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The equivariant coarse Baum–Connes conjecture for metric spaces with proper group actions

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Abstract. The equivariant coarse Baum–Connes conjecture interpolates between the Baum–Connes conjecture for a discrete group and the coarse Baum–Connes conjecture for a proper metric space. In this paper, we study this conjecture under certain assumptions. More precisely, assume that a countable discrete group Γ acts properly and isometrically on a discrete metric space X with bounded geometry, not necessarily cocompact. We show that if the quotient space X/Γ admits a coarse embedding into Hilbert space and Γ is amenable, and that the Γ-orbits in X are uniformly equivariantly coarsely equivalent to each other, then the equivariant coarse Baum–Connes conjecture holds for (X, Γ). Along the way, we prove a K-theoretic amenability statement for the Γ-space X under the same assumptions as above; namely, the canonical quotient map from the maximal equivariant Roe algebra of X to the reduced equivariant Roe algebra of X induces an isomorphism on K-theory.

1. Introduction

The Baum–Connes conjecture [2, 3] provides an algorithm to compute the K-theory of reduced group C^*-algebras, which has important applications in geometry, topology, and analysis (see [2] for a survey). It has been verified for a large class of groups including a-T-menable groups [17] and hyperbolic groups [23].

The coarse Baum–Connes conjecture [26, 27] is a geometric analog of the Baum–Connes conjecture, which also has significant applications in geometry and topology, such as the Novikov conjecture and Gromov's conjecture about Riemannian metric of positive scalar curvature [9, 15, 19, 36, 37]. Many results (cf. [5, 8, 12–14, 20, 21, 25, 30–32]) have been achieved in recent years after Yu's breakthrough to the coarse Baum–Connes conjecture for metric spaces which are coarsely embeddable into Hilbert space [35].

Let X be a discrete metric space. We say that X has *bounded geometry* if for any $R > 0$, there exists $M > 0$ such that any ball in X with radius R contains at most M elements. Let Γ be a countable discrete group. Assume that Γ acts properly and isometrically on X with bounded geometry, not necessarily cocompact. In this case, we call X a Γ-space. There is an equivariant higher index map [10, 28]

$$\mathrm{Ind}^{\Gamma} : \lim_{d\to\infty} K_*^{\Gamma}\big(P_d(X)\big) \to K_*\big(C^*(X)^{\Gamma}\big),$$

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where $K_*^\Gamma(P_d(X))$ is the Γ-equivariant K-homology group of the Rips complex $P_d(X)$ of X on scale $d > 0$, and $K_*(C^*(X)^\Gamma)$ is the K-theory group of the equivariant Roe algebra of the Γ-space X. The equivariant coarse Baum–Connes conjecture states that the equivariant higher index map Ind^Γ above is an isomorphism provided that X has bounded geometry. When the Γ-action is cocompact, $C^*(X)^\Gamma$ is Morita equivalent to the reduced group C^*-algebra of Γ, so that the equivariant coarse Baum–Connes conjecture is a reformulation of the Baum–Connes conjecture (cf. [33]). When the group is trivial, the equivariant coarse Baum–Connes conjecture is plainly the coarse Baum–Connes conjecture.

In [28], Shan proved that the equivariant higher index map is injective when M is a simply connected complete Riemannian manifold with non-positive sectional curvature and Γ is a torsion-free group acting on M properly and isometrically. In [10], Fu and Wang showed that the equivariant coarse Baum–Connes conjecture holds for a Γ-space X with bounded geometry which admits an equivariant coarse embedding into Hilbert space. In [11], Fu, Wang, and Yu proved that if a discrete group Γ acts properly and isometrically on a discrete metric space X with bounded geometry, such that both X/Γ and Γ admit a coarse embedding into Hilbert space, and that the action has bounded distortion, then the equivariant higher index map is injective for the Γ-space X.

In a recent work [1], Arzhantseva and Tessera answer in the negative the following well-known question [7, 16]: does coarse embeddability into Hilbert space is preserved under group extensions of finitely generated groups? Their constructions also provide the first example of a finitely generated group which does not coarsely embed into Hilbert space yet does not contain any weakly embedded expander, answering in the affirmative another open problem [1, 24]. Their examples are certain restricted permutational wreath products

$$\mathbb{Z}_2 \wr_G H := \left(\bigoplus_G \mathbb{Z}_2\right) \rtimes H$$

and

$$\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n) := \left(\bigoplus_G \mathbb{Z}_2\right) \rtimes (H \times \mathbb{F}_n),$$

where G is a *Gromov monster group*, i.e., a finitely generated group which contains an isometrically embedded expander in its Cayley group, H is a *Haagerup monster group*, i.e., a finitely generated group with the Haagerup property but without Yu's property A, and $\mathbb{F}_n$ is a finitely generated free group. Note that both examples are *"abelian-by-Haagerup"* group extensions. The reason why they do not coarsely embed into Hilbert relies on the fact that both groups contain a certain *relative expanders* [1]. If we take $X = \mathbb{Z}_2 \wr_G H$ or $\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n)$, and $\Gamma = \bigoplus_G \mathbb{Z}_2$, then we come across with a situation that Γ is a torsion group, the Γ-action on X does not have bounded distortion, and that the space X does not even coarsely embed into Hilbert space, let alone admits a Γ-equivariant coarse embedding into Hilbert space, namely, a situation which does not satisfy the assumptions in each of the main results in [10, 11, 28] mentioned above.

In this paper, as motivated by these examples by Arzhantseva and Tessera, we will further investigate the equivariant coarse Baum–Connes conjecture along this line. The following is our main result.

Theorem 1.1. *Let Γ be a countable discrete group acting properly and isometrically on a discrete metric space X with bounded geometry, not necessarily co-compact. Assume that all Γ-orbits in X are uniformly equivariantly coarsely equivalent. If the quotient space X/Γ admits a coarse embedding into Hilbert space and Γ is amenable, then the equivariant higher index map*

$$\mathrm{Ind}^{\Gamma} : \lim_{d\to\infty} K_*^{\Gamma}\big(P_d(X)\big) \to K_*\big(C^*(X)^{\Gamma}\big)$$

is an isomorphism.

When the group Γ is trivial, Theorem 1.1 recovers Yu's famous result in [35]. On the other hand, when the Γ-action is cocompact, Theorem 1.1 recovers the amenable group case of the celebrated result of Higson and Kasparov in [17], where they proved the theorem for more general a-T-menable groups. Note that the examples $X = \mathbb{Z}_2 \wr_G H$ or $\mathbb{Z}_2 \wr_G (H \times \mathbb{F}_n)$, and $\Gamma = \bigoplus_G \mathbb{Z}_2$ by Arzhantseva and Tessera discussed above provide us with nontrivial examples of non-cocompact group actions satisfying the assumptions in Theorem 1.1. At this point, it is natural to expect that Theorem 1.1 be true for non-cocompact actions with Γ being a-T-menable. However, technically in our approach to the main theorem, we have to establish an isomorphism between the maximal equivariant twisted Roe algebra and the reduced equivariant twisted Roe algebra (see Proposition 3.12) in which the amenability of Γ plays an essential role. Nevertheless, once we get this isomorphism on the C^*-algebra level, we are able to further get a K-theoretic amenability statement for the Γ-space X, in a way as the proof of Theorem 1.1 by using the idea of an infinite-dimensional geometric Bott periodicity; see also [35]. More precisely, we have the following result:

Theorem 1.2. *With the same assumptions as Theorem 1.1, the map*

$$\lambda_* : K_*\big(C^*_{\max}(X)^{\Gamma}\big) \to K_*\big(C^*(X)^{\Gamma}\big)$$

is an isomorphism, where λ_* is the homomorphism induced by the canonical quotient map λ from the maximal equivariant Roe algebra $C^*_{\max}(X)^{\Gamma}$ to the reduced equivariant Roe algebra $C^*(X)^{\Gamma}$.

When the group Γ is trivial, Theorem 1.2 recovers the main result in [30] by Špakula and Willett. When the group action is co-compact, the statement of Theorem 1.2 actually holds for a more general class of groups (see [6]), including all a-T-menable groups, as proved by Higson and Kasparov in the same paper [17]. Unfortunately, due to limitation of our approach, the amenability of the group Γ is necessary in our proof to Theorem 1.2. For the general case, where the group action is non-cocompact, we do not know whether the conclusion of Theorem 1.2 is true or not for Γ being a-T-menable.

The paper is organized as follows. In Section 2, we recall the reduced and maximal equivariant Roe algebras, and formulate the equivariant coarse Baum–Connes conjecture, for a discrete metric space with bounded geometry which admits a proper and isometric action, not necessarily co-compact, by a countable discrete group. In Section 3, we use the coarse embedding of the quotient space X/Γ into a Hilbert space to define the reduced and maximal equivariant twisted Roe algebras, and their localization counterparts. We prove that the K-theory of the equivariant twisted localization algebra is isomorphic to the K-theory of the equivariant twisted Roe algebra. Moreover, we also prove that the maximal equivariant twisted Roe algebra and the reduced equivariant twisted Roe algebra are isomorphic. In Section 4, we complete the proof of the main theorems by using a geometric analog of Higson–Kasparov–Trout's infinite-dimensional Bott periodicity.

2. The equivariant coarse Baum–Connes conjecture

In this section, we will first recall several notions from coarse geometry, and then define the reduced and maximal equivariant Roe algebras, so as to formulate the equivariant coarse Baum–Connes conjecture for a discrete metric space with bounded geometry which admits a proper and isometric action, not necessarily co-compact, by a countable discrete group.

2.1. Roe algebras

In this subsection, we will recall the notions of the (maximal) equivariant Roe algebras.

Let Z be a proper metric space, which means that every closed ball of Z is compact. Let Γ be a countable discrete group acting on Z properly and isometrically. The Γ-action is proper if for any compact set $K \subseteq Z$, $\{\gamma \in \Gamma : \gamma K \cap K\} \neq \emptyset$ is a finite set. Γ acts on Z isometrically if $d(x, y) = d(\gamma x, \gamma y)$ for all $\gamma \in \Gamma$ and $x, y \in Z$. In this case, Z is called a Γ-*space*.

Recall that a Z-*module* is a separable Hilbert space H on which there is a faithful and non-degenerate $*$-representation $\phi : C_0(Z) \to B(H)$ such that the range of ϕ has no nonzero compact operators, where $C_0(Z)$ is the algebra of all complex-valued continuous functions on Z vanishing at infinity.

Let Z be a Γ-space. We define a Γ-action on $C_0(Z)$ by

$$\gamma(f)(x) = f(\gamma^{-1}x)$$

for all $\gamma \in \Gamma$ and $f \in C_0(Z)$.

Definition 2.1. Let H be a Z-module. We say that H is a *covariant* Z-*module* if there is a unitary action $\rho : \Gamma \to \mathcal{U}(H)$ such that

$$\phi\big(\gamma(f)\big) = \rho(\gamma)\phi(f)\rho(\gamma)^*$$

for $v \in H$, $T \in B(H)$ and $\gamma \in \Gamma$, where $\mathcal{U}(H)$ is the set of unitary operators on H. We call such a triple $(C_0(Z), \Gamma, \phi)$ a *covariant system*.

Definition 2.2 ([21]). A covariant system $(C_0(Z), \Gamma, \phi)$ is *admissible* if there exist separable and infinite-dimensional Γ-Hilbert spaces H_Z and H' such that

(1) the Z-module H is isomorphic to $H_Z \otimes H'$,

(2) $\phi = \phi_0 \otimes I$ for some Γ-equivariant $*$-homomorphism $\phi_0 : C_0(Z) \to B(H_Z)$ such that $\phi_0(f)$ is not a compact operator on H_Z for any nonzero function $f \in C_0(Z)$ and ϕ_0 is non-degenerate in the sense that $\{\phi_0(f)H_Z : f \in C_0(Z)\}$ is dense in H_Z,

(3) for each finite subgroup $F \subset \Gamma$ and every F-invariant Borel subset E of Z, there exists a trivial F-representation $H_{F,E}$ such that

$$\phi(\chi_E)H \cong \phi_0(\chi_E)H_Z \otimes l^2(F) \otimes H_{F,E},$$

where $\ell^2(F)$ is endowed with the left regular representation of F.

Definition 2.3. Let $(C_0(Z), \Gamma, \phi)$ be an admissible covariant system.

(1) For $T \in B(H)$, the *support* $\mathrm{Supp}(T)$ is the complement of the set of points $(x, y) \in Z \times Z$ for which there exist f and f' in $C_0(Z)$ such that

$$\phi(f')T\phi(f) = 0, \quad f(x) \neq 0, \quad f'(y) \neq 0.$$

(2) The *propagation* of a bounded operator $T : H \to H$ is defined to be

$$\mathrm{propagation}(T) := \sup\{d(x, y) : (x, y) \in \mathrm{Supp}(T)\}.$$

We say that T has *finite propagation* if $\mathrm{propagation}(T) < +\infty$.

(3) For any $f \in C_0(Z)$, if $\phi(f)T$ and $T\phi(f)$ are compact, then T is called a *locally compact* operator.

(4) An operator $T \in B(H)$ is Γ-*invariant* if $\gamma(T) = T$ for all $\gamma \in \Gamma$.

Now we are ready to define the equivariant Roe algebra.

Definition 2.4. Let $(C_0(Z), \Gamma, \phi)$ be an admissible covariant system. The *algebraic equivariant Roe algebra* $\mathbb{C}[Z]^\Gamma$ is the $*$-subalgebra of $B(H)$ consisting of all locally compact and Γ-invariant operators with finite propagation. The *equivariant Roe algebra* $C^*(Z)^\Gamma$ is the closure of the $*$-algebra $\mathbb{C}[Z]^\Gamma$ under the operator norm in $B(H)$.

A proper Γ-space Z is said to have *bounded geometry* if there exists a discrete Γ-invariant subset $X \subset Z$ such that

(1) X has bounded geometry as a discrete metric space,

(2) X is a c-net in Z for some $c > 0$ in the sense that

$$Z = N_c(X) = \{z \in Z : d(z, X) \leq c\}.$$

The $*$-algebra $\mathbb{C}[Z]^\Gamma$ can be equipped with a C^*-norm associated with any $*$-representation $\phi : \mathbb{C}[Z]^\Gamma \to H_\phi$. To define the maximal norm $\mathbb{C}[Z]^\Gamma$, we need the following result which is essentially proved in [13].

Lemma 2.5. *Let Z be a proper Γ-space with bounded geometry. For any $T \in \mathbb{C}[Z]^\Gamma$, there exists a constant $C_T > 0$ such that for any $*$-representation ϕ of $\mathbb{C}[Z]^\Gamma$ on a Hilbert space $\mathcal{H}_\phi$, we have*

$$\|\phi(T)\|_{B(\mathcal{H}_\phi)} \leq C_T.$$

It follows from the above lemma that the maximal norm on the $*$-algebra $\mathbb{C}[Z]^\Gamma$ is well defined.

Definition 2.6 ([13]). Let Z be a proper Γ-space. The maximal equivariant Roe algebra, denoted by $C^*_{\max}(Z)^\Gamma$, is the completion of $\mathbb{C}[Z]^\Gamma$ with respect to the C^*-norm

$$\|T\|_{\max} := \sup\{\|\phi(T)\|_{B(\mathcal{H}_\phi)} : \phi : \mathbb{C}[Z]^\Gamma \to B(\mathcal{H}_\phi) \text{ is a } *\text{-representation}\}.$$

We remark that the definition of the (maximal) equivariant Roe algebra does not depend on the choice of admissible covariant systems. The (maximal) equivariant Roe algebra $C^*(Z)^\Gamma$ is Morita equivalent to the (maximal) reduced group C^*-algebra $C^*_r(\Gamma)$ when the quotient space Z/Γ is compact; i.e., Z admits a proper and cocompact Γ-action (cf. [27, Lemma 5.14]). Moreover, equivariant Roe algebras are invariant under the equivariant coarse equivalence in the sense of the following.

Definition 2.7. Let Z and Z' be proper Γ-spaces. A Borel map $f : Z \to Z'$ is said to be an *equivariantly coarse embedding* if there exist non-decreasing functions $\rho_+, \rho_- : \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$ such that

(1) $\lim_{r\to+\infty} \rho_\pm(r) \to +\infty$,

(2) $\rho_-(d(x, y)) \leq d(f(x), f(y)) \leq \rho_+(d(x, y))$, for all $x, y \in Z$,

(3) $f(\gamma x) = \gamma f(x)$ for all $x \in Z$ and $\gamma \in \Gamma$.

Moreover, the Γ-spaces Z and Z' are called *equivariantly coarsely equivalent* if there is an equivariant coarse embedding $f : Z \to Z'$ such that the image $f(Z)$ is *coarsely dense* in Z' in the sense that $Z' = N_S(f(Z))$ for some $S > 0$, where $N_S(f(Z)) = \{y \in Y : d(y, f(Z)) < S\}$. We say that a family of metric spaces $\{Z_i\}_{i\in I}$ are said to be *uniformly equivariantly coarsely equivalent* if any two metric spaces Z_i and Z_j are equivariantly coarsely equivalent with the same functions ρ_+, ρ_- and constant $S > 0$ in the above definitions. If the metric space $Z' = H$ is a Hilbert space, and the spaces Z and H are equipped with trivial actions, then the metric space Z is said to admit a coarse embedding into Hilbert space.

The following proposition is just an equivariant version of the isomorphism of the Roe algebras under the coarse equivalence property. So we omit the proof here.

Proposition 2.8. *Let Z and Z' be proper Γ-spaces. If there exists an equivariantly coarse equivalence $f : Z \to Z'$, then*

$$C^*(Z)^\Gamma \cong C^*(Z')^\Gamma,$$
$$C^*_{\max}(Z)^\Gamma \cong C^*_{\max}(Z')^\Gamma.$$

By the universality of the maximal norm, the identity map on $\mathbb{C}[Z]^\Gamma$ extends to a canonical quotient map

$$\lambda : C^*_{\max}(Z)^\Gamma \to C^*(Z)^\Gamma.$$

We then obtain a homomorphism

$$\lambda_* : K_*\big(C^*_{\max}(Z)^\Gamma\big) \to K_*\big(C^*(Z)^\Gamma\big)$$

induced by the map λ at the K-theory level.

2.2. The assembly map

Let us briefly recall the definition of equivariant K-homology introduced by Kasparov [22, 38].

Definition 2.9. For $i = 0, 1$, the K-homology groups $K_i^\Gamma(Z) = KK_i^\Gamma(C_0(Z), \mathbb{C})$ ($i = 0, 1$) are generated by certain cycles modulo a certain equivalence relation:

(1) each cycle for $K_0^\Gamma(Z)$ is a triple (H, ϕ, F), where $\phi : C_0(Z) \to B(H)$ is a covariant $*$-representation and $F \in B(H)$ is Γ-invariant such that $\phi(f)F - F\phi(f)$, $\phi(f)(FF^* - I)$, and $\phi(f)(F^*F - I)$ are compact for all $f \in C_0(Z), \gamma \in \Gamma$;

(2) each cycle for $K_1^\Gamma(Z)$ is a triple (H, ϕ, F), where $\phi : C_0(Z) \to B(H)$ is a covariant $*$-representation and F is a Γ-invariant and self-adjoint operator on H such that $\phi(f)(F^2 - I)$ and $\phi(f)F - F\phi(f)$ are compact for all $f \in C_0(Z), \gamma \in \Gamma$.

In both cases, the equivalence relation on cycles is given by homotopy of the operator F.

Let us now define the *equivariant index map*

$$\mathrm{Ind}^\Gamma : K_*^\Gamma(Z) \to K_*\big(C^*(Z)^\Gamma\big).$$

Note that every class in the group $K_*^\Gamma(Z)$ can be represented by a cycle (H, ϕ, F) such that $(C_0(Z), \Gamma, \phi)$ is an admissible covariant system [21]. Let (H, ϕ, F) be a cycle in $K_0^\Gamma(Z)$ such that $(C_0(Z), \Gamma, \phi)$ is an admissible covariant system.

Take a locally finite, Γ-equivariant, and uniformly bounded open cover $\{U_i\}_{i\in I}$ of X, and let $\{\psi_i\}_{i\in I}$ be a Γ-equivariant partition of unity subordinate to $\{U_i\}_{i\in I}$. Define

$$F' = \sum_{i\in I} \phi(\sqrt{\psi_i})F\phi(\sqrt{\psi_i}),$$

where the sum converges in strong topology. In fact, (H, ϕ, F') is equivalent to (H, ϕ, F) in $K_0^\Gamma(Z)$ and F' has finite propagation. So F' is a multiplier of $C^*(Z)^\Gamma$ and it is invertible modulo $C^*(Z)^\Gamma$. As a result, F' gives rise to an element in $K_0(C^*(Z)^\Gamma)$, denoted by $\partial([F'])$. We define the equivariant index map

$$\mathrm{Ind}^\Gamma : K_0^\Gamma(Z) \to K_0\big(C^*(Z)^\Gamma\big)$$

by

$$\mathrm{Ind}^\Gamma\big([H, \phi, F]\big) = \partial\big([F']\big)$$

for all $[H, \phi, F] \in K_0^\Gamma(Z)$, where F' is defined as above. Similarly, we can define the equivariant index map

$$\mathrm{Ind}^\Gamma : K_1^\Gamma(Z) \to K_1\big(C^*(Z)^\Gamma\big).$$

2.3. The equivariant coarse Baum–Connes conjecture

In this subsection, we will recall the equivariant coarse Baum–Connes conjecture for a discrete metric space with bounded geometry.

From now on, we always assume that X is a discrete metric space with bounded geometry.

Definition 2.10. For $d > 0$, the Rips complex $P_d(X)$ with scale d is the simplicial polyhedron whose vertex set is X, and a finite subset $Y = \{x_0, x_1, \ldots, x_n\} \subseteq X$ spans a simplex if and only if the diameter of Y is no more than d.

Endow $P_d(X)$ with the spherical metric defined by the following way. Identify a simplex $Y = \{x_0, x_1, \ldots, x_n\}$ with

$$S_+^n := \left\{(s_0, s_1, \ldots, s_n) \in \mathbb{R}^{n+1} : s_i \geq 0,\ \sum_{i=0}^n s_i^2 = 1\right\}$$

through the map

$$\sum_{i=0}^n t_i x_i \mapsto \left(\frac{t_0}{\sqrt{\sum_{i=0}^n t_i^2}}, \frac{t_1}{\sqrt{\sum_{i=0}^n t_i^2}}, \ldots, \frac{t_n}{\sqrt{\sum_{i=0}^n t_i^2}}\right),$$

where S_+^n is endowed with the standard Riemannian metric. For each path connected component of $P_d(X)$, the spherical metric is the maximal metric whose restriction on each simplex is the metric above, and the distance of two points in different connected components of $P_d(X)$ is defined to be infinity. Obviously, each Rips complex $P_d(X)$ admits a proper Γ-action by isometries defined by

$$\gamma \cdot \sum_{i=1}^k c_i x_i = \sum_{i=1}^k c_i (\gamma \cdot x_i)$$

for all $\sum_{i=1}^k c_i x_i \in P_d(X)$ and $\gamma \in \Gamma$.

The following conjecture is called *the equivariant coarse Baum–Connes conjecture*.

Conjecture 2.11 (The equivariant coarse Baum–Connes conjecture). Let X be a discrete metric space with bounded geometry and let Γ be a countable discrete group acting on X properly by isometries. Then the equivariant index map

$$\mathrm{Ind}^\Gamma : \lim_{d\to\infty} K_*^\Gamma\big(P_d(X)\big) \to \lim_{d\to\infty} K_*\big(C^*\big(P_d(X)\big)^\Gamma\big) \cong K_*\big(C^*(X)^\Gamma\big)$$

is an isomorphism.

2.4. Equivariant localization algebras and local index map

In this subsection, we will recall the notions of the localization algebras and the local index map.

Definition 2.12. Let Z be a proper Γ-space. The *equivariant localization algebra* $C_L^*(Z)^\Gamma$ is the supremum norm closure of the algebra of all bounded and uniformly norm-continuous functions

$$f : [0, +\infty) \to C^*(Z)^\Gamma,$$

such that

$$\text{propagation}\big(f(t)\big) \to 0 \quad \text{as } t \to \infty.$$

There exists an equivariant local index map (cf. [34] or [11])

$$\text{Ind}^\Gamma : K_*^\Gamma(Z) \to K_*\big(C_L^*(Z)^\Gamma\big).$$

Moreover, we have the following result which is an equivariant analog of [34, Theorem 3.2].

Theorem 2.13. *Let X be a discrete metric space with bounded geometry and let Γ be a countable discrete group acting on X properly and isometrically. Then the local index map*

$$\text{Ind}_L^\Gamma : K_*^\Gamma\big(P_d(X)\big) \to K_*\big(C_L^*\big(P_d(X)\big)^\Gamma\big)$$

is an isomorphism.

For any $d > 0$, there is a natural evaluation map

$$e : C_L^*\big(P_d(X)\big)^\Gamma \to C^*\big(P_d(X)\big)^\Gamma$$

by

$$e(f) = f(0)$$

for all $f \in C_L^*(P_d(X))^\Gamma$. This is a $*$-homomorphism, thus it induces a homomorphism

$$e_* : K_*\big(C_L^*\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*\big(P_d(X)\big)^\Gamma\big)$$

on K-theory. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} & & \lim_{d\to\infty} K_*\big(C_L^*\big(P_d(X)\big)^\Gamma\big) \\ & \overset{\text{Ind}_L^\Gamma}{\nearrow}_{\cong} & \big\downarrow e_* \\ \lim_{d\to\infty} K_*^\Gamma\big(P_d(X)\big) & \xrightarrow{\text{Ind}^\Gamma} & \lim_{d\to\infty} K_*\big(C^*\big(P_d(X)\big)^\Gamma\big). \end{array}$$

As a result, in order to prove that the map Ind^Γ is an isomorphism, it suffices to show that the map

$$e_* : \lim_{d\to\infty} K_*\big(C_L^*\big(P_d(X)\big)^\Gamma\big) \to \lim_{d\to\infty} K_*\big(C^*\big(P_d(X)\big)^\Gamma\big)$$

induced by the evaluation map on K-theory is an isomorphism.

3. Equivariant twisted Roe algebras and localization algebras

In this section, we will follow the constructions in [35] to define C^*-algebras encoding the geometry of the quotient space X/Γ. These algebras include maximal and reduced equivariant twisted Roe algebras, and equivariant twisted localization algebras. We prove that the K-theory of equivariant twisted localization algebras is isomorphic to the K-theory of equivariant twisted Roe algebras under the assumptions on the geometry of X/Γ and the orbits. We also prove that the canonical quotient map between the maximal equivariant twisted Roe algebras and the reduced equivariant twisted Roe algebras is an isomorphism under the same assumptions.

3.1. The equivariant twisted Roe algebras along the quotient space

Let Γ be a countable discrete group, and let X be a Γ-space with bounded geometry. Assume that $\xi : X/\Gamma \to H$ is a coarse embedding of X into Hilbert space H. Let $\pi : X \to X/\Gamma$ be the quotient map. We will first recall the C^*-algebra associated to an infinite-dimensional Euclidean space introduced by Higson–Kasparov–Trout [18].

Let H be a real (countably infinite-dimensional) Hilbert space. We use V_a, V_b, etc. to denote the finite-dimensional affine subspaces of H. Let V_a^0 be the finite-dimensional linear subspaces of H consisting of differences of elements of V_a. Let $\mathrm{Cliff}(V_a^0)$ be the complexified Clifford algebra of V_a^0, and let

$$\mathcal{C}(V_a) = \{h : V_a \to \mathrm{Cliff}(V_a^0) \text{ is continuous and vanishing at infinity}\}$$

be the $\mathbb{Z}_2$-graded C^*-algebra. Let

$$\mathcal{A}(V_a) = \mathcal{S} \hat{\otimes} \mathcal{C}(V_a)$$

be the graded tensor product, where $\mathcal{S} = C_0(\mathbb{R})$ is graded by odd and even functions. If $V_a \subseteq V_b$, then $V_b = V_{ba}^0 + V_a$, where V_{ba}^0 is the orthogonal complement of V_a^0 in V_b^0. For each $v_b \in V_b$, there is a unique decomposition $v_b = v_{ba} + v_a$, where $v_{ba} \in V_{ba}^0$ and $v_a \in V_a$.

If $V_a \subseteq V_b$, we use C_{ba} to denote the function

$$V_b \to \mathrm{Cliff}(V_b^0), \quad v_b \mapsto v_{ba}.$$

Let $X(f) = xf(x)$ be the unbounded multiplier of $\mathcal{S}$ with degree one, where $f \in \mathcal{S}$. Note that $X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba}$ is an unbounded, essentially self-adjoint operator with degree one [18]. So there is a $*$-homomorphism

$$\beta_{ba} : \mathcal{A}(V_a) \to \mathcal{A}(V_b)$$

defined by

$$\beta_{ba}(g \hat{\otimes} h) = g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba})(1 \hat{\otimes} \tilde{h}),$$

for $g \in \mathcal{S}$, $h \in \mathcal{C}(V_a)$, where $\tilde{h}$ is an extension of h by

$$\tilde{h}(v_{ba} + v_a) = h(v_a),$$

and $g(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba})$ is obtained by functional calculus.

Definition 3.1. Define the C^*-algebra $\mathcal{A}(H)$ by

$$\mathcal{A}(H) = \lim_{\rightarrow} \mathcal{A}(V_a),$$

where the inductive limit is taken over all finite-dimensional affine subspaces of H, using the homomorphisms (β_{ab}).

Endow $\mathbb{R}_+ \times H$ with the weakest topology such that the projection onto H is weakly continuous and the function $(t, v) \mapsto t^2 + \|v\|^2$ is continuous. With this topology, $\mathbb{R}_+ \times H$ is a locally compact Hausdorff space. Note that for $v \in H$ and $r > 0$, the subset

$$B(v, r) = \{(t, w) \in \mathbb{R}_+ \times H : t^2 + \|v - w\|^2 < r^2\}$$

is an open subset of $\mathbb{R}_+ \times H$. For each finite-dimensional subspaces $V_a \subseteq H$, $\mathcal{A}(V_a)$ contains $C_0(\mathbb{R}_+ \times V_a)$ as its center. If $V_a \subseteq V_b$, then β_{ba} maps $C_0(\mathbb{R}_+ \times V_a)$ into $C_0(\mathbb{R}_+ \times V_b)$. With the direct system over all finite-dimensional affine subspace of H and (β_{ba}), we obtain a C^*-algebra $\lim_{\rightarrow} C_0(\mathbb{R}_+ \times V_a)$ which is isomorphic to $C_0(\mathbb{R}_+ \times H)$.

Definition 3.2. The *support* of an element $a \in \mathcal{A}(H)$ is the complement of all $(t, v) \in \mathbb{R}_+ \times H$ such that there exists $g \in C_0(\mathbb{R}_+ \times H)$ with $g(t, v) \neq 0$ and $g \cdot a = 0$, where g acts on $\mathcal{A}(H)$ as a multiplier.

For each $d > 0$, the Γ-action on X induces a proper and isometric Γ-action on $P_d(X)$ by

$$\gamma \cdot \sum_{i=1}^{k} c_i x_i = \sum_{i=1}^{k} c_i (\gamma \cdot x_i)$$

for $\sum_{i=1}^{k} c_i x_i \in P_d(X)$ and $\gamma \in \Gamma$.

Note that the coarse embedding $\xi : X/\Gamma \to H$ induces a coarse embedding

$$\xi : P_d(X)/\Gamma \to H$$

by

$$\xi\left(\pi\left(\sum_{i=1}^{k} c_i x_i\right)\right) = \sum_{i=1}^{k} c_i \xi\big(\pi(x_i)\big),$$

for all $\sum_{i=1}^{k} c_i x_i \in P_d(X)$, where $\pi : P_d(X) \to P_d(X)/\Gamma$ is the quotient map induced by the Γ-action on $P_d(X)$.

For any element $x \in P_d(X)$, we define

$$W_k\big(\pi(x)\big) = \xi\big(\pi(x)\big) + \operatorname{span}\big\{\xi\big(\pi(y)\big) - \xi\big(\pi(x)\big) : y \in X/\Gamma, d\big(\pi(x), \pi(y)\big) \leq k^2\big\}.$$

Since X has bounded geometry, so the subspace $W_k(\pi(x))$ is a finite-dimensional affine subspace of H. For $k < k'$, let

$$\beta_{k',k}\big(\pi(x)\big) : \mathcal{A}\big(W_k\big(\pi(x)\big)\big) \to \mathcal{A}\big(W_{k'}\big(\pi(x)\big)\big)$$

be the map defined by the inclusion of $W_k(\pi(x))$ into $W_{k'}(\pi(x))$. Let

$$\beta_k\big(\pi(x)\big) : \mathcal{A}\big(W_k\big(\pi(x)\big)\big) \to \mathcal{A}(H)$$

be the map defined by the definition of $\mathcal{A}(H)$. We write $\beta(\pi(x))$ for

$$\beta_0\big(\pi(x)\big) : \mathcal{S} \cong \mathcal{A}\big(W_0\big(\pi(x)\big)\big) \to \mathcal{A}(H).$$

For each $d > 0$, fix a countable and Γ-invariant dense subset X_d of $P_d(X)$ such that $X_{d_1} \subseteq X_{d_2}$ if $d_1 \leq d_2$.

Definition 3.3. The *algebraic twisted equivariant Roe algebra* $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ is defined to be the set of all functions $T : X_d \times X_d \to \mathcal{A}(H) \mathbin{\hat{\otimes}} K$ such that the following hold.

(1) There exists $M \geq 0$ such that $\|T(x, y)\| \leq M$ for all $x, y \in X_d$.

(2) There exists an integer N such that

$$T(x, y) \in \beta_N\big(\pi(x)\big)\big(\mathcal{A}\big(W_N\big(\pi(x)\big)\big)\big) \mathbin{\hat{\otimes}} K \subseteq \mathcal{A}(H) \mathbin{\hat{\otimes}} K$$

for all $x, y \in X_d$, where $\beta_N(\pi(x)) : \mathcal{A}(W_N(\pi(x))) \to \mathcal{A}(H)$ is the $*$-homomorphism associated to the inclusion of $W_N(\pi(x))$ into H, and K is the algebra of compact operators on the Hilbert space $H_0 \mathbin{\hat{\otimes}} l^2(\Gamma)$.

(3) There exists $r_1 > 0$ such that if $d(x, y) > r_1$, then $T(x, y) = 0$.

(4) There exists $L > 0$ such that for each $y \in X_d$,

$$\sharp\big\{x : T(x, y) \neq 0\big\} \leq L, \quad \sharp\big\{x : T(y, x) \neq 0\big\} \leq L.$$

(5) For any bounded set $B \subseteq P_d(X)$, the set

$$\big\{(x, y) \in B \times B \cap X_d \times X_d : T(x, y) \neq 0\big\}$$

is finite.

(6) There exists $r_2 > 0$ such that

$$\mathrm{Supp}\,\big(T(x, y)\big) \subseteq \big\{(s, h) \in \mathbb{R}_+ \times H : s^2 + \big\|h - \xi\big(\pi(x)\big)\big\|^2 < r_2^2\big\}$$

for $x, y \in X$.

(7) There exists $c > 0$ such that if $Y = (s, h) \in \mathbb{R}_+ \times W_N(\pi(x))$ with $\|Y\| = \sqrt{s^2 + \|h\|^2} \leq 1$ and if $T_1(x, y)$ satisfies that

$$\big(\beta_N\big(\pi(x)\big) \mathbin{\hat{\otimes}} 1\big)\big(T_1(x, y)\big) = T(x, y),$$

then the derivative in the direction of Y, $D_Y(T_1(x, y))$, of the function $T_1(x, y) : \mathbb{R} \times W_N(\pi(x)) \to \mathrm{Cliff}(W_N(\pi(x))) \mathbin{\hat{\otimes}} K$ exists and

$$\big\|D_Y\big(T_1(x, y)\big)\big\| \leq c$$

for all $x, y \in X$.

(8) $T(\gamma x, \gamma y) = T(x, y)$ for all $x, y \in X$ and $\gamma \in \Gamma$.

We define a multiplication on $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ by

$$(T_1T_2)(x,y) = \sum_{z\in X_d} T_1(x,z)T_2(z,y)$$

and an involution by $T^*(x,y) = (T(y,x))^*$. Then $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ is a $*$-algebra. Let

$$E = \Big\{ \sum_{x\in X_d} a_x[x] : a_x \in \mathcal{A}(H) \mathbin{\hat{\otimes}} K, \sum_{x\in X_d} a_x^* a_x \text{ converge in norm} \Big\}.$$

Then E is a Hilbert module over $\mathcal{A}(H) \mathbin{\hat{\otimes}} K$ with

$$\Big\langle \sum_{x\in X_d} a_x[x], \sum_{x\in X_d} b_x[x] \Big\rangle = \sum_{x\in X_d} a_x^* b_x,$$
$$\Big(\sum_{x\in X_d} a_x[x] \Big) a = \sum_{x\in X_d} a_x a[x]$$

for all $a \in \mathcal{A}(H) \mathbin{\hat{\otimes}} K$ and $\sum_{x\in X_d} a_x[x] \in E$.

Now we can define a $*$-representation of $C^*_{\mathrm{alg}}(X, \mathcal{A}(H))^\Gamma$ on E by

$$T\Big(\sum_{x\in X_d} a_x[x] \Big) = \sum_{y\in X_d} \Big(\sum_{x\in X_d} T(y,x)a_x \Big)[y]$$

for all $T \in C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ and $\sum_{x\in X_d} a_x[x] \in E$.

Definition 3.4. The *reduced equivariant twisted Roe algebra* $C^*(P_d(X), \mathcal{A}(H))^\Gamma$ is the operator norm closure of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ in $B(E)$, where $B(E)$ is the algebra of all bounded adjointable homomorphisms from E to E.

By the arguments as in Lemma 2.5, the maximal norm on the algebraic equivariant twisted Roe algebra is also well defined. Thus, we can define the maximal equivariant twisted Roe algebra.

Definition 3.5. The *maximal equivariant twisted Roe algebra* $C^*_{\max}(P_d(X), \mathcal{A}(H))^\Gamma$ is defined to be the completion of the $*$-algebra $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ with respect to the maximal norm

$$\|T\|_{\max} := \sup \big\{ \big\|\phi(T)\big\| : \phi : C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to B(E_\phi), \text{ a } *\text{-representation} \big\}.$$

Let $C^*_{L,\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ be the set of all bounded, uniformly norm continuous functions

$$g : \mathbb{R}_+ \to C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

such that the following hold.

(1) There exists N such that

$$g(t)(x,y) \in \big(\beta_N\big(\pi(x)\big) \mathbin{\hat{\otimes}} 1\big)\big(\mathcal{A}\big(W_N\big(\pi(x)\big)\big) \mathbin{\hat{\otimes}} K\big) \subseteq \mathcal{A}(H) \mathbin{\hat{\otimes}} K$$

for all $x, y \in X_d$ and $t \in \mathbb{R}_+$.

(2) $\lim_{t\to\infty} \text{propagation}(f(t)) \to 0$ as $t \to \infty$.

(3) There exists $R > 0$ such that $\text{Supp}(g(t)(x, y)) \subseteq B_{\mathbb{R}_+\times H}(\xi(\pi(x)), R)$ for all $x, y \in X_d$ and $t \in \mathbb{R}_+$.

(4) There exists $L > 0$ such that for any $y \in P_d(X)$,

$$\sharp\{x : g(t)(x, y) \neq 0\} < L, \quad \sharp\{x : g(t)(y, x) \neq 0\} < L$$

for any $t \in \mathbb{R}_+$.

(5) There exists $c > 0$ such that if $Y = (s, h) \in \mathbb{R}_+ \times W_N(\pi(x))$ with $\|Y\| = \sqrt{s^2 + \|h\|^2}$ ≤ 1 and if $g_1(t)(x, y)$ satisfies that

$$(\beta_N(\pi(x)) \hat{\otimes} 1)(g(t)_1(x, y)) = T(x, y),$$

then the derivative of the function

$$g_1(t)(x, y) : \mathbb{R} \times W_N(\pi(x)) \to \text{Cliff}(W_N(\pi(x))) \hat{\otimes} K$$

in the direction of Y exists, denoted by $D_Y(g(t)_1(x, y))$, and

$$\|D_Y(g(t)_1(x, y))\| \leq c$$

for all $x, y \in X$ and $t \in \mathbb{R}_+$.

Definition 3.6. The *twisted equivariant localization algebra* $C_L^*(P_d(X), \mathcal{A}(H))^\Gamma$ is the completion of $C^*_{L,\text{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ with respect to the norm

$$\|g\| = \sup_{t\in\mathbb{R}_+} \|g(t)\|_{C^*(P_d(X),\mathcal{A}(H))^\Gamma}.$$

There is an evaluation map

$$e : C_L^*(P_d(X), \mathcal{A}(H))^\Gamma \to C^*(P_d(X), \mathcal{A}(H))^\Gamma,$$

by

$$e(g) = g(0),$$

for all $g \in C_L^*(P_d(X), \mathcal{A}(H))^\Gamma$. We have a homomorphism

$$e_* : K_*(C_L^*(P_d(X), \mathcal{A}(H))^\Gamma) \to K_*(C^*(P_d(X), \mathcal{A}(H))^\Gamma),$$

induced by the evaluation map on K-theory.

In the rest of this subsection, we will prove the following result.

Proposition 3.7. *Let Γ be a countable discrete group acting properly and isometrically on a discrete metric space X with bounded geometry. Assume that all Γ-orbits in X are uniformly equivariantly coarsely equivalent. If the quotient space X/Γ admits a coarse embedding into Hilbert space and Γ is amenable, then the map*

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(X), \mathcal{A}(H))^\Gamma) \to \lim_{d\to\infty} K_*(C^*(P_d(X), \mathcal{A}(H))^\Gamma)$$

is an isomorphism.

To prove the above proposition, we need to analyze the ideals of the twisted Roe algebras and localization algebras associated to the open subsets of $\mathbb{R}_+ \times H$. Let O be an open subset of $\mathbb{R}_+ \times H$. Define $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_O$ to be a $*$-subalgebra of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ by

$$C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_O = \big\{T \in C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma : \mathrm{Supp}\big(T(x,y)\big) \subseteq O, \forall x, y \in X\big\}.$$

We can define $C^*_\rho(P_d(X), \mathcal{A}(H))^\Gamma_O$ and $C^*(P_d(X), \mathcal{A}(H))^\Gamma_O$ to be the norm closure of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_O$ in $C^*_{\max}(P_d(X), \mathcal{A}(H))^\Gamma$ and $C^*(P_d(X), \mathcal{A}(H))^\Gamma$, respectively. We would like to point out that the norm on $C^*_\rho(P_d(X), \mathcal{A}(H))^\Gamma_O$ may not be the maximal norm of the $*$-algebra $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_O$.

Similarly, let

$$C^*_{L,\mathrm{alg}}\big(P_d(X), \mathcal{A}\big)^\Gamma_O := \big\{g \in C^*_{L,\mathrm{alg}}\big(P_d(X), \mathcal{A}\big)^\Gamma : \mathrm{Supp}\big(g(t)\big) \subseteq X_d \times X_d \times O\big\}.$$

Define $C^*_L(P_d(X), \mathcal{A})^\Gamma_O$ to be the norm closure of $C^*_{L,\mathrm{alg}}(P_d(X), \mathcal{A})^\Gamma_O$ in $C^*_L(P_d(X), \mathcal{A})^\Gamma$.

Let

$$B\big(\xi\big(\pi(x)\big), r\big) = \big\{(s, v) \in \mathbb{R}_+ \times H : s^2 + \big\|v - \xi\big(\pi(x)\big) < r^2\big\|\big\},$$

where $\pi : X \to X/\Gamma$ is the quotient map and $\xi : X/\Gamma \to H$ is the coarse embedding. Denote $O_r = \bigcup_{x \in X} B(\xi(\pi(x)), r)$, then we have

$$C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma = \lim_{r \to +\infty} C^*_\rho\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r},$$

$$C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma = \lim_{r \to +\infty} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r},$$

$$C^*_L(P_d(X), \mathcal{A}(H))^\Gamma = \lim_{r \to +\infty} C^*_L\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}.$$

Recall that $\pi : X \to X/\Gamma$ is the natural quotient map associated to the Γ-action on X. For any $r > 0$, since X is discrete and with bounded geometry, so X/Γ has bounded geometry and there exists finitely many mutually disjoint subsets of X/Γ, say $\widetilde{X}_k$ for $1 \leq k \leq k_0$, such that

(1) $X/\Gamma = \bigsqcup_{k=1}^{k_0} \widetilde{X_k}$,

(2) for each $\widetilde{X}_k$ and for all $z, z' \in \widetilde{X}_k$, if $z \neq z'$, then

$$\big\|\xi(z) - \xi(z')\big\| \geq 2r,$$

where $\xi : X/\Gamma \to H$ is the coarse embedding.

Let $Y \subset X$ be a Γ-domain of X in the sense that each orbit intersects with Y at exactly one point. For simplicity, we denote

$$Y_k := (\pi|_Y)^{-1}(\widetilde{X}_k) \subset Y$$

for $k = 1, 2, \ldots, k_0$, where $\pi|_Y : Y \to X/\Gamma$ is the restriction of $\pi : X \to X/\Gamma$ to the Γ-domain $Y \subset X$. For each $k = 1, 2, \ldots, k_0$ and $r > 0$, denote

$$O_{r,k} = \bigcup_{x \in Y_k} B\big(\xi\big(\pi(x)\big), r\big).$$

Lemma 3.8. *With the same assumptions as Proposition 3.7 and $\{O_{r,k}\}_{r,k}$ defined above, the map*

$$e_* : \lim_{d\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}}\big) \to \lim_{d\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}}\big)$$

induced by the evaluation map is an isomorphism for all $r > 0$ and $1 \le k \le k_0$.

Proof. For each $x \in X$, set $O_x = B(\xi(\pi(x)), r)$ for brevity. Define

$$\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x}$$

to be the $*$-subalgebra of the product algebra $\prod_{x\in Y_k} C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^{\Gamma}_{O_x}$, such that for each element

$$\prod_{x\in Y_k} T_x \in \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x},$$

the conditions in Definition 3.3 are satisfied for all T_x with $x \in X_k$ uniformly.

For each element $a \in \mathcal{A}(H)$ with support contained in $O_{r,k}$, it can be decomposed as a sum

$$a = \sum_{x\in Y_k} a_x,$$

where each a_x is a restriction of a to O_x and can be viewed as a function supported in O_x for all $x \in X$. It is obvious that

$$C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}} = \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x}.$$

We define $\prod_{x\in Y_k}^{u} C^*(P_d(X), \mathcal{A}(H))^{\Gamma}_{O_x}$ to be the norm closure of

$$\Big\{ \prod_{x\in Y_k} T_x : T_x \in C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x}, \ \sup_{x\in X_k} \|T_x\| < \infty \Big\}$$

under the supremum norm.

Similarly, we define $\prod_{x\in Y_k}^{u} C_L^*(P_d(X), \mathcal{A}(H))^{\Gamma}_{O_x}$ to be the C^*-subalgebra of

$$\Big\{ \prod_{x\in Y_k} b_x : \prod_{x\in Y_k} b_x(t) \in \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x}, \ \sup_{x\in Y_k} \|b_x\| < \infty \Big\}$$

generated by elements $\prod_{x\in Y_k} b_x$, such that

(1) the function

$$\prod_{x\in Y_k} b_x : \mathbb{R}_+ \to \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_x}$$

is uniformly norm-continuous in $t \in \mathbb{R}_+$,

(2) there exists a bounded function $c(t)$ on $\mathbb{R}_+$ with $\lim_{t\to\infty} c(t) = 0$ such that $b_x(t)(y, y') = 0$ whenever $d(y, y') > c(t)$ for all $x \in Y_k$, $y, y' \in X_d$, and $t \in \mathbb{R}_+$.

By the definition of the algebraic equivariant twisted Roe algebra, we can prove that

$$C^*\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{Or,k} \cong \lim_{S\to\infty} \prod_{x\in Y_k}^{u} C^*\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x},$$

$$C^*_L\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}} \cong \lim_{S\to\infty} \prod_{x\in Y_k}^{u} C^*_L\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x},$$

where $N_S(\Gamma\cdot x)$ is the S-neighborhood of $\Gamma\cdot x$ in X. So

$$\begin{aligned}
\lim_{d\to\infty} C^*\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{Or,k} &\cong \lim_{d\to\infty}\lim_{S\to\infty} \prod_{x\in Y_k}^{u} C^*\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x} \\
&\cong \lim_{S\to\infty}\lim_{d\to\infty} \prod_{x\in Y_k}^{u} C^*\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x}, \\
\lim_{d\to\infty} C^*_L\big(P_d(X), \mathcal{A}(H)\big)^{\Gamma}_{Or,k} &\cong \lim_{d\to\infty}\lim_{S\to\infty} \prod_{x\in Y_k}^{u} C^*_L\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x} \\
&\cong \lim_{S\to\infty}\lim_{d\to\infty} \prod_{x\in Y_k}^{u} C^*_L\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x}.
\end{aligned}$$

It suffices to show that

$$\begin{aligned}
e_* : \lim_{d\to\infty} K_*\Big(\prod_{x\in Y_k}^{u} C^*_L\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x}\Big) \\
\to \lim_{d\to\infty} K_*\Big(\prod_{x\in Y_k}^{u} C^*\big(P_d\big(N_S(\Gamma\cdot x)\big), \mathcal{A}(H)\big)^{\Gamma}_{O_x}\Big)
\end{aligned}$$

induced by the evaluation map on K-theory is an isomorphism for each $S > 0$. It is easy to know that the assumptions of this lemma satisfy the setting of [11, Theorem 4.9]. With the same arguments as in the proof of [11, Theorem 4.9], together with the geometric Dirac-dual-Dirac method, we know that e_* above is an isomorphism. ∎

To prove Proposition 3.7, we also need the following lemma, which can be proved by the same arguments in [35, Lemma 6.3].

Lemma 3.9. *Let X_1 and X_2 be Γ-invariant subsets of X, and let*

$$O_r^1 = \bigcup_{x\in X_1} B_{\mathbb{R}_+\times H}\big(\xi(\pi(x)), r\big), \quad O_r^2 = \bigcup_{x\in X_2} B_{\mathbb{R}_+\times H}\big(\xi(\pi(x)), r\big).$$

Then we have that

$$\begin{aligned}
&\lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1} + \lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^2} \\
&\quad = \lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1\cup O_r^2}, \\
&\lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1} + \lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^2} \\
&\quad = \lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1\cup O_r^2}, \\
&\lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1} \cap \lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^2} \\
&\quad = \lim_{r<r_0, r\to r_0} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1\cap O_r^2}, \\
&\lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1} \cap \lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^2} \\
&\quad = \lim_{r<r_0, r\to r_0} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r^1\cap O_r^2}.
\end{aligned}$$

Proof of Proposition 3.7. Since

$$\begin{aligned}
C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma &= \lim_{r\to+\infty} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}, \\
C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma &= \lim_{r\to+\infty} C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r},
\end{aligned}$$

where $O_r = \bigcup_{1\le k\le k_0} O_{r,k}$, following the similar arguments in [29] we obtain that

$$\begin{aligned}
\lim_{d\to\infty}\lim_{r\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big) &\cong \lim_{r\to\infty}\lim_{d\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big), \\
\lim_{d\to\infty}\lim_{r\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big) &\cong \lim_{r\to\infty}\lim_{d\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big).
\end{aligned}$$

We get the commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) \\
\cong\downarrow & & \cong\downarrow \\
\lim\limits_{d\to\infty}\lim\limits_{r\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty}\lim\limits_{r\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big) \\
\cong\downarrow & & \cong\downarrow \\
\lim\limits_{r\to\infty}\lim\limits_{d\to\infty} K_*\big(C_L^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big) & \xrightarrow{e_*} & \lim\limits_{r\to\infty}\lim\limits_{d\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}\big).
\end{array}$$

The conclusion follows that the bottom horizon map is an isomorphism by using Lemma 3.8, Lemma 3.9, and the Mayer–Vietoris sequence argument. ∎

3.2. The maximal and reduced equivariant twisted Roe algebras

In this subsection, we will prove that the canonical quotient map

$$\lambda : C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

is an isomorphism.

Recall that in Section 3.1 we have defined a $*$-subalgebra $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_O$ of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma$ consisting of the operators with entries supported in O for each open subset O in $\mathbb{R}_+ \times H$, and the C^*-subalgebras $C^*_\rho(P_d(X), \mathcal{A}(H))^\Gamma_O$ and $C^*(P_d(X), \mathcal{A}(H))^\Gamma_O$ are the operator norm closure of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_O$ in $C^*_{\max}(P_d(X), \mathcal{A}(H))^\Gamma$ and $C^*(P_d(X), \mathcal{A}(H))^\Gamma$, respectively. Recall that

$$O_{r,k} = \bigcup_{x\in Y_k} B\big(\xi(\pi(x)), r\big) \subset \mathbb{R}_+ \times H.$$

Lemma 3.10. *Let X be a discrete metric space with bounded geometry, and let Γ be a countable discrete amenable group. Assume that Γ acts on X properly and isometrically such that the Γ-orbits are uniformly equivariantly coarsely equivalent. If the quotient space X/Γ admits a coarse embedding into Hilbert space, then the canonical quotient map*

$$\lambda : C^*_\rho\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_{r,k}} \to C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_{r,k}}$$

extended from the identity map on $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_{O_{r,k}}$ is an isomorphism.

Proof. For brevity, we denote by O the open subset $O_{r,k}$ and denote $O_x = B(\xi(\pi(x)), r)$. Let $\prod^u_{x\in Y_k} C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_{O_x}$ be the $*$-algebra consisting of the operators

$$\prod_{x\in Y_k} T_x \in \prod_{x\in Y_k}^u C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_x},$$

such that the family of operators $\{T_x\}_{x\in Y_k}$ satisfies conditions in Definition 3.3 with uniform constants. Since $O = \bigsqcup_{x\in Y_k} O_x$, we have

$$C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_O = \prod_{x\in Y_k}^u C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_x}.$$

By the definition of each $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_{O_x}$, we have

$$\prod_{x\in Y_k}^u C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_x} = \lim_{R\to\infty} \prod_{x\in Y_k}^u C^*_{\mathrm{alg}}\big(N_R(\Gamma\cdot x), \mathcal{A}(H)\big)^\Gamma_{O_x},$$

where the inductive limit can be viewed as the union of these $*$-algebras. If we use

$$\prod_{x\in Y_k}^{u,\rho} C^*\big(N_R(\Gamma\cdot x), \mathcal{A}(H)\big)^\Gamma_{O_x} \quad \text{and} \quad \prod_{x\in Y_k}^u C^*\big(N_R(\Gamma\cdot x), \mathcal{A}(H)\big)^\Gamma_{O_x}$$

to denote the norm closure of $\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}(N_R(\Gamma\cdot x),\mathcal{A}(H))^{\Gamma}_{O_x}$ in $C^*_{\rho}(P_d(X),\mathcal{A}(H))^{\Gamma}_{O_{r,k}}$ and $C^*(P_d(X),\mathcal{A}(H))^{\Gamma}_{O_{r,k}}$, respectively, then

$$C^*_{\rho}\big(P_d(X),\mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}} = \lim_{R\to\infty}\prod_{x\in Y_k}^{u,\rho} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x},$$

$$C^*\big(P_d(X),\mathcal{A}(H)\big)^{\Gamma}_{O_{r,k}} = \lim_{R\to\infty}\prod_{x\in Y_k}^{u} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x}.$$

So to prove the conclusion, we only need to show that for any $R>0$,

$$\prod_{x\in Y_k}^{u,\rho} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x} \cong \prod_{x\in Y_k}^{u} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x}.$$

Denote by $|\Gamma|$ the metric space of Γ endowed with a proper word-length metric. Let $\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}(|\Gamma|,\mathcal{A}(H))^{\Gamma}_{O_x}$ be the C^*-algebra of all the operators

$$\prod_{x\in Y_k} T_x \in \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x}$$

satisfying that

(1) each T_x is a $\Gamma\times\Gamma$-matrix and $T_x(g,h)=T_x(\gamma g,\gamma h)$ for $g,h,\gamma\in\Gamma$,

(2) there exists a constant $M>0$, such that $\sup_{x\in Y_k,g,h\in\Gamma}\|T_x(g,h)\|\le M$,

(3) $\sup_{x\in Y}\mathrm{propagation}(T_x)<\infty$,

(4) there exists $r>0$ such that

$$\mathrm{Supp}\big(T_x(g,h)\big)\subseteq\big\{(s,h)\in\mathbb{R}_+\times H : s^2+\big\|h-\xi\big(\pi(x)\big)\big\|^2<r^2\big\}$$

for $g,h\in\Gamma$,

(5) there exists $c>0$ such that if $Y=(s,v)\in\mathbb{R}_+\times W_N(\pi(x))$ with $\|Y\|=\sqrt{s^2+\|v\|^2}\le 1$ and if $T'_x(g,h)$ is an element satisfying

$$\big(\beta_N\big(\pi(x_x)\big)\,\hat{\otimes}\,1\big)\big(T'_x(g,h)\big)=T(g,h),$$

then the derivative of the function

$$T'_x(g,h):\mathbb{R}\times W_N\big(\pi(x)\big)\to \mathrm{Cliff}\big(W_N\big(\pi(x)\big)\big)\,\hat{\otimes}\,K$$

in the direction of Y exists, denoted by $D_Y(T'_x(g,h))$, and $\|D_Y(T'_x(g,h))\|\le c$ for all $g,h\in\Gamma$.

Since the Γ-orbits are uniformly equivariantly coarsely equivalent, so all the metric spaces $N_R(\Gamma\cdot x)$ with $x\in Y_k$ are uniformly equivariantly coarsely equivalent to $|\Gamma|$. It

follows that

$$\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x} \cong \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x}.$$

Let $\prod^{u,\rho}_{x\in Y_k} C^*(|\Gamma|,\mathcal{A}(H))^{\Gamma}_{O_x}$ and $\prod^{u}_{x\in Y_k} C^*(|\Gamma|,\mathcal{A}(H))^{\Gamma}_{O_x}$ be the norm closure of

$$\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x}$$

as subalgebras in $C^*_{\rho}(P_d(X),\mathcal{A}(H))_O$ and $C^*(P_d(X),\mathcal{A}(H))_O$, respectively. Then we have

$$\prod_{x\in Y_k}^{u,\rho} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x} \cong \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x},$$
$$\prod_{x\in Y_k}^{u} C^*\big(N_R(\Gamma\cdot x),\mathcal{A}(H)\big)^{\Gamma}_{O_x} \cong \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x}.$$

Since Γ is amenable, there exists a sequence of finitely supported positive-type functions $\phi_n : \Gamma \to \mathbb{C}$ such that ϕ_n converge pointwise to 1 (cf. [4, Theorem 2.6.8]). Denote

$$k_n(g,h) = \phi_n(g^{-1}h)$$

for all $g,h\in\Gamma$. It is obvious that each k_n is a positive definite kernel $k_n : \Gamma\times\Gamma\to[0,1]$ such that

$$k_n(\gamma g,\gamma h) = k_n(g,h),$$

for all $\gamma,g,h\in\Gamma$. Therefore, one can define a sequence of linear maps

$$M^{\mathrm{alg}}_{k_n} : \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x} \to \prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}\big(|\Gamma|,\mathcal{A}(H)\big)^{\Gamma}_{O_x}$$

by

$$M^{\mathrm{alg}}_{k_n}\big(\{T_x\}\big)(g,h) = \big\{k_n(g,h)T_x(g,h)\big\}$$

for all $T=\{T_x\}\in\prod^{u}_{x\in Y_k} C^*_{\mathrm{alg}}(|\Gamma|,\mathcal{A}(H))^{\Gamma}_{O_x}$ and $g,h\in\Gamma$. It is not difficult to check that $M^{\mathrm{alg}}_{k_n}$ is a unital completely map for each k_n above. We claim that, for each k_n, the multiplier $M^{\mathrm{alg}}_{k_n}$ is bounded for any norm $\|\cdot\|_\sim$ on $\prod^{u}_{x\in Y_k} C^*_{\mathrm{alg}}(|\Gamma|,\mathcal{A}(H))^{\Gamma}_{O_x}$.

In fact, for each n, ϕ_n is finitely supported, so we can assume that k_n has finite propagation no more than R. Then $M^{\mathrm{alg}}_{k_n}(T)$ has finite propagation no more than R too. With the same proof as Lemma 2.5, there exists a constant $C_{k_n}>0$ such that

$$\big\|M^{\mathrm{alg}}_{k_n}(T)\big\|_\sim \le C_{k_n}\cdot\sup_{x,g,h}\big\|T_x(g,h)\big\|,$$

where the norm of $T_x(g,h)$ is unique since each coefficient $T_x(g,h)$ is in the nuclear C^*-algebra $\mathcal{A}(H)_{O_x} \hat{\otimes} K(H)$. Let $\|\cdot\|_{\mathrm{red}}$ be the norm by representing $\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}(|\Gamma|, \mathcal{A}(H))_{O_x}^{\Gamma}$ on the Hilbert module $\bigoplus_{x\in Y_k}(l^2(\Gamma) \hat{\otimes} \mathcal{A}(H)_{O_x} \hat{\otimes} K(H))$, then we have

$$\left\| M_{k_n}^{\mathrm{alg}}(T) \right\|_{\sim} \leq C_{k_n} \cdot \sup_{x,g,h} \left\| T_x(g,h) \right\| \leq C_{k_n} \cdot \|T\|_{\mathrm{red}} \leq C_{k_n} \cdot \|T\|_{\sim}.$$

So we know that each operator $M_{k_n}^{\mathrm{alg}}$ can be continuously extended to unital completely positive maps, denoted by

$$M_{k_n}^{\rho} : \prod_{x\in Y_k}^{u,\rho} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} \to \prod_{x\in Y_k}^{u,\rho} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma},$$

$$M_{k_n} : \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} \to \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma}.$$

By the definition of k_n, we know that $M_{k_n}^{\rho}$ and M_{k_n} converge in point-norm to identity maps on $\prod_{x\in Y_k}^{u,\rho} C^*(|\Gamma|, \mathcal{A}(H))_{O_x}^{\Gamma}$ and $\prod_{x\in Y_k}^{u} C^*(|\Gamma|, \mathcal{A}(H))_{O_x}^{\Gamma}$, respectively.

Now we have the commutative diagram

$$\begin{array}{ccc} \prod_{x\in Y_k}^{u,\rho} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} & \xrightarrow{M_{k_n}^{\rho}} & \prod_{x\in Y_k}^{u,\rho} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} \\ \Big\downarrow{\scriptstyle \lambda} & & \\ \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} & \xrightarrow{M_{k_n}} & \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma}. \end{array}$$

For each operator $T \in \mathrm{Ker}(\lambda)$, we have

$$\lambda\big(M_{k_n}^{\rho}(T)\big) = M_{k_n}\big(\lambda(T)\big) = 0.$$

Since $M_{k_n}^{\rho}(T) \in C^*_{\mathrm{alg}}(|\Gamma|, \prod_{x\in Y_k} \mathcal{A}(H))_{O_x}^{\Gamma}$ and the restriction of λ on the $*$-subalgebra $\prod_{x\in Y_k}^{u} C^*_{\mathrm{alg}}(|\Gamma|, \mathcal{A}(H))_{O_x}^{\Gamma}$ is the identity map, we have that $M_{k_n}^{\rho}(T) = 0$. It follows that

$$T = \lim_{n\to\infty} M_{k_n}^{\rho}(T) = 0.$$

Therefore, λ is injective. This means that

$$\prod_{x\in Y_k}^{u,\rho} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma} \cong \prod_{x\in Y_k}^{u} C^*\big(|\Gamma|, \mathcal{A}(H)\big)_{O_x}^{\Gamma},$$

and hence

$$\prod_{x\in Y_k}^{u,\rho} C^*\big(N_R(\Gamma\cdot x), \mathcal{A}(H)\big)_{O_x}^{\Gamma} \cong \prod_{x\in Y_k}^{u} C^*\big(N_R(\Gamma\cdot x), \mathcal{A}(H)\big)_{O_x}^{\Gamma}. \qquad \blacksquare$$

We will glue all the isomorphisms between ideals together using the following result.

Lemma 3.11. *Let A, A' be C^*-algebras and $\varphi : A \to A'$ a $*$-homomorphism. Assume that $A = I + J$, $A' = I' + J'$, where I, J are 2-sided ideals of A, and I', J' are 2-sided ideals of A'. If the restriction maps $\varphi_I : I \to I'$, $\varphi_J : J \to J'$ and $\varphi_{I\cap J} : I \cap J \to I' \cap J'$ are all isomorphisms, then $\varphi : A \to A'$ is an isomorphism.*

Proof. Since $A/I = I + J/I = J/I \cap J$, so the quotient map $\varphi_{A/I} : A/I \to A'/I'$ is an isomorphism. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_I} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_{A/I}} & & \\ 0 & \longrightarrow & I' & \longrightarrow & A' & \longrightarrow & A'/I' & \longrightarrow & 0. \end{array}$$

As a consequence of diagram-chasing, we obtain that φ is an isomorphism. ∎

Proposition 3.12. *Let X be a discrete metric space with bounded geometry and let Γ be a countable discrete group. Assume Γ acts on X properly and isometrically with the Γ-orbits uniformly equivariantly coarsely equivalent. If the quotient space X/Γ is coarsely embeddable into Hilbert space and Γ is amenable, then the homomorphism*

$$\lambda : C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

is an isomorphism.

Proof. Denote

$$O_r = \bigcup_{\pi(x)\in X/\Gamma} B\big(\pi(x), r\big).$$

Note that $C^*_\rho(P_d(X), \mathcal{A}(H))^\Gamma_{O_r}$ and $C^*(P_d(X), \mathcal{A}(H))^\Gamma_{O_r}$ are 2-sided ideals of

$$C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \quad \text{and} \quad C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma,$$

respectively. Moreover, we have that

$$C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma = \lim_{r\to+\infty} C^*_\rho\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r},$$
$$C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma = \lim_{r\to+\infty} C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r}.$$

For each $r > 0$, $O_r = \bigcup_{k=1}^{k_0} O_{r,k}$. By Lemmas 3.9, 3.10, and 3.11, we have that the canonical quotient

$$\lambda : C^*_\rho\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r} \to C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma_{O_r},$$

extended from the identity map on $C^*_{\mathrm{alg}}(P_d(X), \mathcal{A}(H))^\Gamma_{O_r}$, is an isomorphism. As a result, the homomorphism

$$\lambda : C^*_\rho\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C^*(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

is an isomorphism. ∎

4. Proof of the main results

In this section, we will show that the maps

$$e_* : \lim_{d\to\infty} K_*\big(C_L\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*(X)^\Gamma\big),$$
$$\lambda_* : K_*\big(C^*_{\max}(X)^\Gamma\big) \to K_*\big(C^*(X)^\Gamma\big)$$

are isomorphisms under the assumptions in Theorem 1.1. The tool to prove these two results is the geometric Dirac-dual-Dirac method constructed by Yu in [35]. The difference here is that we use the coarse embedding of the quotient space X/Γ to construct the Bott map (Definition 4.5) and the Dirac map (Definition 4.4) for the (maximal) Roe algebra of the metric space X. In this setting, the Bott map and the Dirac map are still asymptotic morphisms by the very similar proof. For the completeness, we just list some necessary definitions and conclusions. More details can be found in [35] or [10].

Recall that H is a separable infinite-dimensional real Hilbert space and that $V \subseteq H$ is a finite-dimensional affine subspace of H. Use V^0 to denote the finite-dimensional linear subspace consisting of differences of elements in V. Let $\mathcal{L}^2(V) = L^2(V, \mathrm{Cliff}(V^0))$ be the graded infinite-dimensional Hilbert space of L^2-maps from V to the complexified Clifford algebra of V^0. Let $V_a \subseteq V_b$ be finite-dimensional affine subspaces of H. There is an algebraic decomposition

$$V_b = V^0_{ba} \oplus V_a,$$

where V^0_{ba} is the orthogonal complement of V^0_a in V^0_b. Take a unit vector $\xi_0 \in \mathcal{L}^2(V^0_{ba})$:

$$\xi_0(w) = \pi^{-\frac{\dim(V_{ba})}{4}} \exp\Big(-\frac{1}{2}\|w\|^2\Big),$$

for all $w \in V^0_{ba}$. Then we can regard $\mathcal{L}^2(V_a)$ as a subspace of $\mathcal{L}^2(V_b)$ via an isometric inclusion

$$i_{ba} : \mathcal{L}^2(V_a) \to \mathcal{L}^2(V^0_{ba}) \hat{\otimes} \mathcal{L}^2(V_a) \cong \mathcal{L}^2(V_b), \quad \xi \mapsto \xi_0 \hat{\otimes} \xi. \tag{4.1}$$

If $V_a \subseteq V_b \subseteq V_c$, we have

$$i_{ca} = i_{cb} \circ i_{ba}.$$

So we obtain an inductive limit

$$\mathcal{L}^2(H) = \varinjlim \mathcal{L}^2(V),$$

over all the finite-dimensional affine subspaces $V \subseteq H$ together with the maps $\{i_{ba}\}$.

Let $\mathcal{S}(V) \subseteq \mathcal{L}^2(H)$ be the subspace of Schwartz class functions from V to $\mathrm{Cliff}(V^0)$. Choose an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ for V^0, and let $\{x_1, x_2, \ldots, x_n\}$ be the dual coordinates to $\{e_1, e_2, \ldots, e_n\}$. We define the Dirac operator D_V, an unbounded operator on $\mathcal{L}^2(H)$ with domain $\mathcal{S}(V)$, by the formula

$$D_V\xi = \sum_{i=1}^{n} (-1)^{\deg\xi} \frac{\partial\xi}{\partial x_i} e_i,$$

where e_i is the Clifford multiplication by $e_i \in V^0 \subseteq \mathrm{Cliff}(V^0)$. Define the Clifford operator $C_{V,v}$ with domain $\mathcal{S}(V)$, by

$$(C_{V,v}\xi)(w) = (w - v) \cdot \xi(w),$$

where $v \in V$ is a fixed base point, and the multiplication is the Clifford multiplication by the vector $w - v \in V^0$.

Now fix $x \in P_d(X)$. Let $W_k(\pi(x))$ be the finite-dimensional subspace as in Definition 3.2, and identify $\mathcal{L}^2(W_k(\pi(x)))$ as a subspace of $\mathcal{L}^2(W_{k+1}(\pi(x)))$ via the isometric inclusion

$$\mathcal{L}^2\big(W_k\big(\pi(x)\big)\big) \to \mathcal{L}^2\big(W_{k+1}\big(\pi(x)\big)\big)$$

defined in (4.1). Note that these inclusions preserve the Schwartz subspaces $\mathcal{S}(W_k(\pi(x)))$. Define a Schwartz subspace of $\mathcal{L}^2(H)$ by taking the algebraic direct limit

$$\mathcal{S}\big(\pi(x)\big) = \varinjlim \mathcal{S}\big(W_k\big(\pi(x)\big)\big).$$

Let $V_0(\pi(x)) = W_1(\pi(x))$ and $V_k(\pi(x)) = W_{k+1}(\pi(x)) \ominus W_k(\pi(x))$ if $k \geq 1$, where $x \in P_d(X)$. We consider the Dirac operator D_k defined by

$$D_k = D_{V_k(\pi(x))}$$

and Clifford operators $C_{k,\pi(x)}$ defined by

$$C_{0,\pi(x)} = C_{V_0(\pi(x)),f(\pi(x))}; \quad C_{k,\pi(x)} = C_{V_k(\pi(x)),0}, \quad k \geq 1,$$

associated to each $V_k(\pi(x))$. For each $n \in \mathbb{N}$ and $t \geq 1$, define an unbounded operator $B_{n,t}(\pi(x))$ on $\mathcal{L}^2(H)$ by

$$B_{n,t}\big(\pi(x)\big) = \sum_{k=0}^{n-1}(1 + kt^{-1})D_k + \sum_{k=n}^{\infty}(1 + kt^{-1})(D_k + C_{k,\pi(x)})$$

associated to the decomposition

$$V_0\big(\pi(x)\big) \oplus V_1\big(\pi(x)\big) \oplus \cdots \oplus V_n\big(\pi(x)\big) \oplus \cdots.$$

Note that the operator $B_{n,t}(\pi(x))$ is well defined on the Schwartz space $\mathcal{S}(\pi(x))$.

Let $\mathcal{K}$ be the algebra of compact operators on $\mathcal{H} = \mathcal{L}^2(H)$. We can define the $*$-algebra $C^*_{\mathrm{alg}}(X, \mathcal{K})^\Gamma$ similarly to $C^*(X)^\Gamma$ by changing the coefficients of $T \in C^*_{\mathrm{alg}}(X)^\Gamma$ from K to $\mathcal{K} \mathbin{\widehat{\otimes}} K$. Let $C^*_{\max}(X, \mathcal{K})^\Gamma$ and $C^*(X, \mathcal{K})^\Gamma$ be the maximal and reduced Roe algebras which are the completions of $C^*_{\mathrm{alg}}(X, \mathcal{K})^\Gamma$ under the maximal and reduced norms, respectively.

Recall that in Section 2 we choose a Γ-invariant countable dense subset $X_d \subset P_d(X)$ for each $d > 0$.

Definition 4.1. Let $C^*_{\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})^\Gamma$ be the set of all functions T from $X_d \times X_d$ to $\mathcal{S}\,\hat{\otimes}\,\mathcal{K}\,\hat{\otimes}\,K$ such that

(1) there exists $M > 0$ such that $\|T(x,y)\| \le M$ for any $(x,y) \in X_d \times X_d$,

(2) there exists $r_1 > 0$ such that $T(x,y) = 0$ if $d(x,y) > r_1$,

(3) there exists $L > 0$ such that for each $y \in X_d$,

$$\sharp\{x : T(x,y) \neq 0\} \le L, \quad \sharp\{x : T(y,x) \neq 0\} \le L,$$

(4) for any bounded set $B \subseteq P_d(X)$, the set $\{(x,y) \in B \times B \cap X_d \times X_d : T(x,y) \neq 0\}$ is finite,

(5) there exists $r_2 > 0$ such that if $|t| > r_2$, then $T(x,y)(t) = 0$ for all $(x,y) \in X \times X$, where $T(x,y) \in S\,\hat{\otimes}\,\mathcal{K}\,\hat{\otimes}\,K$ can be viewed as a $\mathcal{K}\,\hat{\otimes}\,K$-valued function on $\mathbb{R}$,

(6) there exists $c > 0$ such that $\|\frac{d}{dt}T(x,y)\| \le c$ for all $(x,y) \in X \times X$;

(7) $\gamma(T) = T$; i.e. $T(\gamma^{-1}x, \gamma^{-1}y) = T(x,y)$.

Following Definition 3.3, we can view $C^*_{\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})^\Gamma$ as a $*$-algebra acting on the graded Hilbert module

$$\mathcal{E} = \ell^2(X_d)\,\hat{\otimes}\,\ell^2(\Gamma)\,\hat{\otimes}\,\mathcal{S}\,\hat{\otimes}\,\mathcal{K}\,\hat{\otimes}\,K$$

over $\mathcal{S}\,\hat{\otimes}\,\mathcal{K}\,\hat{\otimes}\,K$, where the $\mathbb{Z}_2$-grading on $\ell^2(X_d)$, $\ell^2(\Gamma)$, and K is trivial and $\mathcal{S}$ is graded by the odd and even functions.

Definition 4.2. (1) The C^*-algebra $C^*(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})^\Gamma$ is defined to be the operator norm closure of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})^\Gamma$ in $B(\mathcal{E})$, where $B(\mathcal{E})$ is the C^*-algebra of all adjointable module homomorphisms from $\mathcal{E}$ to $\mathcal{E}$.

(2) The C^*-algebra $C^*_{\max}(P_d(X), \mathcal{S}\,\hat{\otimes}\mathcal{K})^\Gamma$ is the completion of $C^*_{\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\mathcal{K})^\Gamma$ with respect to the C^*-norm

$$\|T\|_{\max} := \sup\left\{\left\|\phi(T)\right\| : \phi : C^*_{\mathrm{alg}}\left(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K}\right)^\Gamma \to B(\mathcal{E}_\phi),\ \text{a } *\text{-representation}\right\},$$

where $\mathcal{E}_\phi$ is a Hilbert module over the C^*-algebra $\mathcal{S}\,\hat{\otimes}\,\mathcal{K}\,\hat{\otimes}\,K$.

Naturally, we have the canonical quotient map

$$\lambda : C^*_{\max}\left(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K}\right) \to C^*\left(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K}\right)$$

extended from the identity map on the $*$-algebra $C^*_{\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})$.

We can similarly define the localization algebra, denoted by $C^*_{L,\mathrm{alg}}(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K})^\Gamma$, to be the $*$-algebra of all uniformly bounded and uniformly continuous functions

$$f : [0,\infty) \to C^*_{\mathrm{alg}}\left(P_d(X), \mathcal{S}\,\hat{\otimes}\,\mathcal{K}\right)^\Gamma$$

such that

(1) propagation$(f(t)) \to 0$ as $t \to \infty$,

(2) the constants in conditions (3), (4), (5), and (6) of Definition 4.1 are independent on $t \in [0, \infty)$.

Definition 4.3. The localization algebra, denoted by $C_L^*(P_d(X), \mathcal{K})^\Gamma$, is the completion of the $*$-algebra $C_{\mathrm{alg}}^*(P_d(X), \mathcal{K})^\Gamma$ under the norm

$$\|f\| = \sup_{t \geq 0} \|f(t)\|.$$

From the definition of $C_{\mathrm{alg}}^*(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K})^\Gamma$ and the nuclearity of $\mathcal{S}$, we have

$$\begin{aligned} C^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma &\cong \mathcal{S} \hat{\otimes} C^*\big(P_d(X), \mathcal{K}\big)^\Gamma, \\ C_L^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma &\cong \mathcal{S} \hat{\otimes} C_L^*\big(P_d(X), \mathcal{K}\big)^\Gamma, \\ C_{\max}^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma &\cong \mathcal{S} \hat{\otimes} C_{\max}^*\big(P_d(X), \mathcal{K}\big)^\Gamma. \end{aligned}$$

For every non-negative integer n and $x \in X$, we define

$$\big(\theta_t^n(x)\big) : \mathcal{A}\big(W_n\big(\pi(x)\big)\big) \hat{\otimes} K \to \mathcal{S} \hat{\otimes} K\big(\mathcal{L}^2(H)\big) \hat{\otimes} K$$

by

$$\big(\theta_t^n(x)\big)\big(g \hat{\otimes} h \hat{\otimes} k\big) = g_t\big(X \hat{\otimes} 1 + 1 \hat{\otimes} B_{n,t}\big(\pi(x)\big)\big)\big(1 \hat{\otimes} M_{h_t}\big) \hat{\otimes} k$$

for all $g \in \mathcal{S}$, $h \in C_0(W_n(\pi(x)), \mathrm{Cliff}(W_n(\pi(x))))$, $k \in K$, $x \in X_d$, where $g_t(s) = g(t^{-1}s)$ for $s \in \mathbb{R}$, $h_t(v) = h(\xi(\pi(x)) + t^{-1}(v - \xi(\pi(x))))$ for $v \in W_n(\pi(x))$, M_{h_t} is the pointwise multiplication operator on $\mathcal{L}^2(V)$ (where $W_n(\pi(x)) \subset V$ is an affine subspace of H) via the formula

$$(M_{h_t}\xi)(v + w) = h_t(v)\xi(v + w)$$

for all $\xi \in \mathcal{L}^2(V)$, $v \in W_n(\pi(x))$, and $w \in V \ominus W_n(\pi(x))$. For every $T \in C_{\mathrm{alg}}^*(X, \mathcal{A}(H))^\Gamma$, let n be a non-negative integer such that for every $(x, y) \in X_d \times X_d$, there exists $T_1(x, y) \in \mathcal{A}(W_n(\pi(x))) \hat{\otimes} K$ satisfying $\beta_n(x)(T_1(x, y)) = T(x, y)$. We define

$$\big(\alpha_t(T)\big)(x, y) = \big(\theta_t^n(x)\big)\big(T_1(x, y)\big)$$

for every $T \in C_{\mathrm{alg}}^*(P_d(X), \mathcal{A}(H))^\Gamma$. By [18, Lemma 5.8], we know that $(\alpha_t(T))(x, y) \in K(\mathcal{L}^2(H)) \hat{\otimes} K$.

Definition 4.4. For each $t \geq 1$, we define

$$\alpha_t : C_{\mathrm{alg}}^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C_{\mathrm{alg}}^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma$$

via the formula

$$\big(\alpha_t(T)\big)(x, y) = \big(\theta_t^n(x)\big)\big(T_1(x, y)\big)$$

for every $T \in C_{\mathrm{alg}}^*(P_d(X), \mathcal{A}(H))^\Gamma$, where n is a non-negative integer such that for every $(x, y) \in X_d \times X_d$, there exists $T_1(x, y) \in \mathcal{A}(W_n(\pi(x))) \hat{\otimes} K$ satisfying

$$\beta_n\big(\pi(x)\big)\big(T_1(x, y)\big) = T(x, y).$$

By the same arguments as in [35, Lemma 7.2] and Proposition 3.12, we have the asymptotic morphisms

$$\alpha_{\max,t} : C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C^*_{\max}\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma \cong \mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X), \mathcal{K}\big)^\Gamma,$$
$$\alpha_t : C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to C^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma \cong \mathcal{S} \hat{\otimes} C^*\big(P_d(X), \mathcal{K}\big)^\Gamma$$

for all $t \geq 1$. Then we have the induced homomorphisms

$$\alpha_{\max,*} : K_*\big(C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) \to K_*\big(\mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X), \mathcal{K}\big)^\Gamma\big),$$
$$\alpha_* : K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) \to K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X), \mathcal{K}\big)^\Gamma\big)$$

on K-theory.

Moreover, we can define an asymptotic morphism

$$\alpha_{L,t} : C^*_L\big(P_d(X), \mathcal{A}(H)\big)^\Gamma \to \mathcal{S} \hat{\otimes} C^*_L\big(P_d(X), \mathcal{K}\big)^\Gamma$$

by

$$\alpha_{L,t}(T_s)(t) = \alpha_t(T_s)$$

for all path $(T_s)_{s\in[0,\infty)} \in C^*_L(P_d(X), \mathcal{A}(H))^\Gamma$ and all $t \geq 1$. In addition, we have the homomorphism

$$\alpha_{L,*} : K_*\big(C^*_L\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) \to K_*\big(\mathcal{S} \hat{\otimes} C^*_L\big(P_d(X), \mathcal{K}\big)^\Gamma\big)$$

induced by the asymptotic morphism $(\beta_{L,t})_{t\in[0,\infty)}$.

Definition 4.5. For each $t \geq 1$, define a map

$$\beta_t : \mathcal{S} \hat{\otimes} C^*_{\mathrm{alg}}\big(P_d(X)\big)^\Gamma \to C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

by

$$\big(\beta_t(g \hat{\otimes} T)\big)(x, y) = \big(\beta\big(\pi(x)\big)\big)(g_t) \hat{\otimes} T(x, y)$$

for all $g \in \mathcal{S}$, $T \in C^*_{\mathrm{alg}}(X)^\Gamma$, and $x, y \in X$, where $g_t(s) = g(t^{-1}s)$ for all $s \in \mathbb{R}$, and $\beta(\pi(x)) : \mathcal{S} = \mathcal{A}(f(\pi(x))) \to \mathcal{A}(H)$ is the $*$-homomorphism defined in Section 3.

Following the arguments in [35, Lemma 7.6], we know that β_t extends to asymptotic morphisms

$$\beta_{\max,t} : \mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X)\big)^\Gamma \to C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma,$$
$$\beta_t : \mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma \to C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

for all $t \geq 0$. Then we have the induced homomorphisms

$$\beta_{\max,*} : K_*\big(\mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big),$$
$$\beta_* : K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big)$$

on K-theory.

Applying the map

$$\beta_t : \mathcal{S} \hat{\otimes} C^*_{\mathrm{alg}}\big(P_d(X)\big)^\Gamma \to C^*_{\mathrm{alg}}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma$$

point-wise gives rise to an asymptotic morphism

$$\beta_{L,t} : \mathcal{S} \hat{\otimes} C^*_L\big(P_d(X)\big)^\Gamma \to C^*_L\big(P_d(X), \mathcal{A}\big)^\Gamma$$

for all $t \geq 1$. Then we have a homomorphism

$$\beta_{L,*} : K_*\big(\mathcal{S} \hat{\otimes} C^*_L\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*_L\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big)$$

induced by the asymptotic morphism $(\beta_{L,t})_{t\in[0,\infty)}$ on K-theory. The proof of the following proposition is very similar to [35, Proposition 7.7]; we omit the proof here.

Proposition 4.6. *The compositions*

$$\begin{aligned}
\alpha_* \circ \beta_* &: K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma\big),\\
\alpha_{\max,*} \circ \beta_{\max,*} &: K_*\big(\mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*_{\max}\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma\big),\\
\alpha_{L,*} \circ \beta_{L,*} &: K_*\big(\mathcal{S} \hat{\otimes} C^*_L\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*_L\big(P_d(X), \mathcal{S} \hat{\otimes} \mathcal{K}\big)^\Gamma\big)
\end{aligned}$$

are identity maps, respectively.

Finally, we are ready to complete the proofs of the main results in this paper.

Proof of Theorem 1.1. We have the commutative diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*\big(\mathcal{S} \hat{\otimes} C^*_L\big(P_d(X)\big)^\Gamma\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big)\\
\downarrow{\scriptstyle \beta_{L,*}} & & \downarrow{\scriptstyle \beta_*}\\
\lim\limits_{d\to\infty} K_*\big(C^*_L\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) & \xrightarrow[\cong]{e_*} & \lim\limits_{d\to\infty} K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big)\\
\downarrow{\scriptstyle \alpha_{L,*}} & & \downarrow{\scriptstyle \alpha_*}\\
\lim\limits_{d\to\infty} K_*\big(\mathcal{S} \hat{\otimes} C^*_L\big(P_d(X)\big)^\Gamma\big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big).
\end{array}$$

By Proposition 4.6, we know that $\alpha_* \circ \beta_*$ and $(\alpha_L)_* \circ (\beta_L)_*$ are identity maps, respectively. It follows immediately from the diagram chasing that the map

$$e_* : \lim_{d\to\infty} K_*\big(C^*_L\big(P_d(X)\big)^\Gamma\big) \to \lim_{d\to\infty} K_*\big(C^*\big(P_d(X)\big)^\Gamma\big)$$

induced by evaluation-at-zero map between the localization algebra and Roe algebra is an isomorphism on K-theory. ∎

Proof of Theorem 1.2. For any $d > 0$, we have the commutative diagram

$$\begin{array}{ccc}
K_*\big(\mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X)\big)^\Gamma\big) & \xrightarrow{\lambda_*} & K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big) \\
\downarrow{\scriptstyle \beta_{\max,*}} & & \downarrow{\scriptstyle \beta_*} \\
K_*\big(C^*_{\max}\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) & \xrightarrow[\cong]{\lambda_*} & K_*\big(C^*\big(P_d(X), \mathcal{A}(H)\big)^\Gamma\big) \\
\downarrow{\scriptstyle \alpha_{\max,*}} & & \downarrow{\scriptstyle \alpha_*} \\
K_*\big(\mathcal{S} \hat{\otimes} C^*_{\max}\big(P_d(X)\big)^\Gamma\big) & \xrightarrow{\lambda_*} & K_*\big(\mathcal{S} \hat{\otimes} C^*\big(P_d(X)\big)^\Gamma\big).
\end{array}$$

By Proposition 4.6, we know that $\alpha_{\max,*} \circ \beta_{\max,*}$ and $\alpha_* \circ \beta_*$ are identity maps, respectively. By Theorem 3.12 and the diagram chasing, we have that the map

$$\lambda_* : K_*\big(C^*_{\max}\big(P_d(X)\big)^\Gamma\big) \to K_*\big(C^*\big(P_d(X)\big)^\Gamma\big)$$

induced by the canonical quotient is an isomorphism on K-theory. Consequently, we have that

$$\lambda_* : K_*\big(C^*_{\max}(X)^\Gamma\big) \to K_*\big(C^*(X)^\Gamma\big)$$

is an isomorphism since $P_d(X)$ is Γ-equivariantly coarsely equivalent to X for a suitably large d. ∎

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