ and $T^m|_U = T^n|_U$ is injective. This injectivity forces $Z(U, m, n, U) = \{(y, p, y) : y \in U\} \subseteq \mathcal{I}_T$. Fix $y \in U$. Then $(y, p, y) \in \mathcal{I}_T$, and so [29, Proposition 2.1] implies that

$$p = c(y, p, y) \in c(\mathcal{I}_T \cap (\mathcal{G}_T)_y^y) = c(\mathcal{I}_T \cap (\mathcal{G}_T)_x^x),$$

and hence $(x, p, x) \in \mathcal{I}_T$. ■

Remark 3.6. Proposition 3.5 is related to the sets Σ_X and $H(T)$ of [40] as follows. Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. In the notation of [40, Section 3], suppose that $\Sigma = \Sigma_X$. Then T is an irreducible action of \mathbb{N}^k on X , and [40, Proposition 3.10] implies that

$$\mathcal{I}_T = \{(x, p, x) : x \in X, p \in H(T)\} \cong X \times H(T).$$

Thus Proposition 3.5 implies that $P_T = H(T)$.

We now present two corollaries of Proposition 3.5.

Corollary 3.7. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Then \mathcal{G}_T is effective if and only if $P_T = \{0\}$.*

Proof. By Proposition 3.5, $\mathcal{I}_T = \{(x, p, x) : x \in X, p \in P_T\}$. Hence

$$\mathcal{G}_T \text{ is effective} \iff \mathcal{I}_T = \mathcal{G}_T^{(0)} \iff P_T = \{0\}. \quad \blacksquare$$

Corollary 3.8. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Let $c: \mathcal{G}_T \rightarrow \mathbb{Z}^k$ be as in Lemma 2.4. Then for each $p \in P_T$, we have $(c|_{\mathcal{I}_T})^{-1}(p) = \{(x, p, x) : x \in X\}$, and $\{c|_{\mathcal{I}_T}^{-1}(p) : p \in P_T\}$ is a collection of mutually disjoint clopen bisections whose union is \mathcal{I}_T .*

Proof. Fix $p \in P_T$. Since c is continuous and \mathbb{Z}^k is discrete, $c|_{\mathcal{I}_T^{-1}(p)}$ is clopen. Fix $x \in X$. If $\alpha, \beta \in c|_{\mathcal{I}_T^{-1}(p)}$ and $r(\alpha) = x = r(\beta)$, then $s(\alpha) = x = s(\beta)$ because $\alpha, \beta \in \text{Iso}(\mathcal{G}_T)$, and hence $\alpha = (x, p, x) = \beta$. So $r|_{c|_{\mathcal{I}_T^{-1}(p)}}$ is injective, and a similar argument shows that $s|_{c|_{\mathcal{I}_T^{-1}(p)}}$ is also injective. Consequently $c|_{\mathcal{I}_T^{-1}(p)}$ is a bisection. By Proposition 3.5, we have $c(\mathcal{I}_T) = P_T$, and the result follows. ■

We now prove that when \mathcal{G}_T is minimal, we can form the quotient groupoid $\mathcal{G}_T/\mathcal{I}_T$. As the anonymous referee correctly points out, a more general result is possible—the salient point is that \mathcal{I}_T is a closed normal subgroupoid of the isotropy—but our application is to simplicity of twisted C^* -algebras associated to \mathcal{G}_T , for which minimality of \mathcal{G}_T is a necessary condition (see Theorem 6.1(a)).

Proposition 3.9. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Then \mathcal{I}_T is a closed subgroupoid of \mathcal{G}_T and acts freely and properly on \mathcal{G}_T by right-multiplication. The set $\mathcal{H}_T := \mathcal{G}_T/\mathcal{I}_T$ is a locally compact Hausdorff étale groupoid, with multiplication given by $[\alpha][\beta] := [\alpha\beta]$ for $(\alpha, \beta) \in \mathcal{G}_T^{(2)}$, inversion given by $[\gamma]^{-1} := [\gamma^{-1}]$ for $\gamma \in \mathcal{G}_T$, and range and source maps given by $r([\gamma]) = [r(\gamma)]$ and $s([\gamma]) = [s(\gamma)]$.*

Proof. Together, Remark 2.6 and Lemma 2.4 allow us to apply [29, Proposition 2.1] to see that \mathcal{I}_T is a closed subgroupoid of \mathcal{G}_T . Therefore, [40, Proposition 2.5(d)] implies that \mathcal{H}_T is a locally compact Hausdorff étale groupoid under the given operations. ■

We conclude this section with two technical lemmas that we use in the proof of our characterisation of simplicity of $C^*(\mathcal{G}_T, \sigma)$ in Section 6.

Lemma 3.10. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Let $c: \mathcal{G}_T \rightarrow \mathbb{Z}^k$ be as in Lemma 2.4. Fix $m, n \in \mathbb{Z}^k$, and let U and V be precompact open bisections of \mathcal{G}_T such that $U \subseteq c^{-1}(m)$ and $V \subseteq c^{-1}(n)$. Then $\overline{\mathcal{I}_T U} \cap \overline{V} \subseteq \mathcal{I}_T \overline{U}$ and $\overline{\mathcal{I}_T V} \cap \overline{U} \subseteq \mathcal{I}_T \overline{V}$.*

Proof. Define $K := r(\overline{U}) \times \{n - m\} \times r(\overline{U})$ and $W := K \cap \mathcal{I}_T$. Since r is continuous, K is compact, and hence closed. Since \mathcal{I}_T is closed by Proposition 3.9, W is closed, and hence is a compact subset of K and of \mathcal{I}_T . We claim that $\mathcal{I}_T U \cap V \subseteq W\overline{U}$. For this, suppose that $\gamma \in \mathcal{I}_T U \cap V$. Then there exist $\xi \in \mathcal{I}_T$ and $\eta \in U \subseteq c^{-1}(m)$ such that $\gamma = \xi\eta \in V \subseteq c^{-1}(n)$. Hence $\xi = \gamma\eta^{-1} \in c^{-1}(n - m)$. We also have $r(\xi) = s(\xi) = r(\eta) \in r(\overline{U})$, and so $\xi \in K \cap \mathcal{I}_T = W$. Hence $\gamma = \xi\eta \in W\overline{U}$, and so $\mathcal{I}_T U \cap V \subseteq W\overline{U}$. Since W and \overline{U} are compact, $W\overline{U}$ is compact, and hence closed. Thus $\overline{\mathcal{I}_T U} \cap \overline{V} \subseteq \mathcal{I}_T \overline{U}$. A symmetric argument shows that $\overline{\mathcal{I}_T V} \cap \overline{U} \subseteq \mathcal{I}_T \overline{V}$. ■

Lemma 3.11. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Let $c: \mathcal{G}_T \rightarrow \mathbb{Z}^k$ be as in Lemma 2.4. Fix $m, n \in \mathbb{Z}^k$, and let U and V be precompact open bisections of \mathcal{G}_T such that $U \subseteq c^{-1}(m)$ and $V \subseteq c^{-1}(n)$. Then $s(\mathcal{I}_T U \cap V) = s(\mathcal{I}_T V \cap U)$. Moreover, if $(V^{-1}\mathcal{I}_T U) \cap \mathcal{I}_T \neq \emptyset$, then $n - m \in P_T$, and for each $\gamma \in (V^{-1}\mathcal{I}_T U) \cap \mathcal{I}_T$, we have $s(\gamma) \in s(\mathcal{I}_T U \cap V)$.*

Proof. We first show that $s(\mathcal{I}_T U \cap V) = s(\mathcal{I}_T V \cap U)$. By symmetry, it suffices to show that $s(\mathcal{I}_T U \cap V) \subseteq s(\mathcal{I}_T V \cap U)$. Suppose that $x \in s(\mathcal{I}_T U \cap V)$. Then there exist $\zeta \in \mathcal{I}_T$ and $\eta \in U$ such that $\zeta\eta \in V$ and $x = s(\zeta\eta) = s(\eta)$. Since $\zeta^{-1} \in \mathcal{I}_T$, we have $\eta = \zeta^{-1}(\zeta\eta) \in \mathcal{I}_T V \cap U$, and hence $x = s(\eta) \in s(\mathcal{I}_T V \cap U)$. Thus $s(\mathcal{I}_T U \cap V) \subseteq s(\mathcal{I}_T V \cap U)$, as required.

For the second statement, suppose that $\gamma \in (V^{-1}\mathcal{I}_T U) \cap \mathcal{I}_T$. Then there exist $\alpha \in U$, $\beta \in V$, and $\xi \in \mathcal{I}_T$ such that $\gamma = \beta^{-1}\xi\alpha$, and hence $c(\gamma) = -n + c(\xi) + m$. Since $\gamma, \xi \in \mathcal{I}_T$, we have $c(\gamma), c(\xi) \in P_T$ by Proposition 3.5, and hence $n - m = c(\xi) - c(\gamma) \in P_T$, because P_T is a group by Proposition 3.3. Since $\gamma \in \mathcal{I}_T$, we have $s(\beta) = r(\gamma) = s(\gamma) = r(\alpha^{-1})$, and hence $(\beta, \alpha^{-1}) \in \mathcal{G}_T^{(2)}$. Since $\xi \in \mathcal{I}_T$, we have $r(\beta\alpha^{-1}) = s(\beta^{-1}) = r(\xi) = s(\xi) = r(\alpha) = s(\beta\alpha^{-1})$. We also have $c(\beta\alpha^{-1}) = n - m \in P_T$, and thus Proposition 3.5 implies that $\beta\alpha^{-1} \in \mathcal{I}_T$. Hence $\beta = (\beta\alpha^{-1})\alpha \in \mathcal{I}_T U \cap V$, and so $s(\gamma) = r(\gamma) = s(\beta) \in s(\mathcal{I}_T U \cap V)$. ■

4. Cohomology of Deaconu–Renault groupoids

In this section, we show that every continuous \mathbb{T} -valued 2-cocycle on a minimal Deaconu–Renault groupoid \mathcal{G}_T is cohomologous to a continuous \mathbb{T} -valued 2-cocycle σ on \mathcal{G}_T that is constant on \mathcal{I}_T (in the sense of Definition 4.1). We also introduce the spectral action θ of $\mathcal{H}_T := \mathcal{G}_T/\mathcal{I}_T$, analogous to [29, Lemma 3.6].

Definition 4.1. Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Suppose that $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. We say that σ is *constant on \mathcal{I}_T* if

$$\sigma((x, m, x), (x, n, x)) = \sigma((y, m, y), (y, n, y)) \quad \text{for all } x, y \in X \text{ and } m, n \in P_T.$$

If $\omega \in Z^2(P_T, \mathbb{T})$ is the 2-cocycle satisfying $\sigma((x, m, x), (x, n, x)) = \omega(m, n)$ for all $x \in X$ and $m, n \in P_T$, then we say that σ is *ω -constant on \mathcal{I}_T* , and we write $\sigma|_{\mathcal{I}_T^{(2)}} = 1_X \times \omega$.

The following proposition and the lemmas used in its proof are extensions of cohomological results from [29, Section 3] about boundary-path groupoids of cofinal, row-finite, source-free k -graphs to the more general setting of Deaconu–Renault groupoids. Proposition 4.2 is a generalisation of [29, Proposition 3.1], but we have adapted it slightly to prove that the bicharacter $\omega \in Z^2(P_T, \mathbb{T})$ can be chosen in such a way that it vanishes on its centre, and hence descends to a bicharacter $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$.

Proposition 4.2. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Suppose that $\rho \in Z^2(\mathcal{G}_T, \mathbb{T})$. For each $x \in X$, define $\rho_x: P_T \times P_T \rightarrow \mathbb{T}$ by*

$$\rho_x(m, n) := \rho((x, m, x), (x, n, x)).$$

Then $\rho_x \in Z^2(P_T, \mathbb{T})$. There exists a bicharacter $\omega \in Z^2(P_T, \mathbb{T})$ such that ω vanishes on Z_ω in each coordinate, and ω is cohomologous to ρ_x for every $x \in X$. For any such bicharacter ω , there exists $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$ such that σ is cohomologous to ρ and is ω -constant

on \mathcal{I}_T (in the sense of Definition 4.1), and there exists a bicharacter $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ such that

$$\tilde{\omega}(p + Z_\omega, q + Z_\omega) = \omega(p, q) \quad \text{for all } p, q \in P_T.$$

In order to prove Proposition 4.2, we need the following two results. The first of these results is an extension of [29, Lemma 3.2] to the setting of Deaconu–Renault groupoids.

Lemma 4.3. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. For each $x \in X$, define $\sigma_x: P_T \times P_T \rightarrow \mathbb{T}$ by*

$$\sigma_x(m, n) := \sigma((x, m, x), (x, n, x)).$$

Then $\sigma_x \in Z^2(P_T, \mathbb{T})$. For $\gamma \in \mathcal{G}_T$ and $y := s(\gamma) \in X$, define $\tau_y^\sigma: P_T \rightarrow \mathbb{T}$ by

$$\tau_y^\sigma(p) := \sigma(\gamma, (y, p, y))\sigma(\gamma(y, p, y), \gamma^{-1})\overline{\sigma(\gamma, \gamma^{-1})}. \tag{4.1}$$

- (a) For all $m, n \in P_T$, the map $x \mapsto \sigma_x(m, n)$ from X to \mathbb{T} is continuous.
- (b) For each $p \in P_T$, the map $\gamma \mapsto \tau_y^\sigma(p)$ from \mathcal{G}_T to \mathbb{T} is continuous.
- (c) Fix $\gamma = (x, m, y) \in \mathcal{G}_T$, $p \in P_T$, and $w, z \in \mathbb{T}$. Under the multiplication and inversion operations on $\mathcal{G}_T \times_\sigma \mathbb{T}$ (as defined in equations (2.1) and (2.2)), we have

$$(\gamma, w)((y, p, y), z)(\gamma, w)^{-1} = ((x, p, x), \tau_y^\sigma(p)z). \tag{4.2}$$

- (d) For all $\gamma \in \mathcal{G}_T$ and $p, q \in P_T$, we have

$$\tau_y^\sigma(p + q) = \sigma_{r(\gamma)}(p, q)\overline{\sigma_{s(\gamma)}(p, q)}\tau_y^\sigma(p)\tau_y^\sigma(q). \tag{4.3}$$

- (e) If ω is a bicharacter of P_T such that σ is ω -constant on \mathcal{I}_T , then $\sigma_x = \sigma_y$ for all $x, y \in X$, and $\gamma \mapsto \tau_y^\sigma$ is a continuous \hat{P}_T -valued 1-cocycle on \mathcal{G}_T .

Proof. Routine calculations show that since σ is normalised and satisfies the 2-cocycle identity, we have $\sigma_x \in Z^2(P_T, \mathbb{T})$ for each $x \in X$.

For part (a), note that for each $m, n \in P_T$, the map $x \mapsto \sigma_x(m, n)$ is the composition of the continuous maps $x \mapsto ((x, m, x), (x, n, x))$ and σ .

For part (b), fix $p \in P_T$. For $\gamma \in \mathcal{G}_T$,

$$\tau_y^\sigma(p) = \sigma(\gamma, (s(\gamma), p, s(\gamma)))\sigma(\gamma(s(\gamma), p, s(\gamma)), \gamma^{-1})\overline{\sigma(\gamma, \gamma^{-1})}.$$

Thus the map $\gamma \mapsto \tau_y^\sigma(p)$ from \mathcal{G}_T to \mathbb{T} is continuous because it is a product of continuous functions.

For part (c), fix $\gamma = (x, m, y) \in \mathcal{G}_T$, $p \in P_T$, and $w, z \in \mathbb{T}$. We have

$$\gamma(y, p, y)\gamma^{-1} = (x, m, y)(y, p, y)(y, -m, x) = (x, p, x),$$

and hence

$$\begin{aligned} &(\gamma, w)((y, p, y), z)(\gamma, w)^{-1} \\ &= (\gamma(y, p, y), \sigma(\gamma, (y, p, y))wz)(\gamma^{-1}, \overline{\sigma(\gamma, \gamma^{-1})}\bar{w}) \end{aligned}$$

$$\begin{aligned} &= (\gamma(y, p, y)\gamma^{-1}, \sigma\gamma(y, p, y), \gamma^{-1})\sigma(\gamma, (y, p, y))\overline{\sigma(\gamma, \gamma^{-1})z} \\ &= ((x, p, x), \tau_y^\sigma(p)z). \end{aligned}$$

For part (d), fix $\gamma = (x, m, y) \in \mathcal{G}_T$ and $p, q \in P_T$. For all $z \in \mathbb{T}$, we have

$$\begin{aligned} ((y, p, y), 1)((y, q, y), z) &= ((y, p, y)(y, q, y), \sigma((y, p, y), (y, q, y)))z \\ &= ((y, p + q, y), \sigma_y(p, q)z), \end{aligned}$$

and so, taking $z = \overline{\sigma_y(p, q)}$, we see that

$$((y, p, y), 1)((y, q, y), \overline{\sigma_y(p, q)}) = ((y, p + q, y), 1). \tag{4.4}$$

Together, equations (4.2) and (4.4) imply that

$$\begin{aligned} &((x, p + q, x), \tau_y^\sigma(p + q)) \\ &= (\gamma, 1)((y, p + q, y), 1)(\gamma, 1)^{-1} \\ &= (\gamma, 1)((y, p, y), 1)((y, q, y), \overline{\sigma_y(p, q)})(\gamma, 1)^{-1} \\ &= (\gamma, 1)((y, p, y), 1)(\gamma, 1)^{-1}(\gamma, 1)((y, q, y), \overline{\sigma_y(p, q)})(\gamma, 1)^{-1} \\ &= ((x, p, x), \tau_y^\sigma(p))((x, q, x), \tau_y^\sigma(q)\overline{\sigma_y(p, q)}) \\ &= ((x, p + q, x), \sigma_x(p, q)\overline{\sigma_y(p, q)}\tau_y^\sigma(p)\tau_y^\sigma(q)), \end{aligned}$$

and hence

$$\tau_y^\sigma(p + q) = \sigma_{r(\gamma)}(p, q)\overline{\sigma_{s(\gamma)}(p, q)}\tau_y^\sigma(p)\tau_y^\sigma(q).$$

For part (e), since σ is ω -constant on \mathcal{I}_T , for all $x, y \in X$ and $p, q \in P_T$, we have

$$\sigma_x(p, q) = \sigma((x, p, x), (x, q, x)) = \omega(p, q) = \sigma((y, p, y), (y, q, y)) = \sigma_y(p, q).$$

So $\sigma_x = \sigma_y$, and for each $\gamma \in \mathcal{G}_T$, equation (4.3) reduces to

$$\tau_y^\sigma(p + q) = \tau_y^\sigma(p)\tau_y^\sigma(q).$$

Thus $\tau_y^\sigma: P_T \rightarrow \mathbb{T}$ is a homomorphism, and so $\tau_y^\sigma \in \widehat{P}_T$ for each $\gamma \in \mathcal{G}_T$.

We now show that the map $\gamma \mapsto \tau_y^\sigma$ is multiplicative. Fix $\alpha = (x, m, u), \beta = (u, n, y) \in \mathcal{G}_T$, and $p \in P_T$. Using equation (4.2), we compute

$$\begin{aligned} ((x, p, x), \tau_\alpha^\sigma(p)\tau_\beta^\sigma(p)) &= (\alpha, 1)((u, p, u), \tau_\beta^\sigma(p))(\alpha, 1)^{-1} \\ &= (\alpha, 1)(\beta, 1)((y, p, y), 1)(\beta, 1)^{-1}(\alpha, 1)^{-1} \\ &= (\alpha\beta, \sigma(\alpha, \beta))((y, p, y), 1)(\alpha\beta, \sigma(\alpha, \beta))^{-1} \\ &= ((x, p, x), \tau_{\alpha\beta}^\sigma(p)). \end{aligned}$$

Hence $\tau_\alpha^\sigma\tau_\beta^\sigma = \tau_{\alpha\beta}^\sigma$, and so $\gamma \mapsto \tau_y^\sigma$ is a \widehat{P}_T -valued 1-cocycle on \mathcal{G}_T .

We conclude by showing that the map $\gamma \mapsto \tau_\gamma^\sigma$ is continuous. Fix a finite subset $F \subseteq P_T$ and an open subset $U \subseteq \mathbb{T}$. The set

$$S_{\widehat{P}_T}(F, U) = \{\phi \in \widehat{P}_T : \phi(F) \subseteq U\}$$

is a typical subbasis element for the compact-open topology on \widehat{P}_T , and so it suffices to show that $\{\gamma \in \mathcal{G}_T : \tau_\gamma^\sigma \in S_{\widehat{P}_T}(F, U)\}$ is an open subset of \mathcal{G}_T . We have

$$\{\gamma \in \mathcal{G}_T : \tau_\gamma^\sigma(F) \subseteq U\} = \bigcap_{p \in F} \{\gamma \in \mathcal{G}_T : \tau_\gamma^\sigma(p) \in U\},$$

which is open by part (b). ■

The following lemma is an extension of [29, Lemma 3.3] to the setting of Deaconu–Renault groupoids.

Lemma 4.4. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. As in Lemma 4.3, for each $x \in X$, define $\sigma_x \in Z^2(P_T, \mathbb{T})$ by*

$$\sigma_x(m, n) := \sigma((x, m, x), (x, n, x)).$$

Then the cohomology class of σ_x does not depend on x .

Proof. By [33, Proposition 3.2], it suffices to show that $\sigma_x \sigma_x^* = \sigma_y \sigma_y^*$ for all $x, y \in X$. By Proposition 3.3, P_T is a subgroup of the finitely generated free abelian group \mathbb{Z}^k , and so $P_T \cong \mathbb{Z}^l$ for some $l \leq k$. Fix free abelian generators g_1, \dots, g_l of P_T . Since each $\sigma_x \sigma_x^*$ is a bicharacter (by [33, Proposition 3.2]), it suffices to show that $(\sigma_x \sigma_x^*)(g_i, g_j) = (\sigma_y \sigma_y^*)(g_i, g_j)$ for all $i, j \in \{1, \dots, l\}$ and $x, y \in X$. To see this, we first show that $\sigma_{r(\gamma)} \sigma_{r(\gamma)}^*(g_i, g_j) = \sigma_{s(\gamma)} \sigma_{s(\gamma)}^*(g_i, g_j)$, for all $\gamma \in \mathcal{G}_T$ and $i, j \in \{1, \dots, l\}$.

Let $P_T \times_\sigma \mathbb{T} := P_T \times \mathbb{T}$ be the semidirect product group, which is equal to $P_T \times \mathbb{T}$ as a set, but has group operation

$$(p, w)(q, z) = (p + q, \sigma(p, q)wz).$$

Define $i_\sigma: \mathbb{T} \rightarrow P_T \times_\sigma \mathbb{T}$ by $i_\sigma(z) = (0, z)$ and $q_\sigma: P_T \times_\sigma \mathbb{T} \rightarrow P_T$ by $q_\sigma(p, z) = p$. Consider the bijection $M: H^2(P_T, \mathbb{T}) \rightarrow \text{Ext}(P_T, \mathbb{T})$ that maps the cohomology class of a 2-cocycle $\sigma \in Z^2(P_T, \mathbb{T})$ to the congruence class of the central extension

$$1 \rightarrow \mathbb{T} \xrightarrow{i_\sigma} P_T \times_\sigma \mathbb{T} \xrightarrow{q_\sigma} P_T \rightarrow 0$$

(see [9, Theorem IV.3.12]). Fix $\gamma \in \mathcal{G}_T$. We aim to prove that $\sigma_{s(\gamma)}$ and $\sigma_{r(\gamma)}$ are cohomologous by showing that their cohomology classes have the same image under M . So we must find a homomorphism

$$\varphi_\gamma: P_T \times_{\sigma_{s(\gamma)}} \mathbb{T} \rightarrow P_T \times_{\sigma_{r(\gamma)}} \mathbb{T}$$

that makes the diagram

$$\begin{array}{ccccc}
 & & P_T \times_{\sigma_{s(\gamma)}} \mathbb{T} & & \\
 & i_{\sigma_{s(\gamma)}} \nearrow & \downarrow \varphi_\gamma & \searrow q_{\sigma_{s(\gamma)}} & \\
 \mathbb{T} & & & & P_T \\
 & i_{\sigma_{r(\gamma)}} \searrow & \downarrow \varphi_\gamma & \nearrow q_{\sigma_{r(\gamma)}} & \\
 & & P_T \times_{\sigma_{r(\gamma)}} \mathbb{T} & &
 \end{array} \tag{4.5}$$

commute. Let $\tau_\gamma^\sigma: P_T \rightarrow \mathbb{T}$ be the map of Lemma 4.3, and define $\varphi_\gamma: P_T \times_{\sigma_{s(\gamma)}} \mathbb{T} \rightarrow P_T \times_{\sigma_{r(\gamma)}} \mathbb{T}$ by $\varphi_\gamma(m, z) := (m, \tau_\gamma^\sigma(m)z)$. Fix $(m, z), (n, w) \in P_T \times_{\sigma_{s(\gamma)}} \mathbb{T}$. Recalling from Lemma 4.3(d) that

$$\tau_\gamma^\sigma(m + n) = \sigma_{r(\gamma)}(m, n) \overline{\sigma_{s(\gamma)}(m, n)} \tau_\gamma^\sigma(m) \tau_\gamma^\sigma(n),$$

we obtain

$$\begin{aligned}
 \varphi_\gamma((m, z)(n, w)) &= \varphi_\gamma(m + n, \sigma_{s(\gamma)}(m, n)zw) \\
 &= (m + n, \tau_\gamma^\sigma(m + n) \sigma_{s(\gamma)}(m, n)zw) \\
 &= (m + n, \sigma_{r(\gamma)}(m, n) \tau_\gamma^\sigma(m) \tau_\gamma^\sigma(n)zw) \\
 &= (m, \tau_\gamma^\sigma(m)z)(n, \tau_\gamma^\sigma(n)w) \\
 &= \varphi_\gamma(m, z) \varphi_\gamma(n, w),
 \end{aligned}$$

and thus φ_γ is a homomorphism. Since σ is normalised, the formula (4.1) from Lemma 4.3 gives $\tau_\gamma^\sigma(0) = 1$, and it follows that the diagram (4.5) commutes. Therefore, $\sigma_{r(\gamma)}$ is cohomologous to $\sigma_{s(\gamma)}$, and so [33, Proposition 3.2] implies that

$$(\sigma_{r(\gamma)} \sigma_{r(\gamma)}^*)(g_i, g_j) = (\sigma_{s(\gamma)} \sigma_{s(\gamma)}^*)(g_i, g_j) \quad \text{for all } i, j \in \{1, \dots, l\}, \tag{4.6}$$

as claimed.

Now fix $x, y \in X$. Since (X, T) is minimal, there is a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in \mathcal{G}_T such that $s(\gamma_n) = x$ for all $n \in \mathbb{N}$, and $r(\gamma_n) \rightarrow y$ as $n \rightarrow \infty$. Fix $i, j \in \{1, \dots, l\}$. By Lemma 4.3(a), the map $u \mapsto \sigma_u(g_i, g_j)$ is continuous, and hence the map $u \mapsto (\sigma_u \sigma_u^*)(g_i, g_j)$ is continuous. So $(\sigma_y \sigma_y^*)(g_i, g_j) = \lim_{n \rightarrow \infty} (\sigma_{r(\gamma_n)} \sigma_{r(\gamma_n)}^*)(g_i, g_j)$. Equation (4.6) gives

$$(\sigma_{r(\gamma_n)} \sigma_{r(\gamma_n)}^*)(g_i, g_j) = (\sigma_x \sigma_x^*)(g_i, g_j)$$

for each $n \in \mathbb{N}$, and so $(\sigma_y \sigma_y^*)(g_i, g_j) = (\sigma_x \sigma_x^*)(g_i, g_j)$. ■

Proof of Proposition 4.2. Lemma 4.4 shows that ρ_x is a \mathbb{T} -valued 2-cocycle on P_T whose cohomology class is independent of x . So there exists a 2-cocycle $\omega \in Z^2(P_T, \mathbb{T})$ whose cohomology class agrees with that of each ρ_x . As discussed in Section 2.1 (see [3, Theorem 2.2.8]), we may assume that ω is a bicharacter that vanishes on Z_ω in each coordinate, and that there is a bicharacter $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ such that

$$\tilde{\omega}(p + Z_\omega, q + Z_\omega) = \omega(p, q) \quad \text{for all } p, q \in P_T.$$

We now construct $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$ such that σ is cohomologous to ρ , and σ is ω -constant on \mathcal{I}_T . For each $x \in X$, the 2-cocycles ρ_x and ω are cohomologous, and so the map $\tilde{c}_x: P_T \times P_T \rightarrow \mathbb{T}$ defined by

$$\tilde{c}_x(p, q) := \overline{\omega(p, q)}\rho_x(p, q)$$

is a 2-coboundary on P_T . Since P_T is a subgroup of \mathbb{Z}^k (by Proposition 3.3), there is an integer $l \in \{1, \dots, k\}$ such that $P_T \cong \mathbb{Z}^l$. Fix free abelian generators g_1, \dots, g_l for P_T . For $m \in P_T$, let m_1, \dots, m_l be the unique integers such that $m = \sum_{i=1}^l m_i g_i$. For each $i \in \{1, \dots, l\}$, we write $\langle g_j : j \leq i \rangle$ for the group generated by the set $\{g_j : 1 \leq j \leq i\}$. We claim that there are maps $b_x: P_T \rightarrow \mathbb{T}$, indexed by $x \in X$, such that $x \mapsto b_x(m)$ is continuous for each $m \in P_T$, and for each $i \in \{1, \dots, l\}$, we have

$$b_x(m)\overline{b_x(m + g_i)} = \tilde{c}_x(g_i, m), \quad \text{whenever } m \in \langle g_j : j \leq i \rangle. \tag{4.7}$$

To see this, for each $x \in X$ define $b_x(0) := 1 \in \mathbb{T}$. The map $x \mapsto b_x(0)$ is trivially continuous. Fix $i \in \{1, \dots, l\}$. Suppose inductively that the maps b_x have been defined on $\langle g_j : j < i \rangle$, and that $x \mapsto b_x(m)$ is continuous for each $m \in \langle g_j : j < i \rangle$. To extend b_x to $\langle g_j : j \leq i \rangle$, first observe that $b_x(m)$ is already defined when $m = \sum_{j=1}^i m_j g_j$ and $m_i = 0$. Now suppose inductively that $b_x(m)$ is defined and $x \mapsto b_x(m)$ is continuous whenever $|m_i| \leq a$ for some $a \in \mathbb{N}$, and that b_x satisfies equation (4.7) whenever $|m_i|, |m_i + 1| \leq a$. Fix $m \in \langle g_j : j \leq i \rangle$ such that $|m_i| = a + 1$. Define

$$b_x(m) := \begin{cases} b_x(m - g_i)\overline{\tilde{c}_x(g_i, m - g_i)} & \text{if } m_i > 0, \\ b_x(m + g_i)\tilde{c}_x(g_i, m) & \text{if } m_i < 0. \end{cases}$$

Since Lemma 4.3(a) implies that the maps $x \mapsto \tilde{c}_x(p, q)$ are continuous for all $p, q \in P_T$, the inductive hypothesis guarantees that $x \mapsto b_x(m)$ is continuous. Moreover, rearranging each of the cases in the definition of $b_x(m)$ shows that equation (4.7) is satisfied. So the claim follows by induction.

Recall the coboundary map δ^1 of Definition 2.2(iv). We claim that $\delta^1 b_x = \tilde{c}_x$. To see this, first choose a normalised 1-cochain $\tilde{b}_x: P_T \rightarrow \mathbb{T}$ such that $\delta^1 \tilde{b}_x = \tilde{c}_x$. (This is possible because \tilde{c}_x is a 2-coboundary on P_T .) Define $a_x: P_T \rightarrow \mathbb{T}$ by

$$a_x(m) := \prod_{i=1}^l \overline{\tilde{b}_x(g_i)^{m_i}}.$$

A straightforward calculation shows that a_x is a 1-cocycle, and so $\delta^1 a_x$ is trivial. Hence $\delta^1(a_x \tilde{b}_x) = \delta^1 \tilde{b}_x = \tilde{c}_x$. Putting $m = 0$ in equation (4.7), we see that for each $i \in \{1, \dots, l\}$, $b_x(g_i) = 1$. Hence

$$(a_x \tilde{b}_x)(0) = \left(\prod_{i=1}^l \overline{\tilde{b}_x(g_i)^0} \right) \tilde{b}_x(0) = 1 = b_x(0),$$

and for each $i \in \{1, \dots, l\}$,

$$(a_x \tilde{b}_x)(g_i) = \left(\prod_{\substack{j=1, \\ j \neq i}}^l \overline{\tilde{b}_x(g_j)^0} \right) \overline{(\tilde{b}_x(g_i)^1) \tilde{b}_x(g_i)} = 1 = b_x(g_i).$$

Thus, for all $i \in \{1, \dots, l\}$ and $m \in \langle g_j : j \leq i \rangle$, we have

$$\begin{aligned} (a_x \tilde{b}_x)(m) \overline{(a_x \tilde{b}_x)(m + g_i)} &= (a_x \tilde{b}_x)(g_i) (a_x \tilde{b}_x)(m) \overline{(a_x \tilde{b}_x)(g_i + m)} \\ &= \delta^1(a_x \tilde{b}_x)(g_i, m) \\ &= \tilde{c}_x(g_i, m) \\ &= b_x(m) \overline{b_x(m + g_i)}. \end{aligned}$$

So b_x and $a_x \tilde{b}_x$ both map 0 and each generator g_i to 1, and they also both satisfy equation (4.7). Hence $a_x \tilde{b}_x = b_x$, and thus $\delta^1 b_x = \delta^1(a_x \tilde{b}_x) = \tilde{c}_x$, as claimed.

Since the maps $(x, p, x) \mapsto x$ and $x \mapsto b_x(p)$ are both continuous for each fixed $p \in P_T$, the map $\tilde{b}: \mathcal{I}_T \rightarrow \mathbb{T}$ given by $\tilde{b}(x, p, x) := b_x(p)$ is a continuous 1-cochain on \mathcal{I}_T . We extend \tilde{b} to a map $b: \mathcal{G}_T \rightarrow \mathbb{T}$ by setting $b(\gamma) := 1$ for all $\gamma \in \mathcal{G}_T \setminus \mathcal{I}_T$. Since \mathcal{I}_T is a clopen subset of \mathcal{G}_T (by Proposition 3.9), this map b is a continuous 1-cochain on \mathcal{G}_T . We have $b(x, 0, x) = b_x(0) = 1$ for all $x \in X$, and so b is normalised. Thus the map $\delta^1 b: \mathcal{G}_T^{(2)} \rightarrow \mathbb{T}$ given by $\delta^1 b(\alpha, \beta) := b(\alpha)b(\beta)b(\alpha\beta)^{-1}$ is a continuous 2-coboundary on \mathcal{G}_T . Define $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$ by $\sigma(\alpha, \beta) := \rho(\alpha, \beta) \overline{\delta^1 b(\alpha, \beta)}$. Since σ and ρ differ by the 2-coboundary $\delta^1 b$, they are cohomologous, and so [36, Proposition II.1.2] implies that $C^*(\mathcal{G}_T, \rho) \cong C^*(\mathcal{G}_T, \sigma)$. Finally, fix $x \in X$ and $p, q \in P_T$. Since $\delta^1 b_x = \tilde{c}_x = \rho_x \omega$, we have

$$\sigma((x, p, x), (x, q, x)) = \rho_x(p, q) \overline{\delta^1 b_x(p, q)} = \omega(p, q),$$

and so σ is ω -constant on \mathcal{I}_T . ■

The following result is an extension of [29, Lemma 3.6] to the setting of Deaconu–Renault groupoids.

Proposition 4.5. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Suppose that $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$, and that $\omega \in Z^2(P_T, \mathbb{T})$ is a bicharacter that vanishes on Z_ω in each coordinate such that σ is ω -constant on \mathcal{I}_T , as in Proposition 4.2. Let $\gamma \mapsto \tau_\gamma^\sigma$ be the continuous \hat{P}_T -valued 1-cocycle on \mathcal{G}_T defined in Lemma 4.3(e). For all $\gamma \in \mathcal{I}_T$ and $p \in Z_\omega$, we have $\tau_\gamma^\sigma(p) = 1$. Let $\mathcal{H}_T = \mathcal{G}_T / \mathcal{I}_T$ be the quotient groupoid of Proposition 3.9. There is a continuous \hat{Z}_ω -valued 1-cocycle $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ on \mathcal{H}_T such that $\tilde{\tau}_{[\gamma]}^\sigma(p) = \tau_\gamma^\sigma(p)$ for all $\gamma \in \mathcal{G}_T$ and $p \in Z_\omega$. There is a continuous action θ of \mathcal{H}_T on $X \times \hat{Z}_\omega$ such that*

$$\theta_{[\gamma]}(s(\gamma), \chi) = (r(\gamma), \tilde{\tau}_{[\gamma]}^\sigma \chi) \quad \text{for all } \gamma \in \mathcal{G}_T \text{ and } \chi \in \hat{Z}_\omega.$$

We call the action θ of Proposition 4.5 the *spectral action* associated to (T, σ) . We denote the orbit of $(x, \chi) \in X \times \hat{Z}_\omega$ under θ by $[x, \chi]_\theta$.

Proof of Proposition 4.5. Fix $\gamma \in \mathcal{I}_T$. Proposition 3.5 implies that there exist $y \in X$ and $m \in P_T$ such that $\gamma = (y, m, y)$. We claim that $\tau_\gamma^\sigma(Z_\omega) = \{1\}$. Fix $p \in Z_\omega$. Using formula (4.1) from Lemma 4.3, and that ω is a bicharacter satisfying $\sigma|_{\mathcal{I}_T^{(2)}} = 1_X \times \omega$, and that $\omega\omega^*$ is an antisymmetric bicharacter, we see that

$$\begin{aligned} \tau_\gamma^\sigma(p) &= \sigma(\gamma, (y, p, y))\sigma(\gamma(y, p, y), \gamma^{-1})\overline{\sigma(\gamma, \gamma^{-1})} \\ &= \sigma((y, m, y), (y, p, y))\sigma((y, m + p, y), (y, -m, y))\overline{\sigma((y, m, y), (y, -m, y))} \\ &= \omega(m, p)\omega(m + p, -m)\overline{\omega(m, -m)} \\ &= \omega(m, p)\overline{\omega(p, m)} \\ &= \overline{(\omega\omega^*)(p, m)}, \end{aligned}$$

which is 1 because $p \in Z_\omega$. Thus $\tau_\gamma^\sigma(Z_\omega) = \{1\}$, as claimed.

For any $\gamma \in \mathcal{G}_T$, we have $\tau_\gamma^\sigma \in \widehat{P}_T$ by Lemma 4.3(e), and so $\tau_\gamma^\sigma|_{Z_\omega} \in \widehat{Z}_\omega$. Suppose that $\alpha, \beta \in \mathcal{G}_T$ satisfy $[\alpha] = [\beta]$. Then $\eta = \beta^{-1}\alpha \in \mathcal{I}_T$ satisfies $\alpha = \beta\eta$. For $p \in Z_\omega$, we have $\tau_\eta^\sigma(p) = 1$, and thus, since $\gamma \mapsto \tau_\gamma^\sigma$ is a 1-cocycle,

$$\tau_\alpha^\sigma(p) = \tau_{\beta\eta}^\sigma(p) = \tau_\beta^\sigma(p)\tau_\eta^\sigma(p) = \tau_\beta^\sigma(p).$$

Therefore, there is a map $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ from \mathcal{H}_T to \widehat{Z}_ω such that $\tilde{\tau}_{[\gamma]}^\sigma(p) = \tau_\gamma^\sigma(p)$ for all $\gamma \in \mathcal{G}_T$ and $p \in Z_\omega$. For $\alpha, \beta \in \mathcal{G}_T$ and $p \in Z_\omega$,

$$\tilde{\tau}_{[\alpha][\beta]}^\sigma(p) = \tilde{\tau}_{[\alpha\beta]}^\sigma(p) = \tau_{\alpha\beta}^\sigma(p) = \tau_\alpha^\sigma(p)\tau_\beta^\sigma(p) = \tilde{\tau}_{[\alpha]}^\sigma(p)\tilde{\tau}_{[\beta]}^\sigma(p),$$

and so $\tilde{\tau}_{[\alpha][\beta]}^\sigma = \tilde{\tau}_{[\alpha]}^\sigma\tilde{\tau}_{[\beta]}^\sigma$. Thus $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ is a \widehat{Z}_ω -valued 1-cocycle on \mathcal{H}_T .

We claim that $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ is continuous on \mathcal{H}_T . Fix a finite subset $F \subseteq Z_\omega$ and an open subset $U \subseteq \mathbb{T}$, so that $S_{\widehat{Z}_\omega}(F, U) := \{\chi \in \widehat{Z}_\omega : \chi(F) \subseteq U\}$ is a typical subbasis element for the topology on \widehat{Z}_ω . It suffices to show that $\{[\gamma] \in \mathcal{H}_T : \tilde{\tau}_{[\gamma]}^\sigma \in S_{\widehat{Z}_\omega}(F, U)\}$ is open in \mathcal{H}_T . Since F is finite,

$$S_{\widehat{P}_T}(F, U) := \{\chi \in \widehat{P}_T : \chi(F) \subseteq U\}$$

is open in \widehat{P}_T . By Lemma 4.3(e), the map $\gamma \mapsto \tau_\gamma^\sigma$ is continuous on \mathcal{G}_T , and hence $\{\gamma \in \mathcal{G}_T : \tau_\gamma^\sigma \in S_{\widehat{P}_T}(F, U)\}$ is open in \mathcal{G}_T . Let $\pi_T: \mathcal{G}_T \rightarrow \mathcal{H}_T$ denote the quotient map $\gamma \mapsto [\gamma]$. Then $\pi_T^{-1}(\{[\gamma] : \tilde{\tau}_{[\gamma]}^\sigma \in S_{\widehat{Z}_\omega}(F, U)\}) = \{\gamma \in \mathcal{G}_T : \tau_\gamma^\sigma(F) \subseteq U\}$ is open. Thus, by the definition of the quotient topology, $\{[\gamma] : \tilde{\tau}_{[\gamma]}^\sigma \in S_{\widehat{Z}_\omega}(F, U)\}$ is open in \mathcal{H}_T .

It remains to show that θ is a continuous action of \mathcal{H}_T on $X \times \widehat{Z}_\omega$. For $\alpha, \beta \in \mathcal{G}_T$ such that $[\alpha] = [\beta]$, we have $\alpha\beta^{-1} \in \mathcal{I}_T$, and hence $r(\alpha) = r(\beta)$ and $s(\alpha) = s(\beta)$. Define $R: X \times \widehat{Z}_\omega \rightarrow \mathcal{H}_T^{(0)}$ by $R(x, \chi) := [x]$. Then R is continuous and surjective. Recall from Definition 2.1 that the fibred product $\mathcal{H}_T \star (X \times \widehat{Z}_\omega)$ is defined by

$$\mathcal{H}_T \star (X \times \widehat{Z}_\omega) = \{([\gamma], (x, \chi)) : x \in X, \chi \in \widehat{Z}_\omega, \gamma \in (\mathcal{G}_T)_x\}.$$

Since $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ is a continuous map from \mathcal{H}_T to \widehat{Z}_ω and $r: \mathcal{H}_T \rightarrow X$ is continuous, the map $([\gamma], (s(\gamma), \chi)) \mapsto \theta_{[\gamma]}(s(\gamma), \chi) = (r(\gamma), \tilde{\tau}_{[\gamma]}^\sigma\chi)$ from $\mathcal{H}_T \star (X \times \widehat{Z}_\omega)$ to $X \times \widehat{Z}_\omega$ is

continuous. To see that θ is an action, we must show that conditions (A1) and (A2) of Definition 2.1 are satisfied.

For (A1), fix $(x, \chi) \in X \times \widehat{Z}_\omega$ and $([\alpha], [\beta]) \in \mathcal{H}_T^{(2)}$ such that $([\beta], (x, \chi)) \in \mathcal{H}_T \star (X \times \widehat{Z}_\omega)$. Then $s([\alpha]) = r([\beta])$, and $s([\beta]) = R(x, \chi) = [x]$. Hence $s([\alpha][\beta]) = s([\beta]) = R(x, \chi)$, and so $([\alpha][\beta], (x, \chi)) \in \mathcal{H}_T \star (X \times \widehat{Z}_\omega)$. Since $s(\beta) = x$,

$$\theta_{[\beta]}(x, \chi) = \theta_{[\beta]}(s(\beta), \chi) = (r(\beta), \tilde{\tau}_{[\beta]}^\sigma \chi).$$

Thus

$$R(\theta_{[\beta]}(x, \chi)) = R(r(\beta), \tilde{\tau}_{[\beta]}^\sigma \chi) = [r(\beta)] = r([\beta]) = s([\alpha]),$$

and so $([\alpha], \theta_{[\beta]}(x, \chi)) \in \mathcal{H}_T \star (X \times \widehat{Z}_\omega)$. Finally, since $[\gamma] \mapsto \tilde{\tau}_{[\gamma]}^\sigma$ is a \widehat{Z}_ω -valued 1-cocycle on \mathcal{H}_T , we have $\tilde{\tau}_{[\alpha]}^\sigma \tilde{\tau}_{[\beta]}^\sigma = \tilde{\tau}_{[\alpha][\beta]}^\sigma = \tilde{\tau}_{[\alpha\beta]}^\sigma$, and hence

$$\begin{aligned} \theta_{[\alpha]}(\theta_{[\beta]}(x, \chi)) &= \theta_{[\alpha]}(\theta_{[\beta]}(s(\beta), \chi)) = \theta_{[\alpha]}(r(\beta), \tilde{\tau}_{[\beta]}^\sigma \chi) = \theta_{[\alpha]}(s(\alpha), \tilde{\tau}_{[\beta]}^\sigma \chi) \\ &= (r(\alpha), \tilde{\tau}_{[\alpha]}^\sigma (\tilde{\tau}_{[\beta]}^\sigma \chi)) = (r(\alpha\beta), \tilde{\tau}_{[\alpha\beta]}^\sigma \chi) = \theta_{[\alpha\beta]}(s(\alpha\beta), \chi) \\ &= \theta_{[\alpha][\beta]}(x, \chi). \end{aligned}$$

Thus, (A1) is satisfied.

For (A2), fix $(x, \chi) \in X \times \widehat{Z}_\omega$. Then $s(R(x, \chi)) = s([x]) = [x] = R(x, \chi)$, and so $(R(x, \chi), (x, \chi)) \in \mathcal{H}_T \star (X \times \widehat{Z}_\omega)$. Since $x \in \mathcal{I}_T$, we have $\tau_x^\sigma(Z_\omega) = \{1\}$. Thus, for all $p \in Z_\omega$, we have $\tilde{\tau}_{[x]}^\sigma(p) = \tau_x^\sigma(p) = 1$, and so $\tilde{\tau}_{[x]}^\sigma \chi = \chi$. Hence

$$\theta_{R(x, \chi)}(x, \chi) = \theta_{[x]}(s(x), \chi) = (r(x), \tilde{\tau}_{[x]}^\sigma \chi) = (x, \chi). \quad \blacksquare$$

5. Realising $C^*(\mathcal{I}_T, \sigma)$ as an induced algebra

In this section, we realise the twisted C*-algebra associated to the interior \mathcal{I}_T of the isotropy of a Deaconu–Renault groupoid \mathcal{G}_T and a continuous 2-cocycle $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$ as an induced algebra. We then describe the ideals of this induced algebra. We begin by introducing a spanning set \mathcal{B}_T for $C_c(\mathcal{G}_T)$ and then giving a tensor-product decomposition of $C^*(\mathcal{I}_T, \sigma)$.

Lemma 5.1. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Let $c: \mathcal{G}_T \rightarrow \mathbb{Z}^k$ be as in Lemma 2.4. Let*

$$\mathcal{B}_T := \{f \in C_c(\mathcal{G}_T) : \text{supp}(f) \text{ is a bisection contained in } c^{-1}(n), \text{ for some } n \in \mathbb{Z}^k\}.$$

Then $C_c(\mathcal{G}_T) = \text{span } \mathcal{B}_T$.

Proof. Fix $f \in C_c(\mathcal{G}_T)$. Since $\text{supp}(f)$ is compact, there is a finite set \mathcal{F} of precompact open bisections that cover $\text{supp}(f)$. Since each $U \in \mathcal{F}$ is precompact, there are only finitely many $n \in \mathbb{Z}^k$ such that $U \cap c^{-1}(n) \neq \emptyset$. Since each $c^{-1}(n) \cap U$ is open, it is a

precompact open bisection, so we can assume that c is constant on each $U \in \mathcal{F}$. Now, as in the proof of [39, Lemma 9.1.3], fix a partition of unity $\{g_U : U \in \mathcal{F}\}$ on $\text{supp}(f)$ subordinate to \mathcal{F} . By the Tietze extension theorem, each g_U extends to an element \tilde{g}_U of $C_c(\mathcal{G}_T)$. Now the pointwise products $f_U := \tilde{g}_U \cdot f$ satisfy $\text{supp}(f_U) \subseteq U$, and $\sum_{U \in \mathcal{F}} f_U = f$. ■

Lemma 5.2. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. For each $h \in C_c(X)$ and $p \in P_T$, define $h \cdot 1_p : \mathcal{I}_T \rightarrow \mathbb{C}$ by*

$$(h \cdot 1_p)(x, m, x) := \delta_{p,m}h(x).$$

Then $h \cdot 1_p \in C_c(\mathcal{I}_T)$ for each $p \in P_T$, and $C_c(\mathcal{I}_T) = \text{span}\{h \cdot 1_p : h \in C_c(X), p \in P_T\}$.

Proof. For each $h \in C_c(X)$ and $p \in P_T$, we have

$$\text{osupp}(h \cdot 1_p) = (\text{osupp}(h) \times \{p\} \times \text{osupp}(h)) \cap \mathcal{I}_T,$$

and hence $h \cdot 1_p \in C_c(\mathcal{I}_T)$. Fix $f \in C_c(\mathcal{I}_T)$. Since $\text{supp}(f)$ is compact, there is a finite set $F \subseteq P_T$ such that $\text{supp}(f) \subseteq \bigcup_{p \in F} c|_{\mathcal{I}_T}^{-1}(p)$. For $p \in F$, define $h_p : X \rightarrow \mathbb{C}$ by $h_p(x) := f(x, p, x)$. Then $\text{osupp}(h_p) = r(c|_{\mathcal{I}_T}^{-1}(p) \cap \text{osupp}(f))$, and hence $h_p \in C_c(X)$. Moreover, $f = \sum_{p \in F} h_p \cdot 1_p$. ■

Proposition 5.3. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Suppose that $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$, and that $\omega \in Z^2(P_T, \mathbb{T})$ is a bicharacter that vanishes on Z_ω in each coordinate and satisfies $\sigma|_{\mathcal{I}_T^{(2)}} = 1_X \times \omega$, as in Proposition 4.2. Let $\{u_p : p \in P_T\}$ be the canonical family of generating unitaries for the twisted group C^* -algebra $C^*(P_T, \omega)$. There is an isomorphism $\Upsilon : C^*(\mathcal{I}_T, \sigma) \rightarrow C_0(X) \otimes C^*(P_T, \omega)$ such that $\Upsilon(h \cdot 1_p) = h \otimes u_p$ for all $h \in C_c(X)$ and $p \in P_T$.*

Proof. The argument used to prove [29, Lemma 4.1] works here—for more detail and an alternative approach to proving injectivity, see [3, Proposition 8.1.3]. ■

Before stating the next theorem, we recall the following facts relating to twisted group C^* -algebras. Define $B := P_T/Z_\omega$. There is a right action of \hat{B} on \hat{P}_T such that

$$(\phi \cdot \chi)(p) = \phi(p)\chi(p + Z_\omega) \quad \text{for all } \phi \in \hat{P}_T, \chi \in \hat{B}, \text{ and } p \in P_T.$$

This action induces a continuous, free, proper, right action of \hat{B} on $X \times \hat{P}_T$ given by $(x, \phi) \cdot \chi := (x, \phi \cdot \chi)$. By [17, Theorem 4.40], the map $\phi \cdot \hat{B} \mapsto \phi|_{Z_\omega}$ is an isomorphism $\hat{P}_T/\hat{B} \cong \hat{Z}_\omega$. Thus $\phi \mapsto \phi|_{Z_\omega}$ is a quotient map from \hat{P}_T to \hat{Z}_ω , and so [13, Theorem 3.3.17] implies that $Q : (x, \phi) \mapsto (x, \phi|_{Z_\omega})$ is a quotient map from $X \times \hat{P}_T$ to $X \times \hat{Z}_\omega$.

Let $\{U_{p+Z_\omega} : p + Z_\omega \in B\}$ be the canonical family of generating unitaries for the twisted group C^* -algebra $C^*(B, \tilde{\omega})$. By the universal property of $C^*(B, \tilde{\omega})$, there is a strongly continuous action β^B of \hat{B} on $C^*(B, \tilde{\omega})$ such that

$$\beta_\chi^B(U_{p+Z_\omega}) = \chi(p + Z_\omega)U_{p+Z_\omega} \quad \text{for all } \chi \in \hat{B} \text{ and } p \in P_T.$$

(See [3, Theorem 4.3.1] for proofs of the existence of these two actions of \widehat{B} .) Recall from Definition A.1 the definition of the induced algebra $\text{Ind}_{\widehat{B}}^{X \times \widehat{P}_T}(C^*(B, \tilde{\omega}), \beta^B)$ associated to the dynamical system $(C^*(B, \tilde{\omega}), \widehat{B}, \beta^B)$.

Theorem 5.4. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$, and let $\omega \in Z^2(P_T, \mathbb{T})$ and $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ be bicharacters chosen as in Proposition 4.2. Define*

$$\begin{aligned} \mathcal{X}_T^\omega &:= \text{Ind}_{\widehat{B}}^{X \times \widehat{P}_T}(C^*(B, \tilde{\omega}), \beta^B) \\ &= \left\{ f \in C_0(X \times \widehat{P}_T, C^*(B, \tilde{\omega})) : f(x, \phi \cdot \chi) = (\beta_\chi^B)^{-1}(f(x, \phi)) \right. \\ &\quad \left. \text{for all } (x, \phi) \in X \times \widehat{P}_T \text{ and } \chi \in \widehat{B} \right\}. \end{aligned}$$

There is an isomorphism $\psi_T: C^*(\mathcal{I}_T, \sigma) \rightarrow \mathcal{X}_T^\omega$ such that

$$\psi_T(h \cdot 1_p)(x, \phi) = h(x)\overline{\phi(p)}U_{p+Z_\omega}$$

for all $h \in C_c(X)$, $p \in P_T$, and $(x, \phi) \in X \times \widehat{P}_T$.

Proof. For brevity, define $\mathcal{Y}_T^\omega := \text{Ind}_{\widehat{B}}^{\widehat{P}_T}(C^*(B, \tilde{\omega}), \beta^B)$. Recall from Proposition 5.3 and Theorem A.2 the definitions of the isomorphisms

$$\Upsilon: C^*(\mathcal{I}_T, \sigma) \rightarrow C_0(X) \otimes C^*(P_T, \omega) \quad \text{and} \quad \Omega: C^*(P_T, \omega) \rightarrow \mathcal{Y}_T^\omega.$$

By [35, Propositions B.13 and B.16], there is an isomorphism

$$\Gamma: C_0(X) \otimes C^*(P_T, \omega) \rightarrow C_0(X, \mathcal{Y}_T^\omega)$$

such that $\Gamma(f \otimes a)(x) = f(x)\Omega(a)$ for all $f \in C_0(X)$, $a \in C^*(P_T, \omega)$, and $x \in X$. Hence

$$\Gamma(\Upsilon(h \cdot 1_p))(x) = \Gamma(h \otimes u_p)(x) = h(x)\Omega(u_p), \tag{5.1}$$

for all $h \in C_0(X)$, $p \in P_T$, and $x \in X$. Applications of [35, Propositions B.13, B.15(b), and B.16, and Corollary B.17] show that there is an isomorphism

$$\Lambda: C_0(X, C(\widehat{P}_T, C^*(B, \tilde{\omega}))) \rightarrow C_0(X \times \widehat{P}_T, C^*(B, \tilde{\omega}))$$

given by $\Lambda(g)(x, \phi) = g(x)(\phi)$. (See the proof of [3, Proposition 8.2.2] for details.) We claim that for each $g \in C_0(X, C(\widehat{P}_T, C^*(B, \tilde{\omega})))$,

$$\Lambda(g) \in \mathcal{X}_{T, \omega} \quad \text{if and only if} \quad g(x) \in \mathcal{Y}_{T, \omega} \text{ for all } x \in X. \tag{5.2}$$

To see this, fix $g \in C_0(X, C(\widehat{P}_T, C^*(B, \tilde{\omega})))$. For all $x \in X$, $\phi \in \widehat{P}_T$, and $\chi \in \widehat{B}$, we have

$$\Lambda(g)(x, \phi \cdot \chi) = g(x)(\phi \cdot \chi) \quad \text{and} \quad (\beta_\chi^B)^{-1}(\Lambda(g)(x, \phi)) = (\beta_\chi^B)^{-1}(g(x)(\phi)),$$

and hence

$$\Lambda(g)(x, \phi \cdot \chi) = (\beta_\chi^B)^{-1}(\Lambda(g)(x, \phi)) \quad \text{if and only if} \quad g(x)(\phi \cdot \chi) = (\beta_\chi^B)^{-1}(g(x)(\phi)).$$

It is now clear from the definitions of $\mathcal{X}_{T,\omega}$ and $\mathcal{Y}_{T,\omega}$ that claim (5.2) holds. Therefore, Λ restricts to an isomorphism $\tilde{\Lambda}: C_0(X, \mathcal{Y}_{T,\omega}) \rightarrow \mathcal{X}_{T,\omega}$, and so

$$\psi_T := \tilde{\Lambda} \circ \Gamma \circ \Upsilon: C^*(\mathcal{I}_T, \sigma) \rightarrow \mathcal{X}_{T,\omega}$$

is an isomorphism. Using equation (5.1) and the definitions of $\tilde{\Lambda}$ and Ω , we see that for all $h \in C_c(X)$, $p \in P_T$, and $(x, \phi) \in X \times \hat{P}_T$,

$$\psi_T(h \cdot 1_p)(x, \phi) = \Gamma(\Upsilon(h \cdot 1_p))(x)(\phi) = h(x)\Omega(u_p)(\phi) = h(x)\overline{\phi(p)}U_{p+Z_\omega}. \quad \blacksquare$$

We now give a useful description of the ideals of the induced algebra \mathcal{X}_T^ω .

Proposition 5.5. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable. Fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$, and let $\omega \in Z^2(P_T, \mathbb{T})$ and $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ be bicharacters chosen as in Proposition 4.2. Define $\mathcal{X}_T^\omega := \text{Ind}_{\hat{B}}^{X \times \hat{P}_T}(C^*(B, \tilde{\omega}), \beta^B)$. If I is an ideal of \mathcal{X}_T^ω , then*

$$K_I := \{(x, \phi) \in X \times \hat{P}_T : f(x, \phi) = 0 \text{ for all } f \in I\}$$

is a closed subset of $X \times \hat{P}_T$, and we have

$$I = \{f \in \mathcal{X}_T^\omega : f|_{K_I} \equiv 0\}.$$

In order to prove Proposition 5.5, we need the following special case of [19, Proposition 32], which Green in turn attributes to a preprint of Kleppner.

Lemma 5.6. *Let G be a countable discrete abelian group with identity e , and let $\zeta \in Z^2(G, \mathbb{T})$ be a bicharacter. Let $\{u_g : g \in G\}$ be the canonical family of generating unitaries for the twisted group C^* -algebra $C^*(G, \zeta)$. Suppose that for all $g \in G$, we have $(\zeta\zeta^*)(\{g\} \times G) = \{1\}$ if and only if $g = e$. Then $C^*(G, \zeta)$ is a simple C^* -algebra with a unique trace $\tau_e: C^*(G, \zeta) \rightarrow \mathbb{C}$, which satisfies $\tau_e(u_g) = \delta_{g,e}$ for all $g \in G$.*

Proof. This is a special case of [19, Proposition 32], but is also proved directly in [3, Proposition 8.2.4]. ■

Proof of Proposition 5.5. We have $K_I = \bigcap_{f \in I} f^{-1}(0)$, which is closed because each $f \in I$ is continuous. It is well known that if $C^*(B, \tilde{\omega})$ is simple, then $I = \{f \in \mathcal{X}_T^\omega : f|_{K_I} \equiv 0\}$ (see [3, Proposition 4.2.1] for a proof). We will use Lemma 5.6 to show that $C^*(B, \tilde{\omega})$ is simple. Fix $p \in P_T$. Proposition 4.2 implies that for all $q \in P_T$, we have

$$(\tilde{\omega}\tilde{\omega}^*)(p + Z_\omega, q + Z_\omega) = \omega(p, q)\overline{\omega(q, p)} = (\omega\omega^*)(p, q). \quad (5.3)$$

By the definition of Z_ω , we have $p \in Z_\omega$ if and only if $(\omega\omega^*)(p, q) = 1$ for all $q \in P_T$. Thus, equation (5.3) implies that $p + Z_\omega$ is the identity element of B if and only if $(\tilde{\omega}\tilde{\omega}^*)(\{p + Z_\omega\} \times B) = \{1\}$, and so Lemma 5.6 implies that $C^*(B, \tilde{\omega})$ is simple. ■

6. Simplicity of twisted C^* -algebras of Deaconu–Renault groupoids

In this section, we characterise simplicity of twisted C^* -algebras of Deaconu–Renault groupoids in terms of the underlying data, using the spectral action defined in Proposition 4.5.

Theorem 6.1. *Let (X, T) be a rank- k Deaconu–Renault system such that X is second-countable. Fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$.*

- (a) *If (X, T) is not minimal, then $C^*(\mathcal{G}_T, \sigma)$ is not simple.*
- (b) *Suppose that (X, T) is minimal. Let $\omega \in Z^2(P_T, \mathbb{T})$ and $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ be bicharacters chosen as in Proposition 4.2. Let θ be the spectral action associated to (T, σ) as in Proposition 4.5. Then $C^*(\mathcal{G}_T, \sigma)$ is simple if and only if θ is minimal.*

Proof of Theorem 6.1(a). This follows from [37, Corollary 4.9] applied to the groupoid dynamical system $(\mathcal{G}_T, \mathcal{G}_T \times_\sigma \mathbb{T}, C_0(\mathcal{G}_T^{(0)}))$, but it is easy to provide a short direct proof. Since (X, T) is not minimal, there exists $x \in X$ such that $\overline{[x]}$ is a proper closed invariant set. Let $\mathcal{H} := \mathcal{G}_T|_{\overline{[x]}} = \{\gamma \in \mathcal{G}_T : s(\gamma) \in \overline{[x]}\}$, and let τ be the restriction of σ to $\mathcal{H}^{(2)}$. Then the restriction map $f \mapsto f|_{\mathcal{H}}$ is a $*$ -homomorphism from $C_c(\mathcal{G}_T, \sigma)$ to $C^*(\mathcal{H}, \tau)$, and so it extends to a homomorphism $R: C^*(\mathcal{G}_T, \sigma) \rightarrow C^*(\mathcal{H}, \tau)$. Since $\ker(R) \cap C_0(\mathcal{G}_T^{(0)}) = C_0(X \setminus \overline{[x]})$ is neither $\{0\}$ nor all of $C_0(\mathcal{G}_T^{(0)})$, we see that $\ker(R)$ is a nonzero proper ideal of $C^*(\mathcal{G}_T, \sigma)$. ■

In order to prove part (b) of Theorem 6.1, we need several preliminary results. Let $\omega \in Z^2(P_T, \mathbb{T})$ and $\tilde{\omega} \in Z^2(P_T/Z_\omega, \mathbb{T})$ be bicharacters chosen as in Proposition 4.2. Define $B := P_T/Z_\omega$, and recall from Theorem 5.4 the definition of the isomorphism

$$\psi_T: C^*(\mathcal{I}_T, \sigma) \rightarrow \mathcal{X}_T^\omega = \text{Ind}_{\hat{B}}^{X \times \hat{P}_T}(C^*(B, \tilde{\omega}), \beta^B).$$

Let $\iota: C^*(\mathcal{I}_T, \sigma) \rightarrow C^*(\mathcal{G}_T, \sigma)$ be the homomorphism of [4, Proposition 6.1], so

$$\iota(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \mathcal{I}_T, \\ 0 & \text{if } \gamma \notin \mathcal{I}_T, \end{cases} \quad \text{for all } f \in C_c(\mathcal{I}_T, \sigma) \text{ and } \gamma \in \mathcal{G}_T.$$

Since \mathcal{I}_T is amenable (by [40, Lemma 3.5] and [1, Proposition 5.1.1]), ι is injective by [4, Proposition 6.1]. Define $M_T^\sigma := \iota(C^*(\mathcal{I}_T, \sigma)) \subseteq C^*(\mathcal{G}_T, \sigma)$.

We begin by showing that there is a bounded linear map on M_T^σ given by conjugation in $C^*(\mathcal{G}_T, \sigma)$ by a fixed element of $C_c(\mathcal{G}_T, [0, 1])$ that is supported on a bisection.

Lemma 6.2. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Let U be an open bisection of \mathcal{G}_T . Suppose that $g \in C_c(\mathcal{G}_T, [0, 1])$ satisfies $\text{supp}(g) \subseteq U$. For all $f \in C_c(\mathcal{I}_T, \sigma)$, we have $g^* \iota(f) g \in \iota(C_c(\mathcal{I}_T, \sigma))$. There is a linear contraction $\Xi_g: M_T^\sigma \rightarrow M_T^\sigma$ given by $\Xi_g(a) := g^* a g$.*

Proof. Fix $f \in C_c(\mathcal{I}_T, \sigma)$. Since U is a bisection containing $\text{supp}(g)$, we have

$$\text{supp}(g^* \iota(f)g) \subseteq U^{-1} \mathcal{I}_T U \subseteq \mathcal{I}_T,$$

and hence $g^* \iota(f)g \in \iota(C_c(\mathcal{I}_T, \sigma))$. Since g has range in $[0, 1]$ and is supported on a bisection, $\|g\| = \|g\|_\infty \leq 1$, and thus

$$\|g^* \iota(f)g\| \leq \|g^*\| \|\iota(f)\| \|g\| \leq \|\iota(f)\|.$$

Therefore, $\iota(f) \mapsto g^* \iota(f)g$ extends to a linear contraction $\Xi_g: M_T^\sigma \rightarrow M_T^\sigma$. ■

In the next lemma we introduce a bounded linear map $\Theta_{U,g}$ on the induced algebra $\mathcal{X}_{T,\omega}$ that is reminiscent of the spectral action θ associated to the pair (T, σ) . This map $\Theta_{U,g}$ is defined in terms of a fixed element g of $C_c(\mathcal{G}_T, [0, 1])$ that is supported on an open bisection U of \mathcal{G}_T , and as we show in Proposition 6.5(a), it simply amounts to conjugation of elements of $M_T^\sigma \cong \mathcal{X}_{T,\omega}$ by g .

Lemma 6.3. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in \mathbb{Z}^2(\mathcal{G}_T, \mathbb{T})$. Let U be an open bisection of \mathcal{G}_T . Suppose that $g \in C_c(\mathcal{G}_T, \sigma)$ satisfies $\text{supp}(g) \subseteq U$ and that $g(U) \subseteq [0, 1]$. For each $x \in s(U)$, let $\alpha_{U,x}$ denote the unique element of U with source x . Let $\gamma \mapsto \tau_\gamma^\sigma$ be the continuous \widehat{P}_T -valued 1-cocycle of Lemma 4.3(e). For $f \in \mathcal{X}_T^\omega$, define $\Theta_{U,g}(f): X \times \widehat{P}_T \rightarrow C^*(B, \widetilde{\omega})$ by*

$$\Theta_{U,g}(f)(x, \phi) := \begin{cases} |g(\alpha_{U,x})|^2 f(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U). \end{cases}$$

Then $\Theta_{U,g}(f) \in \mathcal{X}_T^\omega$, and $\Theta_{U,g}: \mathcal{X}_T^\omega \rightarrow \mathcal{X}_T^\omega$ is a bounded linear map.

Proof. Fix $f \in \mathcal{X}_T^\omega$. Then

$$f(x, \phi \cdot \chi) = (\beta_\chi^B)^{-1}(f(x, \phi)) \quad \text{for all } (x, \phi) \in X \times \widehat{P}_T \text{ and } \chi \in \widehat{B}. \tag{6.1}$$

We first show that $\Theta_{U,g}(f) \in C_0(X \times \widehat{P}_T, C^*(B, \widetilde{\omega}))$. The map $\Theta_{U,g}(f)$ is continuous because $x \mapsto \alpha_{U,x}$ is continuous. We have $\text{supp}(\Theta_{U,g}(f)) \subseteq s(\text{supp}(g)) \times \widehat{P}_T$, and so $\Theta_{U,g}(f)$ has compact support. Hence $\Theta_{U,g}(f) \in C_0(X \times \widehat{P}_T, C^*(B, \widetilde{\omega}))$. We must show that $\Theta_{U,g}(f)$ satisfies equation (6.1). Fix $(x, \phi) \in X \times \widehat{P}_T$ and $\chi \in \widehat{B}$. If $x \notin s(U)$, then

$$\Theta_{U,g}(f)(x, \phi \cdot \chi) = 0 = (\beta_\chi^B)^{-1}(0) = (\beta_\chi^B)^{-1}(\Theta_{U,g}(f)(x, \phi)).$$

Suppose that $x \in s(U)$. Since $f \in \mathcal{X}_T^\omega$, equation (6.1) implies that

$$\begin{aligned} \Theta_{U,g}(f)(x, \phi \cdot \chi) &= |g(\alpha_{U,x})|^2 f(r(\alpha_{U,x}), (\tau_{\alpha_{U,x}}^\sigma \phi) \cdot \chi) \\ &= |g(\alpha_{U,x})|^2 (\beta_\chi^B)^{-1}(f(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi)) \\ &= (\beta_\chi^B)^{-1}(\Theta_{U,g}(f)(x, \phi)). \end{aligned}$$

Therefore, $\Theta_{U,g}(f) \in \mathcal{X}_T^\omega$. Since the range of g is contained in $[0, 1]$, routine calculations show that $\Theta_{U,g}: \mathcal{X}_T^\omega \rightarrow \mathcal{X}_T^\omega$ is a bounded linear map. ■

In the next lemma we show that the set of functions of the form $\iota(h \cdot 1_p)$ (as defined in Lemma 5.2) is invariant under conjugation in $C^*(\mathcal{G}_T, \sigma)$ by a fixed element of $C_c(\mathcal{G}_T, [0, 1])$ that is supported on a bisection.

Lemma 6.4. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Let U be an open bisection of \mathcal{G}_T . Suppose that $g \in C_c(\mathcal{G}_T, \sigma)$ satisfies $\text{supp}(g) \subseteq U$ and that $g(U) \subseteq [0, 1]$. For each $x \in s(U)$, let $\alpha_{U,x}$ denote the unique element of U with source x . Let $\Xi_g: M_T^\sigma \rightarrow M_T^\sigma$ and $\gamma \mapsto \tau_\gamma^\sigma$ be as in Lemmas 6.2 and 4.3(e). For each $x \in s(U)$, let $\alpha_{U,x}$ denote the unique element of U with source x . Fix $h \in C_c(X)$ and $p \in P_T$, and define $H_{g,p}: X \rightarrow \mathbb{C}$ by*

$$H_{g,p}(x) := \begin{cases} |g(\alpha_{U,x})|^2 \overline{\tau_{\alpha_{U,x}}^\sigma(p)} h(r(\alpha_{U,x})) & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U). \end{cases}$$

Then $H_{g,p} \in C_c(X)$, and we have $\Xi_g(\iota(h \cdot 1_p)) = \iota(H_{g,p} \cdot 1_p)$.

Proof. Since $x \mapsto \alpha_{U,x}$ is continuous on $s(U)$, and since $\gamma \mapsto \tau_\gamma^\sigma(p)$ is continuous by Lemma 4.3(b), the map $H_{g,p}$ is continuous. Since $\text{supp}(H_{g,p}) \subseteq s(\text{supp}(g))$, we have $H_{g,p} \in C_c(X)$.

By Lemma 6.2, we have $\Xi_g(\iota(h \cdot 1_p)) \in \iota(C_c(\mathcal{I}_T, \sigma))$. Thus, for all $\gamma \in \mathcal{G}_T \setminus \mathcal{I}_T$, we have

$$\Xi_g(\iota(h \cdot 1_p))(\gamma) = 0 = \iota(H_{g,p} \cdot 1_p)(\gamma).$$

Suppose that $\gamma \in \mathcal{I}_T$. Then by Proposition 3.5, there exist $x \in X$ and $m \in P_T$ such that $\gamma = (x, m, x)$. We have

$$\text{supp}(\Xi_g(\iota(h \cdot 1_p))) \subseteq \text{supp}(g^*) \text{supp}(\iota(h \cdot 1_p)) \text{supp}(g) \subseteq U^{-1} \mathcal{I}_T U.$$

Thus, if $x \notin s(U)$, then $\gamma \notin \text{supp}(\Xi_g(\iota(h \cdot 1_p)))$ and $H_{g,p}(x) = 0$, and hence

$$\Xi_g(\iota(h \cdot 1_p))(x, m, x) = 0 = \iota(H_{g,p} \cdot 1_p)(x, m, x).$$

Suppose that $x \in s(U)$. Since g is supported on the bisection U ,

$$\begin{aligned} \Xi_g(\iota(h \cdot 1_p))(x, m, x) &= \tau_{(\alpha_{U,x})^{-1}}^\sigma(m) g^*(\alpha_{U,x}) \iota(h \cdot 1_p)(r(\alpha_{U,x}), m, r(\alpha_{U,x})) g(\alpha_{U,x}) \\ &= |g(\alpha_{U,x})|^2 \overline{\tau_{\alpha_{U,x}}^\sigma(m)} \delta_{p,m} h(r(\alpha_{U,x})) \\ &= \delta_{p,m} H_{g,p}(x) \\ &= \iota(H_{g,p} \cdot 1_p)(x, m, x). \end{aligned}$$

Therefore, $\Xi_g(\iota(h \cdot 1_p)) = \iota(H_{g,p} \cdot 1_p)$. ■

In the following proposition we describe exactly how the map $\Theta_{U,g}$ defined in Lemma 6.3 relates to the conjugation map Ξ_g defined in Lemma 6.2. We also show that ideals of $\mathcal{X}_{T,\omega}$ induced by ideals of $C^*(\mathcal{G}_T, \sigma)$ are invariant under $\Theta_{U,g}$, which is a key result used in the proof of Theorem 6.1(b).

Proposition 6.5. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Let U be an open bisection of \mathcal{G}_T . Suppose that $g \in C_c(\mathcal{G}_T, [0, 1])$ satisfies $\text{supp}(g) \subseteq U$. Recall the definitions of the bounded linear maps $\Xi_g: M_T^\sigma \rightarrow M_T^\sigma$ from Lemma 6.2 and $\Theta_{U,g}: \mathcal{X}_T^\omega \rightarrow \mathcal{X}_T^\omega$ from Lemma 6.3.*

(a) *For all $a \in C^*(\mathcal{I}_T, \sigma)$, we have*

$$\Theta_{U,g}(\psi_T(a)) = \psi_T(\iota^{-1}(\Xi_g(\iota(a)))).$$

(b) *Suppose that I is an ideal of $C^*(\mathcal{G}_T, \sigma)$, and that J is an ideal of $C^*(\mathcal{I}_T, \sigma)$ such that $\iota(J) = I \cap M_T^\sigma$. Then the ideal $\psi_T(J)$ is invariant under $\Theta_{U,g}$.*

Proof. For part (a), fix $h \in C_c(X)$ and $p \in P_T$. Since all the maps involved are bounded and linear, Lemma 5.2 implies that it suffices to show that

$$\Theta_{U,g}(\psi_T(h \cdot 1_p)) = \psi_T(\iota^{-1}(\Xi_g(\iota(h \cdot 1_p)))).$$

Recall from Lemma 6.4 that there is a function $H_{g,p} \in C_c(X)$ given by

$$H_{g,p}(x) = \begin{cases} |g(\alpha_{U,x})|^2 \overline{\tau_{\alpha_{U,x}}^\sigma(p)} h(r(\alpha_{U,x})) & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U), \end{cases}$$

which satisfies $\Xi_g(\iota(h \cdot 1_p)) = \iota(H_{g,p} \cdot 1_p)$. Thus, for all $(x, \phi) \in X \times P_T$, we have

$$\begin{aligned} \Theta_{U,g}(\psi_T(h \cdot 1_p))(x, \phi) &= \begin{cases} |g(\alpha_{U,x})|^2 \psi_T(h \cdot 1_p)(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U), \end{cases} \\ &= \begin{cases} |g(\alpha_{U,x})|^2 h(r(\alpha_{U,x})) \overline{\tau_{\alpha_{U,x}}^\sigma(p)} \phi(p) U_{p+Z_\omega} & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U), \end{cases} \\ &= H_{g,p}(x) \overline{\phi(p)} U_{p+Z_\omega} \\ &= \psi_T(H_{g,p} \cdot 1_p)(x, \phi) \\ &= \psi_T(\iota^{-1}(\Xi_g(\iota(h \cdot 1_p)))(x, \phi). \end{aligned}$$

For part (b), fix $a \in J$. Then $\iota(a) \in I \cap M_T^\sigma$. Since I is an ideal of $C^*(\mathcal{G}_T, \sigma)$ and the range of Ξ_g is contained in M_T^σ , we have $\Xi_g(\iota(a)) = g^* \iota(a) g \in I \cap M_T^\sigma = \iota(J)$, and so $\iota^{-1}(\Xi_g(\iota(a))) \in J$. Hence part (a) implies that

$$\Theta_{U,g}(\psi_T(a)) = \psi_T(\iota^{-1}(\Xi_g(\iota(a)))) \in \psi_T(J),$$

and thus $\Theta_{U,g}(\psi_T(J)) \subseteq \psi_T(J)$. ■

We now use Proposition 6.5(b) to show that the closed subsets of $X \times \widehat{P}_T$ characterising the ideals of the induced algebra $\mathcal{X}_{T,\omega}$ are invariant under the spectral action θ associated to the pair (T, σ) .

Proposition 6.6. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Suppose that I is an ideal of $C^*(\mathcal{G}_T, \sigma)$, and that J is an ideal of $C^*(\mathcal{I}_T, \sigma)$ such that $\iota(J) = I \cap M_T^\sigma$. Define*

$$K_J := \{(x, \phi) \in X \times \widehat{P}_T : f(x, \phi) = 0 \text{ for all } f \in \psi_T(J)\}.$$

Let $Q: X \times \widehat{P}_T \rightarrow X \times \widehat{Z}_\omega$ be the quotient map $(x, \phi) \mapsto (x, \phi|_{Z_\omega})$. Then $Q^{-1}(Q(K_J)) = K_J$, and $Q(K_J)$ is closed and invariant under the spectral action θ of Proposition 4.5.

Proof. We trivially have $K_J \subseteq Q^{-1}(Q(K_J))$. We must show that $Q^{-1}(Q(K_J)) \subseteq K_J$. Fix $(x, \phi) \in Q^{-1}(Q(K_J))$. Then $(x, \phi|_{Z_\omega}) = Q(x, \phi) \in Q(K_J)$, and so there exists $(y, \rho) \in K_J$ such that $(x, \phi|_{Z_\omega}) = Q(y, \rho) = (y, \rho|_{Z_\omega})$. We have $x = y$ and $\phi|_{Z_\omega} = \rho|_{Z_\omega}$, and hence [17, Theorem 4.40] implies that $\phi \cdot \widehat{B} = \rho \cdot \widehat{B}$. So there exists $\chi \in \widehat{B}$ such that $\phi \cdot \chi = \rho \cdot 1_{\widehat{B}} = \rho$. Since $(x, \phi \cdot \chi) = (y, \rho) \in K_J$, we have $f(x, \phi \cdot \chi) = 0$ for all $f \in \psi_T(J)$. Thus, since $\psi_T(J) \subseteq \mathcal{X}_T^\omega$, we have $f(x, \phi) = \beta_\chi^B(f(x, \phi \cdot \chi)) = 0$ for all $f \in \psi_T(J)$. Hence $(x, \phi) \in K_J$, and so $Q^{-1}(Q(K_J)) = K_J$. Since Q is a quotient map, [13, Proposition 2.4.3] implies that $C \subseteq X \times \widehat{Z}_\omega$ is closed if and only if $Q^{-1}(C) \subseteq X \times \widehat{P}_T$ is closed. Since $Q^{-1}(Q(K_J)) = K_J$ is closed in $X \times \widehat{P}_T$ (by Proposition 5.5), we deduce that $Q(K_J)$ is closed.

We now show that $Q(K_J)$ is invariant under θ . Fix $(x, \zeta) \in Q(K_J)$ and $\gamma \in (\mathcal{G}_T)_x$. Then there exists $\phi \in \widehat{P}_T$ such that $(x, \phi) \in K_J$ and $\phi|_{Z_\omega} = \zeta$. We must show that $\theta_{[\gamma]}(x, \zeta) \in Q(K_J)$. Proposition 4.5 implies that $\tau_\gamma^\sigma|_{Z_\omega} = \tilde{\tau}_{[\gamma]}^\sigma$, and so

$$\theta_{[\gamma]}(x, \zeta) = (r(\gamma), \tilde{\tau}_{[\gamma]}^\sigma \zeta) = Q(r(\gamma), \tau_\gamma^\sigma \phi).$$

Hence it suffices to show that $(r(\gamma), \tau_\gamma^\sigma \phi) \in K_J$. Fix $f \in \psi_T(J)$. We must show that $f(r(\gamma), \tau_\gamma^\sigma \phi) = 0$. Let $U \subseteq \mathcal{G}_T$ be an open bisection containing γ . By Urysohn’s lemma there exists $g \in C_c(\mathcal{G}_T, [0, 1])$ such that $\text{supp}(g) \subseteq U$ and $g(\gamma) = 1$. Let $\Theta_{U,g}: \mathcal{X}_T^\omega \rightarrow \mathcal{X}_T^\omega$ be as in Lemma 6.3. Since $s|_{U^{-1}}(x) = \gamma$ and $g(\gamma) = 1$,

$$\Theta_{U,g}(f)(x, \phi) = f(r(\gamma), \tau_\gamma^\sigma \phi). \tag{6.2}$$

Since $f \in \psi_T(J)$, Proposition 6.5(b) implies that $\Theta_{U,g}(f) \in \psi_T(J)$. Since $\psi_T(J)$ is an ideal of \mathcal{X}_T^ω , Proposition 5.5 implies that

$$\psi_T(J) = \{f \in \mathcal{X}_T^\omega : f|_{K_J} \equiv 0\}.$$

Thus, since $(x, \phi) \in K_J$ and $\Theta_{U,g}(f) \in \psi_T(J)$, we have

$$\Theta_{U,g}(f)(x, \phi) = 0. \tag{6.3}$$

Together, equations (6.2) and (6.3) imply that $f(r(\gamma), \tau_\gamma^\sigma \phi) = 0$, as required. ■

We now prove several technical results that we use in the proof of Theorem 6.1(b) to show that when the spectral action θ is not minimal, the twisted groupoid C*-algebra $C^*(\mathcal{G}_T, \sigma)$ is not simple. We first show that, given an element $(x, \phi) \in X \times \widehat{P}_T$ with non-dense orbit under θ , there is a nonzero element of $\mathcal{X}_{T,\omega} \cong M_T^\sigma$ that is supported off the orbit of (x, ϕ) .

Lemma 6.7. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Let $Q: X \times \widehat{P}_T \rightarrow X \times \widehat{Z}_\omega$ be the quotient map $(x, \phi) \mapsto (x, \phi|_{Z_\omega})$. Suppose that $(x, \phi) \in X \times \widehat{P}_T$ satisfies $\overline{[x, \phi|_{Z_\omega}]_\theta} \neq X \times \widehat{Z}_\omega$. Then $Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta})$ is a proper closed subset of $X \times \widehat{P}_T$, and there exists $f \in M_T^\sigma \setminus \{0\}$ such that*

$$\text{supp}((\psi_T \circ \iota^{-1})(f)) \subseteq (X \times \widehat{P}_T) \setminus Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta}).$$

Proof. Let $C_{(x,\phi)} := Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta})$. Since $\overline{[x, \phi|_{Z_\omega}]_\theta} \neq X \times \widehat{Z}_\omega$ and Q is surjective, $C_{(x,\phi)} \neq X \times \widehat{P}_T$. Since Q is continuous, $C_{(x,\phi)}$ is closed. By Urysohn’s lemma there exists $h \in C_c(X \times \widehat{P}_T, [0, 1]) \setminus \{0\}$ such that $\text{supp}(h) \subseteq (X \times \widehat{P}_T) \setminus C_{(x,\phi)}$. Define $g: X \times \widehat{P}_T \rightarrow C^*(B, \tilde{\omega})$ by

$$g(y, \rho) := \int_{\widehat{B}} h(y, \rho \cdot \chi) \beta_\chi^B(U_{0+Z_\omega}) \, d\chi = \int_{\widehat{B}} h(y, \rho \cdot \chi) U_{0+Z_\omega} \, d\chi.$$

By [35, Lemma 6.17], we have $g \in \mathcal{X}_T^\omega$. Since $h \neq 0$ and $h(y, \rho) \geq 0$ for all $(y, \rho) \in \text{supp}(h)$, we have $g \neq 0$. We claim that $\text{supp}(g) \subseteq (X \times \widehat{P}_T) \setminus C_{(x,\phi)}$. Fix $(y, \rho) \in C_{(x,\phi)}$. Then $Q(y, \rho) \in \overline{[x, \phi|_{Z_\omega}]_\theta}$. It suffices to show that $g(y, \rho) = 0$. Fix $\chi \in \widehat{B}$. For all $m \in Z_\omega$, we have $\chi(m + Z_\omega) = 1$, and hence $(\rho \cdot \chi)(m) = \rho(m)\chi(m + Z_\omega) = \rho(m)$. Thus

$$Q(y, \rho \cdot \chi) = (y, (\rho \cdot \chi)|_{Z_\omega}) = (y, \rho|_{Z_\omega}) = Q(y, \rho) \in \overline{[x, \phi|_{Z_\omega}]_\theta},$$

and hence $(y, \rho \cdot \chi) \in C_{(x,\phi)}$. Since $\text{supp}(h) \subseteq (X \times \widehat{P}_T) \setminus C_{(x,\phi)}$, we have $h(y, \rho \cdot \chi) = 0$ for all $\chi \in \widehat{B}$, and therefore,

$$g(y, \rho) = \int_{\widehat{B}} h(y, \rho \cdot \chi) U_{0+Z_\omega} \, d\chi = 0.$$

Define $f := (\iota \circ \psi_T^{-1})(g) \in M_T^\sigma$. Since $g \neq 0$ and $\iota \circ \psi_T^{-1}$ is injective, we have $f \neq 0$. Since $(\psi_T \circ \iota^{-1})(f) = g$, we have

$$\text{supp}((\psi_T \circ \iota^{-1})(f)) = \text{supp}(g) \subseteq (X \times \widehat{P}_T) \setminus C_{(x,\phi)}. \quad \blacksquare$$

Recall from [4, Lemma 6.2(b)] that since \mathcal{I}_T is closed in \mathcal{G}_T (by Proposition 3.9) and amenable, there is a conditional expectation $\Phi: C^*(\mathcal{G}_T, \sigma) \rightarrow M_T^\sigma$ satisfying $\Phi \circ \iota = \iota$ and $\Phi(f) = \iota(f|_{\mathcal{I}_T})$ for all $f \in C_c(\mathcal{G}_T, \sigma)$.

Lemma 6.8. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Recall from Lemma 5.1 the definition of the spanning set \mathcal{B}_T for $C_c(\mathcal{G}_T, \sigma)$. Given $a, b \in \mathcal{B}_T$ and $f \in M_T^\sigma$, there exist $p, q, g \in \mathcal{B}_T$ such that $gq^*, pg^* \in \iota(C_c(\mathcal{I}_T, \sigma))$, the range of g is contained in $[0, 1]$, and the map Ξ_g of Lemma 6.2 satisfies $\Phi(b^*fa) = \Xi_g(gq^*fpq^*)$.*

Proof. Define $U := \text{osupp}(a)$ and $V := \text{osupp}(b)$. Since $a, b \in \mathcal{B}_T$, both \overline{U} and \overline{V} are compact bisections, and there exist $m, n \in \mathbb{Z}^k$ such that $\overline{U} \subseteq c^{-1}(m)$ and $\overline{V} \subseteq c^{-1}(n)$.

Define $X := \mathcal{I}_T V \cap U$ and $Y := \mathcal{I}_T U \cap V$. Define $p, q: \mathcal{G}_T \rightarrow \mathbb{C}$ by

$$p(\gamma) := \begin{cases} a(\gamma) & \text{if } \gamma \in \overline{X}, \\ 0 & \text{if } \gamma \notin \overline{X}, \end{cases} \quad \text{and} \quad q(\gamma) := \begin{cases} b(\gamma) & \text{if } \gamma \in \overline{Y}, \\ 0 & \text{if } \gamma \notin \overline{Y}. \end{cases}$$

Since $\text{supp}(p) \subseteq \overline{X} \subseteq \overline{U}$ and $\text{supp}(q) \subseteq \overline{Y} \subseteq \overline{V}$, we have $p, q \in \mathcal{B}_T$. Let W be an open bisection of \mathcal{G}_T such that $\overline{V} \subseteq W \subseteq c^{-1}(n)$. By Urysohn’s lemma there exists $g \in C_c(\mathcal{G}_T, [0, 1])$ such that $\text{supp}(g) \subseteq W$ and $g|_{\overline{V}} \equiv 1$. Then $g \in \mathcal{B}_T$, and $gq^*, pg^* \in C_c(\mathcal{G}_T, \sigma)$. We claim that $gq^*, pg^* \in \iota(C_c(\mathcal{I}_T, \sigma))$. To see this, it suffices to show that $\text{osupp}(gq^*) \cup \text{osupp}(pg^*) \subseteq \mathcal{I}_T$. Since q and g are supported on bisections, we have

$$\text{osupp}(gq^*) = \text{osupp}(g)(\text{osupp}(q))^{-1} \subseteq WV^{-1} \subseteq WW^{-1} = r(W) \subseteq \mathcal{I}_T.$$

By Lemma 3.10, $\overline{X} = \overline{\mathcal{I}_T V \cap U} \subseteq \mathcal{I}_T \overline{V} \subseteq \mathcal{I}_T W$, and since p and g are supported on bisections, we deduce that

$$\text{osupp}(pg^*) = \text{osupp}(p)(\text{osupp}(g))^{-1} \subseteq \overline{X}W^{-1} \subseteq \mathcal{I}_T WW^{-1} = \mathcal{I}_T r(W) \subseteq \mathcal{I}_T.$$

Therefore, $gq^*, pg^* \in \iota(C_c(\mathcal{I}_T, \sigma))$, and Lemma 6.2 implies that

$$\text{osupp}(\Xi_g(gq^* fpg^*)) \subseteq \mathcal{I}_T. \tag{6.4}$$

We conclude by showing that $\Phi(b^* fa) = \Xi_g(gq^* fpg^*)$. Since ι, Φ , and Ξ_g are bounded linear maps, Lemma 5.2 implies that it suffices to consider $f = \iota(h \cdot 1_p)$ for some $h \in C_c(X)$ and $p \in P_T$. Define $D := \text{osupp}(f) \subseteq c|_{\mathcal{I}_T}^{-1}(p)$. Then $\text{osupp}(g^*g) \subseteq s(W)$, and so

$$\text{osupp}(\Xi_g(gq^* fpg^*)) = \text{osupp}(g^*gq^* fpg^*g) \subseteq s(W)(V^{-1}DU)s(W) \subseteq V^{-1}DU. \tag{6.5}$$

Together, equations (6.4) and (6.5) imply that

$$\text{osupp}(\Xi_g(gq^* fpg^*)) \subseteq (V^{-1}DU) \cap \mathcal{I}_T = \text{osupp}(b^* fa) \cap \mathcal{I}_T = \text{osupp}(\Phi(b^* fa)).$$

Thus, if $\Phi(b^* fa)(\gamma) = 0$ for some $\gamma \in \mathcal{G}_T$, then $\Xi_g(gq^* fpg^*)(\gamma) = 0$. Suppose that $\gamma \in \mathcal{G}_T$ satisfies $\Phi(b^* fa)(\gamma) \neq 0$. Then $\gamma \in \mathcal{I}_T$, and equation (6.5) implies that there exist $\alpha \in U, \beta \in V$, and $\xi \in D \subseteq \mathcal{I}_T$ such that $\gamma = \beta^{-1}\xi\alpha \in \mathcal{I}_T$. A routine calculation gives

$$\Phi(b^* fa)(\gamma) = (b^* fa)(\beta^{-1}\xi\alpha) = \sigma(\beta^{-1}\xi, \alpha)\sigma(\beta^{-1}, \xi)\overline{\sigma(\beta^{-1}, \beta)b(\beta)}f(\xi)a(\alpha). \tag{6.6}$$

Define $y := s(\gamma)$. Since $\gamma \in \mathcal{I}_T$, we have $s(\beta) = r(\gamma) = y$. Since $\beta \in V$ and $g|_{\overline{V}} \equiv 1$,

$$(g^*g)(y) = (g^*g)(s(\beta)) = |g(\beta)|^2 = 1. \tag{6.7}$$

A routine calculation using equation (6.7) and that σ is normalised gives

$$\begin{aligned} \Xi_g(gq^* fpg^*)(\gamma) &= (g^*gq^* fpg^*g)(y\gamma y) \\ &= \sigma(y, \gamma y)\sigma(\gamma, y)(g^*g)(y)(q^*fp)(\gamma)(g^*g)(y) \end{aligned}$$

$$\begin{aligned}
 &= (q^*fp)(\beta^{-1}\xi\alpha) \\
 &= \sigma(\beta^{-1}\xi, \alpha)\sigma(\beta^{-1}, \xi)\overline{\sigma(\beta^{-1}, \beta)q(\beta)}f(\xi)p(\alpha). \tag{6.8}
 \end{aligned}$$

We claim that $p(\alpha) = a(\alpha)$ and $q(\beta) = b(\beta)$. Since $\gamma \in (V^{-1}\mathcal{I}_T U) \cap \mathcal{I}_T$, Lemma 3.11 implies that

$$y = s(\gamma) \in s(\mathcal{I}_T U \cap V) = s(\mathcal{I}_T V \cap U).$$

So there exist $\eta \in \mathcal{I}_T V \cap U = X \subseteq U$ and $\zeta \in \mathcal{I}_T U \cap V = Y \subseteq V$ such that $s(\eta) = y = s(\zeta)$. Since $s|_U$ and $s|_V$ are homeomorphisms onto their ranges and

$$s(\eta) = s(\alpha) = y = s(\beta) = s(\zeta),$$

we deduce that $\alpha = \eta \in X$ and $\beta = \zeta \in Y$. Hence $p(\alpha) = a(\alpha)$ and $q(\beta) = b(\beta)$. Together, equations (6.6) and (6.8) now give

$$\Phi(b^*fa)(\gamma) = \sigma(\beta^{-1}\xi, \alpha)\sigma(\beta^{-1}, \xi)\overline{\sigma(\beta^{-1}, \beta)b(\beta)}f(\xi)a(\alpha) = \Xi_g(gq^*fpg^*)(\gamma). \blacksquare$$

Proposition 6.9. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Let $Q: X \times \hat{P}_T \rightarrow X \times \hat{Z}_\omega$ be the quotient map $(x, \phi) \mapsto (x, \phi|_{Z_\omega})$, and let $\Phi: C^*(\mathcal{G}_T, \sigma) \rightarrow M_T^\sigma$ be the conditional expectation of [4, Lemma 6.2(b)] that extends restriction of functions to \mathcal{I}_T . Fix $(x, \phi) \in X \times \hat{P}_T$. Suppose that $f \in M_T^\sigma$ satisfies*

$$\text{supp}((\psi_T \circ \iota^{-1})(f)) \subseteq (X \times \hat{P}_T) \setminus Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta}).$$

Then for all $a, b \in C^*(\mathcal{G}_T, \sigma)$, we have

$$(\psi_T \circ \iota^{-1} \circ \Phi)(b^*fa)(x, \phi) = 0.$$

Proof. Let $\text{ev}_{(x, \phi)}: \mathcal{X}_T^\omega \rightarrow C^*(B, \tilde{\omega})$ denote the evaluation map $f \mapsto f(x, \phi)$. Recall from Lemma 5.1 the definition of the spanning set \mathcal{B}_T for $C_c(\mathcal{G}_T, \sigma)$. Let $C_{(x, \phi)} := Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta})$. Fix $a, b \in \mathcal{B}_T$, and suppose that $f \in M_T^\sigma$ satisfies

$$\text{supp}((\psi_T \circ \iota^{-1})(f)) \subseteq (X \times \hat{P}_T) \setminus C_{(x, \phi)}.$$

Since $\text{ev}_{(x, \phi)}$, ψ_T , ι^{-1} , and Φ are all bounded linear maps, it suffices to show that

$$(\text{ev}_{(x, \phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi)(b^*fa) = 0.$$

Let Ξ_g be the bounded linear map defined in Lemma 6.2. By Lemma 6.8 there exist $p, q, g \in \mathcal{B}_T$ such that $gq^*, pg^* \in \iota(C_c(\mathcal{I}_T, \sigma))$, the range of g is contained in $[0, 1]$, and

$$\Phi(b^*fa) = \Xi_g(gq^*fpg^*). \tag{6.9}$$

Let U be an open bisection of \mathcal{G}_T containing $\text{supp}(g)$. For $y \in s(U)$, let $\alpha_{U,y}$ denote the unique element of U with source y . Define $h_q := \psi_T(\iota^{-1}(gq^*))$ and $h_p := \psi_T(\iota^{-1}(pg^*))$. Then

$$\psi_T(\iota^{-1}(gq^*fpg^*)) = h_q\psi_T(\iota^{-1}(f))h_p. \tag{6.10}$$

By Proposition 6.5(a),

$$\psi_T(\iota^{-1}(\Xi_g(gq^* fpg^*))) = \Theta_{U,g}(\psi_T(\iota^{-1}(gq^* fpg^*))). \tag{6.11}$$

Together, equations (6.9) and (6.11) imply that

$$\begin{aligned} & (\text{ev}_{(x,\phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi)(b^* fa) \\ &= \psi_T(\iota^{-1}(\Xi_g(gq^* fpg^*)))(x, \phi) \\ &= \Theta_{U,g}(\psi_T(\iota^{-1}(gq^* fpg^*)))(x, \phi) \\ &= \begin{cases} |g(\alpha_{U,x})|^2 \psi_T(\iota^{-1}(gq^* fpg^*))(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) & \text{if } x \in s(U), \\ 0 & \text{if } x \notin s(U). \end{cases} \end{aligned}$$

Thus, to see that $(\text{ev}_{(x,\phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi)(b^* fa) = 0$, it suffices to show that if $x \in s(U)$, then

$$\psi_T(\iota^{-1}(gq^* fpg^*))(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) = 0.$$

If $x \in s(U)$, then

$$Q(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) = (r(\alpha_{U,x}), \tilde{\tau}_{[\alpha_{U,x}]}^\sigma \phi|_{Z_\omega}) = \theta_{[\alpha_{U,x}]}(x, \phi|_{Z_\omega}),$$

and hence

$$(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) \in Q^{-1}(\theta_{[\alpha_{U,x}]}(x, \phi|_{Z_\omega})) \subseteq C_{(x,\phi)}.$$

Since $\text{supp}((\psi_T \circ \iota^{-1})(f)) \subseteq (X \times \hat{P}_T) \setminus C_{(x,\phi)}$, we obtain

$$\psi_T(\iota^{-1}(f))(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) = 0.$$

Combining this with equation (6.10) gives

$$\psi_T(\iota^{-1}(gq^* fpg^*))(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) = (h_q \psi_T(\iota^{-1}(f)) h_p)(r(\alpha_{U,x}), \tau_{\alpha_{U,x}}^\sigma \phi) = 0. \blacksquare$$

We now construct a state $\kappa_{(x,\phi)}$ of $C^*(\mathcal{G}_T, \sigma)$ defined in terms of a fixed element $(x, \phi) \in X \times \hat{P}_T$. In the proof of Theorem 6.1(b), we show that if some point (x, ϕ) has non-dense orbit under θ (so that θ is not minimal), then the GNS representation associated to $\kappa_{(x,\phi)}$ is nonzero and has nontrivial kernel, and thus $C^*(\mathcal{G}_T, \sigma)$ is not simple.

Lemma 6.10. *Let (X, T) be a minimal rank- k Deaconu–Renault system such that X is second-countable, and fix $\sigma \in Z^2(\mathcal{G}_T, \mathbb{T})$. Fix $(x, \phi) \in X \times \hat{P}_T$. Let $\Phi: C^*(\mathcal{G}_T, \sigma) \rightarrow M_T^\sigma$ be the conditional expectation of [4, Lemma 6.2(b)] that extends restriction of functions to \mathcal{I}_T , and let $\text{ev}_{(x,\phi)}: \mathcal{X}_T^\omega \rightarrow C^*(B, \tilde{\omega})$ be the evaluation map $f \mapsto f(x, \phi)$. Let Tr_T^ω denote the canonical trace on $C^*(B, \tilde{\omega})$ (as defined in Lemma 5.6). Let*

$$\kappa_{(x,\phi)} := \text{Tr}_T^\omega \circ \text{ev}_{(x,\phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi: C^*(\mathcal{G}_T, \sigma) \rightarrow \mathbb{C}.$$

For all $h \in C_c(X)$ such that $h(x) = 1$, we have $\kappa_{(x,\phi)}(h \cdot 1_0) = 1$. Moreover, $\kappa_{(x,\phi)}$ is a state of $C^*(\mathcal{G}_T, \sigma)$.

Proof. Suppose that $h \in C_c(X)$ satisfies $h(x) = 1$. Since $\Phi \circ \iota = \iota$, we have

$$(\text{ev}_{(x,\phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi)(\iota(h \cdot 1_0)) = \psi_T(h \cdot 1_0)(x, \phi) = h(x)\overline{\phi(0)}U_{0+Z_\omega} = U_{0+Z_\omega},$$

and hence

$$\kappa_{(x,\phi)}(\iota(h \cdot 1_0)) = \text{Tr}_T^\omega(U_{0+Z_\omega}) = 1. \tag{6.12}$$

Since Φ , ι^{-1} , ψ_T , $\text{ev}_{(x,\phi)}$, and Tr_T^ω are all positive norm-decreasing linear maps, $\kappa_{(x,\phi)}$ is a positive linear functional, and $\|\kappa_{(x,\phi)}\| \leq 1$. By Urysohn’s lemma there exists $h \in C_c(X)$ such that $h(x) = 1$. Then equation (6.12) implies that $\|\kappa_{(x,\phi)}\| \geq 1$, and thus $\kappa_{(x,\phi)}$ is a state of $C^*(\mathcal{G}_T, \sigma)$. ■

We conclude this section by proving Theorem 6.1(b), which says that if (X, T) is minimal, then $C^*(\mathcal{G}_T, \sigma)$ is simple if and only if the spectral action θ is minimal.

Proof of Theorem 6.1(b). Suppose that θ is minimal. Let I be a nontrivial ideal of $C^*(\mathcal{G}_T, \sigma)$. Then there exists a non-injective homomorphism Ψ of $C^*(\mathcal{G}_T, \sigma)$ such that $I = \ker(\Psi)$. By [4, Theorem 6.3], $J := \ker(\Psi \circ \iota)$ is a nontrivial ideal of $C^*(\mathcal{I}_T, \sigma)$. We have

$$\iota(J) = \{\iota(a) : a \in C^*(\mathcal{I}_T, \sigma), \Psi(\iota(a)) = 0\} = \{b \in M_T^\sigma : \Psi(b) = 0\} = I \cap M_T^\sigma \subseteq I.$$

Thus, to see that $C^*(\mathcal{G}_T, \sigma)$ is simple, it suffices to show that $J = C^*(\mathcal{I}_T, \sigma)$, because then $\iota(C_0(X)) \subseteq \iota(J) \subseteq I$, and (as argued in [3, Theorem 5.3.13]) [14, Proposition 3.18] implies that $I = C^*(\mathcal{G}_T, \sigma)$. Define

$$K_J := \{(x, \phi) \in X \times \hat{P}_T : f(x, \phi) = 0 \text{ for all } f \in \psi_T(J)\}.$$

Since $\psi_T(J)$ is an ideal of \mathcal{X}_T^ω , Proposition 5.5 implies that K_J is a closed subset of $X \times \hat{P}_T$, and

$$\psi_T(J) = \{f \in \mathcal{X}_T^\omega : f|_{K_J} \equiv 0\}.$$

Let $Q_T : X \times \hat{P}_T \rightarrow X \times \hat{Z}_\omega$ be the quotient map $(x, \phi) \mapsto (x, \phi|_{Z_\omega})$. Suppose that $Q(K_J)$ is nonempty, and fix $(x, \zeta) \in Q(K_J)$. By Proposition 6.6, $Q(K_J)$ is closed and invariant under θ , and hence

$$\overline{[x, \zeta]_\theta} = \overline{\{\theta_{[\gamma]}(x, \zeta) : \gamma \in (\mathcal{G}_T)_x\}} \subseteq Q(K_J).$$

Since θ is minimal by assumption, $Q(K_J) = X \times \hat{Z}_\omega$. Thus, Proposition 6.6 implies that

$$K_J = Q^{-1}(Q(K_J)) = Q^{-1}(X \times \hat{Z}_\omega) = X \times \hat{P}_T.$$

Hence

$$\psi_T(J) = \{f \in \mathcal{X}_T^\omega : f|_{K_J} \equiv 0\} = \{0\},$$

which contradicts that J is nontrivial, because ψ_T is injective. Therefore, $Q(K_J) = \emptyset$, forcing $K_J = \emptyset$, and hence $\psi_T(J) = \mathcal{X}_T^\omega$. Since ψ_T is an isomorphism, $J = C^*(\mathcal{I}_T, \sigma)$, and hence $C^*(\mathcal{G}_T, \sigma)$ is simple.

For the converse, we prove the contrapositive. Suppose that θ is not minimal. Then there exists $(x, \phi) \in X \times \widehat{P}_T$ such that

$$\overline{[x, \phi|_{Z_\omega}]_\theta} = \overline{\{\theta_{[\gamma]}(x, \phi|_{Z_\omega}) : \gamma \in (\mathcal{G}_T)_x\}} \neq X \times \widehat{Z}_\omega.$$

Let

$$\kappa_{(x, \phi)} := \text{Tr}_T^\omega \circ \text{ev}_{(x, \phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi : C^*(\mathcal{G}_T, \sigma) \rightarrow \mathbb{C}$$

be the state of $C^*(\mathcal{G}_T, \sigma)$ defined in Lemma 6.10. Let $\kappa := \kappa_{(x, \phi)}$, let

$$N_\kappa := \{f \in C^*(\mathcal{G}_T, \sigma) : \kappa(f^* f) = 0\}$$

be the null space for κ , and let $\pi_\kappa : C^*(\mathcal{G}_T, \sigma) \rightarrow B(\mathcal{H}_\kappa)$ be the GNS representation associated to κ . To see that $C^*(\mathcal{G}_T, \sigma)$ is not simple, it suffices to prove that

$$\{0\} \neq \ker(\pi_\kappa) \neq C^*(\mathcal{G}_T, \sigma).$$

Since $\kappa \neq 0$, we have $\mathcal{H}_\kappa \neq \{0\}$. So since π_κ is nondegenerate, $\ker(\pi_\kappa) \neq C^*(\mathcal{G}_T, \sigma)$. We now show that $\ker(\pi_\kappa) \neq \{0\}$. Define $C_{(x, \phi)} := Q^{-1}(\overline{[x, \phi|_{Z_\omega}]_\theta})$. Since $\overline{[x, \phi|_{Z_\omega}]_\theta} \neq X \times \widehat{Z}_\omega$, Lemma 6.7 shows that $C_{(x, \phi)}$ is a proper closed subset of $X \times \widehat{P}_T$, and there exists $f \in M_T^\sigma \setminus \{0\}$ such that

$$\text{supp}((\psi_T \circ \iota^{-1})(f)) \subseteq (X \times \widehat{P}_T) \setminus C_{(x, \phi)}.$$

Fix $a, b \in C^*(\mathcal{G}_T, \sigma)$. To see that $\pi_\kappa(f) = 0$, it suffices to show that

$$(\pi_\kappa(f)(a + N_\kappa) \mid b + N_\kappa) = 0.$$

Since π_κ is the GNS representation associated to κ , we have

$$(\pi_\kappa(f)(a + N_\kappa) \mid b + N_\kappa) = (fa + N_\kappa \mid b + N_\kappa) = \kappa(b^* fa).$$

By Proposition 6.9, we have $(\psi_T \circ \iota^{-1} \circ \Phi)(b^* fa)(x, \phi) = 0$, and hence

$$\kappa(b^* fa) = (\text{Tr}_T^\omega \circ \text{ev}_{(x, \phi)} \circ \psi_T \circ \iota^{-1} \circ \Phi)(b^* fa) = \text{Tr}_T^\omega(0) = 0.$$

Hence $(\pi_\kappa(f)(a + N_\kappa) \mid b + N_\kappa) = 0$, giving $\ker(\pi_\kappa) \neq \{0\}$. ■

Remark 6.11. If X is the infinite-path space of a cofinal, row-finite k -graph with no sources, and each T^n is the degree- n shift map, then Theorem 6.1 coincides with the simplicity characterisation given in [29, Corollary 4.8].

Remark 6.12. Theorem 5.1 of [7] shows that $C^*(\mathcal{G}_T)$ is simple if and only if \mathcal{G}_T is minimal and effective. We claim that [7, Theorem 5.1], in the special case of Deaconu–Renault groupoids, is equivalent to Theorem 6.1 when σ is trivial. In this case, ω and each $\tilde{\tau}_\gamma^\sigma$ are also trivial, and $Z_\omega = P_T$. So Theorem 6.1 says that $C^*(\mathcal{G}_T, \sigma) = C^*(\mathcal{G}_T)$ is simple if and only if the set

$$[x, \phi]_\theta = \{(r(\gamma), \tilde{\tau}_{[\gamma]}^\sigma \phi) : \gamma \in (\mathcal{G}_T)_x\} = r((\mathcal{G}_T)_x) \times \{\phi\}$$

is dense in $X \times \widehat{P}_T$ for all $(x, \phi) \in X \times \widehat{P}_T$. Since $r((\mathcal{G}_T)_x)$ is dense in X and $X \times \{\phi\}$ is closed, we deduce that θ is minimal if and only if $P_T = \{0\}$. By Corollary 3.7, this occurs precisely when \mathcal{G}_T is effective.

7. An application to some crossed products by \mathbb{Z}

In this section, we apply our theorem to characterise simplicity of crossed products of C^* -algebras of rank-1 Deaconu–Renault groupoids arising from continuous \mathbb{T} -valued functions on the underlying spaces. We then specialise this to the analogue of quasi-free actions on topological-graph C^* -algebras.

7.1. Crossed products as twisted groupoid C^* -algebras

To apply our main theorem to understand crossed products of C^* -algebras of Deaconu–Renault groupoids, we need to realise the latter as twisted groupoid C^* -algebras. This follows from a more general result about crossed products of étale groupoids by actions of \mathbb{Z} induced by \mathbb{T} -valued 1-cocycles that may be of independent interest; so we record the general result first. We thank the referee for suggesting the more general formulation.

We will first need the following folklore result about multipliers of the C^* -algebras of Hausdorff étale groupoids. We write $C_b(Y)$ for the set of continuous, bounded, complex-valued functions on a locally compact Hausdorff space Y .

Lemma 7.1. *Let \mathcal{G} be a Hausdorff étale groupoid, and fix $\sigma \in Z^2(\mathcal{G}, \mathbb{T})$. Suppose that $B \subseteq \mathcal{G}$ is a clopen bisection of \mathcal{G} such that $s(B)$ and $r(B)$ are closed, and fix $f \in C_b(B)$. For $g \in C_c(\mathcal{G}, \sigma)$, the convolution product $f * g$ given by*

$$(f * g)(\gamma) = \sum_{\alpha \in \mathcal{G}r(\gamma)} \sigma(\alpha, \alpha^{-1}\gamma) f(\alpha)g(\alpha^{-1}\gamma)$$

belongs to $C_c(\mathcal{G}, \sigma)$. There is a multiplier M_f of $C^(\mathcal{G}, \sigma)$ such that for $g \in C_c(\mathcal{G}, \sigma)$ we have $M_f(g) = f * g$. If $s(B) = r(B) = \mathcal{G}^{(0)}$ and $f(B) \subseteq \mathbb{T}$, then M_f is a unitary multiplier of $C^*(\mathcal{G}, \sigma)$. If B_1 and B_2 are two clopen bisections such that $r(B_i)$ and $s(B_i)$ are closed for each i , and $f_i \in C_b(B_i)$ for each i , then the convolution product $f_1 * f_2$ belongs to $C_b(B_1 B_2)$, and we have $M_{f_1} \circ M_{f_2} = M_{f_1 * f_2}$; likewise, $f_1^* \in C_b(B_1^{-1})$ and $M_{f_1}^* = M_{f_1^*}$.*

Proof. Fix $g \in C_c(\mathcal{G}, \sigma)$. Since $\text{supp}(g)$ is compact, its image $r(\text{supp}(g))$ under the continuous range map is also compact. Use Urysohn’s lemma to fix a compactly supported function $h \in C_c(\mathcal{G}^{(0)}, [0, 1])$ such that $h|_{r(\text{supp}(g))} \equiv 1$. Then $f * g = f * (h * g) = (f * h) * g$. The function $f * h$ is given by $(f * h)(\gamma) = f(\gamma)h(s(\gamma))$, and since $\text{supp}(f * h) \subseteq s|_B^{-1}(\text{supp}(h) \cap s(B)) \subseteq B$, it follows that $f * h \in C_c(\mathcal{G}, \sigma)$. So $f * g = (f * h) * g \in C_c(\mathcal{G}, \sigma)$.

Using the same function h as above, we see that

$$\|f * g\|^2 = \|((f * h) * g)^* * ((f * h) * g)\| = \|g^* * (h * f^* * f * h) * g\|.$$

In $C^*(\mathcal{G}, \sigma)$, we have

$$g^* * (h * f^* * f * h) * g \leq \|h * f^* * f * h\|_\infty g^* * g \leq \|f\|_\infty^2 g^* * g,$$

and so we deduce that $\|f * g\| \leq \|f\|_\infty \|g\|$. So the map $g \mapsto f * g$ on $C_c(\mathcal{G}, \sigma)$ extends to a bounded linear map M_f of norm at most $\|f\|_\infty$ on $C^*(\mathcal{G}, \sigma)$. Defining $f^*: B^{-1} \rightarrow \mathbb{C}$ by $f^*(\gamma) = \overline{\sigma(\gamma^{-1}, \gamma) f(\gamma^{-1})}$ as usual, associativity of multiplication shows that for $g, h \in C_c(\mathcal{G}, \sigma)$, we have $M_f(g)^* * h = g^* * f^* * h = g^* * M_{f^*}(h)$. Thus M_f is adjointable with respect to the standard inner product on $C^*(\mathcal{G}, \sigma)$, with adjoint M_{f^*} . So M_f is a multiplier, as claimed. If $s(B) = r(B) = \mathcal{G}^{(0)}$ and $f(B) \subseteq \mathbb{T}$, then for $g \in C_c(\mathcal{G}, \sigma)$ we have $M_f^*(M_f(g)) = f^* * f * g = 1_{\mathcal{G}^{(0)}} * g = g$, and similarly, $M_f(M_f^*(g)) = g$. So continuity gives $M_f^* M_f = M_f M_f^* = 1_{\mathcal{M}(C^*(\mathcal{G}, \sigma))}$, and thus M_f is a unitary.

For the final statement, we already saw that $M_{f^*} = M_f^*$ for all f , so we just have to establish the multiplicativity. If B_1 and B_2 are clopen bisections and $f_i \in C_b(B_i)$, then $B_1 B_2$ is an open bisection because multiplication is open. To see that $B_1 B_2$ is also closed, suppose that $\gamma_i \rightarrow \gamma$ and each $\gamma_i \in B_1 B_2$. Then each γ_i can be written as $\alpha_i \beta_i$ with each α_i in B_1 and each β_i in B_2 . Since $\gamma_i \rightarrow \gamma$, we have $r(\alpha_i) = r(\gamma_i) \rightarrow r(\gamma)$, and then since $r|_{B_1}$ is a homeomorphism, $\alpha_i \rightarrow \alpha$ for some $\alpha \in B_1$. Similarly (using s in place of r), we have $\beta_i \rightarrow \beta$ for some $\beta \in B_2$. Since each $s(\alpha_i) = r(\beta_i)$, continuity gives $s(\alpha) = r(\beta)$, and since $\alpha_i \beta_i = \gamma_i \rightarrow \gamma$, continuity also gives $\alpha \beta = \gamma$. So $\gamma \in B_1 B_2$. The convolution formula shows that $\text{supp}(f_1 * f_2) \subseteq B_1 B_2$ and that $\|f_1 * f_2\|_\infty \leq \|f_1\|_\infty \|f_2\|_\infty$. For $g \in C_c(\mathcal{G}, \sigma)$ we have $M_{f_1}(M_{f_2}(g)) = f_1 * (f_2 * g) = (f_1 * f_2) * g = M_{f_1 * f_2}(g)$, and then continuity gives $M_{f_1} M_{f_2} = M_{f_1 * f_2}$. ■

We can now discuss how to realise certain crossed products of étale-groupoid C*-algebras as twisted groupoid C*-algebras.

Let \mathcal{G} be a locally compact Hausdorff groupoid with a Haar system and let $c: \mathcal{G} \rightarrow \mathbb{T}$ be a continuous 1-cocycle. By [36, Proposition II.5.1] there is an action $\alpha = \alpha^c$ of \mathbb{Z} on $C^*(\mathcal{G})$ such that

$$\alpha_n(f)(\gamma) = c(\gamma)^n f(\gamma) \quad \text{for all } n \in \mathbb{Z}, f \in C_c(\mathcal{G}), \text{ and } \gamma \in \mathcal{G}. \tag{7.1}$$

Proposition 7.2. *Let \mathcal{G} be a second-countable Hausdorff étale groupoid. Suppose that $c: \mathcal{G} \rightarrow \mathbb{T}$ is a continuous 1-cocycle, and let $\alpha = \alpha^c$ be the corresponding action of \mathbb{Z} on $C^*(\mathcal{G})$ as in equation (7.1). There is a continuous \mathbb{T} -valued 2-cocycle $\omega = \omega_c$ on $\mathcal{G} \times \mathbb{Z}$ given by $\omega((\beta, m), (\gamma, n)) := c(\beta)^n$, and there is an isomorphism $\phi: C^*(\mathcal{G}) \rtimes_\alpha \mathbb{Z} \rightarrow C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ such that*

$$\phi(i_{C^*(\mathcal{G})}(f) i_{\mathbb{Z}}(n))(\gamma, p) = \delta_{-n, p} f(\gamma),$$

for $f \in C_c(\mathcal{G})$, $n \in \mathbb{Z}$, and $(\gamma, p) \in \mathcal{G} \times \mathbb{Z}$.

Proof. First note that ω is normalised because $c(r(\gamma))^n = c(\gamma)^0 = 1$ for all $\gamma \in \mathcal{G}$ and $n \in \mathbb{Z}$. To see that ω satisfies the 2-cocycle identity, fix a composable triple $((\beta, m), (\gamma, n), (\lambda, p))$ in $\mathcal{G} \times \mathbb{Z}$. Then

$$\begin{aligned} \omega((\beta, m), (\gamma, n)) \omega((\beta, m)(\gamma, n), (\lambda, p)) &= c(\beta)^n c(\beta\gamma)^p = c(\beta)^{n+p} c(\gamma)^p \\ &= \omega((\beta, m), (\gamma, n)(\lambda, p)) \omega((\gamma, n), (\lambda, p)). \end{aligned}$$

For the final statement, first note that for $n \in \mathbb{Z}$, the set $\mathcal{G}^{(0)} \times \{n\}$ is a clopen bisection of $\mathcal{G} \times \mathbb{Z}$ with range and source equal to $(\mathcal{G} \times \mathbb{Z})^{(0)}$. Hence Lemma 7.1 shows that there is a unitary multiplier U_n of $C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ that acts on $C_c(\mathcal{G} \times \mathbb{Z}, \omega)$ by convolution with the characteristic function $1_{\mathcal{G}^{(0)} \times \{-n\}}$. Since c vanishes on $\mathcal{G}^{(0)}$, the final statement of Lemma 7.1 shows that $n \mapsto U_n$ is a unitary representation of \mathbb{Z} .

Since $\mathcal{G} \times \{0\}$ is isomorphic to \mathcal{G} and ω is trivial on $\mathcal{G} \times \{0\}$, the universal property of $C^*(\mathcal{G})$ yields a homomorphism $\pi: C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ such that $\pi(f)(\gamma, m) = \delta_{m,0} f(\gamma)$ for all $f \in C_c(\mathcal{G})$ and $(\gamma, m) \in \mathcal{G} \times \mathbb{Z}$. We claim that π is nondegenerate. To see this, fix $g \in C_c(\mathcal{G} \times \mathbb{Z}, \omega)$, and use Urysohn’s lemma to choose $f \in C_c(\mathcal{G})$ such that $\text{supp}(f) \subseteq \mathcal{G}^{(0)}$ and $f|_{\pi_1(r(\text{supp}(g)))} \equiv 1$, where π_1 is the projection of $\mathcal{G} \times \mathbb{Z}$ onto the first coordinate. A routine calculation shows that $\pi(f)g = g$, and hence π is nondegenerate.

We claim that (π, U) is a covariant representation of $(C^*(\mathcal{G}), \mathbb{Z}, \alpha)$. To see this, fix $f \in C_c(\mathcal{G})$ and $n \in \mathbb{Z}$. For all $(\gamma, m) \in \mathcal{G} \times \mathbb{Z}$, we have

$$\begin{aligned} & (U_n \pi(f) U_n^*)(\gamma, m) \\ &= (U_n \pi(f) U_{-n})(\gamma, m) \\ &= \sum_{(\eta, p)(\beta, q)(\lambda, l) = (\gamma, m)} c(\eta)^{q+l} c(\beta)^l 1_{\mathcal{G}^{(0)} \times \{-n\}}(\eta, p) \pi(f)(\beta, q) 1_{\mathcal{G}^{(0)} \times \{n\}}(\lambda, l). \end{aligned}$$

If $(\eta, p)(\beta, q)(\lambda, l) = (\gamma, m)$ contributes a nonzero term, then $(\eta, p) \in \mathcal{G}^{(0)} \times \{-n\}$ and $(\lambda, l) \in \mathcal{G}^{(0)} \times \{n\}$; thus $(\eta, p) = (r(\gamma), -n)$ and $(\lambda, l) = (s(\gamma), n)$, and hence $(\beta, q) = (\gamma, m)$. So we obtain

$$(U_n \pi(f) U_n^*)(\gamma, m) = c(\gamma)^n \pi(f)(\gamma, m) = \delta_{m,0} c(\gamma)^n f(\gamma) = \pi(\alpha_n(f))(\gamma, m).$$

Therefore, (π, U) is a nondegenerate covariant representation of $(C^*(\mathcal{G}), \mathbb{Z}, \alpha)$, and so the universal property of the crossed product gives a homomorphism $\phi: C^*(\mathcal{G}) \rtimes_{\alpha} \mathbb{Z} \rightarrow C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ such that $\phi(i_{C^*(\mathcal{G})}(f) i_{\mathbb{Z}}(n))(\gamma, p) = (\pi(f) U_n)(\gamma, p) = \delta_{-n,p} f(\gamma)$.

To see that ϕ is injective, it suffices by [10, Proposition 4.5.1] to show that π is injective and that there is a strongly continuous action β of \mathbb{T} on $C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ such that for each $z \in \mathbb{T}$, we have $\beta_z(\pi(f)) = \pi(f)$ for all $f \in C_c(\mathcal{G})$, and the extension $\bar{\beta}_z$ of β_z to the multiplier algebra $\mathcal{M}(C^*(\mathcal{G} \times \mathbb{Z}, \omega))$ satisfies $\bar{\beta}_z(U_n) = z^n U_n$ for all $n \in \mathbb{Z}$.

We first show that π is injective. Let Y denote the right- $C^*(\mathcal{G})$ -module direct sum $\bigoplus_{n \in \mathbb{Z}} C^*(\mathcal{G})$. For $f \in C_c(\mathcal{G} \times \mathbb{Z}, \omega)$ and $n \in \mathbb{Z}$, we write $f_n \in C_c(\mathcal{G})$ for the function such that $f(\gamma, n) = f_n(\gamma)$ for all $\gamma \in \mathcal{G}$. For $f \in C_c(\mathcal{G} \times \mathbb{Z})$ and $\xi \in C_c(\mathbb{Z}, C^*(\mathcal{G})) \subseteq Y$, define $f \cdot \xi \in C_c(\mathbb{Z}, C^*(\mathcal{G}))$ by $(f \cdot \xi)(n) := \sum_{p+q=n} \alpha_p(f_q) \xi(p)$, where the product $\alpha_p(f_q) \xi(p)$ is computed in $C^*(\mathcal{G})$.

We claim that $f \mapsto (\xi \mapsto f \cdot \xi)$ is a $*$ -homomorphism from $C_c(\mathcal{G} \times \mathbb{Z}, \omega)$ to $\mathcal{L}(Y)$. To see that it is multiplicative, fix $f \in C_c(\mathcal{G} \times \{a\}) \subseteq C_c(\mathcal{G} \times \mathbb{Z}, \omega)$ and $g \in C_c(\mathcal{G} \times \{b\}) \subseteq C_c(\mathcal{G} \times \mathbb{Z}, \omega)$. Then $fg \in C_c(\mathcal{G} \times \{a+b\})$, and for $\gamma \in \mathcal{G}$, we have

$$(fg)_{a+b}(\gamma) = \sum_{(\beta, p)(\lambda, q) = (\gamma, a+b)} \omega((\beta, p), (\lambda, q)) f(\beta, p) g(\lambda, q)$$

$$\begin{aligned} &= \sum_{\beta\lambda=\gamma} \omega((\beta, a), (\lambda, b)) f_a(\beta) g_b(\lambda) = \sum_{\beta\lambda=\gamma} c(\beta)^b f_a(\beta) g_b(\lambda) \\ &= (\alpha_b(f_a) g_b)(\gamma). \end{aligned}$$

Thus, for $\xi \in C_c(\mathbb{Z}, C^*(\mathcal{G}))$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned} (f \cdot (g \cdot \xi))(n) &= \alpha_{n-a}(f_a)(g \cdot \xi)(n - a) = \alpha_{n-a}(f_a) \alpha_{n-(a+b)}(g_b) \xi(n - (a + b)) \\ &= \alpha_{n-(a+b)}(\alpha_b(f_a) g_b) \xi(n - (a + b)) \\ &= \alpha_{n-(a+b)}((fg)_{a+b}) \xi(n - (a + b)) \\ &= \sum_{p+q=n} \alpha_p((fg)_q) \xi(p) = ((fg) \cdot \xi)(n). \end{aligned}$$

Hence $f \mapsto (\xi \mapsto f \cdot \xi)$ is multiplicative.

To see that it preserves adjoints, fix $f \in C_c(\mathcal{G} \times \{m\}) \subseteq C_c(\mathcal{G} \times \mathbb{Z}, \omega)$, and $\xi \in C_c(\{p\}, C^*(\mathcal{G})) \subseteq Y$ and $\eta \in C_c(\{q\}, C^*(\mathcal{G})) \subseteq Y$. Then

$$\langle f \cdot \xi, \eta \rangle_{C^*(\mathcal{G})} = (f \cdot \xi)(q)^* \eta(q) = \delta_{p+m,q} \xi(p)^* \alpha_p(f_m)^* \eta(q).$$

Since $\omega((\gamma, -m), (\gamma, -m)^{-1}) = c(\gamma)^m$ for $\gamma \in \mathcal{G}$, a computation shows that $\alpha_m((f^*)_{-m}) = (f_m)^*$. Hence $f^* \in C_c(\mathcal{G} \times \{-m\})$, and

$$\begin{aligned} \langle \xi, f^* \cdot \eta \rangle_{C^*(\mathcal{G})} &= \xi(p)^* (f^* \cdot \eta)(p) = \delta_{q-m,p} \xi(p)^* \alpha_{p+m}((f^*)_{-m}) \eta(q) \\ &= \langle f \cdot \xi, \eta \rangle_{C^*(\mathcal{G})}. \end{aligned}$$

So $f \mapsto (\xi \mapsto f \cdot \xi)$ preserves adjoints, and hence is a $*$ -homomorphism.

The universal property of $C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ therefore implies that there is a homomorphism $\psi: C^*(\mathcal{G} \times \mathbb{Z}, \omega) \rightarrow \mathcal{L}(Y)$ such that $\psi(f)\xi = f \cdot \xi$ for $f \in C_c(\mathcal{G} \times \mathbb{Z}, \omega)$ and $\xi \in Y$.

Let $i_0: C^*(\mathcal{G})_{C^*(\mathcal{G})} \rightarrow Y$ be the inclusion as the 0-submodule. A quick calculation shows that for $a, b \in C_c(\mathcal{G})$, we have $\psi(\pi(a))i_0(b) = i_0(ab)$. Since i_0 is isometric and since the left action of $C^*(\mathcal{G})$ on itself by multiplication is isometric, we deduce that $\psi \circ \pi$ is injective, and hence π is injective.

So we just need to construct the action β . For $z \in \mathbb{T}$, the map $\beta_z: C_c(\mathcal{G} \times \mathbb{Z}, \omega) \rightarrow C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ given by $\beta_z(f)(\gamma, n) := z^n f(\gamma, n)$ is a $*$ -homomorphism, and hence the universal property of $C^*(\mathcal{G} \times \mathbb{Z}, \omega)$ implies that it extends to an endomorphism β_z of $C^*(\mathcal{G} \times \mathbb{Z}, \omega)$. Since $\beta_{\bar{z}} \circ \beta_z$ is the identity map on $C_c(\mathcal{G} \times \mathbb{Z}, \omega)$, each β_z is an automorphism, and since $\beta_z \circ \beta_w$ agrees with β_{zw} on $C_c(\mathcal{G} \times \mathbb{Z}, \omega)$, we see that $z \mapsto \beta_z$ is a homomorphism. For $f \in C_c(\mathcal{G} \times \{n\})$, the map $z \mapsto \beta_z(f)$ is clearly continuous, and an $\frac{\varepsilon}{3}$ -argument then shows that β is a strongly continuous action of \mathbb{T} .

We claim that the extension of each β_z to the multiplier algebra $\mathcal{M}(C^*(\mathcal{G} \times \mathbb{Z}, \omega))$ satisfies $\overline{\beta_z}(U_n) = z^n U_n$ for each $n \in \mathbb{Z}$. To see this, fix $n \in \mathbb{Z}$ and an increasing sequence $K_i \subseteq \mathcal{G}^{(0)}$ of compact sets with $\bigcup_{i \in \mathbb{N}} K_i = \mathcal{G}^{(0)}$, and for each $i \in \mathbb{N}$, fix $h_i \in C_c(\mathcal{G}^{(0)} \times \{n\}, [0, 1])$ such that $h_i|_{K_i \times \{n\}} \equiv 1$. For $f \in C_c(\mathcal{G} \times \mathbb{Z})$ there exists $N \in \mathbb{N}$ large enough so that $r(\text{supp}(f)) \subseteq K_N \times \{0\}$, and then $h_i * f = U_n f$ for all $i \geq N$. So the sequence

$(h_i)_{i \in \mathbb{N}}$ converges strictly to U_n , and since $\beta_z(h_i) = z^n h_i$ for all $z \in \mathbb{T}$, this establishes the claim. Thus ϕ is injective.

It remains only to prove that ϕ is surjective. For this, fix an open bisection B of $\mathcal{G} \times \mathbb{Z}$ and distinct points $\beta, \gamma \in B$. Then $B = B' \times \{n\}$ for some open bisection B' of \mathcal{G} and some $n \in \mathbb{Z}$, and so $\beta = (\beta', n)$ and $\gamma = (\gamma', n)$ for distinct $\beta', \gamma' \in \mathcal{G}$. Fix $f \in C_c(\mathcal{G})$ such that $\text{supp}(f) \subseteq B'$, $f(\beta') = 1$, and $f(\gamma') = 0$. Then the support of $\phi(i_{C^*(\mathcal{G})}(f)i_{\mathbb{Z}}(-n)) = \pi(f)U_{-n}$ is contained in B , and we have $\phi(i_{C^*(\mathcal{G})}(f)i_{\mathbb{Z}}(-n))(\beta', n) = f(\beta') = 1$ and $\phi(i_{C^*(\mathcal{G})}(f)i_{\mathbb{Z}}(-n))(\gamma', n) = f(\gamma') = 0$. So [39, Corollary 9.3.5] shows that ϕ is surjective. ■

7.2. An application of Theorem 6.1 to crossed products

We now make use of Theorem 6.1 and Proposition 7.2 to study simplicity of certain crossed products of C^* -algebras of Deaconu–Renault groupoids.

The following is an immediate corollary of results of Renault [38, Section 4.1] (see also [16]) together with Proposition 7.2; we have written it out primarily to establish our set-up for the rest of the section.

Corollary 7.3. *Let (X, T) be a rank-1 Deaconu–Renault system such that X is second-countable, and let $h: X \rightarrow \mathbb{T}$ be a continuous function.*

- (a) *For each $(x, p, y) \in \mathcal{G}_T \subseteq X \times \mathbb{Z} \times X$, the sequence*

$$\left(\prod_{i=0}^N h(T^i(x)) \prod_{i=0}^{N-p} \overline{h(T^i(y))} \right)_{N=|p|}^{\infty}$$

is eventually constant.

- (b) *There is a continuous 1-cocycle $\tilde{h}: \mathcal{G}_T \rightarrow \mathbb{T}$ such that*

$$\tilde{h}(x, p, y) = \prod_{i=0}^N h(T^i(x)) \prod_{i=0}^{N-p} \overline{h(T^i(y))}$$

for large $N \in \mathbb{N}$.

- (c) *There is an action $\alpha^h: \mathbb{Z} \rightarrow \text{Aut}(C^*(\mathcal{G}_T))$ such that $\alpha_n^h(f)(\gamma) = \tilde{h}(\gamma)^n f(\gamma)$ for all $f \in C_c(\mathcal{G}_T)$ and $\gamma \in \mathcal{G}_T$.*
- (d) *There is a continuous 2-cocycle $c_h: (\mathcal{G}_T \times \mathbb{Z})^{(2)} \rightarrow \mathbb{T}$ given by*

$$c_h((\alpha, m), (\beta, n)) := \tilde{h}(\alpha)^n,$$

and there is an isomorphism $\phi: C^(\mathcal{G}_T) \rtimes_{\alpha^h} \mathbb{Z} \rightarrow C^*(\mathcal{G}_T \times \mathbb{Z}, c_h)$ such that*

$$\phi(i_{C^*(\mathcal{G}_T)}(f)i_{\mathbb{Z}}(n))(\gamma, p) = \delta_{-n,p} f(\gamma),$$

for $f \in C_c(\mathcal{G}_T)$, $n \in \mathbb{Z}$, and $(\gamma, p) \in \mathcal{G}_T \times \mathbb{Z}$.

Proof. Statements (a) and (b) follow from the arguments of [38, Section 4.1] or [16, Proposition 3.10]. The action (c) is the one described in equation (7.1). The final statement is a special case of Proposition 7.2. ■

Theorem 7.4. *Let (X, T) be a rank-1 Deaconu–Renault system such that X is second-countable, and let $h: X \rightarrow \mathbb{T}$ be a continuous function. Let $\tilde{h}: \mathcal{G}_T \rightarrow \mathbb{T}$ be the 1-cocycle of Corollary 7.3(b), and let $\alpha^h: \mathbb{Z} \rightarrow \text{Aut}(C^*(\mathcal{G}_T))$ be the action of Corollary 7.3(c). Write ρ for the action of \mathcal{G}_T on $X \times \mathbb{T}$ given by $\rho_\gamma(s(\gamma), z) := (r(\gamma), \tilde{h}(\gamma)z)$. Suppose that X is an uncountable space. Then the crossed product $C^*(\mathcal{G}_T) \rtimes_{\alpha^h} \mathbb{Z}$ is simple if and only if ρ is minimal.*

In order to prove Theorem 7.4, we need the following lemma.

Lemma 7.5. *Let (X, T) be a minimal rank-1 Deaconu–Renault system such that X is second-countable. If X is uncountable, then \mathcal{G}_T is topologically principal.*

Proof. Since \mathcal{G}_T is second-countable, it suffices by [7, Lemma 3.1] to show that $\mathcal{I}_T = \mathcal{G}_T^{(0)}$. To see this, we suppose that $\mathcal{I}_T \neq \mathcal{G}_T^{(0)}$ and derive a contradiction. Recall from Proposition 3.5 that $\mathcal{I}_T = \{(x, p, x) : p \in P_T\}$. Since \mathcal{I}_T is nontrivial, there exists $p \in \mathbb{Z} \setminus \{0\}$ such that $(x, p, x) \in \mathcal{G}_T$ for all x . By definition of the topology on \mathcal{G}_T , it follows that for each $x \in X$ there is an open neighbourhood U of x and a pair $m > n \in \mathbb{N}$ such that $T^m(x) = T^n(x)$ for all $x \in U$. Since the pairs $m > n \in \mathbb{N}$ are countable and X is not countable, that \mathcal{G}_T is second-countable implies that there exist x, U, m, n as above so that U is not countable. Since X is second-countable and T^n is a local homeomorphism, $(T^n)^{-1}(x)$ is countable for every $x \in X$, and so $V = T^n(U)$ is an uncountable open set and $p = m - n > 0$ satisfies $T^p(x) = x$ for all $x \in V$. Fix $x \in V$. Since V is uncountable, there exists $y \in V$ such that

$$T^q(x) \neq y \quad \text{for all } q \in \mathbb{N}. \tag{7.2}$$

Since \mathcal{G}_T is minimal, there is a sequence $(z_i, m_i, x)_{i=1}^\infty$ in \mathcal{G}_T such that $z_i \rightarrow y$. Write each $m_i = a_i - b_i$ with $a_i, b_i \geq 0$ so that $T^{a_i}(z_i) = T^{b_i}(x)$. For each $i \in \mathbb{N}$, there exists $k > 0$ such that $kp > b_i$; and then $T^{a_i + (kp - b_i)}(z_i) = T^{kp}(x) = x$. So we can assume that each $m_i > 0$ and that $T^{m_i}(z_i) = x$ for all $i \in \mathbb{N}$. By passing to a subsequence, we may assume that each $m_i - m_1$ is divisible by p . Fix $l > 0$ such that $lp > m_1$, let $d := lp - m_1$, let $z := T^d(x)$, and let $n_i := m_i + d$ for all $i \in \mathbb{N}$. Then $T^{n_i}(z_i) = z$ for all $i \in \mathbb{N}$, and each n_i is divisible by p . Since $z_i \rightarrow y$, we eventually have $z_i \in V$, and so we eventually have $z = T^{n_i}(z_i) = z_i$. But this forces $y = z = T^d(x)$, which contradicts condition (7.2). Thus \mathcal{I}_T is trivial, as claimed. ■

Proof of Theorem 7.4. By Corollary 7.3(d), the crossed product $C^*(\mathcal{G}_T) \rtimes_{\alpha^h} \mathbb{Z}$ is isomorphic to the twisted groupoid C*-algebra $C^*(\mathcal{G}_T \times \mathbb{Z}, c_h)$, to which we aim to apply Theorem 6.1. For this, observe first that if \bar{T} is the action of \mathbb{N}^2 given by $\bar{T}^{(m,n)} = T^m$, then $\mathcal{G}_T \times \mathbb{Z} \cong \mathcal{G}_{\bar{T}}$.

First suppose that (X, T) is not minimal. Then $\mathcal{G}_{\overline{T}}$ is also not minimal, and the action ρ is not minimal. So (X, \overline{T}) is not minimal, and hence Theorem 6.1(a) implies that $C^*(\mathcal{G}_T) \rtimes_{\alpha^h} \mathbb{Z} \cong C^*(\mathcal{G}_{\overline{T}}, c_h)$ is not simple. So it suffices to prove the result when (X, T) is minimal.

Now suppose that (X, T) is minimal. Since X is uncountable, Lemma 7.5 implies that $\mathcal{I}_T = \mathcal{G}_T^{(0)}$. The isomorphism $\mathcal{G}_T \times \mathbb{Z} \rightarrow \mathcal{G}_{\overline{T}}$ is given by $((x, m, y), n) \mapsto (x, (m, n), y)$. So the interior $\mathcal{I}_{\overline{T}}$ of the isotropy of $\mathcal{G}_{\overline{T}}$ is precisely $\{(x, (m, n), x) : (x, m, x) \in \mathcal{I}_T, n \in \mathbb{Z}\}$. So the preceding paragraph implies that $\mathcal{I}_{\overline{T}} = \{(x, (0, n), x) : x \in X, n \in \mathbb{Z}\}$. The isomorphism $\mathcal{G}_T \times \mathbb{Z} \rightarrow \mathcal{G}_{\overline{T}}$ intertwines c_h with the 2-cocycle $\sigma \in Z^2(\mathcal{G}_{\overline{T}}, \mathbb{T})$ given by $\sigma((x, (m, n), y), (y, (p, q), z)) := \tilde{h}(x, m, y)^q$. The restriction of this σ to $\mathcal{I}_{\overline{T}}^{(2)}$ satisfies

$$\sigma((x, (0, m), x), (x, (0, n), x)) = \tilde{h}(x, 0, x)^n = 1.$$

Hence σ is ω -constant on $\mathcal{I}_{\overline{T}}$ with $\omega = 1$. It follows that $Z_\omega = P_{\overline{T}} = \{0\} \times \mathbb{Z} \cong \mathbb{Z}$.

The $\widehat{\mathbb{Z}}$ -valued 1-cocycle τ^σ obtained from Lemma 4.3(e) satisfies

$$\begin{aligned} \tau_{(x, (m, n), y)}^\sigma(p) &= \sigma((x, (m, n), y), (y, (0, p), y)) \\ &\quad \cdot \sigma((x, (m, n + p), y), (y, (-m, -n), x)) \\ &\quad \cdot \overline{\sigma((x, (m, n), y), (y, (-m, -n), x))} \\ &= \tilde{h}(x, m, y)^p \tilde{h}(x, m, y)^{-n} \overline{\tilde{h}(x, m, y)^{-n}} \\ &= \tilde{h}(x, m, y)^p. \end{aligned}$$

So the isomorphism $\chi \mapsto \chi(1)$ from $\widehat{\mathbb{Z}}$ to \mathbb{T} carries $\tau_{(x, (m, n), y)}^\sigma$ to $\tilde{h}(x, m, y) \in \mathbb{T}$.

We have

$$\mathcal{H}_{\overline{T}} := \mathcal{G}_{\overline{T}} / \mathcal{I}_{\overline{T}} \cong (\mathcal{G}_T \times \mathbb{Z}) / (\mathcal{G}_T^{(0)} \times \mathbb{Z}) \cong \mathcal{G}_T$$

and the isomorphism is the map $[(x, (m, n), y)] \mapsto (x, m, y)$. So the spectral action θ of $\mathcal{H}_{\overline{T}}$ on $X \times \widehat{Z}_\omega$ of Proposition 4.5 is identified with the action of \mathcal{G}_T on $X \times \mathbb{T}$ given by $\theta_{(x, m, y)}(y, z) := (x, \tilde{h}(x, m, y)z)$, which is precisely the action ρ . So Theorem 6.1 shows that $C^*(\mathcal{G}_{\overline{T}}, c_h)$ is simple if and only if ρ is minimal. ■

In the following result, we write t and o for the terminus (range) and origin (source) map in a topological graph, so as to avoid confusion with the range and source maps r and s in the associated groupoid. We write $X(E)$ for the graph correspondence associated to a topological graph $E = (E^0, E^1, t, o)$, and we write $\mathcal{O}_{X(E)}$ for the associated Cuntz–Pimsner algebra. We write $(j_{C_0(E^0)}, j_{X(E)})$ for the universal Cuntz–Pimsner-covariant representation of $X(E)$ that generates $\mathcal{O}_{X(E)}$. See [21, 31, 42] for background on topological graphs and their C^* -algebras.

Corollary 7.6. *Let $E = (E^0, E^1, t, o)$ be a second-countable topological graph such that the terminus map $t: E^1 \rightarrow E^0$ is proper and surjective, and the infinite-path space E^∞ is uncountable. Suppose that $\ell: E^1 \rightarrow \mathbb{T}$ is a continuous function. There is an action*

$\beta^\ell: \mathbb{Z} \curvearrowright C^*(E)$ such that $\beta_n^\ell(j_{X(E)}(\xi)) = j_{X(E)}(\ell^n \cdot \xi)$ for all $\xi \in C_c(E^1)$. Extend ℓ to a continuous function $\ell: E^* \rightarrow \mathbb{T}$ by defining $\ell(e_1 \cdots e_n) := \prod_{i=1}^n \ell(e_i)$ and $\ell|_{E^0} \equiv 1$, and let $T: E^\infty \rightarrow E^\infty$ be the shift map $T(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$. Then $C^*(E) \rtimes_{\beta^\ell} \mathbb{Z}$ is simple if and only if for every infinite path $x \in E^\infty$, the set

$$\{(\lambda T^n(x), \ell(\lambda) \overline{\ell(x(0, n))}) : n \in \mathbb{N}, \lambda \in E^* t(T^n(x))\} \tag{7.3}$$

is dense in $E^\infty \times \mathbb{T}$.

Proof. The map $\xi \mapsto \ell \cdot \xi$ is a unitary operator U_ℓ on the graph correspondence $X(E)$. If $\xi \in C_c(E^1)$ is a positive-valued function such that o is injective on $\text{supp}(\xi)$, then a quick calculation shows that conjugation by U_ℓ fixes the rank-1 operator $\Theta_{\xi, \xi}$. Using this, it is routine to see that if (ψ, π) is a covariant Toeplitz representation of E as in [31, Definitions 2.2 and 2.10], then so is $(\psi \circ U_\ell, \pi)$. So the universal property of $C^*(E) \cong \mathcal{O}_{X(E)}$ described by [31, Theorem 2.13] yields a unique automorphism β^ℓ that fixes $j_{C_0(E^0)}(C_0(E^0))$ and satisfies $\beta^\ell(j_{X(E)}(\xi)) = j_{X(E)}(\ell \cdot \xi)$ for $\xi \in C_c(E^1)$. The formula $\beta_n^\ell := (\beta^\ell)^n$ then gives the desired action.¹

Since $t: E^1 \rightarrow E^0$ is proper, [5, Propositions 3.11 and 3.16] show that E^∞ is a locally compact Hausdorff space and T is a local homeomorphism. By [42, Theorem 5.2], there is an isomorphism $\phi: C^*(E) \rightarrow C^*(\mathcal{G}_T)$ such that

$$\phi(j_{C_0(E^0)}(f))(x, m, y) = \delta_{x, y} \delta_{m, 0} f(t(x))$$

for $f \in C_0(E^0)$ and

$$\phi(j_{X(E)}(\xi))(x, m, y) = \delta_{T(x), y} \delta_{m, 1} \xi(x_1)$$

for $\xi \in C_c(E^1)$.

Define $h: E^\infty \rightarrow \mathbb{T}$ by $h(x) := \ell(x_1)$. Then h is continuous. Let $\alpha^h \in \text{Aut}(C^*(\mathcal{G}_T))$ be the automorphism α_1^h of Corollary 7.3(c). A routine calculation shows that $\alpha^h \circ \phi$ agrees with $\phi \circ \beta^\ell$ on $j_{C_0(E^0)}(C_0(E^0)) \cup j_{X(E)}(C_c(E^1))$, and hence the uniqueness of the automorphism β^ℓ discussed in the first paragraph shows that $\alpha^h \circ \phi = \phi \circ \beta^\ell$. It therefore suffices to show that $C^*(\mathcal{G}_T) \rtimes_{\alpha^h} \mathbb{Z}$ is simple if and only if the set described in equation (7.3) is dense for each $x \in E^\infty$.

Fix $x \in E^\infty$. Let $\tilde{h}: \mathcal{G}_T \rightarrow \mathbb{T}$ be the 1-cocycle of Corollary 7.3(b). We claim that the set described in equation (7.3) is precisely the orbit of $(x, 1)$ under the action ρ of Theorem 7.4. We have

$$(\mathcal{G}_T)_x = \{(\lambda T^n(x), |\lambda| - n, x) : n \in \mathbb{N}, \lambda \in E^* t(T^n(x))\},$$

and so for each $\gamma \in (\mathcal{G}_T)_x$, we have

$$\rho_\gamma(x, 1) = (\lambda T^n(x), \tilde{h}(\lambda T^n(x), |\lambda| - n, x)), \tag{7.4}$$

¹We could also appeal to the fourth paragraph of [30, page 462].

for some $n \in \mathbb{N}$ and $\lambda \in E^*t(T^n(x))$. Direct calculation shows that

$$\tilde{h}(\mu x, |\mu| - |v|, vx) = \ell(\mu)\overline{\ell(v)}, \tag{7.5}$$

for all $x \in E^\infty$ and $\mu, v \in E^*t(x)$. Together, equations (7.4) and (7.5) imply that the set described in equation (7.3) is the orbit of $(x, 1)$ under ρ . Since ρ commutes with the action of \mathbb{T} on $E^\infty \times \mathbb{T}$ by translation in the second coordinate, the orbit of $(x, 1)$ is dense if and only if the orbit of (x, z) is dense for every $z \in \mathbb{T}$. That is, the set described in equation (7.3) is dense for each $x \in E^\infty$ if and only if every ρ -orbit is dense. So the result follows from Theorem 7.4. ■

To conclude, for the class of topological graphs appearing in Corollary 7.6, we give a sufficient condition phrased purely in terms of the graph without reference to the shift map on its infinite-path space, for simplicity of the crossed product described there. (The hypothesis that E^∞ is uncountable is quite weak, and follows from a number of elementary conditions on the graph: for example, that E^0 is uncountable, or that E has at least one vertex that supports at least two distinct cycles.)

Corollary 7.7. *Let $E = (E^0, E^1, t, o)$ be a second-countable topological graph such that the terminus map $t: E^1 \rightarrow E^0$ is proper and surjective, and the infinite-path space E^∞ is uncountable. Let $\ell: E^1 \rightarrow \mathbb{T}$ be a continuous function. Extend ℓ to E^* by defining $\ell(e_1 \cdots e_n) := \prod_{i=1}^n \ell(e_i)$ and $\ell|_{E^0} \equiv 1$. For each $v \in E^0$, define*

$$\text{Orb}^+(v) := \bigcup_{\mu \in E^*v} (t(\mu), \ell(\mu)) \subseteq E^0 \times \mathbb{T}.$$

If $\text{Orb}^+(v)$ is dense in $E^0 \times \mathbb{T}$ for each $v \in E^0$, then the crossed product $C^(E) \rtimes_{\beta, \ell} \mathbb{Z}$ of Corollary 7.6 is simple.*

Proof. Suppose that $\text{Orb}^+(v)$ is dense in $E^0 \times \mathbb{T}$ for each $v \in E^0$. We aim to invoke Theorem 7.4. Fix $(x, w), (y, z) \in E^\infty \times \mathbb{T}$. Recall from [5, Proposition 3.11 and Lemma 3.13] that for $n \in \mathbb{N}$ and an open neighbourhood $U \subseteq E^n$ of $y(0, n)$ such that $o|_U$ is injective, the set $Z(U) = \{y' \in E^\infty : y'(0, n) \in U\}$ is a basic open neighbourhood of y . Let d be the metric on \mathbb{T} induced by the usual metric on \mathbb{R} via the local homeomorphism $t \mapsto e^{it}$ from \mathbb{R} to \mathbb{T} . Let ρ be the action of \mathcal{G}_T on $E^\infty \times \mathbb{T}$ from Theorem 7.4. It suffices to fix a neighbourhood U as above and an $\varepsilon > 0$ and show that there exists $\gamma \in (\mathcal{G}_T)_x$ such that $\rho_\gamma(x, w) \in Z(U) \times B_d(z; \varepsilon)$. Let $\mu_y := y(0, n) \in U$. Since ℓ is continuous, by shrinking U if necessary, we may assume that

$$d(\ell(\mu), \ell(\mu_y)) < \frac{\varepsilon}{2} \quad \text{for all } \mu \in U. \tag{7.6}$$

Since $o: E^1 \rightarrow E^0$ is a local homeomorphism, it is an open map, and so $o(U)$ is open. Since $\text{Orb}^+(t(x))$ is dense in $E^0 \times \mathbb{T}$, we can find $\lambda \in E^*t(x)$ such that

$$(t(\lambda), \ell(\lambda)) \in o(U) \times B_d\left(\overline{z\ell(\mu_y)w}; \frac{\varepsilon}{2}\right).$$

Let $\mu_{t(\lambda)}$ be the unique element of U such that $o(\mu_{t(\lambda)}) = t(\lambda)$. Since d is rotation-invariant, condition (7.6) implies that

$$\begin{aligned} d(\ell(\mu_{t(\lambda)}\lambda)w, \ell(\mu_y)\ell(\lambda)w) &= d(\ell(\mu_{t(\lambda)})\ell(\lambda)w, \ell(\mu_y)\ell(\lambda)w) \\ &= d(\ell(\mu_{t(\lambda)}), \ell(\mu_y)) < \frac{\varepsilon}{2}. \end{aligned} \tag{7.7}$$

Moreover, since d is rotation-invariant and $\ell(\lambda) \in B_d(z\overline{\ell(\mu_y)w}; \frac{\varepsilon}{2})$, we have

$$d(\ell(\mu_y)\ell(\lambda)w, z) = d(\ell(\lambda), z\overline{\ell(\mu_y)w}) < \frac{\varepsilon}{2}. \tag{7.8}$$

Together, equations (7.7) and (7.8) imply that

$$\begin{aligned} d(\ell(\mu_{t(\lambda)}\lambda)w, z) &\leq d(\ell(\mu_{t(\lambda)}\lambda)w, \ell(\mu_y)\ell(\lambda)w) + d(\ell(\mu_y)\ell(\lambda)w, z) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{7.9}$$

Now equations (7.5) and (7.9) imply that

$$\rho_{(\mu_{t(\lambda)}\lambda x, n+|\lambda|, x)}(x, w) = (\mu_{t(\lambda)}\lambda x, \ell(\mu_{t(\lambda)}\lambda)w) \in Z(U) \times B_d(z; \varepsilon),$$

as required. ■

A. Realising twisted group C*-algebras as induced algebras

In this appendix we describe how to realise twisted group C*-algebras as induced algebras, which is a key step in the proof of Theorem 5.4. These results are fairly well known and a detailed treatment is given in [3, Theorem 4.3.1], so we give relatively little detail here. We assume knowledge of $C(X)$ -algebras (or, more generally, $C_0(X)$ -algebras). See [41, Section C.1] for the definition and relevant results.

We first recall the definition of the induced algebra of a dynamical system. (See [35, Section 6.3] for more details.)

Definition A.1. Let G be a compact Hausdorff group acting continuously on the right of a locally compact Hausdorff space X , and let α be a strongly continuous action of G on a C*-algebra D . The *induced algebra* of the dynamical system (D, G, α) is defined by

$$\text{Ind}_G^X(D, \alpha) := \{f \in C_0(X, D) : f(x \cdot g) = \alpha_g^{-1}(f(x)) \text{ for all } x \in X \text{ and } g \in G\}.$$

Theorem A.2. Let A be a countable discrete abelian group. Suppose that $\omega \in Z^2(A, \mathbb{T})$ is a bicharacter that vanishes on Z_ω , in the sense that $\omega(Z_\omega, A) \cup \omega(A, Z_\omega) = \{1\}$. Define $B := A/Z_\omega$, and let $\tilde{\omega} \in Z^2(B, \mathbb{T})$ be the bicharacter satisfying $\tilde{\omega}(p + Z_\omega, q + Z_\omega) = \omega(p, q)$ for all $p, q \in A$. Let $\{u_p : p \in A\}$ be the canonical family of generating unitaries for the twisted group C*-algebra $C^*(A, \omega)$, and let $\{U_{p+Z_\omega} : p + Z_\omega \in B\}$ be the canonical family of generating unitaries for the twisted group C*-algebra $C^*(B, \tilde{\omega})$.

(a) *There is a continuous, free, proper right action of \widehat{B} on \widehat{A} given by*

$$(\phi \cdot \chi)(p) := \phi(p)\chi(p + Z_\omega) \quad \text{for all } \phi \in \widehat{A}, \chi \in \widehat{B}, \text{ and } p \in A.$$

The orbit space \widehat{A}/\widehat{B} is compact.

(b) *There is a strongly continuous action β^B of \widehat{B} on $C^*(B, \tilde{\omega})$ such that*

$$\beta_\chi^B(U_{p+Z_\omega}) = \chi(p + Z_\omega)U_{p+Z_\omega} \quad \text{for all } \chi \in \widehat{B} \text{ and } p \in A.$$

(c) *There is an isomorphism $\Omega: C^*(A, \omega) \rightarrow \text{Ind}_{\widehat{B}}^{\widehat{A}}(C^*(B, \tilde{\omega}), \beta^B)$ such that*

$$\Omega(u_p)(\phi) = \overline{\phi(p)}U_{p+Z_\omega} \quad \text{for all } p \in A \text{ and } \phi \in \widehat{A}.$$

In particular, $C^(A, \omega)$ is a $C(\widehat{A}/\widehat{B})$ -algebra.*

A detailed proof of Theorem A.2 can be found in [3, Theorem 4.3.1]. Parts (a) and (b) are routine, but we reproduce some of the details of part (c) below. For this, we need the following preliminary result.

Lemma A.3. *Let A be a countable discrete abelian group, and let $\omega \in Z^2(A, \mathbb{T})$ be a bicharacter.*

(a) *There is a strongly continuous action β^A of \widehat{A} on $C^*(A, \omega)$ such that $\beta_\phi^A(u_p) = \phi(p)u_p$ for all $\phi \in \widehat{A}$ and $p \in A$.*

(b) *There is a faithful conditional expectation $\Phi^A: C^*(A, \omega) \rightarrow \mathbb{C}1_{C^*(A, \omega)}$ such that*

$$\Phi^A(x) = \int_{\widehat{A}} \beta_\phi^A(x) \, d\phi \quad \text{for all } x \in C^*(A, \omega).$$

(c) *Suppose that Y is a nonzero unital C^* -algebra and $\Psi: C^*(A, \omega) \rightarrow Y$ is a unital homomorphism. If $x \in C^*(A, \omega)$ satisfies $\Psi(\beta_\phi^A(x)) = 0$ for all $\phi \in \widehat{A}$, then $x = 0$.*

Proof. Parts (a) and (b) follow from standard arguments (see [3, Lemmas 4.3.2 and 4.3.4]). For part (c), fix $x \in C^*(A, \omega)$ such that $\Psi(\beta_\phi^A(x)) = 0$ for all $\phi \in \widehat{A}$. Then

$$\Psi(\beta_\phi^A(x^*x)) = \Psi(\beta_\phi^A(x))^* \Psi(\beta_\phi^A(x)) = 0,$$

and [35, Lemma C.3] implies that

$$\Psi(\Phi^A(x^*x)) = \Psi\left(\int_{\widehat{A}} \beta_\phi^A(x^*x) \, d\phi\right) = \int_{\widehat{A}} \Psi(\beta_\phi^A(x^*x)) \, d\phi = 0.$$

Since Ψ is unital, it is injective on $\mathbb{C}1_{C^*(A, \omega)}$, and so we deduce that $\Phi^A(x^*x) = 0$. Hence $x = 0$, because Φ^A is faithful. ■

Proof of Theorem A.2(c). Let $\mathcal{Y}_{A,\omega} := \text{Ind}_{\widehat{B}}^{\widehat{A}}(C^*(B, \widetilde{\omega}), \beta^B)$. We aim to use the universal property of $C^*(A, \omega)$ to find a homomorphism $\Omega: C^*(A, \omega) \rightarrow \mathcal{Y}_{A,\omega}$ such that $\Omega(u_p)(\phi) = \overline{\phi(p)}U_{p+Z_\omega}$ for all $p \in A$ and $\phi \in \widehat{A}$. For each $p \in A$, define $v_p: \widehat{A} \rightarrow C^*(B, \widetilde{\omega})$ by $v_p(\phi) := \overline{\phi(p)}U_{p+Z_\omega}$. A routine argument shows that each v_p is continuous and that $v_p(\phi \cdot \chi) = (\beta_\chi^B)^{-1}(v_p(\phi))$ for all $\phi \in \widehat{A}$ and $\chi \in \widehat{B}$, and hence $v_p \in \mathcal{Y}_{A,\omega}$. It is clear that each v_p is a unitary. For all $p, q \in A$, we have

$$U_{p+Z_\omega}U_{q+Z_\omega} = \widetilde{\omega}(p + Z_\omega, q + Z_\omega)U_{p+q+Z_\omega} = \omega(p, q)U_{p+q+Z_\omega},$$

and hence for all $\phi \in \widehat{A}$, we have

$$(v_p v_q)(\phi) = \overline{\phi(p)}U_{p+Z_\omega}\overline{\phi(q)}U_{q+Z_\omega} = \omega(p, q)\overline{\phi(p+q)}U_{p+q+Z_\omega} = \omega(p, q)v_{p+q}(\phi).$$

Therefore, $v_p v_q = \omega(p, q)v_{p+q}$, and so the universal property of $C^*(A, \omega)$ implies that there is a homomorphism $\Omega: C^*(A, \omega) \rightarrow \mathcal{Y}_{A,\omega}$ such that $\Omega(u_p) = v_p$ for each $p \in A$.

We first show that Ω is surjective. Let $Z(\mathcal{Y}_{A,\omega})$ denote the centre of $\mathcal{Y}_{A,\omega}$. By [41, Proposition 3.49], the unital C^* -algebra $\mathcal{Y}_{A,\omega}$ is a $C(\widehat{A}/\widehat{B})$ -algebra with respect to the nondegenerate homomorphism $\Phi_{\mathcal{Y}_{A,\omega}}: C(\widehat{A}/\widehat{B}) \rightarrow Z(\mathcal{Y}_{A,\omega})$ given by

$$\Phi_{\mathcal{Y}_{A,\omega}}(f)(\phi) = f(\phi \cdot \widehat{B})1_{C^*(B, \widetilde{\omega})}.$$

For each $\phi \in \widehat{A}$, the set

$$I_{\phi \cdot \widehat{B}} := \overline{\text{span}}\{\Phi_{\mathcal{Y}_{A,\omega}}(f)g : f \in C(\widehat{A}/\widehat{B}), g \in \mathcal{Y}_{A,\omega}, f(\phi \cdot \widehat{B}) = 0\}$$

is an ideal of $\mathcal{Y}_{A,\omega}$. Define

$$\mathcal{A} := \bigsqcup_{\phi \cdot \widehat{B} \in \widehat{A}/\widehat{B}} \mathcal{Y}_{A,\omega}/I_{\phi \cdot \widehat{B}},$$

and let $\rho: \mathcal{A} \rightarrow \widehat{A}/\widehat{B}$ be the surjective map given by $\rho(g + I_{\phi \cdot \widehat{B}}) := \phi \cdot \widehat{B}$. An application of [41, Proposition C.10(a) and Theorem C.25] shows that there is a unique topology on \mathcal{A} such that $(\mathcal{A}, \rho, \widehat{A}/\widehat{B})$ is an upper semicontinuous C^* -bundle, and that for each $g \in \mathcal{Y}_{A,\omega}$, the section $\phi \cdot \widehat{B} \mapsto g + I_{\phi \cdot \widehat{B}}$ is continuous. Define

$$\Gamma(\mathcal{A}) := \{h: \widehat{A}/\widehat{B} \rightarrow \mathcal{A} : h \text{ is continuous, and } \rho(h(\phi \cdot \widehat{B})) = \phi \cdot \widehat{B} \text{ for all } \phi \in \widehat{A}\},$$

and let $F: \mathcal{Y}_{A,\omega} \rightarrow \Gamma(\mathcal{A})$ be the map given by $F(g)(\phi \cdot \widehat{B}) := g + I_{\phi \cdot \widehat{B}}$, for all $g \in \mathcal{Y}_{A,\omega}$ and $\phi \in \widehat{A}$. By [41, Theorem C.26], F is a $C(\widehat{A}/\widehat{B})$ -linear isomorphism of $\mathcal{Y}_{A,\omega}$ onto the $C(\widehat{A}/\widehat{B})$ -algebra $\Gamma(\mathcal{A})$. An application of [41, Proposition C.24] shows that $F(\Omega(C^*(A, \omega)))$ is a dense subspace of $\Gamma(\mathcal{A})$. Since F is an isomorphism, it follows that $\Omega(C^*(A, \omega))$ is dense in $\mathcal{Y}_{A,\omega}$, and hence Ω is surjective.

To see that Ω is injective, let β^A be the strongly continuous action of Lemma A.3(a). For each $\phi \in \widehat{A}$, the map $v_p \mapsto \phi(p)v_p$ extends to an automorphism α_ϕ^A of $\mathcal{Y}_{A,\omega}$ satisfying $\alpha_\phi^A \circ \Omega = \Omega \circ \beta_\phi^A$. If $x \in C^*(A, \omega)$ satisfies $\Omega(x) = 0$, then for all $\phi \in \widehat{A}$, we have $\Omega(\beta_\phi^A(x)) = \alpha_\phi^A(\Omega(x)) = 0$, and then Lemma A.3(c) gives $x = 0$. Hence Ω is injective. Thus Ω is an isomorphism, and $C^*(A, \omega)$ is a $C(\widehat{A}/\widehat{B})$ -algebra with respect to the homomorphism $\Omega^{-1} \circ \Phi_{\mathcal{Y}_{A,\omega}}$. ■

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References

- [1] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*. Monogr. Enseign. Math. 36, L'Enseignement Mathématique, Geneva, 2000 Zbl [0960.43003](#) MR [1799683](#)
- [2] B. Armstrong, *Twisted topological higher-rank graphs and their C^* -algebras*. Honours thesis, University of Wollongong, Australia, 2014
- [3] B. Armstrong, *Simplicity of twisted C^* -algebras of topological higher-rank graphs*. Ph.D. thesis, The University of Sydney, Australia, 2019
- [4] B. Armstrong, [A uniqueness theorem for twisted groupoid \$C^*\$ -algebras](#). *J. Funct. Anal.* **283** (2022), no. 6, article no. 109551 Zbl [1504.46063](#) MR [4433049](#)
- [5] B. Armstrong and N. Brownlowe, [Product-system models for twisted \$C^*\$ -algebras of topological higher-rank graphs](#). *J. Math. Anal. Appl.* **466** (2018), no. 2, 1443–1475 Zbl [1392.05052](#) MR [3825450](#)
- [6] L. Baggett and A. Kleppner, [Multiplier representations of abelian groups](#). *J. Functional Analysis* **14** (1973), 299–324 Zbl [0286.22005](#) MR [364537](#)
- [7] J. Brown, L. O. Clark, C. Farthing, and A. Sims, [Simplicity of algebras associated to étale groupoids](#). *Semigroup Forum* **88** (2014), no. 2, 433–452 Zbl [1304.46046](#) MR [3189105](#)
- [8] J. H. Brown, L. O. Clark, A. Sierakowski, and A. Sims, [Purely infinite simple \$C^*\$ -algebras that are principal groupoid \$C^*\$ -algebras](#). *J. Math. Anal. Appl.* **439** (2016), no. 1, 213–234 Zbl [1355.46035](#) MR [3474359](#)
- [9] K. S. Brown, *Cohomology of groups*. Grad. Texts in Math. 87, Springer, New York, 1982 Zbl [0584.20036](#) MR [672956](#)
- [10] N. P. Brown and N. Ozawa, *C^* -algebras and finite-dimensional approximations*. Grad. Stud. Math. 88, American Mathematical Society, Providence, RI, 2008 Zbl [1160.46001](#) MR [2391387](#)
- [11] T. M. Carlsen, E. Ruiz, A. Sims, and M. Tomforde, [Reconstruction of groupoids and \$C^*\$ -rigidity of dynamical systems](#). *Adv. Math.* **390** (2021), article no. 107923 Zbl [1484.46052](#) MR [4292963](#)
- [12] V. Deaconu, [Groupoids associated with endomorphisms](#). *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1779–1786 Zbl [0826.46058](#) MR [1233967](#)
- [13] R. Engelking, *General topology*. Second edn., Sigma Ser. Pure Math. 6, Heldermann Verlag, Berlin, 1989 Zbl [0684.54001](#) MR [1039321](#)
- [14] R. Exel, [Inverse semigroups and combinatorial \$C^*\$ -algebras](#). *Bull. Braz. Math. Soc. (N.S.)* **39** (2008), no. 2, 191–313 Zbl [1173.46035](#) MR [2419901](#)
- [15] R. Exel and J. Renault, [Semigroups of local homeomorphisms and interaction groups](#). *Ergodic Theory Dynam. Systems* **27** (2007), no. 6, 1737–1771 Zbl [1133.46040](#) MR [2371594](#)

- [16] C. Farsi, L. Huang, A. Kumjian, and J. Packer, [Cocycles on groupoids arising from \$\mathbb{N}^k\$ -actions](#). *Ergodic Theory Dynam. Systems* **42** (2022), no. 11, 3325–3356 Zbl [07624339](#) MR [4492626](#)
- [17] G. B. Folland, *A course in abstract harmonic analysis*. Second edn., Textb. Math., CRC Press, Boca Raton, FL, 2016 Zbl [1342.43001](#) MR [3444405](#)
- [18] G. Goehle, *Groupoid crossed products*. Ph.D. thesis, Dartmouth College, Hanover, NH, 2009
- [19] P. Green, [The local structure of twisted covariance algebras](#). *Acta Math.* **140** (1978), no. 3–4, 191–250 Zbl [0407.46053](#) MR [493349](#)
- [20] A. an Huef, A. Kumjian, and A. Sims, [A Dixmier–Douady theorem for Fell algebras](#). *J. Funct. Anal.* **260** (2011), no. 5, 1543–1581 Zbl [1251.46028](#) MR [2749438](#)
- [21] T. Katsura, [A class of \$C^*\$ -algebras generalizing both graph algebras and homeomorphism \$C^*\$ -algebras. I. Fundamental results](#). *Trans. Amer. Math. Soc.* **356** (2004), no. 11, 4287–4322 Zbl [1049.46039](#) MR [2067120](#)
- [22] A. Kleppner, [Multipliers on abelian groups](#). *Math. Ann.* **158** (1965), 11–34 Zbl [0135.06604](#) MR [174656](#)
- [23] A. Kumjian and D. Pask, [Higher rank graph \$C^*\$ -algebras](#). *New York J. Math.* **6** (2000), 1–20 Zbl [0946.46044](#) MR [1745529](#)
- [24] A. Kumjian, D. Pask, and I. Raeburn, [Cuntz–Krieger algebras of directed graphs](#). *Pacific J. Math.* **184** (1998), no. 1, 161–174 Zbl [0917.46056](#) MR [1626528](#)
- [25] A. Kumjian, D. Pask, I. Raeburn, and J. Renault, [Graphs, groupoids, and Cuntz–Krieger algebras](#). *J. Funct. Anal.* **144** (1997), no. 2, 505–541 Zbl [0929.46055](#) MR [1432596](#)
- [26] A. Kumjian, D. Pask, and A. Sims, [Homology for higher-rank graphs and twisted \$C^*\$ -algebras](#). *J. Funct. Anal.* **263** (2012), no. 6, 1539–1574 Zbl [1253.55006](#) MR [2948223](#)
- [27] A. Kumjian, D. Pask, and A. Sims, [On the \$K\$ -theory of twisted higher-rank-graph \$C^*\$ -algebras](#). *J. Math. Anal. Appl.* **401** (2013), no. 1, 104–113 Zbl [1279.46053](#) MR [3011251](#)
- [28] A. Kumjian, D. Pask, and A. Sims, [On twisted higher-rank graph \$C^*\$ -algebras](#). *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 5177–5216 Zbl [1329.46050](#) MR [3335414](#)
- [29] A. Kumjian, D. Pask, and A. Sims, [Simplicity of twisted \$C^*\$ -algebras of higher-rank graphs and crossed products by quasifree actions](#). *J. Noncommut. Geom.* **10** (2016), no. 2, 515–549 Zbl [1362.46053](#) MR [3519045](#)
- [30] M. Laca and S. Neshveyev, [KMS states of quasi-free dynamics on Pimsner algebras](#). *J. Funct. Anal.* **211** (2004), no. 2, 457–482 Zbl [1060.46049](#) MR [2056837](#)
- [31] H. Li, D. Pask, and A. Sims, [An elementary approach to \$C^*\$ -algebras associated to topological graphs](#). *New York J. Math.* **20** (2014), 447–469 Zbl [1302.46042](#) MR [3218785](#)
- [32] X. Li, [Every classifiable simple \$C^*\$ -algebra has a Cartan subalgebra](#). *Invent. Math.* **219** (2020), no. 2, 653–699 Zbl [1450.46047](#) MR [4054809](#)
- [33] D. Olesen, G. K. Pedersen, and M. Takesaki, [Ergodic actions of compact abelian groups](#). *J. Operator Theory* **3** (1980), no. 2, 237–269 Zbl [0456.46053](#) MR [578942](#)
- [34] I. Raeburn and J. L. Taylor, [Continuous trace \$C^*\$ -algebras with given Dixmier–Douady class](#). *J. Austral. Math. Soc. Ser. A* **38** (1985), no. 3, 394–407 Zbl [0627.46068](#) MR [779202](#)
- [35] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^* -algebras*. Math. Surveys Monogr. 60, American Mathematical Society, Providence, RI, 1998 Zbl [0922.46050](#) MR [1634408](#)
- [36] J. Renault, *A groupoid approach to C^* -algebras*. Lecture Notes in Math. 793, Springer, Berlin, 1980 Zbl [0433.46049](#) MR [584266](#)
- [37] J. Renault, [The ideal structure of groupoid crossed product \$C^*\$ -algebras](#). *J. Operator Theory* **25** (1991), no. 1, 3–36 Zbl [0786.46050](#) MR [1191252](#)

- [38] J. Renault, AF equivalence relations and their cocycles. In *Operator algebras and mathematical physics (Constanța, 2001)*, pp. 365–377, Theta, Bucharest, 2003 Zbl [1247.46059](#) MR [2018241](#)
- [39] A. Sims, Hausdorff étale groupoids and their C^* -algebras. In *Operator algebras and dynamics: groupoids, crossed products, and Rokhlin dimension*, edited by F. Perera, Adv. Courses Math., CRM Barcelona, Birkhäuser/Springer, Cham, 2020 Zbl [1495.46055](#) MR [4321941](#)
- [40] A. Sims and D. P. Williams, [The primitive ideals of some étale groupoid \$C^*\$ -algebras](#). *Algebr. Represent. Theory* **19** (2016), no. 2, 255–276 Zbl [1357.46051](#) MR [3489096](#)
- [41] D. P. Williams, *Crossed products of C^* -algebras*. Math. Surveys Monogr. 134, American Mathematical Society, Providence, RI, 2007 Zbl [1119.46002](#) MR [2288954](#)
- [42] T. Yeend, [Topological higher-rank graphs and the \$C^*\$ -algebras of topological 1-graphs](#). In *Operator theory, operator algebras, and applications*, pp. 231–244, Contemp. Math. 414, Amer. Math. Soc., Providence, RI, 2006 Zbl [1111.46036](#) MR [2277214](#)
- [43] T. Yeend, Groupoid models for the C^* -algebras of topological higher-rank graphs. *J. Operator Theory* **57** (2007), no. 1, 95–120 Zbl [1119.46312](#) MR [2301938](#)

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Becky Armstrong

School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand; becky.armstrong@vuw.ac.nz

Nathan Brownlowe

School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia; nathan.brownlowe@sydney.edu.au

Aidan Sims

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia; asims@uow.edu.au