J. Noncommut. Geom. 18 (2024), 93–121
DOI 10.4171/JNCG/523

Subproduct systems with quantum group symmetry

Erik Habbestad and Sergey Neshveyev

Abstract. We introduce a class of subproduct systems of finite dimensional Hilbert spaces whose fibers are defined by the Jones–Wenzl projections in Temperley–Lieb algebras. The quantum symmetries of a subclass of these systems are the free orthogonal quantum groups. For this subclass, we show that the corresponding Toeplitz algebras are nuclear C*-algebras that are KK-equivalent to $\mathbb{C}$ and obtain a complete list of generators and relations for them. We also show that their gauge-invariant subalgebras coincide with the algebras of functions on the end compactifications of the duals of the free orthogonal quantum groups. Along the way we prove a few general results on equivariant subproduct systems, in particular, on the behavior of the Toeplitz and Cuntz–Pimsner algebras under monoidal equivalence of quantum symmetry groups.

Introduction

The notion of a subproduct system, introduced by Shalit and Solel [22] and Bhat and Mukherjee [7], lies at the intersection of two lines of research. One is dilation theory for semigroups of completely positive maps, the other is noncommutative function theory for row contractions. Recall that a row contraction is an m-tuple of operators $(S_1, \dots, S_m)$ such that $\sum_{i=1}^m S_i S_i^* \le 1$. By a result of Popescu [20], a universal, in a suitable sense, model for row contractions is provided by the creation operators $T_1, \dots, T_m$ on the full Fock space $\mathcal{F}(\mathbb{C}^m)$. If we are interested in the row contractions satisfying in addition algebraic relations $P_j(S_1, \dots, S_m) = 0$ for some homogeneous polynomials P_j in m noncommuting variables, then such a model is often obtained by taking the compressions $S_i = e_{\mathcal{H}} T_i e_{\mathcal{H}}$ of the operators T_i to the subspace $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^\infty H_n \subset \mathcal{F}(\mathbb{C}^m)$, where $H_n = I_n^\perp \subset (\mathbb{C}^m)^{\otimes n}$ and I_n is the degree n component of the ideal $I \subset \mathbb{C}\langle X_1, \dots, X_m\rangle$ generated by P_j [2,4]. The collection $\mathcal{H} = (H_n)_{n=0}^\infty$ has the property $H_{k+l} \subset H_k \otimes H_l$, which in the terminology of [22] means that it is a standard subproduct system of finite dimensional Hilbert spaces.

Our main object of interest in this paper is the structure of the Toeplitz algebra $\mathcal{T}_{\mathcal{H}} = C^*(S_1, \dots, S_m) \subset B(\mathcal{F}_{\mathcal{H}})$ and its quotient $\mathcal{O}_{\mathcal{H}} = \mathcal{T}_{\mathcal{H}}/\mathcal{K}(\mathcal{F}_{\mathcal{H}})$, called the Cuntz–Pimsner algebra of $\mathcal{H}$ by Viselter [29]. Although the definition of these algebras is similar to such much more studied constructions as Cuntz–Krieger algebras, graph algebras and Pimsner

2020 Mathematics Subject Classification. Primary 46L05; Secondary 46L67, 19K35, 30H20.
Keywords. Subproduct systems, Toeplitz operators, Cuntz–Pimsner algebras, quantum groups, K-theory.

algebras, a number of basic properties of the latter algebras do not have obvious analogues for $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{H}}$ [29], and a more or less complete understanding of $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{O}_{\mathcal{H}}$ has been achieved only in a few cases. For example, Kakariadis and Shalit [16] made a comprehensive analysis of the case where the ideal I is generated by monomials. In the present paper we consider principal ideals $I = \langle P \rangle$ generated by one quadratic polynomial. Furthermore, we require that the corresponding rank one projection $e = [\mathbb{C}P] \in B(\mathbb{C}^m \otimes \mathbb{C}^m)$ defines representations of Temperley–Lieb algebras $TL_n(\lambda^{-1})$ on $(\mathbb{C}^m)^{\otimes n}$ for some $\lambda \geq 4$; recall that $TL_n(\lambda^{-1})$ is generated by projections e_j, $1 \leq j \leq n-1$, satisfying the relations

$$e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2, \qquad e_i e_{i\pm 1} e_i = \frac{1}{\lambda} e_i.$$

Concretely, we consider the polynomials $P = \sum_{i,j=1}^m a_{ij} X_i X_j$ $(m \geq 2)$ such that $A\bar{A}$ is a scalar multiple of a unitary matrix, where $A = (a_{ij})_{i,j}$. We call such polynomials Temperley–Lieb. The simplest example of such a polynomial is $X_1X_2 - X_2X_1$, which defines Arveson's symmetric subproduct system SSP_2 [4]. In this case $\lambda = 4$. Since the Temperley–Lieb algebras for $\lambda \geq 4$ are all isomorphic, the entire collection that we consider can be thought of as a deformation/quantum analogue of SSP_2, and indeed, as we will see, the corresponding C*-algebras $\mathcal{T}_P$ and $\mathcal{O}_P$ share a lot of properties. To be precise, our main results are proved for the subclass of polynomials such that $A\bar{A} = \pm 1$. They have well-studied quantum symmetries – the free orthogonal quantum groups [26]. The general case will be analyzed in detail in a separate publication.

Particular cases of the subproduct systems associated with Temperley–Lieb polynomials have already appeared in the literature. In addition to SSP_2, the case of polynomials $X_1X_2 - qX_2X_1$ $(q \in \mathbb{C}^*)$ has been studied in [2, 6, 22], although without a detailed analysis of the associated C*-algebras. Our main class of examples, corresponding to the case $A\bar{A} = \pm 1$, has been studied by Andersson [1], with an emphasis on the gauge-invariant part of $\mathcal{T}_P$. It should be said that his paper contains interesting ideas, but suffers from numerous imprecise and outright wrong constructions and arguments. Recently, the case of polynomials $\sum_{i=1}^m (-1)^i X_i X_{m-i+1}$ has been studied by Arici and Kaad [3], and some of our results strengthen and generalize the results in their paper.

The contents of the paper is as follows. In Section 1, after a brief reminder on subproduct systems and associated C*-algebras, we discuss the class of Temperley–Lieb polynomials. Using known formulas for the Jones–Wenzl projections in the Temperley–Lieb algebras we find some nontrivial relations in $\mathcal{T}_P$.

In Section 2 we make a digression into the general theory of quantum group equivariant subproduct systems. By this we mean that each space H_n is equipped with a unitary representation of a compact quantum group G and the embedding maps $H_{k+l} \to H_k \otimes H_l$ are equivariant. A technical question that we study is under which conditions the action of G on the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ is reduced, meaning that the averaging over G defines a faithful cp map $\mathcal{O}_{\mathcal{H}} \to \mathcal{O}_{\mathcal{H}}^G$. Another question is what happens when we consider a monoidally equivalent compact quantum group $\widetilde{G}$ and transform $\mathcal{H} = (H_n)_{n=0}^\infty$

into a $\widetilde{G}$-equivariant subproduct system $\widetilde{\mathscr{H}}$. We show that the Toeplitz algebras $\mathcal{T}_{\mathscr{H}}$ and $\mathcal{T}_{\widetilde{\mathscr{H}}}$ correspond to each other under the equivalence of categories of G- and $\widetilde{G}$-C*-algebras defined in [11]. The same is true for the reduced forms of the Cuntz–Pimsner algebras.

We want to make it clear that none of the results of this section is strictly speaking needed for the subsequent sections, but they provide a conceptual framework for the results in those sections.

In Section 3 we return to the Temperley–Lieb polynomials and from this point onward concentrate on the polynomials $P = \sum_{i,j=1}^{m} a_{ij} X_i X_j$ such that $A\bar{A} = \pm 1$. As we already mentioned, every such polynomial has a large quantum symmetry group O_P^+, where by large we mean that every O_P^+-module H_n is irreducible. We show that by starting from Arveson's system SSP_2, corresponding to $P = X_1X_2 - X_2X_1$, then moving to the polynomials $q^{-1/2}X_1X_2 \pm q^{1/2}X_2X_1$ $(q > 0)$ and then considering the general case, the results of Section 2 quickly lead to the conclusion that the Cuntz–Pimsner algebra $\mathcal{O}_P$ is O_P^+-equivariantly isomorphic to the linking algebra $B(\mathrm{SU}_{\tau q}(2), O_P^+)$ (for suitable $q \in (0, 1]$ and $\tau = \pm 1$), which defines an equivalence between the representation categories of $\mathrm{SU}_{\tau q}(2)$ and O_P^+. We then find an explicit such isomorphism. It is worth stressing that the reason we know that $\mathcal{O}_P$ cannot be smaller is that the action of O_P^+ on $B(\mathrm{SU}_{\tau q}(2), O_P^+)$ is reduced and ergodic, and therefore it does not admit any proper O_P^+-equivariant quotients. This is exactly the same argument as was used by Arveson for SSP_2 to conclude that the Cuntz–Pimsner algebra in his case is $C(S^3)$ [4]. This property can be viewed as a replacement of the gauge-invariant uniqueness theorem, which fails for general subproduct systems [29]; see the recent paper by Dor-on [13] for a related discussion. Once $\mathcal{O}_P$ has been computed, it is not difficult to show that the relations in $\mathcal{T}_P$ that we found in Section 1 are complete.

In Section 4 we consider the gauge-invariant part $\mathcal{T}_P^{(0)}$ of $\mathcal{T}_P$. Since H_n, $n \geq 0$, exhaust the irreducible O_P^+-modules up to isomorphism, $\mathcal{T}_P^{(0)}$ can be considered as an algebra of functions on a compactification of the dual discrete quantum group $\mathbb{F}O_P$ of O_P^+. A compactification $\overline{\mathbb{F}O_P}$ of $\mathbb{F}O_P$ has been constructed for the polynomials $P = q^{-1/2}X_1X_2 \pm q^{1/2}X_2X_1$ by Tuset and the second author [18] using an analogy with the algebra of equivariant pseudo-differential operators of order zero on $\mathrm{SU}(2)$, and for general P by Vaes and Vergnioux [25] as a quantum analogue of the end compactification of a free group. We show that, excluding possibly the cases $O_P^+ \cong \mathrm{SU}_{\pm 1}(2)$, we have $\mathcal{T}_P^{(0)} = C(\overline{\mathbb{F}O_P})$. As a consequence, we now have an explicit isomorphism $C(\partial \mathbb{F}O_P) \cong {}^{\mathbb{T}}B(\mathrm{SU}_{\tau q}(2), O_P^+)$. The existence of such an isomorphism has been known, but in an indirect way, via an identification of both sides with the Martin boundary of $\mathbb{F}O_P$ [11, 18, 24, 25].

In Section 5 we study K-theoretic properties of $\mathcal{T}_P$. If we look again at SSP_2, then it follows already from an index theorem of Vegunopalkrishna [27] that the embedding map $\mathbb{C} \to \mathcal{T}_{X_1X_2 - X_2X_1}$ is a KK-equivalence. At this point it is natural to expect that the same is true for general $\mathcal{T}_P$. For the polynomials $P = \sum_{i=1}^{m} (-1)^i X_i X_{m-i+1}$ this has indeed been shown by Arici and Kaad [3]. They constructed an explicit inverse $\mathcal{T}_P \to \mathbb{C}$

in the KK-category, which is a highly nontrivial task for $m \geq 3$ (and for $m = 2$ as well, if we want in addition SU(2)-equivariance). It is plausible that with some modification their construction and homotopy arguments work for general P. We take, however, a different route and use the Baum–Connes conjecture for $\mathbb{F}O_P$, as formulated by Meyer and Nest [17] and proved by Voigt [30], to reduce the problem to comparing the K-theory of crossed products. We thus show that the embedding $\mathbb{C} \to \mathcal{T}_P$ is a $KK^{O_P^+}$-equivalence without explicitly providing an inverse.

1. Temperley–Lieb subproduct systems

Recall, following [7, 22], that a subproduct system $\mathcal{H}$ of finite dimensional Hilbert spaces (over the additive monoid $\mathbb{Z}_+$) is a sequence of Hilbert spaces $(H_n)_{n=0}^{\infty}$ together with isometries $w_{k,l}\colon H_{k+l} \to H_k \otimes H_l$ such that

$$\dim H_0 = 1, \qquad \dim H_1 = m < \infty, \qquad (w_{k,l} \otimes 1)w_{k+l,n} = (1 \otimes w_{l,n})w_{k,l+n}.$$

By [22, Lemma 6.1], we can assume that $H_0 = \mathbb{C}$, $H_{k+l} \subset H_k \otimes H_l$ and the isometries $w_{k,l}$ are simply the embedding maps. The subproduct systems satisfying this stronger property are called standard. For such subproduct systems we denote by f_n the projection $H_1^{\otimes n} \to H_n$.

Remark 1.1. We could also start with formally weaker axioms for subproduct systems: it is enough to require that the identities $(w_{k,l} \otimes 1)w_{k+l,n} = (1 \otimes w_{l,n})w_{k,l+n}$ hold only up to phase factors. Indeed, the same arguments as in [22, Lemma 6.1] show that we can still construct a standard subproduct system out of such a datum, and this standard system remains the same if we multiply $w_{k,l}$ by phase factors. On a more conceptual level, the possibility of getting rid of phase factors in an essentially unique way follows from triviality of the cohomology groups $H^2(\mathbb{Z}_+;\mathbb{T})$, $H^3(\mathbb{Z}_+;\mathbb{T})$ (see [9, Proposition X.4.1]).

Given a subproduct system $\mathcal{H} = (H_n)_{n=0}^{\infty}$, the associated Fock space is defined by

$$\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} H_n.$$

For every $\xi \in H_1$, we define an operator

$$S_\xi\colon \mathcal{F}_{\mathcal{H}} \to \mathcal{F}_{\mathcal{H}} \quad \text{by} \quad S_\xi \zeta = w_{1,n}^*(\xi \otimes \zeta) \quad \text{for } \zeta \in H_n.$$

We will usually fix an orthonormal basis $(\xi_i)_{i=1}^m$ in H_1 and write S_i for S_{ξ_i}. The Toeplitz algebra $\mathcal{T}_{\mathcal{H}}$ of $\mathcal{H}$ is defined as the unital C*-algebra generated by $S_1, \dots, S_m$.

If $\mathcal{H}$ is standard and $H = H_1$, it is convenient to identify $\mathcal{F}_{\mathcal{H}}$ with a subspace of the full Fock space

$$\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}.$$

Consider the operators $T_i : \mathcal{F}(H) \to \mathcal{F}(H)$, $T_i \zeta = \xi_i \otimes \zeta$, and the projection

$$e_{\mathcal{H}} : \mathcal{F}(H) \to \mathcal{F}_{\mathcal{H}}.$$

We obviously have $S_i = e_{\mathcal{H}} T_i |_{\mathcal{F}_{\mathcal{H}}}$. Then also $S_i^* = e_{\mathcal{H}} T_i^* |_{\mathcal{F}_{\mathcal{H}}}$, but since $H_{n+1} \subset H_1 \otimes H_n$, we actually have the stronger property

$$S_i^* = T_i^* |_{\mathcal{F}_{\mathcal{H}}}. \tag{1.1}$$

Since $1 - \sum_{i=1}^m T_i T_i^*$ is the projection onto $H^{\otimes 0} = \mathbb{C}$, we then get

$$e_0 = 1 - \sum_{i=1}^{m} S_i S_i^*, \tag{1.2}$$

where e_0 is the projection $\mathcal{F}_{\mathcal{H}} \to H_0$. As the vacuum vector $\Omega = 1 \in H_0$ is cyclic for $\mathcal{T}_{\mathcal{H}}$, it follows that $\mathcal{K}(\mathcal{F}_{\mathcal{H}}) \subset \mathcal{T}_{\mathcal{H}}$. The Cuntz–Pimsner algebra of $\mathcal{H}$ [29] is defined by

$$\mathcal{O}_{\mathcal{H}} = \mathcal{T}_{\mathcal{H}} / \mathcal{K}(\mathcal{F}_{\mathcal{H}}).$$

Once we fix an orthonormal basis $(\xi_i)_{i=1}^m$ in H, it is convenient to identify the tensor algebra $T(H)$ with the algebra $\mathbb{C}\langle X_1, \dots, X_m \rangle$ of polynomials in m noncommuting variables. When we do this, we omit the symbol $\otimes$ for the product in $T(H)$, so we write $X_i \xi$ instead of $\xi_i \otimes \xi$.

By [22, Proposition 7.2], there is a one-to-one correspondence between the standard subproduct systems with $H_1 = H$ and the homogeneous ideals I in $\mathbb{C}\langle X_1, \dots, X_m \rangle$ such that the degree one homogeneous component I_1 of I is zero. Namely, given such an ideal I, we define

$$H_n = I_n^{\perp} \subset H^{\otimes n}.$$

In this work we mainly consider ideals $I = \langle P \rangle$ generated by one homogeneous polynomial of degree 2. We denote by $\mathcal{H}_P$ the corresponding standard subproduct system, but then normally use the subscript P instead of $\mathcal{H}_P$, so we write $\mathcal{F}_P$, $\mathcal{T}_P$, etc.

We concentrate on polynomials having a particular symmetry.

Definition 1.2. Given a Hilbert space H of finite dimension $m \geq 2$, we say that a nonzero vector $\xi \in H \otimes H$ is *Temperley–Lieb*, if the corresponding projection $e = [\mathbb{C}\xi] \in B(H \otimes H)$ satisfies

$$(e \otimes 1)(1 \otimes e)(e \otimes 1) = \frac{1}{\lambda} e \otimes 1 \quad \text{in } B(H \otimes H \otimes H) \tag{1.3}$$

for some $\lambda > 0$.

As the following lemma shows, we then also have

$$(1 \otimes e)(e \otimes 1)(1 \otimes e) = \frac{1}{\lambda} 1 \otimes e.$$

Lemma 1.3. *Assume e_1 and e_2 are projections in a C^*-algebra A with a faithful tracial state τ such that*

$$\tau(e_1) = \tau(e_2) \quad \textit{and} \quad e_1 e_2 e_1 = \frac{1}{\lambda} e_1 \quad \textit{for some } \lambda > 0.$$

Then we also have $e_2 e_1 e_2 = \frac{1}{\lambda} e_2$.

Proof. Consider the element $p = e_2 - \lambda e_2 e_1 e_2$. Then $p^2 = p$ and $\tau(p) = 0$. Hence $p = 0$. ∎

In order to describe explicitly the Temperley–Lieb tensors, it is convenient to view the space $H \otimes H$ as the space of anti-linear operators $H \to H$, with $\zeta \otimes \eta$ corresponding to the operator $(\eta, \cdot)\zeta$. Equivalently, the tensor $\xi_A \in H \otimes H$ corresponding to an anti-linear operator $A \colon H \to H$ is given by

$$\xi_A = \sum_i \xi_i \otimes A\xi_i,$$

where $(\xi_i)_i$ is any orthonormal basis in H.

Lemma 1.4. *Given $A \neq 0$, the tensor $\xi_A \in H \otimes H$ is Temperley–Lieb if and only if A^2 is unitary up to a scalar factor, so that $(A^2)^* A^2 = \alpha 1$ for some $\alpha > 0$, and then λ satisfying (1.3) is given by $\lambda = \alpha^{-1}(\operatorname{Tr} A^* A)^2$.*

Proof. Consider the polar decomposition $A = U|A|$. A priori U is only an anti-linear partial isometry. We extend it to an anti-unitary $\widetilde{U}$. Let $(\xi_i)_i$ be an orthonormal basis consisting of eigenvectors of $|A|$, $|A|\xi_i = \lambda_i \xi_i$. Put $\zeta_i = \widetilde{U}\xi_i$. Let $(e_{ij})_{i,j}$ and $(f_{ij})_{i,j}$ be the matrix units in $B(H)$ corresponding to $(\xi_i)_i$ and $(\zeta_i)_i$, respectively. As $\xi_A = \sum_i \lambda_i \xi_i \otimes \zeta_i$, for the projection $e = [\mathbb{C}\xi_A]$ we have

$$e = \beta \sum_{i,j} \lambda_i \lambda_j e_{ij} \otimes f_{ij}, \quad \text{where } \beta = (\operatorname{Tr} A^* A)^{-1}.$$

Since $f_{i_1 j_1} e_{i_2 j_2} f_{i_3 j_3} = (\xi_{i_2}, \zeta_{j_1})(\zeta_{i_3}, \xi_{j_2}) f_{i_1 j_3}$, we have

$$\begin{aligned}(e \otimes 1)&(1 \otimes e)(e \otimes 1)\\ &= \beta^3 \sum_{i_1, i_2, i_3, j_1, j_2, j_3} \delta_{j_1, i_3} \lambda_{i_1} \lambda_{j_1} \lambda_{i_2} \lambda_{j_2} \lambda_{i_3} \lambda_{j_3} (\xi_{i_2}, \zeta_{j_1})(\zeta_{i_3}, \xi_{j_2}) e_{i_1 j_3} \otimes f_{i_1 j_3} \otimes f_{i_2 j_2}.\end{aligned}$$

It follows that (1.3) holds if and only if the following identities are satisfied:

$$\beta^3 \lambda_{i_1} \lambda_{j_3} \sum_i \lambda_i^2 \lambda_{i_2} \lambda_{j_2} (\xi_{i_2}, \zeta_i)(\zeta_i, \xi_{j_2}) = \delta_{i_2, j_2} \frac{\beta}{\lambda} \lambda_{i_1} \lambda_{j_3}.$$

As $\lambda_{i_1} \lambda_{j_3} \neq 0$ at least for some indices, it follows that if we introduce the unitary matrix $V = ((\zeta_j, \xi_i))_{i,j}$ and the diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$, then we must have

$$\lambda \beta^2 \Lambda V \Lambda^2 V^* \Lambda = 1.$$

This is equivalent to invertibility of Λ together with the identity $\lambda\beta^2 V\Lambda^2 = \Lambda^{-2}V$. In terms of $|A|$ and U this means that $\lambda\beta^2 U|A|^2 = |A|^{-2}U$, or equivalently, $\lambda\beta^2 AA^* = (A^*A)^{-1}$, which proves the lemma. ∎

Using this lemma it is not difficult to parameterize the Temperley–Lieb tensors up to scalar factors and unitary transformations on H. Namely, since $(V \otimes V)\xi_A = \xi_{VAV^*}$ for any unitary V on H, we want to classify up to unitary conjugacy anti-linear operators $A\colon H \to H$ such that A^2 is unitary. If $A = U|A|$ is the polar decomposition of such an A, then $U|A| = |A|^{-1}U$. It follows that the spectrum $\sigma(|A|)$ of $|A|$ is closed under the transformation $\beta \mapsto \beta^{-1}$, and if $H_\beta \subset H$ is the spectral subspace for $|A|$ corresponding to β, then $UH_\beta = H_{\beta^{-1}}$. This implies that the collection of pairs (β, Z_β), where $\beta \in S := \sigma(|A|) \cap (0, 1]$ and Z_β are the eigenvalues of U^2 on H_β counting with multiplicity, is an invariant of the unitary conjugacy class of A. We claim that this is a complete invariant and the only restrictions on the sets S and Z_β are that Z_1 is closed under complex conjugation and

$$\sum_{\beta\in S\cap(0,1)} 2|Z_\beta| + |Z_1| = m.$$

Indeed, it suffices to explain how to choose an orthonormal basis in which A has a normal form. For every $\beta \in S \cap (0, 1)$ we choose an orthonormal basis $(\xi_{\beta i})_i$ in H_β consisting of eigenvectors of U^2, $U^2\xi_{\beta i} = z_{\beta i}\xi_{\beta i}$. Then the vectors $\zeta_{\beta i} = U\xi_{\beta i}$ form an orthonormal basis in $H_{\beta^{-1}}$. Each two-dimensional space spanned by $\xi_{\beta i}$ and $\zeta_{\beta i}$ is invariant under U and $|A|$, hence under A and A^*, and the matrix of the restriction of A to this space is

$$\begin{pmatrix} 0 & z_{\beta i}\beta^{-1} \\ \beta & 0 \end{pmatrix}.$$

Next, assuming 1 is an eigenvector of $|A|$, consider the space H_1. It is invariant under U, so here we can use the well-known description of anti-unitary matrices due to Wigner. It will be convenient to slightly modify it though. If $H_{1z} \subset H_1$ is the spectral subspace for U^2 corresponding to $z \in \mathbb{T}$, then $UH_{1z} = H_{1\bar{z}}$. It follows that, similarly to the previous paragraph, the direct sum of the spaces H_{1z} for $z \neq \pm 1$ decomposes into a direct sum of two-dimensional U-invariant subspaces such that on each of them U has the form

$$\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}, \quad 0 < \arg(z) \le \pi.$$

The same is true for $z = -1$, since if $\xi \in H_{1,-1}$, then $U\xi \perp \xi$, as

$$(\xi, U\xi) = (UU\xi, U\xi) = -(\xi, U\xi).$$

Finally, consider H_{11}. The real subspace of vectors fixed by U is a real form of H_{11}. Choose an orthonormal basis $g_1, \dots, g_k$ in this Euclidean subspace. If $k = 2l$ is even, we define an orthonormal basis $(\xi_j)_j$ in H_{11} by

$$\xi_j = \frac{1}{\sqrt{2}}(g_j + ig_{k-j+1}) \quad \text{and} \quad \xi_{k-j+1} = \frac{1}{\sqrt{2}}(g_j - ig_{k-j+1}), \quad 1 \le j \le l.$$

If $k = 2l + 1$, then, in order to get a basis, in addition to the above vectors we need to take $\xi_{l+1} = g_{l+1}$. In both cases the matrix of $U|_{H_{11}}$ in the basis $(\xi_j)_j$ is

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

This finishes the proof of our claim. The discussion can be summarized as follows.

Proposition 1.5. *Assume H is a Hilbert space of dimension $m = 2l + r$ ($l \in \mathbb{N}$, $r \in \{0, 1\}$) and $A\colon H \to H$ is an anti-linear operator such that A^2 is unitary. Then there is an orthonormal basis $(\xi_i)_{i=1}^m$ in which A has the form*

$$\begin{pmatrix} 0 & & a_m \\ & \iddots & \\ a_1 & & 0 \end{pmatrix} \quad \text{for some } a_i \in \mathbb{C}^*, \qquad |a_i a_{m-i+1}| = 1 \quad \text{for all } i,$$

and then λ satisfying (1.3) for $\xi = \xi_A = \sum_i a_i \xi_i \otimes \xi_{m-i+1}$ is given by

$$\lambda^{1/2} = \sum_{i=1}^m |a_i|^2 \geq m.$$

The numbers $a_1, \dots, a_m$ are uniquely determined by A up to permutations of pairs of coefficients (a_i, a_{m-i+1}) ($1 \leq i \leq l$) if we in addition require $0 < a_i \leq 1$ ($1 \leq i \leq l$), $a_{l+1} = 1$ if m is odd, and $0 \leq \arg(a_{m-i+1}) \leq \pi$ whenever $a_i = 1$ ($1 \leq i \leq l$).

Identifying H with the space of homogeneous linear polynomials in m variables, we can equivalently say that a homogeneous quadratic polynomial

$$\sum_{i,j=1}^m a_{ij} X_i X_j$$

is Temperley–Lieb if and only if the matrix $A = (a_{ij})_{i,j}$ is such that $A\bar{A}$ is a nonzero scalar multiple of a unitary matrix, where $\bar{A} = (\bar{a}_{ij})_{i,j}$. By a rescaling and a unitary change of variables, every such polynomial can be written as

$$\sum_{i=1}^m a_i X_i X_{m-i+1}, \quad \text{with} \quad |a_i a_{m-i+1}| = 1 \quad \text{for all } i,$$

and such a presentation is unique up to permutations of pairs (a_i, a_{m-i+1}) if we put restrictions on the coefficients a_i as in Proposition 1.5.

Given a Temperley–Lieb polynomial $P = \sum_{i,j=1}^m a_{ij} X_i X_j$, consider the corresponding standard subproduct system $\mathcal{H}_P = (H_n)_{n=0}^\infty$. If $e \in B(H \otimes H)$ is the projection onto $\mathbb{C}P$, then

$$H_n = f_n H^{\otimes n}, \qquad f_n = 1 - \bigvee_{i=0}^{n-2} 1^{\otimes i} \otimes e \otimes 1^{\otimes(n-i-2)} \quad \text{for } n \geq 2,$$

and $f_0 = 1 \in \mathbb{C}$, $f_1 = 1 \in B(H)$. Since the projections $1^{\otimes i} \otimes e \otimes 1^{\otimes(n-i-2)}$ satisfy the Temperley–Lieb relations, by computations of Jones [15] and Wenzl [31] (see [19, Lemma 2.5.8] and its proof), we have

$$f_{n+1} = 1 \otimes f_n - [2]_q \phi(n)(1 \otimes f_n)(e \otimes 1^{\otimes(n-1)})(1 \otimes f_n), \quad n \geq 0, \tag{1.4}$$

where $q \in (0, 1]$ is such that $\lambda^{1/2} = q + q^{-1}$, $[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}$ (with the convention $[k]_1 = k$) and

$$\phi(n) = \frac{[n]_q}{[n+1]_q} = \frac{q^n - q^{-n}}{q^{n+1} - q^{-n-1}}. \tag{1.5}$$

Note that $\phi(n) \to q$ as $n \to +\infty$.

Lemma 1.6. *For all $n \geq 0$, we have* $\dim H_n = [n+1]_t$, *where $t \in (0, 1]$ is such that $t + t^{-1} = m$.*

Proof. It is known that the projection $[2]_q \phi(n)(1 \otimes f_n)(e \otimes 1^{\otimes(n-1)})(1 \otimes f_n)$ in (1.4) is equivalent to $e \otimes f_{n-1}$, see, e.g., again [19, Lemma 2.5.8]. Hence the dimensions of the spaces $H_n = f_n H^{\otimes n}$ satisfy the recurrence relation

$$\dim H_{n+1} = m \dim H_n - \dim H_{n-1}.$$

This gives the result. ∎

We next want to describe certain relations in $\mathcal{T}_P$. As we will see later, these relations are complete, at least for Temperley–Lieb polynomials of a particular type.

Let us first introduce some notation. Since $\mathcal{K}(\mathcal{F}_P) \subset \mathcal{T}_P$, we can view the algebra $c = C(\mathbb{Z}_+ \cup \{+\infty\})$ of converging sequences as a subalgebra of $\mathcal{T}_P$, with the projection $e_n \in c$ corresponding to the projection $\mathcal{F}_P \to H_n$. Denote by γ the shift to the left on c.

Proposition 1.7. *Consider a polynomial $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ such that $|a_i a_{m-i+1}| = 1$ for all i ($m \geq 2$). Let $q \in (0, 1]$ be such that*

$$\sum_{i=1}^m |a_i|^2 = q + q^{-1}.$$

Then we have the following relations in $\mathcal{T}_P$:

$$f S_i = S_i \gamma(f) \quad (f \in c,\ 1 \leq i \leq m), \qquad \sum_{i=1}^m S_i S_i^* = 1 - e_0, \qquad \sum_{i=1}^m a_i S_i S_{m-i+1} = 0,$$
$$S_i^* S_j + a_i \bar{a}_j \phi S_{m-i+1} S_{m-j+1}^* = \delta_{ij} 1 \quad (1 \leq i, j \leq m),$$

where the element $\phi = (\phi(n))_{n=0}^\infty \in c$ is defined by (1.5).

Proof. The first relation simply reflects the fact that $S_i H_n \subset H_{n+1}$. The second relation is (1.2), it holds in any subproduct system of finite dimensional Hilbert spaces. The third relation is an immediate consequence of the definition of $\mathcal{H}_P$. So it is only the last relation that is not obvious.

Denote by e_{ij} the matrix units in $B(H)$. The projection $e \in B(H \otimes H)$ onto $\mathbb{C}P$ is

$$e = \frac{1}{[2]_q} \sum_{i,j=1}^{m} a_i \bar{a}_j e_{ij} \otimes e_{m-i+1,m-j+1}.$$

Therefore by (1.4) we have

$$f_{n+1} = 1 \otimes f_n - \phi(n) \sum_{i,j} a_i \bar{a}_j e_{ij} \otimes f_n (e_{m-i+1,m-j+1} \otimes 1^{\otimes(n-1)}) f_n.$$

From this, for every $\xi \in H_n$, we get

$$S_i^* S_j \xi = \delta_{ij} \xi - \phi(n) a_i \bar{a}_j f_n (e_{m-i+1,m-j+1} \otimes 1^{\otimes(n-1)}) f_n \xi.$$

By observing that by (1.1) we have

$$f_n(e_{kl} \otimes 1^{\otimes(n-1)}) f_n = f_n T_k T_l^* f_n = S_k S_l^*$$

on H_n, we obtain the last relation in $\mathcal{T}_P$. ■

2. Equivariant subproduct systems

As a preparation for a more thorough study of $\mathcal{T}_P$, in this section we consider subproduct systems that are equivariant with respect to a compact quantum group.

Let us fix the notation and recall some basic notions, see [19] for more details. By a compact quantum group G we mean a Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$ that is generated by matrix coefficients of finite dimensional unitary right comodules. We have a one-to-one correspondence between such comodules and the finite dimensional unitary representations of G, that is, unitaries $U \in B(H_U) \otimes \mathbb{C}[G]$ such that $(\iota \otimes \Delta)(U) = U_{12} U_{13}$. Namely, the right comodule structure on H_U is given by

$$\delta_U : H_U \to H_U \otimes \mathbb{C}[G], \quad \delta_U(\xi) = U(\xi \otimes 1).$$

The tensor product $U \otimes V$ of two unitary representations (denoted also by $U \mathbin{\text{⦶}} V$ or $U \times V$) is defined by $U_{13} V_{23}$. We denote by $\mathrm{Irr}(G)$ the set of the isomorphism classes of irreducible unitary representations of G. For every $s \in \mathrm{Irr}(G)$ we fix a representative $U_s \in B(H_s) \otimes \mathbb{C}[G]$.

Denote by h the Haar state on $\mathbb{C}[G]$ and by $L^2(G)$ the corresponding GNS-space. We view $\mathbb{C}[G]$ as a subalgebra of $B(L^2(G))$. The (reduced) C*-algebra $C(G)$ of continuous functions on G is defined as the norm closure of $\mathbb{C}[G]$.

Consider the $*$-algebra $\mathcal{U}(G) = \mathbb{C}[G]^*$ dual to the coalgebra $(\mathbb{C}[G], \Delta)$, with the involution $\omega^*(x) = \overline{\omega(S(x)^*)}$. Every finite dimensional unitary representation U of G defines a $*$-representation

$$\pi_U : \mathcal{U}(G) \to B(H_U), \quad \pi_U(\omega) = (\iota \otimes \omega)(U).$$

The representations π_s defined by U_s, $s \in \mathrm{Irr}(G)$, give rise to an isomorphism

$$\mathcal{U}(G) \cong \prod_{s \in \mathrm{Irr}(G)} B(H_s),$$

and in what follows we are not going to distinguish between these two algebras. Consider the subalgebra

$$c_c(\widehat{G}) = \bigoplus_{s \in \mathrm{Irr}(G)} B(H_s) \subset \mathcal{U}(G).$$

It coincides with the subalgebra of $C(G)^* \subset \mathcal{U}(G)$ spanned by the linear functionals $h(\cdot\, x)$ for $x \in \mathbb{C}[G]$. We remark that $c_c(\widehat{G})$ is weakly* dense in $C(G)^*$, since the state h is faithful on $C(G)$ and $\mathbb{C}[G]$ is dense in $C(G)$.

Assume now that A is a C*-algebra and $\alpha: A \to A \otimes C(G)$ is a $*$-homomorphism such that $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$. We say that α is a (right) action of G on A, or that A is a G-C*-algebra, if either of the following equivalent conditions is satisfied:

(i) the linear space $\alpha(A)(1 \otimes C(G))$ is a dense subspace of $A \otimes C(G)$ (the Podleś condition);

(ii) there is a dense $*$-subalgebra $\mathcal{A} \subset A$ on which α defines a coaction of the Hopf algebra $(\mathbb{C}[G], \Delta)$, that is, $\alpha(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathbb{C}[G]$ and $(\iota \otimes \varepsilon)\alpha(a) = a$ for all $a \in \mathcal{A}$,

where $\otimes_{\mathrm{alg}}$ denotes the purely algebraic tensor product. Then the largest subalgebra as in (ii) is given by

$$\mathcal{A} = \mathrm{span}\{(\iota \otimes h)(\alpha(a)(1 \otimes x)) : a \in A,\ x \in \mathbb{C}[G]\},$$

its elements are called regular. To put it differently, we have a left $c_c(\widehat{G})$-module structure on A defined by

$$\omega \blacktriangleright a = (\iota \otimes \omega)\alpha(a),$$

and then $\mathcal{A} = c_c(\widehat{G}) \blacktriangleright A$.

An action is called reduced if α is injective, or equivalently, by faithfulness of h on $C(G)$, if the conditional expectation

$$E = (\iota \otimes h)\alpha : A \to A^G = \{a \in A : \alpha(a) = a \otimes 1\}$$

is faithful. For reduced actions the subalgebra of regular elements can also be described as

$$\mathcal{A} = \{a \in A : \alpha(a) \in A \otimes_{\mathrm{alg}} \mathbb{C}[G]\}, \tag{2.1}$$

since if $\alpha(a) \in A \otimes_{\mathrm{alg}} \mathbb{C}[G]$, then $\alpha(a - (\iota \otimes \varepsilon)\alpha(a)) = 0$.

Given any action $\alpha\colon A \to A \otimes C(G)$, we get a reduced action on $A_r = A/\ker\alpha$, since

$$\ker\alpha = \{a \in A : E(a^*a) = 0\}.$$

We say that A_r is the reduced form of the G-C*-algebra A. Note that A and A_r have the same subalgebra $\mathcal{A}$ of regular elements.

If G is coamenable, that is, the counit ε on $\mathbb{C}[G] \subset C(G)$ is bounded, then $(\iota \otimes \varepsilon)\alpha = \iota$ on A, so all actions of G are reduced.

Lemma 2.1. *Assume $\alpha\colon A \to A \otimes C(G)$ is a reduced action and $X \subset A$ is a finite dimensional G-invariant subspace, so that $\alpha(X) \subset X \otimes C(G)$. Then all elements of X are regular.*

Proof. By assumption, X is a $c_c(\widehat{G})$-submodule of A. As X is finite dimensional, the action of $B(H_s)$ must be zero for all but a finite number of $s \in \operatorname{Irr}(G)$. In other words, there is a finite subset $F \subset \operatorname{Irr}(G)$ such that for all $\phi \in A^*$ the space $(\phi \otimes \iota)\alpha(X) \subset C(G) \subset L^2(G)$ is orthogonal to the matrix coefficients of U_s for all $s \notin F$. It follows that this space is contained in the space $\mathbb{C}[G]_F$ spanned by the matrix coefficients of representations U_s, $s \in F$. But then $\alpha(X) \subset A \otimes \mathbb{C}[G]_F$, so X consists of regular elements by (2.1). ■

Everything above of course also applies to the left actions $\alpha\colon A \to C(G) \otimes A$.

Assume now we are given two compact quantum groups G and $\widetilde{G}$. They are said to be monoidally equivalent if their categories of finite dimensional unitary representations are equivalent as C*-tensor categories. By [8, 23], every such equivalence, up to a natural isomorphism, is defined by a bi-Galois object/linking algebra. This is a unital C*-algebra $B = B(G, \widetilde{G})$ equipped with two commuting reduced actions

$$\delta\colon B \to C(G) \otimes B, \quad \tilde{\delta}\colon B \to B \otimes C(\widetilde{G})$$

that are ergodic and free; the precise meaning of the last property will not be important to us. Then an equivalence $F\colon \operatorname{Rep} G \to \operatorname{Rep} \widetilde{G}$ is defined by mapping a finite dimensional unitary right comodule (H_U, δ_U) for $(\mathbb{C}[G], \Delta)$ into the cotensor product

$$H_U \,\square_G\, B = \{X \in H_U \otimes B : (\delta_U \otimes \iota)(X) = (\iota \otimes \delta)(X)\},$$

with the right $(\mathbb{C}[\widetilde{G}], \Delta)$-comodule structure given by $\iota \otimes \tilde{\delta}$ and the scalar product

$$(X, Y) = Y^*X \in {}^{G}B = \mathbb{C}1,$$

where we interpret $\zeta^*\xi$ for $\xi, \zeta \in H_U$ as (ξ, ζ). The tensor structure on F is defined by

$$(H_U \,\square_G\, B) \otimes (H_V \,\square_G\, B) \to H_{U \otimes V} \,\square_G\, B, \quad X \otimes Y \mapsto X_{13}Y_{23}.$$

In a similar way one can define an equivalence between the categories of reduced G-C*-algebras and $\widetilde{G}$-C*-algebras, but there is a small technical issue that has to be taken care of [11]. Let $\mathcal{B} = \mathcal{B}(G, \widetilde{G}) \subset B$ be the subalgebra of regular elements with respect to the G-action, equivalently, with respect to the $\tilde{G}$-action. Given a reduced action $\alpha\colon A \to A \otimes C(G)$, consider the subalgebra $\mathcal{A} \subset A$ of regular elements and define

$$A \,\square_G\, B = \overline{\{X \in \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B} : (\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X)\}} \subset A \otimes B.$$

The right action of $\tilde{G}$ on $A \,\square_G\, B$ is again given by $\iota \otimes \tilde{\delta}$.

In some cases it is possible to use exactly the same definition of $A \,\square_G\, B$ as for finite dimensional comodules.

Lemma 2.2. *Assume $\alpha\colon A \to A \otimes C(G)$ is a reduced action of a compact quantum group G on a C^*-algebra A and there is a net of completely bounded G-equivariant maps $\theta_i\colon A \to A$ of finite rank such that $\theta_i(a) \to_i a$ for all $a \in A$ and $\sup_i \|\theta_i\|_{\mathrm{cb}} < \infty$. Then, for any compact quantum group $\tilde{G}$ monoidally equivalent to G, we have*

$$A \,\square_G\, B(G, \tilde{G}) = \{X \in A \otimes B(G, \tilde{G}) : (\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X)\}.$$

Proof. If $(\alpha \otimes \iota)(X) = (\iota \otimes \delta)(X)$, then the elements $X_i = (\theta_i \otimes \iota)(X)$ have the same property and converge to X. Therefore it suffices to show that $X_i \in \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}(G, \tilde{G})$. Lemma 2.1 implies that the finite dimensional space $\theta_i(A)$ is contained in $\mathcal{A}$, hence $X_i \in \mathcal{A} \otimes_{\mathrm{alg}} B(G, \tilde{G})$ and then, using again that $(\alpha \otimes \iota)(X_i) = (\iota \otimes \delta)(X_i)$, we must have $X_i \in \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{B}(G, \tilde{G})$. ∎

Consider a reduced action $\alpha\colon A \to A \otimes C(G)$ and a unitary, not necessarily finite dimensional, representation $U \in M(\mathcal{K}(H) \otimes C(G))$ of G. Assume we are given a covariant representation $\pi\colon A \to B(H)$, meaning that

$$(\pi \otimes \iota)\alpha(a) = U(\pi(a) \otimes 1)U^*.$$

Given a monoidally equivalent compact quantum group $\tilde{G}$, we can extend the functor

$$F = \cdot \,\square_G\, B(G, \tilde{G})\colon \mathrm{Rep}\, G \to \mathrm{Rep}\, \tilde{G}$$

to all unitary representations, since they decompose into irreducible ones. Therefore we get a Hilbert space $H \,\square_G\, B(G, \tilde{G})$. By taking any state ψ on $B(G, \tilde{G})$ and considering the associated GNS-representation $\pi_\psi\colon B(G, \tilde{G}) \to B(H_\psi)$, we can view $H \,\square_G\, B(G, \tilde{G})$ as a subspace of $H \otimes H_\psi$. It follows that $\pi \otimes \pi_\psi$ defines by restriction a representation $\pi \,\square\, \iota$ of $A \,\square_G\, B(G, \tilde{G})$ on $H \,\square_G\, B(G, \tilde{G})$.

Lemma 2.3. *If the representation π is faithful, then $\pi \,\square\, \iota$ is faithful as well.*

Proof. Since the representation $\pi \,\square\, \iota$ is covariant and the $\tilde{G}$-action on $A \,\square_G\, B(G, \tilde{G})$ is reduced, it suffices to check that the representation is faithful on

$$(A \,\square_G\, B(G, \tilde{G}))^{\tilde{G}} = A^G \otimes 1.$$

Decompose H into isotypical components, $H \cong \bigoplus_{s\in\mathrm{Irr}(G)} L_s \otimes H_s$, where L_s are some Hilbert spaces. Then the representation π restricted to A^G has the form $\pi(a) = (\theta_s(a)\otimes 1)_{s\in\mathrm{Irr}(G)}$ for some representations $\theta_s\colon A^G \to B(L_s)$. But then

$$H \,\square_G\, B(G,\tilde{G}) \cong \bigoplus_{s\in\mathrm{Irr}(G)} L_s \otimes (H_s \,\square_G\, B(G,\tilde{G})),$$

and the representation $\pi \,\square\, \iota$ on $A^G \otimes 1$ is simply

$$(\pi \,\square\, \iota)(a\otimes 1) = (\theta_s(a)\otimes 1\otimes 1)_{s\in\mathrm{Irr}(G)},$$

which is obviously faithful. ■

As we already mentioned, for the coamenable compact quantum groups all actions are reduced. As soon as G is noncoamenable, there are nonreduced actions, for example, the action by right translations on the universal completion $C_u(G)$ of $\mathbb{C}[G]$. An interesting question is whether a quotient of a reduced action is reduced. We do not know an answer, but here is a partial result.

Lemma 2.4. *Assume $\alpha\colon A \to A\otimes C(G)$ is a reduced action of a compact quantum group G on a C*-algebra A, $I \subset A$ is a closed G-invariant ideal. Assume also that the following conditions are satisfied:*

- (i) *G is monoidally equivalent to a coamenable compact quantum group;*
- (ii) *there is a net of completely bounded G-equivariant maps $\theta_i\colon I \to I$ of finite rank such that $\theta_i(a) \to_i a$ for all $a \in I$ and $\sup_i \|\theta_i\|_{\mathrm{cb}} < \infty$.*

Then the action of G on A defines reduced actions of G on I and A/I.

For the proof we need the observation from [8] that if $\tilde{G}$ is coamenable, then $B(G,\tilde{G})$ is a nuclear C*-algebra. This follows from a general result in [12] stating that for coamenable $\tilde{G}$ a $\tilde{G}$-C*-algebra is nuclear if and only if the fixed point algebra is nuclear.

Proof of Lemma 2.4. If $\mathcal{A} \subset A$ is the subalgebra of regular elements, then $\theta_i(I) \subset \mathcal{A}$ by Lemma 2.1. As $\theta_i(a) \to_i a$ for all $a \in I$, it follows that $\mathcal{I} = I \cap \mathcal{A}$ is dense in I. Since $\alpha|_{\mathcal{I}}$ is a coaction of $(\mathbb{C}[G],\Delta)$ on $\mathcal{I}$, we conclude that $\alpha|_I$ is an action of G on I. This action is obviously reduced.

Now, consider a coamenable compact quantum group $\tilde{G}$ monoidally equivalent to G. Let us fix an equivalence of the representation categories of G and $\tilde{G}$ and consider the corresponding bi-Galois objects $B(G,\tilde{G})$ and $B(\tilde{G},G)$. We then have bi-equivariant isomorphisms

$$B(G,\tilde{G}) \,\square_{\tilde{G}}\, B(\tilde{G},G) \cong C(G), \quad B(\tilde{G},G) \,\square_G\, B(G,\tilde{G}) \cong C(\tilde{G}),$$

since the bi-Galois object corresponding to the identity functor on $\mathrm{Rep}\, G$ is $C(G)$. Then, modulo these isomorphisms, we have a natural isomorphism $A \cong A \,\square_G\, B(G,\tilde{G}) \,\square_{\tilde{G}}\, B(\tilde{G},G)$ for any reduced action $\alpha\colon A \to A\otimes C(G)$, defined by

$$A \to A \,\square_G\, C(G), \quad a \mapsto \alpha(a).$$

Put

$$\tilde{I} = I \,\square_G\, B(G,\widetilde{G}), \quad \widetilde{A} = A \,\square_G\, B(G,\widetilde{G}), \quad \widetilde{C} = \widetilde{A}/\tilde{I}.$$

Denote by $\widetilde{\alpha}$ the action of $\widetilde{G}$ on $\widetilde{A}$. The action of $\widetilde{G}$ on $\widetilde{C}$ is reduced by coamenability. As $B(\widetilde{G},G)$ is a nuclear C*-algebra, we have a short exact sequence

$$0 \to \tilde{I} \otimes B(\widetilde{G},G) \to \widetilde{A} \otimes B(\widetilde{G},G) \to \widetilde{C} \otimes B(\widetilde{G},G) \to 0.$$

By identifying A with $\widetilde{A} \,\square_{\widetilde{G}}\, B(\widetilde{G},G)$ as explained above and denoting $\widetilde{C} \,\square_{\widetilde{G}}\, B(\widetilde{G},G)$ by C, we then get an exact sequence

$$0 \to (\tilde{I} \otimes B(\widetilde{G},G)) \cap (\widetilde{A} \,\square_{\widetilde{G}}\, B(\widetilde{G},G)) \to A \to C \to 0.$$

We claim that $(\tilde{I} \otimes B(\widetilde{G},G)) \cap (\widetilde{A} \,\square_{\widetilde{G}}\, B(\widetilde{G},G)) = I$. It is clear that

$$\begin{aligned}\tilde{I} \,\square_{\widetilde{G}}\, B(\widetilde{G},G) &\subset (\tilde{I} \otimes B(\widetilde{G},G)) \cap (\widetilde{A} \,\square_{\widetilde{G}}\, B(\widetilde{G},G)) \\ &\subset \{x \in \tilde{I} \otimes B(\widetilde{G},G) : (\widetilde{\alpha} \otimes \iota)(x) = (\iota \otimes \tilde{\delta})(x)\},\end{aligned}$$

where $\widetilde{\alpha}$ and $\tilde{\delta}$ denote the actions of $\widetilde{G}$ on $\widetilde{A}$ and $B(\widetilde{G},G)$. The last set coincides with $\tilde{I} \,\square_{\widetilde{G}}\, B(\widetilde{G},G) = I$ by Lemma 2.2 applied to $\widetilde{G}$ and the cb maps $\widetilde{\theta}_i = \theta_i \otimes \iota|_{I \square_G B(G,\widetilde{G})}$ on $\tilde{I}$. This proves our claim.

Therefore we have a G-equivariant short exact sequence $0 \to I \to A \to C \to 0$ where all actions are reduced, which means that the action of G on A/I is reduced. ■

Remark 2.5. What we tacitly used in the above argument is that if $A \to C$ is a surjective homomorphism of G-C*-algebras, then we get a surjective map $\mathcal{A} \to \mathcal{C}$ between the subalgebras of regular elements, and then, for any compact quantum group $\widetilde{G}$ monoidally equivalent to G, the map $\mathcal{A} \,\square_G\, \mathcal{B}(G,\widetilde{G}) \to \mathcal{C} \,\square_G\, \mathcal{B}(G,\widetilde{G})$ is surjective as well. This implies that if $0 \to I \to A \to C \to 0$ is a short exact sequence of reduced G-C*-algebras, then the sequence of $\widetilde{G}$-C*-algebras

$$0 \to I \,\square_G\, B(G,\widetilde{G}) \to A \,\square_G\, B(G,\widetilde{G}) \to C \,\square_G\, B(G,\widetilde{G}) \to 0$$

might not be exact in the middle, but at least the reduced form of $(A \,\square_G\, B(G,\widetilde{G}))/(I \,\square_G\, B(G,\widetilde{G}))$ is $C \,\square_G\, B(G,\widetilde{G})$.

Let us finally turn to equivariant subproduct systems. Assume G is a compact quantum group and $\mathcal{H} = (H_n)_{n=0}^\infty$ is a subproduct system of finite dimensional G-modules. By this we mean that we are given unitary representations U_n of G on H_n and the structure maps $H_{k+l} \to H_k \otimes H_l$ are G-intertwiners.

In this case we have a unitary representation

$$U_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} U_n \in M(\mathcal{K}(\mathcal{F}_{\mathcal{H}}) \otimes C(G))$$

of G on $\mathcal{F}_{\mathcal{H}}$. It defines reduced right actions of G on $\mathcal{K}(\mathcal{F}_{\mathcal{H}})$ and $\mathcal{T}_{\mathcal{H}}$ by

$$\alpha(T) = U_{\mathcal{H}}(T \otimes 1)U_{\mathcal{H}}^*.$$

To see that we indeed get an action on the Toeplitz algebra, fix an orthonormal basis in H_1 and let u_{ij} be the matrix coefficients of U_1 in this basis. Then

$$\alpha(S_j) = \sum_i S_i \otimes u_{ij}. \tag{2.2}$$

This implies that α defines a coaction of $(\mathbb{C}[G], \Delta)$ on the unital $*$-algebra generated by the operators S_i, hence it defines an action of G on $\mathcal{T}_{\mathcal{H}}$.

We then get an action of G on $\mathcal{O}_{\mathcal{H}}$. It is not clear to us whether this action is always reduced, but by Lemma 2.4 we at least have the following.

Proposition 2.6. *If $\mathcal{H} = (H_n)_{n=0}^{\infty}$ is a subproduct system of finite dimensional G-modules and G is monoidally equivalent to a coamenable compact quantum group, then the action of G on $\mathcal{O}_{\mathcal{H}}$ is reduced.*

Assume now that $\tilde{G}$ is a compact quantum group monoidally equivalent to G and fix a bi-Galois object $B = B(G, \tilde{G})$. Given a subproduct system $\mathcal{H} = (H_n)_{n=0}^{\infty}$ of finite dimensional G-modules, we get a subproduct system $\tilde{\mathcal{H}} = (H_n \mathbin{\square}_G B)_{n=0}^{\infty}$ of finite dimensional $\tilde{G}$-modules. Note that even if $\mathcal{H}$ is standard, $\tilde{\mathcal{H}}$ is usually not.

By definition we have a canonical unitary isomorphism $\mathcal{F}_{\tilde{\mathcal{H}}} \cong \mathcal{F}_{\mathcal{H}} \mathbin{\square}_G B$. It follows that we get a representation of $\mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B$ on $\mathcal{F}_{\tilde{\mathcal{H}}}$. This representation is faithful by Lemma 2.3.

Proposition 2.7. *Assume $\mathcal{H} = (H_n)_{n=0}^{\infty}$ is a subproduct system of finite dimensional G-modules and $\tilde{G}$ is monoidally equivalent to G. Consider the subproduct system $\tilde{\mathcal{H}} = (H_n \mathbin{\square}_G B(G, \tilde{G}))_{n=0}^{\infty}$. Then the canonical representation $\mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G}) \to B(\mathcal{F}_{\tilde{\mathcal{H}}})$ defines $\tilde{G}$-equivariant isomorphisms*

$$\mathcal{K}(\mathcal{F}_{\mathcal{H}}) \mathbin{\square}_G B(G, \tilde{G}) \cong \mathcal{K}(\mathcal{F}_{\tilde{\mathcal{H}}}), \quad \mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G}) \cong \mathcal{T}_{\tilde{\mathcal{H}}}.$$

Proof. The first isomorphism is a particular case of [30, Proposition 8.4]. The second will be obtained by similar arguments.

Since the representation $\mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G}) \to B(\mathcal{F}_{\tilde{\mathcal{H}}})$ is faithful, we can view $\mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G})$ as a subalgebra of $B(\mathcal{F}_{\tilde{\mathcal{H}}})$. If $X = \sum_i \xi_i \otimes b_i \in \tilde{H}_1 := H_1 \mathbin{\square}_G B(G, \tilde{G})$, then by definition the element $\sum_i S_{\xi_i} \otimes b_i \in \mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G})$ is the operator $S_X \in \mathcal{T}_{\tilde{\mathcal{H}}}$. Therefore $\mathcal{T}_{\tilde{\mathcal{H}}} \subset \mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G})$. It follows that

$$\mathcal{T}_{\tilde{\mathcal{H}}} \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G) \subset \mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G}) \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G) \subset B(\mathcal{F}_{\tilde{\mathcal{H}}} \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G)).$$

By identifying $\mathcal{F}_{\tilde{\mathcal{H}}} \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G)$ with $\mathcal{F}_{\mathcal{H}}$, this means that $\mathcal{T}_{\tilde{\mathcal{H}}} \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G) \subset \mathcal{T}_{\mathcal{H}}$. By swapping the roles of G and $\tilde{G}$ we conclude that $\mathcal{T}_{\tilde{\mathcal{H}}} = \mathcal{T}_{\mathcal{H}} \mathbin{\square}_G B(G, \tilde{G})$ and $\mathcal{T}_{\tilde{\mathcal{H}}} \mathbin{\square}_{\tilde{G}} B(\tilde{G}, G) = \mathcal{T}_{\mathcal{H}}$. ■

Together with Remark 2.5 this gives the following.

Corollary 2.8. *If $\mathcal{O}_{\mathcal{H},r}$ is the reduced form of the G-C^*-algebra $\mathcal{O}_{\mathcal{H}}$, then the $\tilde{G}$-C^*-algebra $\mathcal{O}_{\mathcal{H},r} \mathbin{\square}_G B(G, \tilde{G})$ is isomorphic to the reduced form of $\mathcal{O}_{\tilde{\mathcal{H}}}$.*

3. Toeplitz and Cuntz–Pimsner algebras of Temperley–Lieb subproduct systems with large symmetry

We now return to the subproduct systems defined by Temperley–Lieb polynomials and from now on consider only the polynomials $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ such that

$$a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\} \quad \text{for all } 1 \leq i \leq m.$$

A distinguishing property of these polynomials is that they have large quantum symmetries, the free orthogonal quantum groups [26]. Namely, for every P as above consider the free orthogonal quantum group $O_P^+ = O_{F_P}^+$ defined by the matrix

$$F_P = \begin{pmatrix} 0 & & a_m \\ & \iddots & \\ a_1 & & 0 \end{pmatrix}. \tag{3.1}$$

The algebra $\mathbb{C}[O_P^+]$ of regular functions on O_P^+ is a universal unital $*$-algebra with generators u_{ij}, $1 \leq i, j \leq m$, and relations

$$u_{ij}^* = a_i^{-1} a_j u_{m-i+1,m-j+1}, \quad \text{the matrix } (u_{ij})_{i,j} \text{ is unitary.}$$

The comultiplication is given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Consider the standard subproduct system $\mathcal{H}_P = (H_n)_{n=0}^\infty$ defined by P. By definition, the quantum group O_P^+ has a unitary representation $U_1 = (u_{ij})_{i,j}$ on H_1 and the representation $U_1 \otimes U_1$ leaves $P \in H_1 \otimes H_1$ invariant. Therefore we get unitary representations U_n of O_P^+ on $H_n = f_n H_1^{\otimes n}$. The spin $\frac{n}{2}$ representations U_n, $n \geq 0$, exhaust all irreducible representations of O_P^+ up to equivalence, see [5] or [19, Section 2.5].

Remark 3.1. The quantum group O_P^+ is a genuine group only when $m = 2$, $|a_1| = |a_2| = 1$ and $\tau = 1$, in which case it is SU(2). The classical part of O_P^+, that is, the stabilizer of P in $U(m)$, is generically rather small: if the numbers $|a_i|$ are all different, it consists of the unitary diagonal matrices $\operatorname{diag}(z_1, \dots, z_m)$ such that $z_i z_{m-i+1} = 1$.

Let us try to understand the Cuntz–Pimsner algebra $\mathcal{O}_P$. In the simplest case $P = X_1X_2 - X_2X_1$, the subproduct system $\mathcal{H}_P$ is Arveson's symmetric subproduct system SSP_2 and $\mathcal{O}_P$ is isomorphic to $C(S^3) \cong C(\mathrm{SU}(2))$ by [4, Theorem 5.7].

Next, take $q \in (0, 1]$, $\tau = \pm 1$ and consider the polynomial

$$P = q^{-1/2} X_1 X_2 - \tau q^{1/2} X_2 X_1.$$

From Proposition 1.7 we see that the following relations are satisfied in $\mathcal{O}_P$:

$$S_1 S_1^* + S_2 S_2^* = 1, \qquad q^{-1/2} S_1 S_2 - \tau q^{1/2} S_2 S_1 = 0,$$
$$S_1^* S_1 + S_2 S_2^* = 1, \qquad S_2^* S_2 + q^2 S_1 S_1^* = 1, \qquad S_1^* S_2 - \tau q S_2 S_1^* = 0.$$

These are the relations in $\mathbb{C}[\mathrm{SU}_{\tau q}(2)] = \mathbb{C}[O_P^+]$ for the generators u_{21} and u_{22}, which are usually denoted by γ and α^*, respectively. In view of (2.2) and coamenability of $\mathrm{SU}_{\tau q}(2)$ we conclude that there is a well-defined $\mathrm{SU}_{\tau q}(2)$-equivariant surjective $*$-homomorphism $C(\mathrm{SU}_{\tau q}(2)) \to \mathcal{O}_P$, $\gamma \mapsto S_1$, $\alpha \mapsto S_2^*$. This homomorphism must be injective, since the action of $\mathrm{SU}_{\tau q}(2)$ on $C(\mathrm{SU}_{\tau q}(2))$ defined by Δ is reduced and ergodic. Thus, $\mathcal{O}_P \cong C(\mathrm{SU}_{\tau q}(2))$, which is a quantum analogue of Arveson's result for $q = \tau = 1$.

Consider now the general case $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$. Let $q \in (0, 1]$ be such that

$$\sum_{i=1}^m |a_i|^2 = q + q^{-1}. \tag{3.2}$$

Then there is a unique up to a natural isomorphism monoidal equivalence between O_P^+ and $\mathrm{SU}_{\tau q}(2)$, see [8] or again [19, Section 2.5]. Such an equivalence F maps the spin $\frac{n}{2}$ representations of O_P^+ into spin $\frac{n}{2}$ representations of $\mathrm{SU}_{\tau q}(2)$. In both representation categories there is a unique up to a scalar factor morphism $H_{k+l} \to H_k \otimes H_l$. By Remark 1.1 it follows that F maps $\mathcal{H}_P$ into an $\mathrm{SU}_{\tau q}(2)$-equivariant subproduct system isomorphic to $\mathcal{H}_{q^{-1/2}X_1X_2 - \tau q^{1/2} X_2 X_1}$. By Proposition 2.6 and Corollary 2.8 we conclude that

$$\mathcal{O}_P \cong C(\mathrm{SU}_{\tau q}(2)) \,\square_{\mathrm{SU}_{\tau q}(2)}\, B(\mathrm{SU}_{\tau q}(2), O_P^+) \cong B(\mathrm{SU}_{\tau q}(2), O_P^+).$$

Once we know that such an isomorphism is to be expected, it is not difficult to construct it from scratch using known generators and relations of $B(\mathrm{SU}_{\tau q}(2), O_P^+)$. Namely, consider the matrices F_P, given by (3.1), and

$$F_{q,\tau} = \begin{pmatrix} 0 & -\tau q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix}.$$

Then by [8, Theorem 5.5 and Remark 5.7], $B(\mathrm{SU}_{\tau q}(2), O_P^+)$ is a universal unital C*-algebra with generators y_{ij}, $1 \le i \le 2$, $1 \le j \le m$, and relations

$$Y = F_{q,\tau} \bar{Y} F_P^{-1}, \qquad Y \text{ is unitary},$$

where $Y = (y_{ij})_{i,j}$, $\bar{Y} = (y_{ij}^*)_{i,j}$. The right action

$$\delta\colon B(\mathrm{SU}_{\tau q}(2), O_P^+) \to B(\mathrm{SU}_{\tau q}(2), O_P^+) \otimes C(O_P^+)$$

is given by $\delta(y_{ij}) = \sum_{k=1}^m y_{ik} \otimes u_{kj}$, and the left action of $\mathrm{SU}_{\tau q}(2)$ is defined in a similar way.

Consider the elements $y_j = y_{2j}$. The relation $Y = F_{q,\tau} \bar{Y} F_P^{-1}$ means that

$$y_{1j} = q^{1/2} \bar{a}_j y_{m-j+1}^*,$$

and then a simple computation shows that unitarity of Y is equivalent to the relations

$$\sum_{i=1}^m y_i y_i^* = 1, \qquad \sum_{i=1}^m a_i y_i y_{m-i+1} = 0,$$

$$y_i^* y_j + q a_i \bar{a}_j y_{m-i+1} y_{m-j+1}^* = \delta_{ij} 1 \quad (1 \le i, j \le m).$$

These are exactly the relations in $\mathcal{O}_P$ that we get from Proposition 1.7. We thus arrive at the following result.

Proposition 3.2. *Assume $P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all i ($m \geq 2$), and let $q \in (0, 1]$ be given by (3.2). Then we have an O^+_P-equivariant isomorphism $B(\mathrm{SU}_{\tau q}(2), O^+_P) \cong \mathcal{O}_P$, $y_i \mapsto S_i$ ($1 \leq i \leq m$).*

Proof. By the above discussion, the homomorphism $B(\mathrm{SU}_{\tau q}(2), O^+_P) \to \mathcal{O}_P$, $y_i \mapsto S_i$, is well defined, surjective and O^+_P-equivariant. It must be injective, since the action of O^+_P on the linking algebra $B(\mathrm{SU}_{\tau q}(2), O^+_P)$ is reduced and ergodic. ■

Since $B(\mathrm{SU}_{\tau q}(2), O^+_P)$ is nuclear by coamenability of $\mathrm{SU}_{\tau q}(2)$, we get the following corollary.

Corollary 3.3. *The C*-algebras $\mathcal{T}_P$ and $\mathcal{O}_P$ are nuclear.*

We are now ready to describe relations in $\mathcal{T}_P$. Recall from Section 1 that we can view the C*-algebra $c = C(\mathbb{Z}_+ \cup \{+\infty\})$ of converging sequences as a subalgebra of $\mathcal{T}_P$ and we denote by γ the shift to the left on c.

Theorem 3.4. *Assume $P = \sum_{i=1}^{m} a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all i ($m \geq 2$), and let $q \in (0, 1]$ be given by (3.2). Then $\mathcal{T}_P$ is a universal C*-algebra generated by $S_1, \dots, S_m$ and c satisfying the relations*

$$f S_i = S_i \gamma(f) \quad (f \in c,\ 1 \leq i \leq m), \quad \sum_{i=1}^{m} S_i S_i^* = 1 - e_0, \quad \sum_{i=1}^{m} a_i S_i S_{m-i+1} = 0,$$
$$S_i^* S_j + a_i \bar{a}_j \phi S_{m-i+1} S_{m-j+1}^* = \delta_{ij} 1 \quad (1 \leq i, j \leq m),$$

where the element $\phi \in c$ is defined by $\phi(n) = \frac{[n]_q}{[n+1]_q}$.

Proof. By Proposition 1.7 we already know that the above relations are satisfied in $\mathcal{T}_P$. Consider a universal C*-algebra $\widetilde{\mathcal{T}}_P$ generated $\widetilde{S}, \dots, \widetilde{S}_m$ and c satisfying these relations. Let $\pi: \widetilde{\mathcal{T}}_P \to \mathcal{T}_P$ be the quotient map.

Every element of the $*$-algebra generated by $\widetilde{S}, \dots, \widetilde{S}_m$ and c can be written as a linear combination of elements of the form $f \widetilde{S}_{i_1} \cdots \widetilde{S}_{i_k} \widetilde{S}_{j_1}^* \cdots \widetilde{S}_{j_l}^*$, where $f \in c$ and $k, l \geq 0$. As $e_0 \widetilde{S}_i = 0$, it follows $e_0 \widetilde{\mathcal{T}}_P e_0 = \mathbb{C} e_0$, that is, e_0 is a minimal projection in $\widetilde{\mathcal{T}}_P$. Hence the closed ideal $I = \langle e_0 \rangle$ generated by e_0 is isomorphic to the algebra of compact operators on some Hilbert space.

Next, since

$$\sum_{i=1}^{m} \widetilde{S}_i e_n \widetilde{S}_i^* = e_{n+1} \sum_{i=1}^{m} \widetilde{S}_i \widetilde{S}_i^* = e_{n+1}(1 - e_0) = e_{n+1},$$

by induction on n we conclude that $e_n \in I$ for all $n \geq 0$, that is, $c_0 \subset I$. It follows that the images of $\widetilde{S}_i$ in $\widetilde{\mathcal{T}}_P / I$ satisfy the defining relations of $B(\mathrm{SU}_{\tau q}(2), O^+_P) \cong \mathcal{O}_P$. On the other hand, by construction, $\mathcal{O}_P$ is a quotient of $\widetilde{\mathcal{T}}_P / I$. It follows that $\pi: \widetilde{\mathcal{T}}_P \to \mathcal{T}_P$ gives rise to an isomorphism $\widetilde{\mathcal{T}}_P / I \cong \mathcal{O}_P$.

Therefore we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \widetilde{\mathcal{T}}_P & \longrightarrow & \mathcal{O}_P & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \pi|_I} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{F}_P) & \longrightarrow & \mathcal{T}_P & \longrightarrow & \mathcal{O}_P & \longrightarrow & 0 \end{array}$$

with exact rows and surjective π. As I is a simple C*-algebra, the map $\pi|_I$ must be an isomorphism, hence $\pi\colon \widetilde{\mathcal{T}}_P \to \mathcal{T}_P$ is an isomorphism as well. ∎

Remark 3.5. Since $\mathcal{T}_P$ is generated by $S_1, \dots, S_m$, the relations in $\mathcal{T}_P$ can be written without using the subalgebra c. Namely, instead of the first two relations we could require the elements p_n defined by

$$p_n = \sum_{i_1,\dots,i_n=1}^{m} S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*, \quad n \geq 1,$$

to be projections, and then define the element ϕ for the last relation by

$$\phi = q p_1 + \sum_{n=1}^{\infty} (\phi(n) - q)(p_n - p_{n+1}),$$

which makes sense, as $p_1 \geq p_2 \geq \cdots$ and therefore the projections $p_n - p_{n+1}$ are mutually orthogonal.

Indeed, then the projections p_n together with the unit generate a copy of c, with $e_n = p_n - p_{n+1}$ for $n \geq 0$, where $p_0 = 1$. As $p_{n+1} = \sum_i S_i p_n S_i^*$, we have

$$\begin{aligned}(1 - p_{n+1}) S_i p_n S_i^* (1 - p_{n+1}) &\leq \sum_j (1 - p_{n+1}) S_j p_n S_j^* (1 - p_{n+1}) \\ &= (1 - p_{n+1}) p_{n+1} (1 - p_{n+1}) = 0,\end{aligned}$$

so that $(1 - p_{n+1}) S_i p_n = 0$. A similar computation gives $p_{n+1} S_i (1 - p_n) = 0$. It follows that $p_{n+1} S_i = S_i p_n$ for all $n \geq 0$. This is equivalent to our first relation $f S_i = S_i \gamma(f)$.

4. Gauge action and compactifications of the dual discrete quantum groups

We continue to consider the Temperley–Lieb polynomials $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$.

As the O_P^+-modules H_n exhaust all irreducible representations of O_P^+ up to equivalence, the algebra of bounded functions on the dual discrete quantum group $\mathbb{F} O_P = \widehat{O_P^+}$ is by definition

$$\ell^\infty(\mathbb{F} O_P) = \ell^\infty\text{-}\bigoplus_{n=0}^{\infty} B(H_n) \subset B(\mathcal{F}_P).$$

For $x \in \ell^\infty(\mathbb{F} O_P)$, we denote by x_n its component in $B(H_n)$.

Consider the unitary representation $z \mapsto V_z$ of $\mathbb{T}$ on $\mathcal{F}_P$, where V_z is the unitary that acts on H_n by the scalar z^n. Then we get an action $\operatorname{Ad} V$ of $\mathbb{T}$ on $B(\mathcal{F}_P)$, and $\ell^\infty(\mathbb{F}O_P)$ coincides with the fixed point subalgebra of $B(\mathcal{F}_P)$ with respect to this action. The automorphisms $\operatorname{Ad} V_z$ leave $\mathcal{T}_P$ globally invariant and define the gauge action σ of $\mathbb{T}$ on $\mathcal{T}_P$, given by

$$\sigma_z(S_i) = zS_i.$$

Note that on $\mathcal{O}_P \cong B(\mathrm{SU}_{\tau q}(2), O_P^+)$ the gauge action coincides with the action of the maximal torus $\mathbb{T} \subset \mathrm{SU}_{\tau q}(2)$. More precisely, the maximal torus is defined by the homomorphism

$$\pi\colon C(\mathrm{SU}_{\tau q}(2)) \to C(\mathbb{T}), \quad \pi(u_{21}) = 0, \quad \pi(u_{22})(z) = \bar{z}.$$

Therefore for the left action

$$\tilde{\delta}\colon B(\mathrm{SU}_{\tau q}(2), O_P^+) \to C(\mathrm{SU}_{\tau q}(2)) \otimes B(\mathrm{SU}_{\tau q}(2), O_P^+), \quad \tilde{\delta}(y_{ij}) = \sum_k u_{ik} \otimes y_{kj},$$

we have

$$(\pi \otimes \iota)\tilde{\delta}(y_{2j}) = (z \mapsto \bar{z} y_{2j}) \in C(\mathbb{T}; B(\mathrm{SU}_{\tau q}(2), O_P^+)) = C(\mathbb{T}) \otimes B(\mathrm{SU}_{\tau q}(2), O_P^+).$$

Consider the gauge-invariant subalgebra

$$\mathcal{T}_P^{(0)} := \mathcal{T}_P^\sigma = \mathcal{T}_P \cap \ell^\infty(\mathbb{F}O_P)$$

of $\mathcal{T}_P$. Since it is unital and contains

$$c_0(\mathbb{F}O_P) := c_0\text{-}\bigoplus_{n=0}^{\infty} B(H_n) = \mathcal{K}(\mathcal{F}_P)^\sigma,$$

it can be thought of as an algebra of continuous functions on a compactification of $\mathbb{F}O_P$.

Assume $\sum_i |a_i|^2 > 2$. Following [25], consider the inductive system of O_P^+-equivariant ucp maps

$$\psi_{n,n+k}\colon B(H_n) \to B(H_{n+k}), \quad T \mapsto f_{n+k}(T \otimes 1)f_{n+k},$$

where we use that $H_{n+k} \subset H_n \otimes H_k$, and define

$$\begin{aligned} C(\overline{\mathbb{F}O_P}) &= \overline{\{x \in \ell^\infty(\mathbb{F}O_P) : \psi_{n,n+k}(x_n) = x_{n+k} \text{ for all } n \text{ large enough and } k \geq 0\}} \\ &= \{x \in \ell^\infty(\mathbb{F}O_P) : \lim_{n\to+\infty} \sup_{k\geq 0} \|\psi_{n,n+k}(x_n) - x_{n+k}\| = 0\}. \end{aligned}$$

Clearly, $c_0(\mathbb{F}O_P) \subset C(\overline{\mathbb{F}O_P})$. It is shown in [25] that $C(\overline{\mathbb{F}O_P})$ is a C*-algebra. This construction is a quantum analogue of the end compactification of a free group. Our goal in this section is to prove the following.

Theorem 4.1. *Assume $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ is a polynomial such that $a_i \bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all i, and $\sum_{i=1}^m |a_i|^2 > 2$. Then $\mathcal{T}_P^{(0)} = C(\overline{\mathbb{F} O_P})$.*

In the proof we are not going to use that $C(\overline{\mathbb{F} O_P})$ is an algebra, so as a byproduct we will reprove that this is indeed the case.

Let $q \in (0, 1)$ be given by (3.2). The proof of the theorem relies on the following key known estimate.

Lemma 4.2. *There is a constant $C > 0$ depending only on q such that*

$$\| f_{n+1} - (1 \otimes f_n)(f_n \otimes 1)\| \le C q^n \quad \text{for all } n \ge 0.$$

A stronger result is proved in [25, Lemma 8.4] using a generalization of Wenzl's recursive formula for f_n established in [14]. Let us show that the particular case we need is a consequence of basic properties of the representation theory of $\mathrm{SU}_q(2)$.

Proof of Lemma 4.2. It suffices to prove the estimate in the category $\operatorname{Rep} \mathrm{SU}_{\tau q}(2)$ equivalent to $\operatorname{Rep} O_P^+$. Furthermore, since $\operatorname{Rep} \mathrm{SU}_{-q}(2)$ can be obtained from $\operatorname{Rep} \mathrm{SU}_q(2)$ by introducing new associativity morphisms such that $(H_1 \otimes H_1) \otimes H_1 \to H_1 \otimes (H_1 \otimes H_1)$ is the multiplication by -1, it is enough to consider $\operatorname{Rep} \mathrm{SU}_q(2)$. In other words, we may assume that $P = q^{-1/2} X_1 X_2 - q^{1/2} X_2 X_1$.

As $H_{n+1} \subset (H_1 \otimes H_n) \cap (H_n \otimes H_1)$, we just need to show that the restriction of $1 \otimes f_n$ to $(H_n \otimes H_1) \ominus H_{n+1}$ has norm $\le C q^n$. Since the $\mathrm{SU}_q(2)$-module $(H_n \otimes H_1) \ominus H_{n+1} \cong H_{n-1}$ is irreducible, this norm is equal to $\|(1 \otimes f_n)\xi\|$ for any unit vector ξ in this module. As the vector ξ we take a highest weight vector. A formula for this vector is not difficult to find, up to a scalar factor it is

$$[n]_q^{1/2} X_1^n X_2 - q^{(n+1)/2} \zeta_n X_1,$$

where ζ_n is a unit vector of weight $\frac{n}{2} - 1$ in H_n, see the first paragraph of [19, Section 2.5]. Therefore we see that there is a unit vector $\xi_n \in (H_n \otimes H_1) \ominus H_{n+1}$ such that

$$\|\xi_n - X_1^n X_2\| \le a q^n$$

for a constant a depending only on q. But then

$$\|(1 \otimes f_n)\xi_n\| \le a q^n + \|X_1 f_n(X_1^{n-1} X_2)\| \le a q^n + a q^{n-1} + \|f_n \xi_{n-1}\| = a q^n + a q^{n-1},$$

which proves the lemma. ∎

Similarly to the operators S_i defined using the multiplication on the left, we can use the multiplication on the right and define

$$R_i \colon \mathcal{F}_P \to \mathcal{F}_P, \quad R_i \xi = f_{n+1}(\xi X_i) \quad \text{for} \quad \xi \in H_n.$$

Lemma 4.3. *For all $1 \le i, j \le m$, we have $[S_i, R_j] = 0$. There is a constant C depending only on q such that*

$$\|[S_i^*, R_j]|_{H_n}\| \le C q^n \quad \text{for all } n \ge 0.$$

Proof. For $\xi \in H_n$, we have $S_i R_j \xi = f_{n+2}(X_i \xi X_j) = R_j S_i \xi$, which proves the first claim. To prove the second claim, take a unit vector $\xi \in H_n$ for some $n \geq 1$. Then

$$S_i^* R_j \xi = S_i^* f_{n+1}(\xi X_j) = T_i^* f_{n+1}(\xi X_j),$$

where we used (1.1). By Lemma 4.2 the last expression is Cq^n-close to

$$T_i^*(1 \otimes f_n)(\xi X_j) = f_n(T_i^*(\xi) X_j) = R_j S_i^* \xi,$$

which gives the required estimate. ■

Corollary 4.4. *For every $S \in \mathcal{T}_P$ and every R in the C^*-algebra $C^*(R_1, \dots, R_m)$, we have $[S, R] \in \mathcal{K}(\mathcal{F}_P)$.*

Remark 4.5. The last corollary is true for $q = 1$ as well, since in this case the proof of Lemma 4.2 gives $\| f_{n+1} - (1 \otimes f_n)(f_n \otimes 1) \| \leq C n^{-1/2}$.

Similarly to (1.2), we have $\sum_{i=1}^m R_i R_i^* = 1 - e_0$. Define a contractive cp map

$$\Theta \colon B(\mathcal{F}_P) \to B(\mathcal{F}_P), \quad \Theta(T) = \sum_{i=1}^m R_i T R_i^*.$$

Since $\mathcal{K}(\mathcal{F}_P) \subset \mathcal{T}_P$, by Corollary 4.4 this map leaves $\mathcal{T}_P$ globally invariant. It also leaves $\ell^\infty(\mathbb{F} O_P)$ globally invariant and we have

$$\Theta^k(x)_{n+k} = \psi_{n,n+k}(x_n) \quad \text{for all } x \in \ell^\infty(\mathbb{F} O_P),\ n, k \geq 0. \tag{4.1}$$

Denote by $\mathcal{A}_P$ the unital $*$-subalgebra of $\mathcal{T}_P$ generated by the elements S_i, $1 \leq i \leq m$. Let $\mathcal{A}_P^{(0)}$ be the gauge-invariant part of $\mathcal{A}_P$.

Lemma 4.6. *For every $x \in \mathcal{A}_P^{(0)}$, there exists a constant $C > 0$ such that*

$$\| x_{n+k} - \Theta^k(x)_{n+k} \| \leq C q^n \quad \textit{for all } n, k \geq 0.$$

Proof. By Lemma 4.3 we can find a constant $\tilde{C}$ depending on x such that

$$\| x_n - \Theta(x)_n \| \leq \tilde{C} q^n \quad \text{for all } n \geq 0.$$

As Θ^l is a contraction mapping $B(H_{n+k-l})$ into $B(H_{n+k})$, it follows that

$$\| \Theta^l(x)_{n+k} - \Theta^{l+1}(x)_{n+k} \| \leq \tilde{C} q^{n+k-l} \quad \text{for all } n \geq 0,\ k \geq l \geq 0.$$

Summing up over $l = 0, \dots, k-1$ we get the required estimate, with $C = \tilde{C} q (1-q)^{-1}$. ■

Proof of Theorem 4.1. The inclusion $\mathcal{A}_P^{(0)} \subset C(\overline{\mathbb{F} O_P})$ follows from (4.1) and Lemma 4.6. Hence $\mathcal{T}_P^{(0)} \subset C(\overline{\mathbb{F} O_P})$.

In order to prove the opposite inclusion, take any $x \in \ell^\infty(\mathbb{F}O_P)$ such that

$$x_{n_0+k} = \psi_{n_0,n_0+k}(x_{n_0})$$

for some n_0 and all $k \geq 0$. Since the maps ψ_{n_0,n_0+k} are O_P^+-equivariant, we may assume that x_{n_0} lies in a spin l spectral component of $B(H_{n_0})$. Note that as $H_{n_0} \otimes H_{n_0} \cong H_0 \oplus H_2 \oplus \cdots \oplus H_{2n_0}$, l must be an integer $\leq n_0$.

As $\mathcal{O}_P^{(0)} \cong {}^{\mathbb{T}}B(\mathrm{SU}_{\tau q}(2), O_P^+)$, the multiplicity of the spin l component of $\mathcal{O}_P^{(0)}$ is 1. Fix a nonzero, hence injective, O_P^+-equivariant map $H_l \to \mathcal{O}_P^{(0)}$ and lift it to an O_P^+-equivariant map $\pi\colon H_l \to \mathcal{A}_P^{(0)}$. By Lemma 4.6 applied to $\pi(\xi)$ for the elements ξ of a basis of H_l, we can find $C > 0$ such that

$$\|\pi(\xi)_{n+k} - \Theta^k(\pi(\xi))_{n+k}\| \leq Cq^n\|\xi\| \quad \text{for all } \xi \in H_l \text{ and } n,k \geq 0. \tag{4.2}$$

Next, we claim that there exist $\alpha > 0$ and $n_1 \geq 0$ such that $\|\pi(\xi)_n\| \geq \alpha\|\xi\|$ for all $\xi \in H_l$ and $n \geq n_1$. If this is not true, then by compactness of the unit sphere of H_l we can find a unit vector $\xi \in H_l$ such that $\liminf_n \|\pi(\xi)_n\| = 0$. But then by (4.2) we must have $\lim_n \|\pi(\xi)_n\| = 0$, that is, the image of $\pi(\xi)$ in $\mathcal{O}_P^{(0)}$ is zero, which is a contradiction. Thus, our claim is proved.

Now, fix $\varepsilon > 0$ and choose $n \geq \max\{n_0, n_1\}$ such that $Cq^n\|x\| < \varepsilon\alpha$. For the unique $\xi \in H_l$ such that $\pi(\xi)_n = x_n$ we have $\|\xi\| \leq \alpha^{-1}\|x\|$. For all $k \geq 0$ we have

$$x_{n+k} = \Theta^k(x)_{n+k} = \Theta^k(\pi(\xi))_{n+k}.$$

Hence, applying again (4.2), we get

$$\|x_{n+k} - \pi(\xi)_{n+k}\| \leq Cq^n\|\xi\| < \varepsilon.$$

Therefore, modulo the compacts, x is ε-close to $\mathcal{A}_P^{(0)}$. Hence $x \in \mathcal{T}_P^{(0)}$. ∎

5. K-theory

We continue to consider the same class of polynomials $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ as in the previous two sections.

Since $\mathcal{O}_P \cong B(\mathrm{SU}_{\tau q}, O_P^+)$, by [10, Example 7.5] we have

$$K_0(\mathcal{O}_P) \cong \mathbb{Z}/(m-2)\mathbb{Z}, \quad K_1(\mathcal{O}_P) = \begin{cases} \mathbb{Z}, & \text{if } m = 2, \\ 0, & \text{if } m \geq 3. \end{cases}$$

This already gives a lot of information about the K-theory of $\mathcal{T}_P$ (for example, $K_1(\mathcal{T}_P) = 0$ if $m \geq 3$), but is not quite enough to fully determine it. Our goal is to prove the following.

Theorem 5.1. *Assume $P = \sum_{i=1}^m a_i X_i X_{m-i+1}$ is a polynomial such that $a_i\bar{a}_{m-i+1} = -\tau \in \{-1, 1\}$ for all i ($m \geq 2$). Then the embedding map $\mathbb{C} \to \mathcal{T}_P$ is a $KK^{O_P^+}$-equivalence.*

For the polynomials $P = \sum_i (-1)^i X_i X_{m-i+1}$ this upgrades the $KK^{\mathrm{SU}(2)}$-equivalence of Arici–Kaad [3] to the equivariant KK-category of the whole quantum symmetry group.

Our proof relies on the strong Baum–Connes conjecture for the dual of $\mathrm{SU}_q(2)$ established by Voigt [30]. More precisely, we need the following standard consequence of this result.

Proposition 5.2. *Assume $q \in [-1, 1] \setminus \{0\}$, and A and B are separable $\mathrm{SU}_q(2)$-C^*-algebras. Then a class $x \in KK^{\mathrm{SU}_q(2)}(A, B)$ is a $KK^{\mathrm{SU}_q(2)}$-equivalence if and only if it defines a KK-equivalence between $A \rtimes \mathrm{SU}_q(2)$ and $B \rtimes \mathrm{SU}_q(2)$.*

Proof. A class x is a $KK^{\mathrm{SU}_q(2)}$-equivalence if and only if it induces an isomorphism

$$KK^{\mathrm{SU}_q(2)}(C, A) \cong KK^{\mathrm{SU}_q(2)}(C, B) \tag{5.1}$$

for every separable $\mathrm{SU}_q(2)$-C*-algebra C. The fact that $\widehat{\mathrm{SU}_q(2)}$ is torsion-free and satisfies the strong Baum–Connes conjecture implies that the localizing subcategory of the triangulated category $KK^{\mathrm{SU}_q(2)}$ generated by the separable C*-algebras with trivial $\mathrm{SU}_q(2)$-action coincides with $KK^{\mathrm{SU}_q(2)}$ [30]. As the functors $KK^{\mathrm{SU}_q(2)}(\cdot, A)$ and $KK^{\mathrm{SU}_q(2)}(\cdot, B)$ are cohomological and transform the c_0-direct sums into direct products [17], it follows that in order to check (5.1) it suffices to consider C with trivial $\mathrm{SU}_q(2)$-action. For such C we have the Green–Julg isomorphisms

$$KK^{\mathrm{SU}_q(2)}(C, A) \cong KK(C, A \rtimes \mathrm{SU}_q(2))$$

and

$$KK^{\mathrm{SU}_q(2)}(C, B) \cong KK(C, B \rtimes \mathrm{SU}_q(2)),$$

see [28, Theorem 5.7]. Therefore x is a $KK^{\mathrm{SU}_q(2)}$-equivalence if and only if it induces an isomorphism $KK(C, A \rtimes \mathrm{SU}_q(2)) \cong KK(C, B \rtimes \mathrm{SU}_q(2))$ for every separable C*-algebra C, that is, if and only if it defines a KK-equivalence between $A \rtimes \mathrm{SU}_q(2)$ and $B \rtimes \mathrm{SU}_q(2)$. ■

Combining this with the Universal Coefficient Theorem [21], we get the following.

Corollary 5.3. *Assume A and B are separable $\mathrm{SU}_q(2)$-C^*-algebras such that $A \rtimes \mathrm{SU}_q(2)$ and $B \rtimes \mathrm{SU}_q(2)$ satisfy the UCT. Then a class $x \in KK^{\mathrm{SU}_q(2)}(A, B)$ is a $KK^{\mathrm{SU}_q(2)}$-equivalence if and only if it induces an isomorphism $K_*(A \rtimes \mathrm{SU}_q(2)) \cong K_*(B \rtimes \mathrm{SU}_q(2))$.*

Let us now fix our conventions and notation for the crossed products. Given a compact quantum group G, consider its right regular representation $V \in M(\mathcal{K}(L^2(G)) \otimes C(G))$, defined by

$$V(a\xi_h \otimes \xi) = \Delta(a)(\xi_h \otimes \xi), \quad a \in C(G),\ \xi \in L^2(G),$$

where $\xi_h = 1 \in C(G) \subset L^2(G)$. Then $V(a \otimes 1)V^* = \Delta(a)$. The integrated form of V is the representation

$$\rho = \pi_V : C^*(G) = c_0(\hat{G}) = c_0\text{-}\bigoplus_{s \in \mathrm{Irr}(G)} B(H_s) \to B(L^2(G)),$$

$$\rho(\omega) = (\iota \otimes \omega)(V) \quad \text{for } \omega \in c_c(\hat{G}) \subset C^*(G).$$

Explicitly, letting $\omega * a = (\iota \otimes \omega)\Delta(a) \in C(G)$ for $\omega \in c_c(\hat{G})$ and $a \in C(G)$, we have

$$\rho(\omega)a\xi_h = (\omega * a)\xi_h.$$

Given an action $\alpha: A \to A \otimes C(G)$, the crossed product is defined by

$$A \rtimes G = \overline{\alpha(A)(1 \otimes \rho(C^*(G)))} \subset M(A \otimes \mathcal{K}(L^2(G))).$$

Proof of Theorem 5.1. Let $q \in (0, 1]$ be given by (3.2). By [30, Theorem 8.5], the functor $A \mapsto A \,\square_{\mathrm{SU}_{\tau q}(2)}\, B(\mathrm{SU}_{\tau q}(2), O_P^+)$ extends to an equivalence of the equivariant KK-categories. Therefore by Proposition 2.7 it suffices to prove the theorem for the polynomials

$$P = q^{-1/2} X_1 X_2 - \tau q^{1/2} X_2 X_1, \quad q \in (0, 1],\ \tau = \pm 1.$$

Consider the short exact sequence $0 \to \mathcal{K}(\mathcal{F}_P) \to \mathcal{T}_P \to \mathcal{O}_P \to 0$. Passing to crossed products we get a short exact sequence

$$0 \to \mathcal{K}(\mathcal{F}_P) \rtimes \mathrm{SU}_{\tau q}(2) \to \mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2) \to \mathcal{O}_P \rtimes \mathrm{SU}_{\tau q}(2) \to 0. \tag{5.2}$$

Since $\mathcal{O}_P \cong C(\mathrm{SU}_{\tau q}(2))$ by Proposition 3.2, by the Takesaki–Takai duality we have $\mathcal{O}_P \rtimes \mathrm{SU}_{\tau q}(2) \cong \mathcal{K}(L^2(\mathrm{SU}_{\tau q}(2)))$. Since the action of $\mathrm{SU}_{\tau q}(2)$ on $\mathcal{K}(\mathcal{F}_P)$ is implemented by the unitary representation $U_P = \bigoplus_{n=0}^{\infty} U_n$, we also have an isomorphism

$$\mathcal{K}(\mathcal{F}_P) \rtimes \mathrm{SU}_{\tau q}(2) \cong \mathcal{K}(\mathcal{F}_P) \otimes \rho(C^*(\mathrm{SU}_{\tau q}(2))), \quad X \mapsto U_P^* X U_P.$$

As $C^*(\mathrm{SU}_{\tau q}(2)) = c_0\text{-}\bigoplus_{n=0}^{\infty} B(H_n)$, it follows that we can write (5.2) as

$$0 \to c_0\text{-}\bigoplus_{n=0}^{\infty} \mathcal{K}(\mathcal{F}_P \otimes H_n) \to \mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2) \to \mathcal{K}(L^2(\mathrm{SU}_{\tau q}(2))) \to 0. \tag{5.3}$$

In this picture the canonical homomorphisms of $C^*(\mathrm{SU}_{\tau q}(2))$ into $M(\mathcal{K}(\mathcal{F}_P) \rtimes \mathrm{SU}_{\tau q}(2))$ and $\mathcal{O}_P \rtimes \mathrm{SU}_{\tau q}(2)$ are $(\pi_{U_P \otimes U_n})_{n=0}^{\infty}$ (since $(U_P^*)_{12} V_{23} (U_P)_{12} = (U_P)_{13} V_{23}$) and ρ, respectively.

From (5.3) it is clear that $\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)$ is a type I C*-algebra, hence it satisfies the UCT, and $K_1(\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)) = 0$. Since $C^*(\mathrm{SU}_{\tau q}(2))$ is also of type I with trivial K_1-group, by Corollary 5.3 we just need to show that the canonical embedding

$$\pi: C^*(\mathrm{SU}_{\tau q}(2)) \to \mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2), \quad x \mapsto 1 \otimes \rho(x),$$

induces an isomorphism of the K_0-groups.

Further, we identify $K_0(C^*(\mathrm{SU}_{\tau q}(2)))$ with the representation ring $R(\mathrm{SU}_{\tau q}(2)) = \bigoplus_{n=0}^{\infty} \mathbb{Z}[U_n]$; in other words, $[U_n] \in K_0(C^*(\mathrm{SU}_{\tau q}(2)))$ denotes the class of a rank one projection in $B(H_n)$. From (5.3) we see that $K_0(\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2))$ is a free abelian group with generators $[p_n]$, $n \in \mathbb{Z}_+ \cup \{\infty\}$, where p_n is a rank one projection in $\mathcal{K}(\mathcal{F}_P \otimes H_n)$ for $n \in \mathbb{Z}_+$ and p_∞ is a projection in $\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)$ such that its image in $\mathcal{K}(L^2(\mathrm{SU}_{\tau q}(2)))$ is a rank one projection. As p_∞ we can take the image of $1 \in B(H_0)$ under $\pi : C^*(\mathrm{SU}_{\tau q}(2)) \to \mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)$, so that

$$\pi_*([U_0]) = [p_\infty].$$

Now, let us fix $n \geq 1$ and compute $\pi_*([U_n])$. We have

$$\pi_*([U_n]) = c[p_\infty] + \sum_{k=0}^{\infty} c_k [p_k] \tag{5.4}$$

for some $c, c_k \in \mathbb{Z}$, with only finitely many nonzero coefficients. The homomorphism $\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2) \to \mathcal{K}(L^2(\mathrm{SU}_{\tau q}(2)))$ kills all the projections p_n, $n \geq 0$. On the other hand, its composition with π is the right regular representation ρ. As the multiplicity of U_n in V is $\dim H_n = n + 1$, we conclude that $c = n + 1$.

For $k \geq 0$, consider the representation of $\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)$ on $\mathcal{F}_P \otimes H_k$. Since the multiplicity of every isotopical component of $U_P \otimes U_k$ is finite, this is a representation by compact operators. Thus, we get a homomorphism

$$K_0(\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2)) \to K_0(\mathcal{K}(\mathcal{F}_P \otimes H_k)).$$

Applying it to (5.4) we obtain

$$(\pi_{U_P \otimes U_k})_*([U_n]) = [n+1](\pi_{U_P \otimes U_k})_*([U_0]) + c_k[p_k] \quad \text{in } K_0(\mathcal{K}(\mathcal{F}_P \otimes H_k)).$$

Therefore, if we denote by m_{lk} the multiplicity of U_l in $U_P \otimes U_k$, then

$$c_k = m_{nk} - (n+1)m_{0k}.$$

Since $U_P = \bigoplus_{i=0}^{\infty} U_i$ and $U_l \otimes U_k \cong U_{|l-k|} \oplus U_{|l-k|+2} \oplus \cdots \oplus U_{l+k}$, we have

$$m_{lk} = \frac{1}{2}(l + k - |l-k|) + 1, \quad \text{hence} \quad c_k = \frac{1}{2}(k - n - |k-n|).$$

To summarize, for all $n \geq 0$ we have

$$\pi_*([U_n]) = (n+1)[p_\infty] + \sum_{k=0}^{n-1} (k-n)[p_k].$$

This shows that $\pi_* : R(\mathrm{SU}_{\tau q}(2)) \to K_0(\mathcal{T}_P \rtimes \mathrm{SU}_{\tau q}(2))$ is indeed an isomorphism. ∎

Acknowledgements. We are grateful to Stefaan Vaes and Christian Voigt for illuminating comments on linking algebras and KK-theory.

Funding. Supported by the NFR project 300837 “Quantum Symmetry”.

References

[1] A. Andersson, Berezin quantization of noncommutative projective varieties. 2018, arXiv:1506.01454v8

[2] A. Arias and G. Popescu, Noncommutative interpolation and Poisson transforms. *Israel J. Math.* **115** (2000), 205–234 Zbl 0967.47045 MR 1749679

[3] F. Arici and J. Kaad, Gysin sequences and $SU(2)$-symmetries of C^*-algebras. *Trans. London Math. Soc.* **8** (2021), no. 1, 440–492 MR 4451552

[4] W. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.* **181** (1998), no. 2, 159–228 Zbl 0952.46035 MR 1668582

[5] T. Banica, Théorie des représentations du groupe quantique compact libre $\mathrm{O}(n)$. *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), no. 3, 241–244 Zbl 0862.17010 MR 1378260

[6] B. V. R. Bhat and T. Bhattacharyya, A model theory for q-commuting contractive tuples. *J. Operator Theory* **47** (2002), no. 1, 97–116 Zbl 1019.47014 MR 1905815

[7] B. V. R. Bhat and M. Mukherjee, Inclusion systems and amalgamated products of product systems. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **13** (2010), no. 1, 1–26 Zbl 1198.46050 MR 2646788

[8] J. Bichon, A. De Rijdt, and S. Vaes, Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups. *Comm. Math. Phys.* **262** (2006), no. 3, 703–728 Zbl 1122.46046 MR 2202309

[9] H. Cartan and S. Eilenberg, *Homological algebra*. Princeton University Press, Princeton, NJ, 1956 MR 77480

[10] K. De Commer and M. Yamashita, Tannaka–Kreĭn duality for compact quantum homogeneous spaces II. Classification of quantum homogeneous spaces for quantum SU(2). *J. Reine Angew. Math.* **708** (2015), 143–171 Zbl 1334.46050 MR 3420332

[11] A. De Rijdt and N. Vander Vennet, Actions of monoidally equivalent compact quantum groups and applications to probabilistic boundaries. *Ann. Inst. Fourier (Grenoble)* **60** (2010), no. 1, 169–216 Zbl 1331.46063 MR 2664313

[12] S. Doplicher, R. Longo, J. E. Roberts, and L. Zsidó, A remark on quantum group actions and nuclearity. *Rev. Math. Phys.* **14** (2002), no. 7-8, 787–796 Zbl 1030.46091 MR 1932666

[13] A. Dor-On, Toeplitz quotient C^*-algebras and ratio limits for random walks. *Doc. Math.* **26** (2021), 1529–1556 Zbl 1475.60082 MR 4353332

[14] I. B. Frenkel and M. G. Khovanov, Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$. *Duke Math. J.* **87** (1997), no. 3, 409–480 Zbl 0883.17013 MR 1446615

[15] V. F. R. Jones, Index for subfactors. *Invent. Math.* **72** (1983), no. 1, 1–25 Zbl 0508.46040 MR 696688

[16] E. T. A. Kakariadis and O. M. Shalit, Operator algebras of monomial ideals in noncommuting variables. *J. Math. Anal. Appl.* **472** (2019), no. 1, 738–813 Zbl 1425.46047 MR 3906397

[17] R. Meyer and R. Nest, The Baum–Connes conjecture via localisation of categories. *Topology* **45** (2006), no. 2, 209–259 Zbl 1092.19004 MR 2193334

[18] S. Neshveyev and L. Tuset, The Martin boundary of a discrete quantum group. *J. Reine Angew. Math.* **568** (2004), 23–70 Zbl 1130.46041 MR 2034922

[19] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*. Cours Spéc. 20, Société Mathématique de France, Paris, 2013 Zbl 1316.46003 MR 3204665

[20] G. Popescu, von Neumann inequality for $(B(\mathcal{H})^n)_1$. *Math. Scand.* **68** (1991), no. 2, 292–304 Zbl 0774.46033 MR 1129595

[21] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. *Duke Math. J.* **55** (1987), no. 2, 431–474 Zbl 0644.46051 MR 894590

[22] O. M. Shalit and B. Solel, Subproduct systems. *Doc. Math.* **14** (2009), 801–868 Zbl 1189.46056 MR 2608451

[23] K.-H. Ulbrich, Fibre functors of finite-dimensional comodules. *Manuscripta Math.* **65** (1989), no. 1, 39–46 Zbl 0674.16006 MR 1006625

[24] S. Vaes and N. Vander Vennet, Identification of the Poisson and Martin boundaries of orthogonal discrete quantum groups. *J. Inst. Math. Jussieu* **7** (2008), no. 2, 391–412 Zbl 1139.46044 MR 2400727

[25] S. Vaes and R. Vergnioux, The boundary of universal discrete quantum groups, exactness, and factoriality. *Duke Math. J.* **140** (2007), no. 1, 35–84 Zbl 1129.46062 MR 2355067

[26] A. Van Daele and S. Wang, Universal quantum groups. *Internat. J. Math.* **7** (1996), no. 2, 255–263 Zbl 0870.17011 MR 1382726

[27] U. Venugopalkrishna, Fredholm operators associated with strongly pseudoconvex domains in C^n. *J. Functional Analysis* **9** (1972), 349–373 Zbl 0241.47023 MR 315502

[28] R. Vergnioux, *KK-théorie équivariante et opérateur de julg–valette pour les groupes quantiques*. Ph.D. thesis, Université Paris–Diderot, Paris, 2002

[29] A. Viselter, Cuntz–Pimsner algebras for subproduct systems. *Internat. J. Math.* **23** (2012), no. 8, article no. 1250081 Zbl 1260.46039 MR 2949219

[30] C. Voigt, The Baum–Connes conjecture for free orthogonal quantum groups. *Adv. Math.* **227** (2011), no. 5, 1873–1913 Zbl 1279.19002 MR 2803790

[31] H. Wenzl, On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada* **9** (1987), no. 1, 5–9 Zbl 0622.47019 MR 873400

Received 17 December 2021.

Erik Habbestad
Mathematics Institute, University of Oslo, PB 1053 Blindern, 0316 Oslo, Norway;
erikhab@math.uio.no

Sergey Neshveyev
Mathematics Institute, University of Oslo, PB 1053 Blindern, 0316 Oslo, Norway;
sergeyn@math.uio.no