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Discrete quantum structures I: Quantum predicate logic

Andre Kornell

Abstract. A discrete quantum structure is a discrete quantum space that is equipped with relations and functions of various finite arities. Discrete quantum spaces are identified with hereditarily atomic von Neumann algebras, their relations with projection operators, and their functions with unital normal $*$-homomorphisms. The propositional quantum logic of Birkhoff and von Neumann has been extended to a predicate quantum logic by Weaver; we investigate this predicate quantum logic as the internal logic of discrete quantum structures. We extend this predicate quantum logic to include function symbols and an equality symbol. Overall, we recover the basic structures of discrete quantum mathematics from physical first principles. More complicated structures will be recovered similarly in part II of this paper.

1. Introduction

1.1. Equality

This paper establishes a connection between quantum logic and discrete noncommutative mathematics. The study of quantum logic was initiated by Birkhoff and von Neumann, who drew an analogy between the lattice of projection operators in a von Neumann algebra and the lattice of measurable subsets of a measure space, modulo null sets [6, Secs. 5, 6], providing our interpretation of the Boolean connectives $\neg$, $\wedge$, and $\vee$. The lattice of projection operators was then investigated as a propositional logic, providing our interpretation of the Boolean connective $\rightarrow$ [13, 19, 32, 44]. Weaver extended this quantum propositional logic to a quantum predicate logic, providing our interpretation of the quantifiers $\forall$ and $\exists$ [56, Sec. 2.6]. Motivated by the same physical and logical considerations, we continue this line of research by suggesting an interpretation of the equality relation.

Noncommutative mathematics in the sense of noncommutative geometry may be said to originate with the observation of Gelfand and Naĭmark that commutative unital C*-algebras are in duality with compact Hausdorff spaces [15, Lem. 1]. The notion of a locally compact quantum space, i.e., a pseudospace, as an object that is formally dual to a C*-algebra was first clearly enunciated by Woronowicz [59, Sec. 1]. The notion of a discrete quantum space as an object that is formally dual to a c_0-direct sum of full matrix algebras then appeared implicitly in the work of Podleś and Woronowicz on Pontryagin duality for

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compact quantum groups [42, Sec. 2]. The discrete quantum structures considered in this paper are all essentially discrete quantum spaces equipped with additional structure.

Discrete quantum structures generalize the structures of many-sorted first-order logic [46], which consist of sets equipped with functions and relations, e.g., groups, graphs, and vector spaces. Discrete quantum structures have not been previously considered in full generality, but their definition is already implicit in the established generalizations of noncommutative mathematics. For technical simplicity, we generalize sets to von Neumann algebras that are ℓ^∞-direct sums of full matrix algebras, rather than to C*-algebras that are c_0-direct sums of full matrix algebras. These are the hereditarily atomic von Neumann algebras [30, Prop. 5.4]. We generalize the Cartesian product to the spatial tensor product, we generalize relations to projections, and we generalize functions to unital normal $*$-homomorphisms in the opposite direction.

The only complication to this straightforward narrative is that sometimes the order of multiplication in a von Neumann algebra M is unexpectedly reversed. For example, the multiplication of a discrete quantum group is a unital normal $*$-homomorphism $M \to M \mathbin{\overline{\otimes}} M$, but the adjacency relation of a discrete quantum graph is a projection in

$$M \mathbin{\overline{\otimes}} M^{\mathrm{op}}.$$

Of course, if M is commutative, then $M^{\mathrm{op}} = M$, so this complication is a phenomenon that is peculiar to the quantum setting.

For each von Neumann algebra M, we interpret the equality relation to be the largest projection $\delta_M \in M \mathbin{\overline{\otimes}} M^{\mathrm{op}}$ that is orthogonal to $p \otimes (1-p)$ for every projection $p \in M$. If $M = \ell^\infty(A)$ for some set A, then δ_M is the projection that corresponds to the diagonal of the Cartesian square $A \times A$. However, if $M = L^\infty(\mathbb{R})$, then $\delta_M = 0$. The intuitive explanation for this phenomenon is that the diagonal of the Cartesian square $\mathbb{R} \times \mathbb{R}$ has Lebesgue measure zero. The equality relation δ_M is suitably nondegenerate for precisely the class of hereditarily atomic von Neumann algebras. Indeed, the following are equivalent:

(1) δ_M is not orthogonal to $p \otimes p$ for any nonzero projection $p \in M$;

(2) M is hereditarily atomic.

This equivalence is proved in Appendix A.1, and it provides an additional justification for our focus on this class of von Neumann algebras. It also provides an additional characterization of this class [30, Prop. 5.4]. In contrast, it is routine to verify that 0 is the only projection in $M_2(\mathbb{C}) \mathbin{\overline{\otimes}} M_2(\mathbb{C})$ that is orthogonal to $p \otimes (1-p)$ for every projection $p \in M_2(\mathbb{C})$.

1.2. Physical intuition

The projection δ_M may be equivalently defined as the infimum of all projections of the form $p \otimes p + (1-p) \otimes (1-p)$ for a projection $p \in M$. To interpret this definition physically, we regard von Neumann algebras as abstract physical systems. Then, the von Neumann algebra is hereditarily atomic if and only if the physical system is discrete in the sense that each observable admits a complete set of pairwise orthogonal eigenstates [30, Prop. 5.4]. The normal states of the von Neumann algebra are the states of the physical system, the projections in the von Neumann algebra are the Boolean observables of the physical system, and the tensor product of two von Neumann algebras is the composite of two spatially separated physical systems. In the case of hereditarily atomic von Neumann algebras, the categorical tensor product is also the spatial tensor product [17, Prop. 8.6].

Let M be a hereditarily atomic von Neumann algebra. From the physical perspective, δ_M is a Boolean observable on the composite system $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ that guarantees equal outcomes for pairs of equal Boolean observables on M and M^{op}. Furthermore, every Boolean observable with this property implies δ_M: if the former is measured and found to be true, then a measurement of the latter will also find it to be true. Thus, δ_M may be characterized as the Boolean observable that is true in exactly those states of the system $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ that guarantee equal outcomes for pairs of equal Boolean observables. Such states are used in perfect quantum strategies for synchronous games [2, Sec. 5.2] and [9, 37, 40].

Viewed as abstract physical systems, M and M^{op} have exactly the same states and observables. Probed separately, their physics is indistinguishable. However, the composite systems $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ and $M \mathbin{\bar{\otimes}} M$ exhibit different physics. Of course, $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ and $M \mathbin{\bar{\otimes}} M$ are isomorphic as von Neumann algebras because M is hereditarily atomic, but there is generally no isomorphism between them that fixes the projections of both tensor factors, as the example $M = M_2(\mathbb{C})$ demonstrates. If $M \neq 0$, then there exists at least one state on the system $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ that guarantees equal outcomes for pairs of equal Boolean observables, but there need not be a state on the system $M \mathbin{\bar{\otimes}} M$ with this property.

Fancifully, we might regard M and M^{op} as otherwise isomorphic physical systems that are oriented oppositely in time. To motivate this intuition, we consider the example of an electron-positron pair that is produced by a neutral pion decay [10]. We may model the spin of the electron by $M_2(\mathbb{C})$ and the spin of the positron by $M_2(\mathbb{C})^{\mathrm{op}}$. The conservation of angular momentum then implies that their magnetic moments are equal along any axis of measurement. Thus, the composite system $M_2(\mathbb{C}) \mathbin{\bar{\otimes}} M_2(\mathbb{C})^{\mathrm{op}}$ is in the unique state such that a measurement of $\delta_{M_2(\mathbb{C})}$ is guaranteed to yield 1. Furthermore, this observation demonstrates that the spin of the positron must be modeled by the opposite operator algebra, because the composite system must have a state with zero total angular momentum. Of course, a positron is sometimes regarded as an electron traveling backward in time [12, 47].

1.3. Quantum sets

The development in this paper proceeds in terms of quantum sets, their functions, and their binary relations. The reader may choose to view each quantum set $\mathcal{X}$ as an object that is formally dual to a hereditarily atomic von Neumann algebra in the same way that a pseudospace is formally dual to a C*-algebra [59]. In this account, the category of quantum sets and functions is defined to be the opposite of the category of hereditarily atomic von Neumann algebras and unital normal $*$-homomorphisms, and the category of quantum sets and binary relations is defined to be the category of hereditarily atomic von Neumann algebras and Weaver's quantum relations [57]. The former category is then included into the latter category via the equivalence in [29].

Formally, we instead define a quantum set $\mathcal{X}$ to be an object whose data consists of a set $\mathrm{At}(\mathcal{X})$ of nonzero finite-dimensional Hilbert spaces called the atoms of $\mathcal{X}$ [30]. The corresponding hereditarily atomic von Neumann algebra $\ell^\infty(\mathcal{X})$ is then defined to be the ℓ^∞-direct sum of the factors $L(X)$ for $X \in \mathrm{At}(\mathcal{X})$. Each quantum set $\mathcal{X}$ has a dual $\mathcal{X}^*$ that is obtained by dualizing all the atoms of $\mathcal{X}$, and $\ell^\infty(\mathcal{X}^*)$ is naturally isomorphic to $\ell^\infty(\mathcal{X})^{\mathrm{op}}$. Each pair of quantum sets, $\mathcal{X}$ and $\mathcal{Y}$, has a Cartesian product $\mathcal{X} \times \mathcal{Y}$ that is obtained by forming all possible tensor products of an atom of $\mathcal{X}$ with an atom of $\mathcal{Y}$, and $\ell^\infty(\mathcal{X} \times \mathcal{Y})$ is naturally isomorphic to $\ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Y})$.

A binary relation R from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ is just a choice of subspaces $R(X, Y) \leq L(X, Y)$ for all $X \in \mathrm{At}(\mathcal{X})$ and $Y \in \mathrm{At}(\mathcal{Y})$. Binary relations from $\mathcal{X}$ to $\mathcal{Y}$ are in one-to-one correspondence with projections in the hereditarily atomic von Neumann algebra $\ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Y})^{\mathrm{op}}$ [57, Prop. 2.23]. Thus, binary relations from $\mathcal{X}$ to $\mathcal{Y}$ form an orthomodular lattice [24]. The one-to-one correspondence between binary relations from $\mathcal{X}$ to $\mathcal{Y}$ and quantum relations from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$ in Weaver's sense is verified in Appendix A.2. The category **qRel** of quantum sets and binary relations is dagger compact [30, Thm. 3.6], i.e., strongly compact, and this enables our extensive use of the graphical calculus [1, 41].

The quantum sets in this paper are essentially just discrete quantum spaces, which first arose in the study of compact quantum groups [42]. Finite quantum sets, which are formally dual to finite-dimensional C*-algebras, were identified even earlier [59]. Finite quantum sets appear naturally in the study of quantum symmetry [55] and quantum information [38]. In [38], finite quantum sets appear as special symmetric dagger Frobenius algebras in the category of finite-dimensional Hilbert spaces [54]. The class of finite quantum sets is sufficient and convenient for many applications, but the class of all quantum sets has two significant advantages: discrete quantum groups are generally infinite, and the category of all quantum sets and functions is closed monoidal [30, Thm. 9.1].

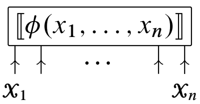
Two closely related quantum generalizations of sets have been proposed. Giles generalized sets essentially to atomic von Neumann algebras, calling them q-spaces [16]. Rump has also recently proposed a striking geometric definition of quantum sets [43]. For both of these definitions and for our definition, a quantum set is an object that may be partitioned into irreducible components, and up to isomorphism, the predicates on each component are the closed subspaces of some Hermitian space [25], forming a complete atomic orthomodular lattice. For the moment, our choice of definition appears to be the most compatible with the established body of noncommutative generalizations [30, 42]. Other quantum analogs of set theory exist, but they are less closely related [45, 49].

The notion of a quantum relation is defined for arbitrary von Neumann algebras, but in effect, this paper treats only hereditarily atomic von Neumann algebras. This feature of the approach is discussed in Section 1.6.

1.4. Results

The semantics that we define in this paper assigns an interpretation to each nonduplicating term and to each nonduplicating formula in a language of many-sorted first-order logic that draws its nonlogical symbols from the category **qRel**. The qualifier "nonduplicating" refers to a syntactic constraint that reflects the absence of a diagonal function for quantum sets and for quantum spaces more generally [59] and the impossibility of broadcasting quantum states [4]. The sorts of our language are quantum sets. Its relation symbols are binary relations into the monoidal unit **1**, and its function symbols are binary relations that are functions [30, Def. 4.1]. The quantum set **1** consists of a single one-dimensional atom, which we take to be $\mathbb{C}$. Equality for sort $\mathcal{X}$ is a binary relation $E_{\mathcal{X}}$ from $\mathcal{X} \times \mathcal{X}^*$ to **1**. It is both the counit of the dagger compact structure of **qRel** and the binary relation into **1** that corresponds canonically to the projection $\delta_{\ell^\infty(\mathcal{X})}$ that was defined in Section 1.1.

The semantics interprets each nonduplicating formula $\phi(x_1, \ldots, x_n)$, whose distinct free variables $x_1, \ldots, x_n$ are of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively, as a binary relation $[\![\phi(x_1, \ldots, x_n)]\!]$ from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to **1**. In the graphical calculus, this is depicted as follows:

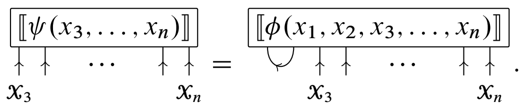


Our core computational device relates the equality relation to the graphical calculus.

Theorem 1.4.1 (Also Theorem 3.3.2). *Let $\phi(x_1, x_2, x_3, \ldots, x_n)$ be a nonduplicating formula, whose distinct free variables $x_1, x_2, x_3, \ldots, x_n$ are of sorts $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n$, respectively. Assume that $\mathcal{X}_2 = \mathcal{X}_1^*$, and let $\psi(x_3, \ldots, x_n)$ be the nonduplicating formula*

$$(\exists x_1)\,(\exists x_2)\,(\phi(x_1, x_2, x_3, \ldots, x_n)\ \&\ E_{\mathcal{X}_1}(x_1, x_2)).$$

Then,



In the graphical calculus, the object $\mathcal{X}_1^*$ may be depicted as an upward-directed wire labeled $\mathcal{X}_1^*$ or as a downward-directed wire labeled $\mathcal{X}_1$. Thus, we may connect the wire depicting $\mathcal{X}_1$ with the wire depicting $\mathcal{X}_2 = \mathcal{X}_1^*$, as shown. We use the symbol & for the Sasaki projection connective, which is defined by $P \,\&\, Q = (P \vee \neg Q) \wedge Q$ [44, Def. 5.1].

Theorem 1.4.1 provides an unexpected connection between matrix multiplication and quantum logic. Each nonzero finite-dimensional Hilbert space H is associated with a quantum set $\mathcal{H}$, whose only atom is H. Furthermore, each operator r on H is associated with a binary relation R from $\mathcal{H} \times \mathcal{H}^*$ to $\mathbb{C}$, whose only component is the span of the functional $H \otimes H^* \to \mathbb{C}$ that is canonically obtained from r. Theorem 1.4.1 has the following corollary.

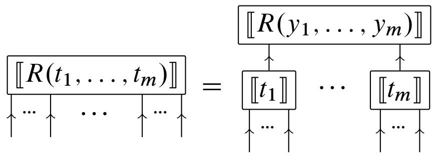
Corollary 1.4.2 (Also Corollary 3.3.3). *Let H be a nonzero finite-dimensional Hilbert space, and let $\mathcal{H}$ be the associated quantum set. Let r, s, and t be operators on H, and let R, S, and T be the associated binary relations from $\mathcal{H} \times \mathcal{H}^*$ to* $\mathbf{1}$. *Then, the following are equivalent:*

(1) $[\![(\exists x_2)\,(\exists x_3)\,((R(x_1, x_2) \wedge S(x_3, x_4))\ \&\ E_{\mathcal{H}}(x_3, x_2))]\!] = T$;

(2) *sr and t are scalar multiples of each other.*

Thus, matrix multiplication is implicit in the ortholattice structure of subspaces of finite-dimensional Hilbert spaces and their tensor products.

This first part of the paper concludes with a detailed treatment of the semantics of terms. Each nonduplicating term t of sort $\mathcal{Y}$, whose distinct free variables $x_1, \ldots, x_n$ are of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively, is interpreted as a function $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$. Such a function is equivalently a unital normal $*$-homomorphism $\ell^\infty(\mathcal{Y}) \to \ell^\infty(\mathcal{X}_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X}_n)$. Formally, such a function is defined to be a binary relation from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to $\mathcal{Y}$ of a particular kind [30, Def. 4.1]. This convention highlights the compositionality of the semantics:



for each binary relation R from $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$ to $\mathbf{1}$ and terms $t_1, \ldots, t_m$ of sorts $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$, respectively; see Lemma 3.5.2.

Functions are introduced indirectly via function graphs. While a function $\mathcal{X} \to \mathcal{Y}$ is formally a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, the corresponding function graph is a binary relation from $\mathcal{X} \times \mathcal{Y}^*$ to $\mathbf{1}$. Such function graphs may be defined in the semantics of this paper.

Theorem 1.4.3 (Also Theorem 3.4.2). *Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. There is a canonical bijective correspondence between the following sets:*

(1) *unital normal $*$-homomorphisms $\ell^\infty(\mathcal{Y}) \to \ell^\infty(\mathcal{X})$,*

(2) *binary relations G from $\mathcal{X} \times \mathcal{Y}^*$ to* $\mathbf{1}$ *such that*

  (a) $[\![(\forall x)\,(\exists y)\, G(x, y)]\!]$ *and*

  (b) $[\![(\forall y_1)\,(\forall y_2)\,((\exists x_1)\,(\exists x_2)\,((G_*(x_1, y_1) \wedge G(x_2, y_2))\ \&\ E_{\mathcal{X}}(x_2, x_1)) \to E_{\mathcal{Y}}(y_1, y_2))]\!]$

  are both equal to the larger of the two binary relations on $\mathbf{1}$.

The purpose of introducing functions in this indirect way is to minimize the conceptual assumptions that lead to the structures that we consider. This first part of the paper motivates the basic ingredients of discrete quantum mathematics essentially just from the formalization of propositions by the closed subspaces of a Hilbert space and the formalization of composite quantum systems by tensor-product Hilbert spaces.

1.5. Motivation

The many quantum generalizations that underlie noncommutative mathematics are motivated by diverse considerations, but they are nevertheless mutually compatible, and this is true even within discrete noncommutative mathematics. For example, discrete quantum groups arose in the program of extending Pontryagin duality to include noncommutative groups, more than two decades after its inception [26, 27, 42, 48, 52], and the notion of quantum isomorphism between simple graphs arose quite independently in the study of quantum nonlocality originating from the Kochen–Specker theorem [2, 9, 14, 22, 28, 36], but these two quantum generalizations are now understood to be related [3, 5, 8, 34, 39].

This compatibility between quantum generalizations of disparate origins demands an explanation, and the simplest possible explanation is that the quantum generalizations that underlie noncommutative mathematics are all instances of a single quantum generalization. Such an explanation requires a general notion of quantum structure and furthermore a method for extending each class of ordinary structures to a class of quantum structures. For comparison, in [59], Woronowicz not only defines the general notion of a locally compact quantum space, i.e., a pseudospace, but also extends the classes of finite, finite-dimensional, and compact locally compact spaces to classes of finite, finite-dimensional, and compact locally compact quantum spaces, respectively. The problem is to extend this approach to encompass all possible quantum structures and all possible properties.

This paper proposes a solution to this problem in the special case of discrete structures. We may define a discrete structure to consist of sets, relations, and functions, and similarly, we may define a discrete quantum structure to consist of quantum sets, relations, and functions [30]. The role of quantum predicate logic with equality is then to extend each class of ordinary structures to a class of quantum structures. The nonduplicating formulas are the classes that we are extending, and the semantics is the method by which we are extending these classes. Formally, a formula of many-sorted first-order logic is defined to be *nonduplicating* if no variable appears more than once in any atomic subformula. This restriction is weaker than it might appear; it is routine to show that every formula of many-sorted first-order logic with equality is equivalent over discrete structures to a nonduplicating formula.

Many established classes of discrete quantum structures are unified in this way. The second part of this paper will treat quantum graphs [11], quantum metric spaces[33], quantum posets [57, 58], quantum graph homomorphisms [36], quantum graph isomorphisms [2], quantum permutations [55], and quantum groups [53], all discrete in the sense that the underlying von Neumann algebra is hereditarily atomic.

1.6. Arbitrary von Neumann algebras

The notion of a quantum relation is defined for all von Neumann algebras [57] and not just those that are hereditarily atomic. However, this paper treats only hereditarily atomic von Neumann algebras. There is no established notion of orthocomplement for quantum relations in the general case. As a consequence, there is no obvious interpretation of negation or, more generally, of implication. Thus, Theorem 1.4.1 cannot even be stated.

The naive approach to defining the orthocomplement of a quantum relation on an arbitrary von Neumann algebra M fails immediately. For illustration, let $M = L(H)$ for a separable infinite-dimensional Hilbert space H. The identity quantum relation on M is the span of the identity operator on H, and the orthocomplement of this identity quantum relation should consist of operators that are in some sense orthogonal to the identity operator. Hence, naively, the orthocomplement should consist of trace-zero operators. However, the trace-zero operators on H do not form an ultraweakly closed subspace of $L(H)$, and furthermore their ultraweak closure is $L(H)$ itself. Thus, we are led to define an orthocomplement that is not even a complement.

In addition to this technical obstacle, there is a significant conceptual obstacle to the generalization of predicate logic to arbitrary von Neumann algebras. In noncommutative mathematics, von Neumann algebras are a quantum generalization of well-behaved measure spaces and not of sets. There is no established generalization of predicate logic to measure spaces. Thus, a quantum predicate logic for arbitrary von Neumann algebras would be a quantum generalization of something that has not yet been defined classically.

The classical measure-theoretic setting appears to offer little in the way of simplification. The projection operators in a commutative von Neumann algebra form a complete Boolean algebra, which suggests the use of classical logic, but it is again the binary relations that are an obstacle. Whether the identity binary relation on $L^\infty(\mathbb{R})$ is viewed as a quantum relation or as a measurable relation [57], it appears to have no negation that is suitable even for intuitionistic logic.

Intuitionistic logic has previously been investigated as an internal logic of physical systems in the context of the Bohrification program [18, 20]. In this approach, a proposition is a suitable choice of propositions for each classical context, i.e., a suitable choice of projection operators for each commutative subalgebra of an operator algebra [21]. Intuitionistic logic and quantum logic correspond to related but ultimately distinct conceptions of noncontextuality in quantum systems, and it is the latter conception that forms the conceptual foundation of noncommutative geometry.

In summary, while there does exist a notion of a quantum structure that accommodates arbitrary von Neumann algebras, we have no corresponding interpretation of nonduplicating formulas with equality, and moreover, we have no conceptual reason to expect one.

1.7. Many-sorted logic

Some quantum groups are commutative, and others are not. Some quantum graphs are complete, and others are not. The core definition of this paper unifies these two notions and many others; it specifies whether or not a quantum structure possesses a classical property for a large class of quantum structures that includes both discrete groups and discrete graphs and for a large class of classical properties that includes both commutativity and completeness. Specifically, it applies to discrete quantum structures, which consist of quantum sets, binary relations, and functions, and to the classical properties that are formalized by the *nonduplicating* sentences of many-sorted first-order logic. The notion of a nonduplicating formula is defined in Definition 2.7.3. Thus, the purpose of logical formulas in this paper is to speak precisely about classical properties, which enables us to state and prove theorems at this level of generality.

More formally, we quantize the semantics of many-sorted first-order logic [46]. We presently review this semantics in a form that is convenient to this goal. For simplicity, we work with a single fixed many-sorted structure that consists of all sets and the relations and functions between them. Our language includes infinitely many *variables* for each set. The class of all *terms* is defined recursively: a variable of sort A is a term of sort A, and for each function $f\colon A_1 \times \cdots \times A_m \to B$, if $t_1, \ldots, t_m$ are terms of sorts $A_1, \ldots, A_m$, respectively, then the expression $f(t_1, \ldots, t_m)$ is a term of sort B. An *atomic formula* is then defined to be an expression of the form $R(t_1, \ldots, t_n)$, where $R \subseteq A_1 \times \cdots \times A_n$ and $t_1, \ldots, t_n$ are terms of sorts $A_1, \ldots, A_n$, respectively. The natural numbers m and n may be equal to 0. Finally, the class of all *formulas* is defined recursively: an atomic formula is a formula, and if ϕ and ψ are formulas and v is a variable, then the expressions $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $(\forall v)\,\phi$, and $(\exists v)\,\phi$ are formulas. A *sentence* is defined to be simply a formula with no free variables.

The sentences that we have defined are mathematical objects like groups and topological spaces. Tarski's analysis of semantics [50, 51] leads to the definition of truth as a property of sentences, in the same sense that commutativity and compactness are properties of groups and topological spaces, respectively. To formulate this definition, we first formalize the specification of subsets by properties. For each sequence of variables $v_1, \ldots, v_n$ of sorts $A_1, \ldots, A_n$, respectively, and each formula ϕ whose free variables are among $v_1, \ldots, v_n$, we define the subset

$$[\![(v_1, \ldots, v_n) \mid \phi]\!] \subseteq A_1 \times \cdots \times A_n,$$

the set of all tuples $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ that *satisfy* ϕ. This definition proceeds by recursion on the class of all formulas, and in the end, we obtain a partial class function whose first argument is a tuple of variables and whose second argument is a formula.

A sentence ϕ is then defined to be *true* if $[\![() \mid \phi]\!] = \{()\}$. In effect, we have that

$$[\![(v_1, \ldots, v_n) \mid \phi]\!] = \{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n \mid \phi(a_1, \ldots, a_n) \text{ is true}\}.$$

This equation emphasizes that the formula ϕ is a formula of the object language, a mathematical object about which we may state theorems, rather than a formula of the metalanguage, the language in which we state our theorems. The notation $\phi(a_1, \ldots, a_n)$ expresses the substitution of the elements $a_1, \ldots, a_n$ for the variables $v_1, \ldots, v_n$, where each element $a_i \in A_i$ is regarded as a function from the singleton set $\{*\} := \{()\}$ to the set A_i.

1.8. Conventions

Each variable is of a unique sort, which is a quantum set. However, as an aid to memory and intuition, we write $(\exists x \in \mathcal{X})\,\phi$ in place of $(\exists x)\,\phi$ and $[\![x \in \mathcal{X} \mid \phi]\!]$ in place of $[\![(x) \mid \phi]\!]$, where $\mathcal{X}$ is the sort of x. If ϕ is a formula, then we write $\phi(x_1, \ldots, x_n)$ to indicate that the free variables of ϕ are among $x_1, \dots, x_n$ and that variables $x_1, \dots, x_n$ are pairwise distinct. We do likewise for terms.

Let H and K be Hilbert spaces. We write $L(H, K)$ for the set of all bounded operators from H to K, we write $L(H)$ for the set $L(H, H)$ of all bounded operators on H, and we write H^* for the set $L(H, \mathbb{C})$ of all bounded functionals on H. Let a be a linear operator from H to K. We write $a^\dagger \in L(K, H)$ for the Hermitian adjoint of a, we write $a^* \in L(K^*, H^*)$ for the Banach space transpose of a, and we write $a_* \in L(H^*, K^*)$ for the "conjugate" $(a^\dagger)^*$ of a. Note that if A is an operator algebra on H, then A^* is canonically isomorphic to the opposite of A, that is, to the algebra A with the order of multiplication reversed. We retain the stock term "$*$-homomorphism" to mean a homomorphism that respects the Hermitian adjoint operation $a \mapsto a^\dagger$.

We write $\mathbb{C}a$ for the linear span of a bounded operator a. If a and b are bounded operators, then we write $a \cdot b$ for the product of a and b, in order to separate the factors visually and to make the operator product readily distinguishable from function application. If V and W are subspaces of bounded operators, then we write $V \cdot W$ for the linear span of $\{v \cdot w \mid v \in V,\, w \in W\}$.

Let A and B be sets. We regard each binary relation from A to B foremost as a morphism from A to B in the category of sets and binary relations. Similarly, we regard a relation of arity $(A_1, \dots, A_n)$ foremost as morphism from $A_1 \times \cdots \times A_n$ to $\{*\}$, the monoidal unit of the canonical monoidal structure on the category of sets and binary relations. Thus, a binary relation on A is essentially the same thing as a relation of arity (A, A), but we regard the former as morphism from A to A, and we regard the latter as a morphism from $A \times A$ to $\{*\}$. In the same vein, we may regard any element $a \in A$ as a morphism from $\{*\}$ to A.

We use the adjective "ordinary" to emphasize that we are using a noun in its standard mathematical sense. Thus, an ordinary set is just a set.

2. Definition

We now expound the interpretation of nonduplicating first-order formulas over quantum sets. We recall quantum sets in Section 2.1, and we define their relations in Section 2.2. We define the interpretation of primitive formulas in Section 2.3, and we extend this interpretation to all nonduplicating relational formulas in Section 2.4. Then, we define quantifiers over the diagonal in Section 2.5, and we use these quantifiers to define function graphs in Section 2.6. Finally, we define the interpretation of arbitrary nonduplicating formulas in Section 2.7.

2.1. Quantum sets

A quantum set is essentially just a set of nonzero finite-dimensional Hilbert spaces, intuitively, a union of indecomposable quantum sets. This section is a brief summary of some relevant definitions from [30].

Definition 2.1.1. A *quantum set* $\mathcal{X}$ is uniquely determined by a set $\mathrm{At}(\mathcal{X})$ of nonzero finite-dimensional Hilbert spaces, called the *atoms* of $\mathcal{X}$.

Each quantum set $\mathcal{X}$ is associated to the von Neumann algebra

$$\ell^\infty(\mathcal{X}) = \bigoplus_{X \in \mathrm{At}(\mathcal{X})} L(X),$$

which intuitively consists of all bounded complex-valued functions on $\mathcal{X}$. This algebra is typically not commutative, and thus the elements of $\mathcal{X}$ are figures of speech, rather like the points of a quantum space. Formally, $\mathcal{X}$ is equal to $\mathrm{At}(\mathcal{X})$, but intuitively, they are distinct objects, and this notational distinction affects the meaning of our expressions. For example, $\ell^\infty(\mathcal{X})$ is generally not isomorphic to $\ell^\infty(\mathrm{At}(\mathcal{X}))$. Indeed, the former von Neumann algebra is generally not commutative, but the latter von Neumann algebra is always commutative, because $\mathrm{At}(\mathcal{X})$ is just an ordinary set, which happens to consist of Hilbert spaces. This explains the circuitous language in Definition 2.1.1.

In quantum mathematics, we should recover the classical theory whenever the relevant operator algebras are all commutative. This is the definitional feature of any quantum generalization in the sense of noncommutative geometry. In our case, we observe that $\ell^\infty(\mathcal{X})$ is commutative if and only if each atom of $\mathcal{X}$ is one-dimensional. Intuitively, such atoms correspond to those elements of $\mathcal{X}$ which exist individually, apart from the other elements. This gloss clarifies how ordinary sets should be incorporated into the picture.

Definition 2.1.2. To each ordinary set A, we associate a quantum set $`A$ whose atoms are one-dimensional Hilbert spaces, with one such atom for each element of A. More generally, we say that a quantum set $\mathcal{X}$ is *classical* if and only if each of its atoms is one-dimensional.

We may gloss the first sentence of Definition 2.1.2 by the equation

$$\mathrm{At}(`A) = \{\mathbb{C}_a \mid a \in A\},$$

where $\mathbb{C}_a$ denotes a one-dimensional Hilbert space that is somehow labeled by the element a. The exact formalization of this labeling is inconsequential; it is only important that distinct elements a_1 and a_2 correspond to distinct Hilbert spaces $\mathbb{C}_{a_1}$ and $\mathbb{C}_{a_2}$ so that we have a canonical bijection $A \to \mathrm{At}(`A)$.

A property of a quantum set $\mathcal{X}$ generalizes a property of an ordinary set A if we obtain the latter from the former by replacing $\mathcal{X}$ by $`A$, and it is likewise for operations. For example, the Cartesian product of quantum sets generalizes the Cartesian product of ordinary sets.

Definition 2.1.3. The *Cartesian product* $\mathcal{X} \times \mathcal{Y}$ of quantum sets $\mathcal{X}$ and $\mathcal{Y}$ is defined by $\mathrm{At}(\mathcal{X} \times \mathcal{Y}) = \{X \otimes Y \mid X \in \mathrm{At}(X),\ Y \in \mathrm{At}(\mathcal{Y})\}$.

This is not an exact quantum generalization because $`A \times `B$ may be formally distinct from $`(A \times B)$. However, $`A \times `B$ is isomorphic to $`(A \times B)$ in the obvious sense.

The Cartesian product of quantum sets corresponds to the spatial tensor of von Neumann algebras in the sense that

$$\ell^\infty(\mathcal{X} \times \mathcal{Y}) \cong \ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Y})$$

for all quantum sets $\mathcal{X}$ and $\mathcal{Y}$.

Each quantum set $\mathcal{X}$ has a *dual* $\mathcal{X}^*$ that is defined by the equation

$$\mathrm{At}(\mathcal{X}^*) = \{X^* \mid X \in \mathrm{At}(\mathcal{X})\}$$

[30, Def. 3.4]. Dualization in this sense corresponds to reversing the order of multiplication in a von Neumann algebra: $\ell^\infty(\mathcal{X})^{\mathrm{op}}$ is canonically isomorphic to $\ell^\infty(\mathcal{X}^*)$ via the map $a \mapsto a^*$, where a^* is the transpose rather than the adjoint of a bounded operator a. Formally, we define $a^*(X) = a(X)^*$ for each atom $X \in \mathrm{At}(\mathcal{X})$. The quantum sets $\mathcal{X}$ and $\mathcal{X}^*$ are distinct but closely related. The example of pair production in Section 1.1 suggests the intuition that the elements of $\mathcal{X}^*$ are the antielements of $\mathcal{X}$. Like the elements of $\mathcal{X}$, the antielements of $\mathcal{X}$ are just figures of speech; they are not mathematical objects.

2.2. Relations on quantum sets

Intuitively, we may view each quantum set as the phase space of an abstract physical system that is discrete in the sense that each observable admits an orthonormal basis of eigenvectors [30, Prop. 5.4]. Thus, the predicates, i.e., unary relations on a quantum set $\mathcal{X}$, should be in bijection with the projections in $\ell^\infty(\mathcal{X})$. Such a projection is formally a family of projections $p_X \in L(X)$, for $X \in \mathrm{At}(\mathcal{X})$, so we may define a predicate P on $\mathcal{X}$ to be simply a family of subspaces $P(X) \leq X$, for $X \in \mathrm{At}(\mathcal{X})$, as it is done in [30, App. B]. However, for technical and intuitive reasons, we prefer to work with subspaces of the dual Hilbert spaces.

Definition 2.2.1. Let $\mathcal{X}$ be a quantum set. A *predicate* P on $\mathcal{X}$ is a function assigning a subspace $P(X) \leq L(X, \mathbb{C})$ to each atom X of $\mathcal{X}$.

The canonical one-to-one correspondence between predicates P on $\mathcal{X}$ and projections p in $\ell^\infty(\mathcal{X})$ is defined by $P(X) = L(X, \mathbb{C}) \cdot p(X)$, for $X \in \mathrm{At}(\mathcal{X})$.

For each atom X, the subspaces of $L(X, \mathbb{C})$ form a modular orthomodular lattice, and thus, the predicates on $\mathcal{X}$ themselves form a modular orthomodular lattice $\mathrm{Pred}(\mathcal{X})$, with its operations defined atomwise. This is essentially the orthomodular lattice of projections in $\ell^\infty(\mathcal{X})$. We use the standard notations $\wedge$ and $\vee$ for the meets and the joins, respectively, as well as $\bot_\mathcal{X}$ and $\top_\mathcal{X}$ for the smallest and largest predicates on a quantum set $\mathcal{X}$, respectively, but we notate the orthocomplementation by $\neg$.

Each predicate P on $\mathcal{X}$ has a *conjugate* P_*, a predicate on $\mathcal{X}^*$ that is defined by $P_*(X^*) = \{\xi_* \mid \xi \in P(X)\}$ for each atom $X \in \mathrm{At}(\mathcal{X})$. The functional ξ_* is formally defined by $\xi_* = (\xi^*)^\dagger$, and since $\xi \in X^*$, we may also characterize ξ_* by the equation $\xi_*(\eta) = \langle \xi | \eta \rangle$, for $\eta \in X^*$. Intuitively, P_* holds of those elements of $\mathcal{X}^*$ such that P

holds of their counterparts in $\mathcal{X}$. We thus obtain an isomorphism of orthomodular lattices $\mathrm{Pred}(\mathcal{X}) \to \mathrm{Pred}(\mathcal{X}^*)$.

We now define the *Cartesian product* of two predicates, generalizing the Cartesian product of two subsets to the quantum setting.

Definition 2.2.2. If P and Q are predicates on quantum sets $\mathcal{X}$ and $\mathcal{Y}$, respectively, then the predicate $P \times Q$ on $\mathcal{X} \times \mathcal{Y}$ is defined by $(P \times Q)(X \otimes Y) = P(X) \otimes Q(Y)$, for $X \in \mathrm{At}(\mathcal{X})$ and $Y \in \mathrm{At}(\mathcal{Y})$.

Both $P(X)$ and $Q(Y)$ are vector spaces of functionals, and $P(X) \otimes Q(Y)$ denotes another vector space of functionals, so we have suppressed the canonical isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Thus, $(P \times Q)(X \otimes Y)$ is essentially just the span of bilinear functionals

$$(x, y) \mapsto \xi(x)\eta(y),$$

for $\xi \in P(X)$ and $\eta \in Q(Y)$. The construction $(P, Q) \mapsto P \times Q$ corresponds to the tensor product of two projections, i.e., to the conjunction of two Boolean observables on the composite of two abstract physical systems.

Finally, we define relations on quantum sets, generalizing the relations of ordinary many-sorted logic.

Definition 2.2.3. Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets. A *relation* of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ is a predicate on the Cartesian product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, with $n \geq 0$.

Thus, a relation of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ essentially just assigns a vector space of multilinear functionals $X_1 \times \cdots \times X_n \to \mathbb{C}$ to each choice of atoms $X_1 \in \mathrm{At}(\mathcal{X}_1)$, $X_2 \in \mathrm{At}(\mathcal{X}_2)$, etc. We write $\mathrm{Rel}(\mathcal{X}_1, \dots, \mathcal{X}_n)$ for the set of all relations of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$. Permuting the quantum sets $\mathcal{X}_1, \dots, \mathcal{X}_n$ according to some permutation π of the index set $\{1, \dots, n\}$, we expect and obtain a bijection between $\mathrm{Rel}(\mathcal{X}_1, \dots, \mathcal{X}_n)$ and $\mathrm{Rel}(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(n)})$.

Definition 2.2.4. Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let π be a permutation of the index set $\{1, \dots, n\}$. For each relation R of arity $(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(n)})$, define the relation $\pi_\#(R)$ of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ by

$$\pi_\#(R)(X_1 \otimes \cdots \otimes X_n) = R(X_{\pi(1)} \otimes \cdots \otimes X_{\pi(n)}) \cdot u_\pi,$$

for all $X_1 \in \mathrm{At}(\mathcal{X}_1)$, $X_2, \in \mathrm{At}(\mathcal{X}_2)$, etc., where $u_\pi \colon X_1 \otimes \cdots \otimes X_n \to X_{\pi(1)} \otimes \cdots \otimes X_{\pi(n)}$ is the unitary operator that permutes the tensor factors according to π.

The construction $R \mapsto \pi_\#(R)$ is clearly a bijection

$$\mathrm{Rel}(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(n)}) \to \mathrm{Rel}(\mathcal{X}_1, \dots, \mathcal{X}_n),$$

with inverse $S \mapsto (\pi^{-1})_\#(S)$. Furthermore, it is an isomorphism of orthomodular lattices. Its effect on the projections corresponding to these relations is given by the canonical unital normal $*$-homomorphism

$$\ell^\infty(\mathcal{X}_{\pi(1)}) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \ell^\infty(\mathcal{X}_{\pi(n)}) \to \ell^\infty(\mathcal{X}_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \ell^\infty(\mathcal{X}_n).$$

2.3. Interpreting primitive formulas

We work with the language of many-sorted first-order logic whose sorts are the quantum sets of Section 2.1 and whose relation symbols are the relations of Section 2.2. Each sort, that is, each quantum set, is assigned an infinite stock of variables, intuitively ranging over that quantum set. Within formulas, we write $x \in \mathcal{X}$ to annotate that x has sort $\mathcal{X}$, replacing the more traditional notation $x : \mathcal{X}$. We will incorporate function symbols in Section 2.6.

Definition 2.3.1. A *primitive atomic formula* is an expression of the form $R(x_1, \dots, x_n)$, where the R is a relation of some arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ and $x_1, \dots, x_n$ are distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. The class of *primitive formulas* is defined recursively: each primitive atomic formula is a primitive formula, and if ϕ and ψ are primitive formulas and x is a variable of some sort $\mathcal{X}$, then the expressions $\neg\phi$, $\phi \wedge \psi$ and $(\forall x \in \mathcal{X})\, \phi$ are primitive formulas. A *primitive sentence* is a primitive formula with no free variables. We will sometimes write $\phi(x_1, \dots, x_n)$ in place of ϕ to indicate that the free variables of ϕ are among $x_1, \dots, x_n$ and that the variables $x_1, \dots, x_n$ are pairwise distinct.

For each sequence of distinct variables $x_1, \dots, x_n$ of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively, and each primitive formula $\phi(x_1, \dots, x_n)$, we now define a relation

$$[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]$$

of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$ to be our *interpretation* of ϕ in the *context* $x_1 \in \mathcal{X}_1, \dots, x_n \in \mathcal{X}_n$. We will occasionally simply write $[\![\phi(x_1, \dots, x_n)]\!]$ when the context is obvious.

The notation $[\![x_1 : X_1, \dots, x_n : X_n \vdash \phi(x_1, \dots, x_n)]\!]$ is more or less standard to categorical logic, but it is not as intuitive in this setting. The chosen notation is intended to suggest the standard notation for defining subsets, e.g., $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\}$. We use brackets rather than braces because the familiar bijection between subsets and predicates does not survive the quantum generalization [30, Sec. 10]. We are defining predicates.

Definition 2.3.2. Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let $x_1, \dots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. For each permutation π of $\{1, \dots, n\}$, and each relation R of arity $(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(m)})$, for some $m \leq n$, we define

$$\begin{aligned} &[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid R(x_{\pi(1)}, \dots, x_{\pi(m)})]\!] \\ &\quad = \pi_\#(R \times \top_{\mathcal{X}_{\pi(m+1)}} \times \dots \times \top_{\mathcal{X}_{\pi(n)}}). \end{aligned}$$

It is then straightforward to verify that this relation depends only on the values of π on $\{1, \dots, m\}$. Furthermore, for arbitrary primitive formulas $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$, we define

(1) $[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \neg\phi(x_1, \dots, x_n)]\!]$
$= \neg[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]$;

(2) $[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n) \wedge \psi(x_1, \dots, x_n)]\!]$
$= [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]$
$\wedge [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!]$;

(3) $[\![(x_2, \dots, x_n) \in \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \mid (\forall x_1 \in \mathcal{X}_1)\, \phi(x_1, \dots, x_n)]\!]$
$= \sup\{R \in \mathrm{Rel}(\mathcal{X}_2, \dots, \mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times R$
$\leq [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]\}$.

The quantum sets $\mathcal{X}_1, \dots, \mathcal{X}_n$ in Definition 2.3.2 are arbitrary, as are the variables $x_1, \dots, x_n$, so we have defined the interpretation of all primitive formulas by recursion over that class.

Our interpretation of universal quantification may be justified by observing that it is a straightforward generalization of the classical interpretation. A further argument was given by Weaver when he introduced this definition [56]. In essence, Weaver drew an analogy between the elements of a set and the pure normal states of a type I factor, and we could do the same here. The same analogy was later drawn by Rump [43]. However, we do not view pure states as a direct analog of elements, instead holding fast to the orthodox understanding of elements and points in noncommutative mathematics that is expressed in Section 2.1.

The interpretation of a primitive formula in the empty context is a relation of arity $()$, i.e., a predicate on the empty Cartesian product of quantum sets. By convention, this empty Cartesian product is the quantum set $\mathbf{1}$ whose only atom is the field $\mathbb{C}$ of complex numbers, considered as a one-dimensional Hilbert space. It has exactly two predicates, the predicate $\top = \top_{\mathbf{1}}$, defined by $\top(\mathbb{C}) = L(\mathbb{C}, \mathbb{C})$, and $\bot = \bot_{\mathbf{1}}$, defined by $\bot(\mathbb{C}) = 0$. It is natural to say that a formula $\phi()$, which has no free variables, is *true* if $[\![\phi()]\!] = \top$.

Proposition 2.3.3 (Also Proposition A.3.2). *Let $\mathcal{X}_1, \dots, \mathcal{X}_p$ be quantum sets, and let $x_1, \dots, x_p$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_p$, respectively. For each permutation σ of $\{1, \dots, p\}$ and each primitive formula $\phi(x_1, \dots, x_n)$, with $n \leq p$, we have that*

$$[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \cdots \times \mathcal{X}_{\sigma(p)} \mid \phi(x_1, \dots, x_n)]\!]$$
$$= (\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p}).$$

This is the expected but necessary observation that permuting the context corresponds exactly to permuting the arity of the resulting relation and that additionally any unused variable x of some sort $\mathcal{X}$ corresponds to a factor of $\top_{\mathcal{X}}$. This behavior is built into the definition of our interpretation of primitive atomic formulas, but an inductive argument is necessary to show that it persists for primitive formulas of higher syntactic complexity. The proof is relegated to Appendix A.3.

2.4. Defined logical symbols

As in classical logic, the disjunction connective $\vee$ and the existential quantifier $\exists$ may be expressed in terms of their duals.

Definition 2.4.1. For primitive formulas $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$, we write

$$\psi(x_1, \dots, x_n) \vee \psi(x_1, \dots, x_n)$$

as an abbreviation for $\neg(\neg\psi(x_1, \ldots, x_n) \wedge \neg\psi(x_1, \ldots, x_n))$, and we write

$$(\exists x_1 \in \mathcal{X}_1)\, \phi(x_1, \ldots, x_n)$$

as an abbreviation for $\neg(\forall x_1 \in \mathcal{X}_1) \neg\phi(x_1, \ldots, x_n)$.

Proposition 2.4.2. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be quantum sets, and let $x_1, \ldots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. For primitive formulas $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$,*

(1) $[\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \ldots, x_n) \vee \psi(x_1, \ldots, x_n)]\!]$
$= [\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \ldots, x_n)]\!]$
$\vee\, [\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_1, \ldots, x_n)]\!]$;

(2) $[\![(x_2, \ldots, x_n) \in \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \mid (\exists x_1 \in \mathcal{X}_1)\, \phi(x_1, \ldots, x_n)]\!]$
$= \inf\{R \in \mathrm{Rel}(\mathcal{X}_2, \ldots, \mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times R$
$\geq [\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \ldots, x_n)]\!]\}$.

Proof. Straightforward. ∎

In classical logic, an implication $\phi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)$ may be viewed as abbreviating the formula $\neg\phi(x_1, \ldots, x_n) \vee \psi(x_1, \ldots, x_n)$, but it is now widely understood both that this expression is entirely unsatisfactory to the quantum setting and that no such expression is entirely satisfactory. Hardegree observed [19] that there are exactly three polynomials $P \to Q$ in propositional variables P and Q, for the operations $\neg$, $\wedge$, and $\vee$, that satisfy the following requirements in every orthomodular lattice:

(1) $P \wedge (P \to Q) \leq Q$,

(2) $(P \to Q) \wedge \neg Q \leq \neg P$,

(3) $P \to Q = \top$ if and only if $P \leq Q$.

They are as follows:

(1) $\neg P \vee (P \wedge Q)$,

(2) $(\neg P \wedge \neg Q) \vee Q$,

(3) $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

None of these expressions is entirely satisfactory because none of them satisfies the expected transitivity law $(P \to Q) \wedge (Q \to R) \leq P \to R$. In this paper, we interpret the implication $P \to Q$ to be the Sasaki arrow $\neg P \vee (P \wedge Q)$.

This choice may be motivated by physical considerations. If P and Q are propositions about a physical system [6] and $\neg P \vee (P \wedge Q)$ is true of the initial state with probability one, then a positive outcome in a measurement of the truth value of P guarantees a positive outcome in a successive measurement of the truth value of Q. Moreover, $\neg P \vee (P \wedge Q)$ is the weakest such proposition in the sense that any other proposition with this property implies it [13, 44]. This choice of implication may also be pragmatically justified by its role in the proof of Proposition 3.3.1.

Definition 2.4.3. For primitive formulas $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$, we write

$$\phi(x_1, \dots, x_n) \to \psi(x_1, \dots, x_n)$$

as an abbreviation for $\neg\phi(x_1, \dots, x_n) \vee (\phi(x_1, \dots, x_n) \wedge \psi(x_1, \dots, x_n))$. We also write

$$\phi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n)$$

as an abbreviation for

$$(\phi(x_1, \dots, x_n) \to \psi(x_1, \dots, x_n)) \wedge (\psi(x_1, \dots, x_n) \to \phi(x_1, \dots, x_n)).$$

Contradiction $\perp$ and equality $=$ are commonly regarded as logical symbols on the basis that their interpretation does not really depend on the structure being considered. This distinction between logical and nonlogical relations is not meaningful within our approach of interpreting primitive formulas in a single many-sorted structure, the class of all quantum sets equipped with all their relations. Each relation symbol is a relation that denotes itself, and we do not consider other structures in which that symbol may denote some other relation. However, with this semantics in hand, it is entirely straightforward to define the notion of a discrete quantum model that accommodates both a logical equality symbol and various nonlogical relations symbols [23].

Contradiction $\perp$ is a relation of arity $()$; it was defined in Section 2.3. The equality relation $E_{\mathcal{X}}$ on a quantum set $\mathcal{X}$ is a relation of arity $(\mathcal{X}, \mathcal{X}^*)$, which we now define.

Definition 2.4.4. Let $\mathcal{X}$ be a quantum set. The equality relation on $\mathcal{X}$ is the relation $E_{\mathcal{X}}$ of arity $(\mathcal{X}, \mathcal{X}^*)$ defined by $E_{\mathcal{X}}(X \otimes X^*) = \mathbb{C}\varepsilon_X$ for all atoms $X \in \mathrm{At}(\mathcal{X})$ and $E_{\mathcal{X}}(X_1 \otimes X_2^*) = 0$ for distinct atoms $X_1, X_2 \in \mathrm{At}(\mathcal{X})$, where ε_X is the evaluation operator $X \otimes X^* \to \mathbb{C}$.

The equality projection $\delta_{\ell^\infty(\mathcal{X})}$ that was defined in Section 1.1 may be regarded as an element of $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$; see Section 2.1. Viewed in this way, it is the projection that corresponds to the predicate $E_{\mathcal{X}}$ in the sense that $E_{\mathcal{X}}(X_1 \otimes X_2^*) = L(X_1 \otimes X_2^*, \mathbb{C}) \cdot \delta_{\ell^\infty(\mathcal{X})}$ for all $X_1, X_2 \in \mathrm{At}(\mathcal{X})$. This is proved in Appendix A.4. From the intuitive perspective that $\mathcal{X}^*$ consists of the antielements of $\mathcal{X}$, the relation $E_{\mathcal{X}}$ identifies elements of $\mathcal{X}$ with antielements of $\mathcal{X}$, and there is generally no way to identify the elements of one copy of $\mathcal{X}$ with the elements of another. As in Section 2.1, the elements and antielements of a quantum set are just figures of speech.

2.5. Quantifying over the diagonal

The characteristic feature of the equality relation in the quantum setting is its mixed arity. For this reason, axioms often quantify over both the underlying quantum set of a discrete quantum structure and over its dual, and the relations that constitute that structure often have mixed arity. For example, the reflexivity of a relation R equipping a quantum set $\mathcal{X}$ is naturally expressed by the nonduplicating sentence

$$(\forall x_1 \in \mathcal{X})\, (\forall x_2 \in \mathcal{X}^*)\, (E_{\mathcal{X}}(x_1, x_2) \to R(x_1, x_2)).$$

The variables x_1 and x_2 must have sorts $\mathcal{X}$ and $\mathcal{X}^*$, respectively, because $E_{\mathcal{X}}$ has arity $(\mathcal{X}, \mathcal{X}^*)$. It follows that R should also have arity $(\mathcal{X}, \mathcal{X}^*)$. Thus, a reflexive relation on $\mathcal{X}$ should have arity $(\mathcal{X}, \mathcal{X}^*)$.

The formulation of reflexivity given in the above paragraph suggests a device for expressing the quantification of a variable ranging simultaneously over a quantum set $\mathcal{X}$ and over its dual $\mathcal{X}^*$. For greater convenience, we might modify our conventions to allow a single bound variable to appear once as an $\mathcal{X}$-sorted argument and once as an $\mathcal{X}^*$-sorted argument in any atomic formula. However, to avoid the risk of confusion and the cost of time borne by introducing this notation, we make do with a minor addition to our syntax that canonizes this device as a defined quantifier. We do so in part because this quantifier occurs frequently in the axiomatizations of already established quantum generalizations of discrete structures; see the second part of this paper.

Definition 2.5.1. Let $\mathcal{X}$ be a quantum set. Let $\phi(x_1, x_2, x_3, \ldots, x_n)$ be a primitive formula with x_1 and x_2 of sorts $\mathcal{X}$ and $\mathcal{X}^*$, respectively. We write

$$(\forall(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x_1, \ldots, x_n)$$

as an abbreviation for $(\forall x_2 \in \mathcal{X}^*)\, (\forall x_1 \in \mathcal{X})\, (E_{\mathcal{X}}(x_1, x_2) \to \phi(x_1, \ldots, x_n))$. We also write

$$(\exists(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x_1, \ldots, x_n)$$

as an abbreviation for $\neg(\forall(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*)\, \neg\phi(x_1, \ldots, x_n)$.

For clarity, we will often decorate a variable that ranges over the dual of a given quantum set with an asterisk as a part of that symbol. For two variables that are paired by the quantifier that we have just defined, it is convenient for the variables to differ by exactly the asterisk, e.g., $(\forall(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, R(x, x_*)$, for R a relation of arity $(\mathcal{X}, \mathcal{X}^*)$. The variables x and x_* are entirely distinct.

The example of pair production in Section 1.1 suggests a vivid intuition for the universal diagonal quantifier. We reframe this example in terms of quantum sets: the phase space of the electron's spin is a quantum set $\mathcal{X}$ that consists of a single two-dimensional atom X. The phase space of the positron's spin is then $\mathcal{X}^*$, and the phase space of the composite system is $\mathcal{X} \times \mathcal{X}^*$. Each relation R of arity $(\mathcal{X}, \mathcal{X}^*)$ corresponds to a projection r in $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$, i.e., to a Boolean observable on the composite system $\ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{X})^{\mathrm{op}}$. We will soon show that $[\![(\forall(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, R(x, x_*)]\!] = \top$ if and only if $\delta_{\ell^\infty(\mathcal{X})} \leq r$; see Proposition 3.2.2. Therefore, $[\![(\forall(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, R(x, x_*)]\!] = \top$ if and only if a measurement of r is guaranteed to yield 1 whenever the composite system is prepared via a neutral pion decay.

The suggested intuition is that we may analogously produce element-antielement pairs from any nonempty quantum set $\mathcal{X}$ and that the universal diagonal quantifier refers to all element-antielements pairs produced in this way. To the extent that a variable x of sort $\mathcal{X}$ may be regarded as naming an element of $\mathcal{X}$, the variable x_* of sort $\mathcal{X}^*$ may be regarded as naming the corresponding antielement. However, the variables x and x_* are formally unrelated, and the elements and antielements of $\mathcal{X}$ are just figures of speech.

2.6. Function graphs

Functions may be treated logically as relations. Classically, we may identify each function f from a set X to a set Y with its graph relation $[\![(x, y) \in X \times Y \mid f(x) = y]\!]$. We follow the same approach in the quantum setting.

Let us suppose that F is a function from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ in some appropriate sense. As the variable x ranges over $\mathcal{X}$, the term $F(x)$ ranges in $\mathcal{Y}$, so the graph relation of F is a relation defined by the formula $E_{\mathcal{Y}}(F(x), y)$. The equality relation $E_{\mathcal{Y}}$ has arity $(\mathcal{Y}, \mathcal{Y}^*)$, so the variable y must range over $\mathcal{Y}^*$, not $\mathcal{Y}$. Thus, the graph relation of a function F from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ should be a relation of arity $(\mathcal{X}, \mathcal{Y}^*)$. Therefore, we define a function graph from $\mathcal{X}$ to $\mathcal{Y}$ to be a relation G of arity $(\mathcal{X}, \mathcal{Y}^*)$ that is univalent in $\mathcal{X}$ and total in $\mathcal{X}$, expressing both properties by primitive formulas.

Definition 2.6.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. A relation G of arity $(\mathcal{X}, \mathcal{Y}^*)$ is said to be a *function graph* if it satisfies the following:

(1) $[\![(\forall x \in \mathcal{X})\, (\exists y_* \in \mathcal{Y}^*)\, G(x, y_*)]\!] = \top$;

(2) $[\![(\forall y_1 \in \mathcal{Y})\, (\forall y_{2*} \in \mathcal{Y}^*)\, ((\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, (G_*(x_*, y_1) \wedge G(x, y_{2*})) \to E_{\mathcal{Y}}(y_1, y_{2*}))]\!] = \top$.

This definition is recognizable from ordinary logic. The placement of asterisks in the second formula is essentially dictated by the arity of $E_{\mathcal{Y}}$. We reason that the variables y_1 and y_{2*} must be of sorts $\mathcal{Y}$ and $\mathcal{Y}^*$, respectively, so the first atomic subformula must use the conjugate relation G_* in place of G. This implies that the variable x_* in the first atomic subformula must be of sort $\mathcal{X}^*$. Likewise, the variable x in the second atomic subformula must be of sort $\mathcal{X}$.

For each function graph G of arity $(\mathcal{X}, \mathcal{Y}^*)$, we extend the language by adding a function symbol $\breve{G}$. Anticipating Theorem 3.4.2, we formally define $\breve{G}$ to be the binary relation $(G \times I_{\mathcal{Y}}) \circ (I_{\mathcal{X}} \times E^*_{\mathcal{Y}})$ from $\mathcal{X}$ to $\mathcal{Y}$ [30, Sec. 3]. Reasoning graphically, as described in Section 3.1, it is easy to see that

$$R = E_{\mathcal{Y}} \circ (\breve{R} \times I_{\mathcal{Y}^*})$$

for each relation R of arity $(\mathcal{X}, \mathcal{Y}^*)$. Thus, the mapping $G \mapsto \breve{G}$ is injective, and we introduce no ambiguity by defining our function symbols in this way.

We remark that a relation may have more than one arity. Finite Cartesian products are formally defined associating to the left, so a function graph G of arity $(\mathcal{X}_1 \times \cdots \times \mathcal{X}_m, \mathcal{Y}^*)$ is also a relation of arity $(\mathcal{X}_1, \ldots, \mathcal{X}_m, \mathcal{Y}^*)$. Hence, we will use the function symbol $\breve{G}$ both with one argument and with n arguments, Definition 2.6.1 notwithstanding. This is made precise in Section 2.7.

2.7. Interpreting nonduplicating formulas

We now define the class of nonduplicating formulas, and we extend the semantics of Section 2.3 to this class.

Definition 2.7.1. The class of *nonduplicating terms* is defined recursively: a variable of sort $\mathcal{X}$ is a nonduplicating term of sort $\mathcal{X}$, and for each function graph G of arity $(\mathcal{X}_1, \dots, \mathcal{X}_m, \mathcal{Y}^*)$, if $s_1, \dots, s_m$ are nonduplicating terms of sorts $\mathcal{X}_1, \dots, \mathcal{X}_m$, respectively, and no two of these terms have a variable in common, then the expression $\breve{G}(s_1, \dots, s_m)$ is a nonduplicating term of sort $\mathcal{Y}$. Furthermore, for each relation R of arity $(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$, if $t_1, \dots, t_n$ are nonduplicating terms of sorts $\mathcal{Y}_1, \dots, \mathcal{Y}_n$, respectively, and no two of these terms have a variable in common, then the expression $R(t_1, \dots, t_n)$ is a *nonduplicating atomic formula*.

Definition 2.7.2. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be quantum sets; let R be a relation of arity $(\mathcal{Y}_1, \dots, \mathcal{Y}_n)$. If $R(t_1, \dots, t_n)$ is a nonduplicating atomic formula that is not primitive, then it abbreviates the nonduplicating formula

$$(\exists (y_n = y_{n*}) \in \mathcal{Y}_n \times \mathcal{Y}_n^*) \cdots (\exists (y_1 = y_{1*}) \in \mathcal{Y}_1 \times \mathcal{Y}_1^*) \\ (R(y_1, \dots, y_n) \wedge t_1 \curvearrowright y_{1*} \wedge \cdots \wedge t_n \curvearrowright y_{n*}),$$

where the variables $y_1, \dots, y_n$ and $y_{1*}, \dots, y_{n*}$ are all new in the sense that they do not occur in the formula $R(t_1, \dots, t_n)$. In this context, a formula of the form $t \curvearrowright y_*$, for t of sort $\mathcal{Y}$ and y_* of sort $\mathcal{Y}^*$, abbreviates $E_{\mathcal{Y}}(t, y_*)$ if t is a variable and $G(s_1, \dots, s_m, y_*)$ if t is of the form $\breve{G}(s_1, \dots, s_m)$, for some terms $s_1, \dots, s_m$.

Thus, every nonduplicating atomic formula that is not primitive abbreviates a primitive formula. For example, the formula $P(\breve{G}(x))$, with G a function graph of arity $(\mathcal{X}, \mathcal{Y}^*)$, abbreviates the formula $(\exists (y = y_*) \in \mathcal{Y} \times \mathcal{Y}^*)\, (P(y) \wedge G(x, y_*))$, which in turn abbreviates the formula $(\exists y_* \in \mathcal{Y}^*)\, (\exists y \in \mathcal{Y})\, (E_{\mathcal{Y}}(y, y_*) \to (P(y) \wedge G(x, y_*)))$, which finally abbreviates the primitive formula

$$(\exists y_* \in \mathcal{Y}^*)\, (\forall y \in \mathcal{Y})\, (\neg E_{\mathcal{Y}}(y, y_*) \vee (E_{\mathcal{Y}}(y, y_*) \wedge (P(y) \wedge G(x, y_*)))).$$

This observation extends easily to the class of all nonduplicating formulas.

Definition 2.7.3. The class of *nonduplicating formulas* is defined recursively: each nonduplicating atomic formula is a nonduplicating formula, and if ϕ and ψ are nonduplicating formulas and x is variable of some sort $\mathcal{X}$, then the expressions $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $(\forall x \in \mathcal{X})\, \phi$, and $(\exists x \in \mathcal{X})\, \phi$ are nonduplicating formulas.

The abbreviations that we have defined in Section 2 together define a translation, i.e., a class function from nonduplicating formulas to primitive formulas, which fixes all of the primitive formulas. This extends our interpretation of primitive formulas to all nonduplicating formulas. Formally, for each nonduplicating formula $\phi(x_1, \dots, x_n)$, we define

$$[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!] \\ = [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \widetilde{\phi}(x_1, \dots, x_n)]\!],$$

where $\widetilde{\phi}(x_1, \dots, x_n)$ is the translation of $\phi(x_1, \dots, x_n)$. Note that the translation has exactly the same free variables as the original formula.

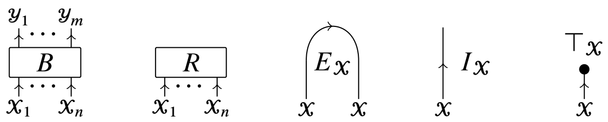


Figure 1. Some depicted binary relations.

For the sake of the exposition, quantification over the diagonal remains an informal abbreviation; i.e., $(\forall(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1, \ldots, y_n)$ is the formula

$$(\forall x_* \in \mathcal{X}^*)\, (\forall x \in \mathcal{X})\, (E_{\mathcal{X}}(x, x_*) \to \phi(x, x_*, y_1, \ldots, y_n))$$

for each nonduplicating formula $\phi(x, x_*, y_1, \ldots, y_n)$, and it is likewise for $(\exists(x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1, \ldots, y_n)$. Similarly, equivalence remains an informal abbreviation; i.e., $\phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)$ is the formula $(\phi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)) \wedge (\psi(x_1, \ldots, x_n) \to \phi(x_1, \ldots, x_n))$ for all nonduplicating formulas $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$.

3. Computation

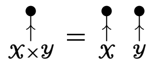
Computation with the relations that we have defined is most easily performed with the aid of wire diagrams. A relation R of some arity $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ is also a binary relation from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to $\mathbf{1}$ in the sense of [30]. Hence, $R(X_1 \otimes \cdots \otimes X_n) = R(X_1 \otimes \cdots \otimes X_n, \mathbb{C})$ for all atoms $X_1 \in \mathrm{At}(\mathcal{X}_1)$, $X_2 \in \mathrm{At}(\mathcal{X}_2)$, etc. The category of quantum sets and binary relations is compact closed and therefore supports a graphical calculus in which binary relations are depicted as boxes and quantum sets are depicted as wires [1].

3.1. Wire diagrams

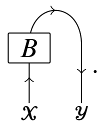
A binary relation B from a product $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to a product $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$ is depicted as a box with n wires entering the box from the bottom, each associated to one of the quantum sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$, and with m wires leaving the box from the top, each associated to one of the quantum sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$. A relation R of arity $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ is therefore depicted as a box with wires coming just from below. See Figure 1.

We orient each wire, with downward-oriented wires corresponding to dual quantum sets. In other words, a downward-oriented wire labeled $\mathcal{X}$ corresponds to the quantum set $\mathcal{X}^*$. The advantage of this notation is that the equality relation, which is also the counit of the dagger compact structure on the category of quantum sets and binary relations, can be depicted simply as an arc. For each quantum set $\mathcal{X}$, the identity binary relation $I_{\mathcal{X}}$ on $\mathcal{X}$ is depicted simply as a wire, and the maximum predicate $\top_{\mathcal{X}}$ is depicted by a "loose end", which we will sometimes "pull away", that is, completely omit. See Figure 1.

In this diagrammatic calculus, the monoidal product of two morphisms, i.e., of two binary relations, is depicted by placing the corresponding diagrams side by side. Thus, for all quantum sets $\mathcal{X}$ and $\mathcal{Y}$, we have the equation



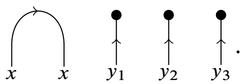
because $\top_{\mathcal{X}\times\mathcal{Y}} = \top_{\mathcal{X}} \times \top_{\mathcal{Y}}$. Similarly, the composition of binary relations is depicted by placing one diagram above the other and tying together the corresponding wires. For example, if B is a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, then the binary relation $\hat{B} := E_{\mathcal{Y}} \circ (B \times I_{\mathcal{Y}^*})$ from $\mathcal{X} \times \mathcal{Y}^*$ to $\mathbf{1}$ is depicted in the following diagram:



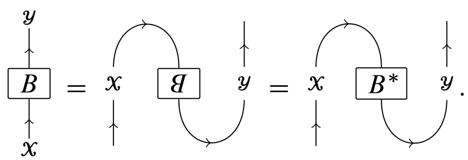
We will often use variables to label the wires of a diagram in order to distinguish various occurrences of the same quantum set, particularly when depicting the interpretation of a formula. For example, the relation

$$[\![(x, x_*, y_1, y_2, y_3) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \mid E_{\mathcal{X}}(x, x_*)]\!]$$

is depicted in the following diagram:



The defining properties of a dagger compact category are such that wires may be deformed in the intuitive way. Boxes may be moved around or even turned upside down, which corresponds to dualization in the sense of the dagger compact structure. Thus, for any binary relation B from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$, we have the following:



The expression $\mathrm{Rel}(\mathcal{X}; \mathcal{Y})$ denotes the set of all binary relations from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ in [30] and in the present paper. For quantum sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}_1, \ldots, \mathcal{Y}_m$, monoidal closure yields a canonical bijection between $\mathrm{Rel}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n; \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m)$ and $\mathrm{Rel}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n \times \mathcal{Y}_1^* \times \cdots \times \mathcal{Y}_m^*; \mathbf{1})$, so whether a wire leaves the diagram upward or downward has no fundamental significance beyond sorting the factors between the domain and the codomain. Thus, binary relations are all essentially predicates. Similarly, our wire diagrams are essentially diagrams in a space with no top or bottom; each box denotes some predicate, and each emanates wires according to the arity of that predicate.

One significant advantage of diagrammatic computation is the ease with which we can permute the variables of a context. Formally, we appeal to the following proposition, which is proved in Appendix A.5.

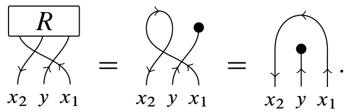
Proposition 3.1.1 (Also Proposition A.5.1). *Let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be quantum sets, and let π be a permutation of $\{1, \dots, n\}$. Let U_π be the canonical isomorphism* [35, Thm. XI.1.1] *from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to $\mathcal{X}_{\pi(1)} \times \cdots \times \mathcal{X}_{\pi(n)}$ in the symmetric monoidal category of quantum sets and binary relations* [30, Sec. 3]. *Then, $\pi_\#(R) = R \circ U_\pi$ for all relations R of arity $(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(n)})$.*

Together, Propositions 2.3.3 and 3.1.1 allow us to quickly compute simplified diagrams depicting interpreted formulas. For example, writing

$$R := [\![(x_1, x_{2*}, y) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{Y} \mid E_{\mathcal{X}}(x_1, x_{2*})]\!],$$

we may compute that

$$[\![(x_{2*}, y, x_1) \in \mathcal{X}^* \times \mathcal{Y} \times \mathcal{X} \mid E_{\mathcal{X}}(x_1, x_{2*})]\!] =$$



The weave of wires below the box depicting the relation R depicts the canonical isomorphism from $\mathcal{X}^* \times \mathcal{Y} \times \mathcal{X}$ to $\mathcal{X} \times \mathcal{X}^* \times \mathcal{Y}$ that is derived from the symmetric monoidal structure of the category of quantum sets and binary relations.

3.2. Standard quantifiers

We establish two basic propositions about the standard quantifiers $\forall$ and $\exists$.

Lemma 3.2.1. *Let $\phi(x_1, \dots, x_n)$ be a nonduplicating formula, with $x_1, \dots, x_n$ of sorts $\mathcal{X}_1, \dots, \mathcal{X}_n$, respectively. For all $m \in \{0, \dots, n\}$, we have*

$$\begin{aligned}
&[\![(x_{m+1}, \dots, x_n) \in \mathcal{X}_{m+1} \times \cdots \times \mathcal{X}_n \mid (\forall x_m \in \mathcal{X}_m) \cdots (\forall x_1 \in \mathcal{X}_1)\, \phi(x_1, \dots, x_n)]\!] \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{X}_{m+1}, \dots, \mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_m} \times R \\
&\qquad\qquad \leq [\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]\}.
\end{aligned}$$

The special case $m = n$ shows that $[\![(\forall x_n \in \mathcal{X}_n) \cdots (\forall x_1 \in \mathcal{X}_1)\, \phi(x_1, \dots, x_n)]\!] = \top$ if and only if $[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!]$ is the maximum relation of arity $(\mathcal{X}_1, \dots, \mathcal{X}_n)$.

Proof. For each $m \in \{0, \dots, n\}$, write R_m for the left side of the equality. Hence, we are to show that for all $m \in \{0, \dots, n\}$, we have that

$$R_m = \sup\{R \in \mathrm{Rel}(\mathcal{X}_{m+1}, \dots, \mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_m} \times R \leq R_0\}.$$

We proceed by induction on m. The base case is just the obvious equality

$$R_0 = \sup\{R \in \mathrm{Rel}(\mathcal{X}_1, \ldots, \mathcal{X}_n) \mid R \leq R_0\}.$$

For the induction step, we assume that the desired equality holds for some natural $m - 1 \in \{0, \ldots, n-1\}$, and thus, we have the following:

$$\begin{aligned} R_{m-1} &= \sup\{R \in \mathrm{Rel}(\mathcal{X}_m, \ldots, \mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_{m-1}} \times R \leq R_0\}, \\ R_m &= \sup\{R \in \mathrm{Rel}(\mathcal{X}_{m+1}, \ldots, \mathcal{X}_n) \mid \top_{\mathcal{X}_m} \times R \leq R_{m-1}\}. \end{aligned}$$

In particular, we have that

$$\top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_{m-1}} \times \top_{\mathcal{X}_m} \times R_m \leq \top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_{m-1}} \times R_{m-1} \leq R_0.$$

Now, suppose that R is any other relation that satisfies

$$\top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_{m-1}} \times \top_{\mathcal{X}_m} \times R \leq R_0.$$

It follows that $\top_{\mathcal{X}_m} \times R \leq R_{m-1}$ by the first equation and then that $R \leq R_m$ by the second equation. Therefore, R_m is indeed the supremum of the relations R satisfying $\top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_m} \times R \leq R_0$. ■

Proposition 3.2.2. *Let $\phi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ be nonduplicating formulas, with $x_1, \ldots, x_n$ of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. Then,*

$$[\![(\forall x_n \in \mathcal{X}_n) \cdots (\forall x_1 \in \mathcal{X}_1)\, (\phi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n))]\!] = \top$$

if and only if

$$\begin{aligned} &[\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \ldots, x_n)]\!] \\ &\qquad \leq [\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \psi(x_1, \ldots, x_n)]\!]. \end{aligned}$$

Proof. By Lemma 3.2.1, the equation is true if and only if

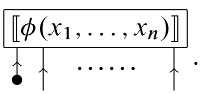
$$[\![\phi(x_1, \ldots, x_n)]\!] \to [\![\psi(x_1, \ldots, x_n)]\!]$$

is the maximum relation of arity $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$. This condition is equivalent to the claimed inequality by a fundamental property of the Sasaki arrow [19]. ■

Proposition 3.2.3. *Let $\phi(x_1, \ldots, x_n)$ be a nonduplicating formula, with $x_1, \ldots, x_n$ of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$. Then,*

$$\begin{aligned} &[\![(x_2, \ldots, x_n) \in \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \mid (\exists x_1 \in \mathcal{X}_1)\, \phi(x_1, \ldots, x_n)]\!] \\ &\quad = [\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \ldots, x_n)]\!] \circ (\top_{\mathcal{X}_1}^\dagger \times I_{\mathcal{X}_2} \times \cdots \times I_{\mathcal{X}_n}) \end{aligned}$$

=



.

Proof. Write

$$[\![\phi(x_1,\dots,x_n)]\!] = [\![(x_1,\dots,x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1,\dots,x_n)]\!].$$

We refer to [31, App. C] for the relationship between the adjoint † and the orthogonality relation $\perp$. We calculate that

$$\begin{aligned}
&[\![(x_2,\dots,x_n) \in \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \mid (\exists x_1 \in \mathcal{X}_1)\, \phi(x_1,\dots,x_n)]\!] \\
&\quad = [\![(x_2,\dots,x_n) \in \mathcal{X}_2 \times \cdots \times \mathcal{X}_n \mid \neg(\forall x_1 \in \mathcal{X}_1)\, \neg\phi(x_1,\dots,x_n)]\!] \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times R \le [\![\neg\phi(x_1,\dots,x_n)]\!]\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid \top_{\mathcal{X}_1} \times R \perp [\![\phi(x_1,\dots,x_n)]\!]\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid [\![\phi(x_1,\dots,x_n)]\!] \circ (\top_{\mathcal{X}_1} \times R)^\dagger = \perp\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid [\![\phi(x_1,\dots,x_n)]\!] \circ (\top^\dagger_{\mathcal{X}_1} \times R^\dagger) = \perp\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid [\![\phi(x_1,\dots,x_n)]\!] \circ (\top^\dagger_{\mathcal{X}_1} \times I_{\mathcal{X}_2} \times \cdots \times I_{\mathcal{X}_n}) \\
&\qquad\qquad \circ R^\dagger = \perp\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{X}_2,\dots,\mathcal{X}_n) \mid R \perp [\![\phi(x_1,\dots,x_n)]\!] \\
&\qquad\qquad \circ (\top^\dagger_{\mathcal{X}_1} \times I_{\mathcal{X}_2} \times \cdots \times I_{\mathcal{X}_n})\} \\
&\quad = \neg\neg([\![\phi(x_1,\dots,x_n)]\!] \circ (\top^\dagger_{\mathcal{X}_1} \times I_{\mathcal{X}_2} \times \cdots \times I_{\mathcal{X}_n})) \\
&\quad = [\![\phi(x_1,\dots,x_n)]\!] \circ (\top^\dagger_{\mathcal{X}_1} \times I_{\mathcal{X}_2} \times \cdots \times I_{\mathcal{X}_n}).
\end{aligned}$$

∎

Applying diagrammatic reasoning, we find that existential quantifiers commute, as a corollary of Proposition 3.2.3, and therefore, so do universal quantifiers. If $\mathcal{X}_1 = `A$ for some ordinary set A, then existential quantification over $\mathcal{X}_1$ is equivalent to a disjunction over A, essentially because the maximum binary relation from a singleton $\{*\}$ to A is the disjunction of the elements of A, each considered as a binary relation from $\{*\}$ to A. See Lemma A.6.1.

3.3. Diagonal quantifiers

We now characterize our two defined quantifiers over the diagonal.

Proposition 3.3.1. *Let $\phi(x, x_*, y_1,\dots,y_n)$ be a nonduplicating formula, with x of sort $\mathcal{X}$, with x_* of sort $\mathcal{X}^*$, and with $y_1,\dots,y_n$ of sorts $\mathcal{Y}_1,\dots,\mathcal{Y}_n$, respectively. Write $[\![\phi(x, x_*, y_1,\dots,y_n)]\!]$ as an abbreviation for*

$$[\![(x, x_*, y_1,\dots,y_n) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid \phi(x, x_*, y_1,\dots,y_n)]\!].$$

Then,

$$\begin{aligned}
&[\![(y_1,\dots,y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\forall (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1,\dots,y_n)]\!] \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1,\dots,\mathcal{Y}_n) \mid E_{\mathcal{X}} \times R \le [\![\phi(x, x_*, y_1,\dots,y_n)]\!]\}.
\end{aligned}$$

Proof. We calculate that

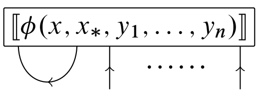
$$
\begin{aligned}
&[\![(y_1, \ldots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\forall (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\quad = [\![(y_1, \ldots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\forall x_* \in \mathcal{X}^*)\, (\forall x \in \mathcal{X}) \\
&\qquad\qquad (E_{\mathcal{X}}(x, x_*) \to \phi(x, x_*, y_1, \ldots, y_n))]\!] \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid \top_{\mathcal{X}} \times \top_{\mathcal{X}^*} \times R \\
&\qquad\qquad \leq [\![E_{\mathcal{X}}(x, x_*) \to \phi(x, x_*, y_1, \ldots, y_n)]\!]\} \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid \top_{\mathcal{X}} \times \top_{\mathcal{X}^*} \times R \\
&\qquad\qquad \leq [\![E_{\mathcal{X}}(x, x_*)]\!] \to [\![\phi(x, x_*, y_1, \ldots, y_n)]\!]\} \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid (\top_{\mathcal{X}} \times \top_{\mathcal{X}^*} \times R) \,\&\, [\![E_{\mathcal{X}}(x, x_*)]\!] \\
&\qquad\qquad \leq [\![\phi(x, x_*, y_1, \ldots, y_n)]\!]\} \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid (\top_{\mathcal{X}} \times \top_{\mathcal{X}^*} \times R) \,\&\, (E_{\mathcal{X}} \times \top_{\mathcal{Y}_1} \times \cdots \times \top_{\mathcal{Y}_n}) \\
&\qquad\qquad \leq [\![\phi(x, x_*, y_1, \ldots, y_n)]\!]\} \\
&\quad = \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid E_{\mathcal{X}} \times R \leq [\![\phi(x, x_*, y_1, \ldots, y_n)]\!]\}.
\end{aligned}
$$

In this context, $\&$ denotes the Sasaki projection connective, which is defined by $P \,\&\, Q = (P \vee \neg Q) \wedge Q$. For every relation Q, the mapping $P \mapsto P \,\&\, Q$ is left adjoint to the mapping $P \mapsto Q \to P$ [13]. ■

The following theorem serves as a bridge between the semantics defined in Section 2 and the interpretation of wire diagrams in the dagger compact category of quantum sets and binary relations [30, Sec. 3].

Theorem 3.3.2. *From the assumptions of Proposition 3.3.1, we have the following equality:*

$$
\begin{aligned}
&[\![(y_1, \ldots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\quad = [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \circ (E_{\mathcal{X}}^\dagger \times I_{\mathcal{Y}_1} \times \cdots \times I_{\mathcal{Y}_n})
\end{aligned}
$$

$=$ 

.

Proof. Appealing to Proposition 3.3.1 for the second equality, we reason that

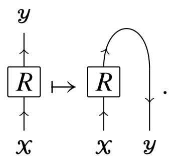
$$
\begin{aligned}
&[\![(y_1, \ldots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\quad = [\![(y_1, \ldots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid \neg(\forall (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, \neg\phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid E_{\mathcal{X}} \times R \leq [\![\neg\phi(x, x_*, y_1, \ldots, y_n)]\!]\} \\
&\quad = \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid E_{\mathcal{X}} \times R \perp [\![\phi(x, x_*, y_1, \ldots, y_n)]\!]\}
\end{aligned}
$$

$$\begin{aligned}
&= \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \circ (E_{\mathcal{X}} \times R)^\dagger = \bot\} \\
&= \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \circ (E_{\mathcal{X}}^\dagger \times R^\dagger) = \bot\} \\
&= \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\qquad\qquad \circ (E_{\mathcal{X}}^\dagger \times I_{\mathcal{Y}_1} \times \cdots \times I_{\mathcal{Y}_n}) \circ R^\dagger = \bot\} \\
&= \neg \sup\{R \in \mathrm{Rel}(\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \mid R \perp [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \\
&\qquad\qquad \circ (E_{\mathcal{X}}^\dagger \times I_{\mathcal{Y}_1} \times \cdots \times I_{\mathcal{Y}_n})\} \\
&= \neg\neg([\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \circ (E_{\mathcal{X}}^\dagger \times I_{\mathcal{Y}_1} \times \cdots \times I_{\mathcal{Y}_n})) \\
&= [\![\phi(x, x_*, y_1, \ldots, y_n)]\!] \circ (E_{\mathcal{X}}^\dagger \times I_{\mathcal{Y}_1} \times \cdots \times I_{\mathcal{Y}_n}). \qquad \blacksquare
\end{aligned}$$

If $\mathcal{X} = `A$ for some ordinary set A, then existential quantification over the diagonal of $\mathcal{X} \times \mathcal{X}^*$ is equivalent to a disjunction over A. See Lemma A.6.2.

Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. There is a natural bijective correspondence between binary relations from $\mathcal{X}$ to $\mathcal{Y}$ and relations of arity $(\mathcal{X}, \mathcal{Y}^*)$. It is given by

$$R \mapsto E_{\mathcal{Y}} \circ (R \times I_{\mathcal{Y}^*}),$$



Because we regard the distinction between domain wires and codomain wires to be simply an aid to computation, we view R and $\hat{R} := E_{\mathcal{Y}} \circ (R \times I_{\mathcal{Y}^*})$ to be essentially identical. Thus, the following corollary of Theorem 3.3.2 expresses a close connection between the existential diagonal quantifier and the composition of binary relations between quantum sets.

Corollary 3.3.3. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets. Let R be a binary relation from $\mathcal{X}$ to $\mathcal{Y}$, let S be a binary relation from $\mathcal{Y}$ to $\mathcal{Z}$, and let T be the binary relation from $\mathcal{X}$ to $\mathcal{Z}$ defined by $T = S \circ R$. Then,*

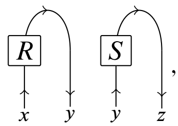
$$[\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid (\exists (y = y_*) \in \mathcal{Y} \times \mathcal{Y}^*)\, (\hat{R}(x, y_*) \wedge \hat{S}(y, z_*))]\!] = \hat{T}.$$

This conclusion clearly recalls the composition of binary relations between ordinary sets.

Proof of Corollary 3.3.3. We apply Definition 2.3.2, Proposition 3.1.1, and Theorem 3.3.2:

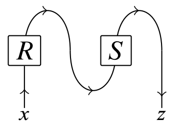
$$\begin{aligned}
&[\![(x, y_*, y, z_*) \in \mathcal{X} \times \mathcal{Y}^* \times \mathcal{Y} \times \mathcal{Z}^* \mid \hat{R}(x, y_*) \wedge \hat{S}(y, z_*)]\!] \\
&\quad = ([\![(x, y_*) \in \mathcal{X} \times \mathcal{Y}^* \mid \hat{R}(x, y_*)]\!] \times \top_{\mathcal{Y}} \times \top_{\mathcal{Z}^*}) \\
&\qquad \wedge (\top_{\mathcal{X}} \times \top_{\mathcal{Y}^*} \times [\![(y, z_*) \in \mathcal{Y} \times \mathcal{Z}^* \mid \hat{S}(y, z_*)]\!])
\end{aligned}$$

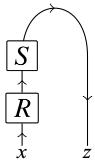
$$= [\![(x, y_*) \in \mathcal{X} \times \mathcal{Y}^* \mid \hat{R}(x, y_*)]\!] \times [\![(y, z_*) \in \mathcal{Y} \times \mathcal{Z}^* \mid \hat{S}(y, z_*)]\!]$$

$=$ 

,

$$[\![(x, z_*) \in \mathcal{X} \times \mathcal{Z}^* \mid (\exists (y = y_*) \in \mathcal{Y} \times \mathcal{Y}^*)\, (\hat{R}(x, y_*) \wedge \hat{S}(y, z_*))]\!]$$

$=$ 

$=$ 

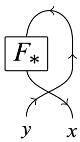
$= \hat{T}.$ ■

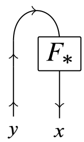
3.4. Functions

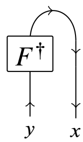
Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. We show that the mapping $R \mapsto \hat{R}$ restricts to a bijective correspondence between functions from $\mathcal{X}$ to $\mathcal{Y}$ in the sense of [30, Def. 4.1] and function graphs of arity $(\mathcal{X}, \mathcal{Y}^*)$ in the sense of Definition 2.6.1.

Lemma 3.4.1. *Let F be a partial function from $\mathcal{X}$ to $\mathcal{Y}$ in the sense of* [30]*, i.e., a binary relation from $\mathcal{X}$ to $\mathcal{Y}$ satisfying the inequality $F \circ F^\dagger \leq I_{\mathcal{Y}}$. Then, $\hat{F}$ is a relation of arity $(\mathcal{X}, \mathcal{Y}^*)$ that satisfies condition (2) of Definition* 2.6.1*. Furthermore, this construction is bijective.*

Proof. Let $R = F^\dagger$, let $S = F$, and let $T = S \circ R = F \circ F^\dagger$. First, we observe that the formulas $\hat{R}(y, x_*)$ and $\hat{F}_*(x_*, y)$ have the same interpretation in any context. It is clearly sufficient to show that they have the same interpretation in the context $(y, x_*) \in \mathcal{Y} \times \mathcal{X}^*$:

$[\![(y, x_*) \in \mathcal{Y} \times \mathcal{X}^* \mid \hat{F}_*(x_*, y)]\!] =$ 

$=$ 

$=$ 

$$= [\![(y, x_*) \in \mathcal{Y} \times \mathcal{X}^* \mid \hat{R}(y, x_*)]\!].$$

It follows directly from Definition 2.3.2 that we may replace $\hat{F}_*(x_*, y)$ by $\hat{R}(y, x_*)$ in any formula without altering the interpretation of that formula. Hence, we may apply Proposition 3.2.2 and Corollary 3.3.3 to reason as follows:

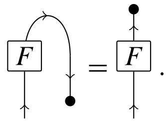
$$\begin{aligned}
&[\![(\forall y_1 \in \mathcal{Y})\, (\forall y_{2*} \in \mathcal{Y}^*)\, ((\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)(\hat{F}_*(x_*, y_1) \wedge \hat{F}(x, y_{2*})) \\
&\qquad\qquad\qquad\qquad \to E_{\mathcal{Y}}(y_1, y_{2*}))]\!] = \top \\
\iff\; & [\![(y_1, y_{2*}) \in \mathcal{Y} \times \mathcal{Y}^* \mid (\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, (\hat{F}_*(x_*, y_1) \wedge \hat{F}(x, y_{2*}))]\!] \leq E_{\mathcal{Y}} \\
\iff\; & [\![(y_1, y_{2*}) \in \mathcal{Y} \times \mathcal{Y}^* \mid (\exists (x = x_*) \in \mathcal{X} \times \mathcal{X}^*)\, (\hat{R}(y_1, x_*) \wedge \hat{S}(x, y_{2*}))]\!] \leq E_{\mathcal{Y}} \\
\iff\; & \hat{T} \leq E_{\mathcal{Y}} = \hat{I}_{\mathcal{Y}} \iff T \leq I_{\mathcal{Y}} \iff F \circ F^\dagger \leq I_{\mathcal{Y}}.
\end{aligned}$$

■

Theorem 3.4.2. *Let F be a function from $\mathcal{X}$ to $\mathcal{Y}$ in the sense of* [30]*, i.e., a binary relation from $\mathcal{X}$ to $\mathcal{Y}$ satisfying the inequalities $F^\dagger \circ F \geq I_\mathcal{X}$ and $F \circ F^\dagger \leq I_\mathcal{Y}$. Then, $\hat{F}$ is a function graph. Furthermore, this construction is bijective. Applying* [30, Thm. 7.4]*, we obtain a canonical bijection between function graphs of arity $(\mathcal{X}, \mathcal{Y}^*)$ and unital normal $*$-homomorphisms from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$.*

Proof. With Lemma 3.4.1 in hand, it remains only to show that F satisfies $F^\dagger \circ F \geq I_\mathcal{X}$ if and only if $[\![(\forall x \in \mathcal{X})\,(\exists y_* \in \mathcal{Y})\, \hat{F}(x, y_*)]\!] = \top$. As we observed in Section 3.2, this equality holds if and only if $[\![x \in \mathcal{X} \mid (\exists y_* \in \mathcal{Y})\, \hat{F}(x, y_*)]\!]$ is the maximum predicate $\top_\mathcal{X}$. Reasoning diagrammatically, we have that

$$[\![x \in \mathcal{X} \mid (\exists y_* \in \mathcal{Y})\, \hat{F}(x, y_*)]\!] =$$



We conclude that

$$[\![(\forall x \in \mathcal{X})\,(\exists y_* \in \mathcal{Y})\, \hat{F}(x, y_*)]\!] = \top$$

if and only if $\top_\mathcal{Y} \circ F = \top_\mathcal{X}$.

Thus, the construction $F \mapsto \hat{F}$ is a bijection between partial functions F satisfying $\top_\mathcal{Y} \circ F = \top_\mathcal{X}$ and function graphs. By [30, Lem. B.4], $\top_\mathcal{Y} \circ F = \top_\mathcal{X}$ if and only if the normal $*$-homomorphism $F^\star$ is unital, and by [30, Lem. 6.4], the latter condition holds if and only if $F^\dagger \circ F \geq I_\mathcal{X}$. Therefore, the construction $F \mapsto \hat{F}$ restricts to a bijection from functions to function graphs. ∎

Having established this one-to-one correspondence between functions and function graphs, it becomes natural to use functions for function symbols. Indeed, this is what we have been doing. For each function F from a quantum set $\mathcal{X}$ to a quantum set $\mathcal{Y}$ and each function graph G of arity $(\mathcal{X}, \mathcal{Y}^*)$, the equation $G = \hat{F}$ is easily seen to be equivalent to the equation $F = \breve{G}$ via the graphical calculus. We did not directly define our function symbols to be functions in Section 2.6 to delay drawing from [30], in order to demonstrate that this notion may be motivated from elementary physical and logical considerations.

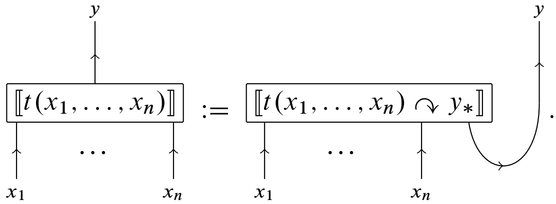
3.5. Terms

One effect of Definition 2.7.2 is that nonduplicating terms may be interpreted as compositions of functions in the expected way.

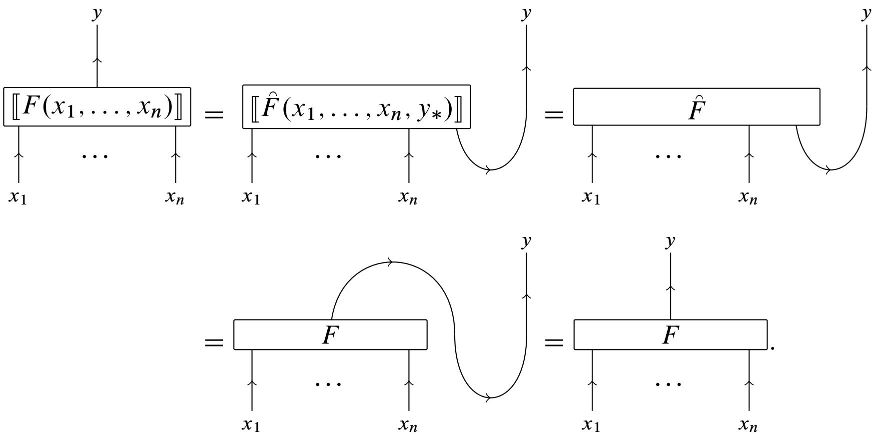
Definition 3.5.1. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be quantum sets, and let $x_1, \ldots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. Let $\mathcal{Y}$ be a quantum set, and let $t(x_1, \ldots, x_n)$ be a term of sort $\mathcal{Y}$. We define $[\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid t(x_1, \ldots, x_n)]\!]$ to be

$$([\![(x_1, \ldots, x_n, y_*) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \times \mathcal{Y}^* \mid t(x_1, \ldots, x_n) \curvearrowright y_*]\!] \times I_\mathcal{Y}) \circ (I_{\mathcal{X}_1} \times \cdots \times I_{\mathcal{X}_n} \times E_\mathcal{Y}^*),$$

where the formula $t(x_1, \ldots, x_n) \curvearrowright y_*$ is defined in Definition 2.7.2. Graphically,

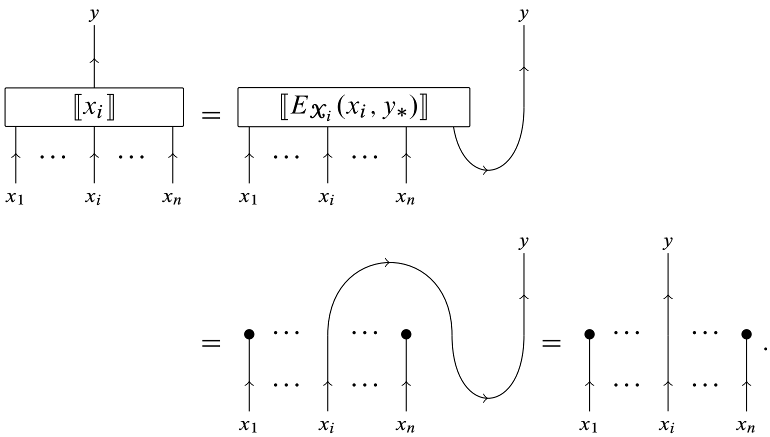


Let F be a function $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$. By definition, the formula $F(x_1, \ldots, x_n) \curvearrowright y_*$ abbreviates the nonduplicating formula $\hat{F}(x_1, \ldots, x_n, y_*)$, and therefore,



Thus, $[\![(x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid F(x_1, \ldots, x_n)]\!] = F$.

Let $i \in \{1, \ldots, n\}$. The variable x_i is a term of sort $\mathcal{X}_i$. The formula $x_i \curvearrowright y_*$ abbreviates the nonduplicating formula $E_{\mathcal{X}_i}(x_i, y_*)$, and therefore,



Thus,

$$[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid x_i]\!] = \top_{\mathcal{X}_1} \times \cdots \times \top_{\mathcal{X}_{i-1}} \times I_{\mathcal{X}_i} \times \top_{\mathcal{X}_{i+1}} \times \cdots \times \top_{\mathcal{X}_n}.$$

This is the projection function $P_i \colon \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{X}_i$, which is dual to the canonical inclusion unital normal $*$-homomorphism $P_i^\star \colon \ell^\infty(\mathcal{X}_i) \hookrightarrow \ell^\infty(\mathcal{X}_1) \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} \ell^\infty(\mathcal{X}_n)$ [30, Sec. 10] and [31, App. B].

Lemma 3.5.2. *Let $\mathcal{Y}_1, \dots, \mathcal{Y}_m$ be quantum sets, and also let R be a relation of arity $(\mathcal{Y}_1, \dots, \mathcal{Y}_m)$. For each index $i \in \{1, \dots, m\}$, let t_i be a term of sort $\mathcal{Y}_i$, whose distinct variables $x_{i,1}, \dots, x_{i,n_i}$ are of sorts $\mathcal{X}_{1,i}, \dots, \mathcal{X}_{i,n_i}$, respectively. For each index $i \in \{1, \dots, m\}$, let $[\![t_i]\!]$ be an abbreviation for $[\![(x_{i,1}, \dots, x_{i,n_i}) \in \mathcal{X}_{i,1} \times \cdots \times \mathcal{X}_{i,n_i} \mid t_i]\!]$. Similarly, let $[\![R(t_1, \dots, t_m)]\!]$ be an abbreviation for*

$$[\![(x_{1,1}, \dots, x_{1,n_1}, \dots, x_{m,1}, \dots, x_{m,n_m}) \\ \in \mathcal{X}_{1,1} \times \cdots \times \mathcal{X}_{1,n_1} \times \cdots \times \mathcal{X}_{m,1} \times \cdots \times \mathcal{X}_{m,n_m} \mid R(t_1, \dots, t_m)]\!].$$

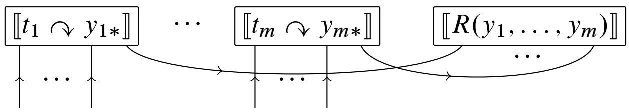
If $R(t_1, \dots, t_m)$ is nonduplicating, then

$$[\![R(t_1, \dots, t_m)]\!] = R \circ ([\![t_1]\!] \times \cdots \times [\![t_m]\!]).$$

Proof. Assume that $R(t_1, \dots, t_m)$ is nonduplicating. We calculate that

$$[\![R(t_1, \dots, t_m)]\!] \\ = [\![(\exists (y_m = y_{m*}) \in \mathcal{Y} \times \mathcal{Y}^*) \cdots (\exists (y_1 = y_{1*}) \in \mathcal{Y} \times \mathcal{Y}^*) \\ (R(y_1, \dots, y_m) \wedge t_1 \curvearrowright y_{1*} \wedge \cdots \wedge t_m \curvearrowright y_{m*})]\!]$$

=



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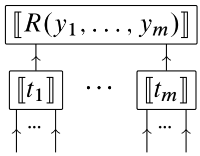


$= R \circ ([\![t_1]\!] \times \cdots \times [\![t_m]\!]).$ ∎

Lemma 3.5.3. *Let $\mathcal{Y}_1, \dots, \mathcal{Y}_m$ and $\mathcal{Z}$ be quantum sets, and let F be a function from $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$ to $\mathcal{Z}$. For each index $i \in \{1, \dots, m\}$, let t_i be a term of sort $\mathcal{Y}_i$, whose distinct variables $x_{i,1}, \dots, x_{i,n_i}$ are of sorts $\mathcal{X}_{1,i}, \dots, \mathcal{X}_{i,n_i}$, respectively. For each index $i \in \{1, \dots, m\}$, let $[\![t_i]\!]$ be an abbreviation for $[\![(x_{i,1}, \dots, x_{i,n_i}) \in \mathcal{X}_{i,1} \times \cdots \times \mathcal{X}_{i,n_i} \mid t_i]\!]$. Similarly, let $[\![F(t_1, \dots, t_m)]\!]$ be an abbreviation for*

$$[\![(x_{1,1}, \dots, x_{1,n_1}, \dots, x_{m,1}, \dots, x_{m,n_m}) \\ \in \mathcal{X}_{1,1} \times \cdots \times \mathcal{X}_{1,n_1} \times \cdots \times \mathcal{X}_{m,1} \times \cdots \times \mathcal{X}_{m,n_m} \mid F(t_1, \dots, t_m)]\!].$$

If $F(t_1, \ldots, t_m)$ is nonduplicating, then

$$\llbracket F(t_1, \ldots, t_m) \rrbracket = F \circ (\llbracket t_1 \rrbracket \times \cdots \times \llbracket t_m \rrbracket).$$

Proof. Assume that $F(t_1, \ldots, t_m)$ is nonduplicating. Let $\mathcal{X} = \mathcal{X}_{1,1} \times \cdots \times \mathcal{X}_{1,n_1} \times \cdots \times \mathcal{X}_{m,1} \times \cdots \times \mathcal{X}_{m,n_m}$, and let $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_m$. We apply Lemma 3.5.2 to calculate that

$$\begin{aligned}
\llbracket F(t_1, \ldots, t_m) \rrbracket &= (\llbracket F(t_1, \ldots, t_m) \curvearrowright z_* \rrbracket \times I_{\mathcal{Z}}) \circ (I_{\mathcal{X}} \times E^*_{\mathcal{Z}}) \\
&= (\llbracket \hat{F}(t_1, \ldots, t_m, z_*) \rrbracket \times I_{\mathcal{Z}}) \circ (I_{\mathcal{X}} \times E^*_{\mathcal{Z}}) \\
&= ((\hat{F} \circ (\llbracket t_1 \rrbracket \times \cdots \times \llbracket t_m \rrbracket \times \llbracket z_* \rrbracket)) \times I_{\mathcal{Z}}) \circ (I_{\mathcal{X}} \times E^*_{\mathcal{Z}}) \\
&= ((\hat{F} \circ (\llbracket t_1 \rrbracket \times \cdots \times \llbracket t_m \rrbracket \times I_{\mathcal{Z}^*})) \times I_{\mathcal{Z}}) \circ (I_{\mathcal{X}} \times E^*_{\mathcal{Z}}) \\
&= (\hat{F} \times I_{\mathcal{Z}}) \circ (I_{\mathcal{Y}} \times E^*_{\mathcal{Z}}) \circ (\llbracket t_1 \rrbracket \times \cdots \times \llbracket t_m \rrbracket) \\
&= F \circ (\llbracket t_1 \rrbracket \times \cdots \times \llbracket t_m \rrbracket).
\end{aligned}$$ ■

We may conjugate a term t by conjugating each function symbol and variable that appears in that term. Formally, if t is of the form $F(t_1, \ldots, t_m)$, then we define t_* to be $F_*(t_{1*}, \ldots, t_{m*})$, and if t is a variable x, then we define t_* to be x_*, the conjugate variable. If the term t has variables among $x_1, \ldots, x_n$, of sorts quantum sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively, then we may apply Lemma 3.5.3 to show that

$$\llbracket (x_{1*}, \ldots, x_{n*}) \in \mathcal{X}_1^* \times \cdots \times \mathcal{X}_n^* \mid t_* \rrbracket = \llbracket (x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid t \rrbracket_*.$$

Proposition 3.5.4. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ and $\mathcal{Y}$ be quantum sets. Let $x_1, \ldots, x_n$ be distinct variables of sorts $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively, and let $x_{1*}, \ldots, x_{n*}$ be distinct variables of sorts $\mathcal{X}_{1*}, \ldots, \mathcal{X}_{n*}$, respectively. Let s and t be terms of sort $\mathcal{Y}$ whose free variables are among $x_1, \ldots, x_n$. Then,*

$$\llbracket (x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid s \rrbracket = \llbracket (x_1, \ldots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid t \rrbracket$$

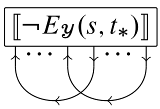
if and only if

$$\llbracket (\forall (x_n = x_{n*}) \in \mathcal{X}_n \times \mathcal{X}_n^*) \cdots (\forall (x_1 = x_{1*}) \in \mathcal{X}_1 \times \mathcal{X}_1^*)\, E_{\mathcal{Y}}(s, t_*) \rrbracket = \top. \qquad (\dagger)$$

Proof. By the duality of diagonal quantifiers (Definition 2.5.1), equation (†) holds if and only if

$$\llbracket (\exists (x_n = x_{n*}) \in \mathcal{X}_n \times \mathcal{X}_n^*) \cdots (\exists (x_1 = x_{1*}) \in \mathcal{X}_1 \times \mathcal{X}_1^*) \neg E_{\mathcal{Y}}(s, t_*) \rrbracket = \bot$$

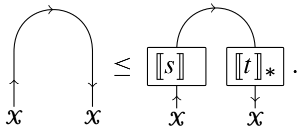
or equivalently



$= \bot.$

We recognize the diagram on the left as depicting $\llbracket \neg E_{\mathcal{Y}}(s, t_*) \rrbracket \circ E^\dagger_{\mathcal{X}}$, for $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. Thus, equation (†) holds if and only if $(\neg \llbracket E_{\mathcal{Y}}(s, t_*) \rrbracket) \circ E^\dagger_{\mathcal{X}} = \bot$ or equivalently $E_{\mathcal{X}} \perp \neg \llbracket E_{\mathcal{Y}}(s, t_*) \rrbracket$ or equivalently $E_{\mathcal{X}} \leq \llbracket E_{\mathcal{Y}}(s, t_*) \rrbracket$. By Lemma 3.5.2, we have that

$[\![Ey(s, t_*)]\!] = Ey \circ ([\![s]\!] \times [\![t]\!]_*)$, and therefore, equation (†) is equivalent to the inequality



Straightening the wires, we conclude that equation (†) is equivalent to $I_{\mathcal{X}} \leq [\![t]\!]^\dagger \circ [\![s]\!]$.

It is a basic fact about functions between quantum sets that $I_{\mathcal{X}} \leq [\![t]\!]^\dagger \circ [\![s]\!]$ if and only if $[\![s]\!] = [\![t]\!]$. Indeed, $I_{\mathcal{X}} \leq [\![t]\!]^\dagger \circ [\![s]\!]$ implies that $[\![t]\!] \leq [\![t]\!] \circ [\![t]\!]^\dagger \circ [\![s]\!] \leq [\![s]\!]$, and similarly, it implies that $[\![s]\!]^\dagger \leq [\![t]\!]^\dagger \circ [\![s]\!] \circ [\![s]\!]^\dagger \leq [\![t]\!]^\dagger$, so $[\![s]\!] \leq [\![t]\!]$. Altogether, $I_{\mathcal{X}} \leq [\![t]\!]^\dagger \circ [\![s]\!]$ implies that $[\![s]\!] = [\![t]\!]$. Conversely, if $[\![s]\!] = [\![t]\!]$, then $I_{\mathcal{X}} \leq [\![t]\!]^\dagger \circ [\![s]\!]$, by the definition of a function between quantum sets. Thus, we conclude that equation (†) is equivalent to $[\![s]\!] = [\![t]\!]$. ∎

A. Appendix

A.1. Nondegenerate equality

We show that the equality relation on a von Neumann algebra is nondegenerate if and only if that von Neumann algebra is hereditarily atomic.

Lemma A.1.1. *Let M be a commutative von Neumann algebra that contains no minimal projections. There exists no normal state φ on the spatial tensor product $M \mathbin{\bar{\otimes}} M$ such that $\varphi(p \otimes (1 - p)) = 0$ for every projection $p \in M$.*

Proof. Suppose that we have such a normal state φ on $M \mathbin{\bar{\otimes}} M$. Let φ_1 and φ_2 be the normal states on M defined by $\varphi_1(a) = \varphi(a \otimes 1)$ and $\varphi_2(a) = \varphi(1 \otimes a)$ for all $a \in M$. For both $i \in \{1, 2\}$, let p_i be the support projection of φ_i, in other words, the smallest projection in M such that $\varphi_i(p_i) = 1$. It is easy to see that φ_i is faithful on $p_i M$. Indeed, for any projection $q \leq p_i$, if $\varphi_i(q) = 0$, then $\varphi_i(p_i - q) = 1$, which implies that $q = 0$, by the minimality of p_i. We conclude that $\varphi_1 \mathbin{\bar{\otimes}} \varphi_2$ is a faithful normal state on $p_1 M \mathbin{\bar{\otimes}} p_2 M$ [7, Proposition III.2.2.29].

Our given normal state φ factors through $p_1 M \mathbin{\bar{\otimes}} p_2 M$, as we now show. Indeed, by our choice of p_1 and p_2, we have that $\varphi(p_1 \otimes 1) = 1 = \varphi(1 \otimes p_2)$. Writing φ as a countable linear combination of vector states, we find that $\varphi(p_1 \otimes p_2) = 1$ and furthermore that $\varphi((p_1 \otimes p_2)b) = \varphi(b)$ for all $b \in M \mathbin{\bar{\otimes}} M$. Thus, φ does factor through $p_1 M \mathbin{\bar{\otimes}} p_2 M$, as claimed.

Finite partitions of the identity $1 \in M$ into pairwise orthogonal projections form a directed set Λ, with finer partitions appearing higher in the order. For each such partition $\lambda \in \Lambda$, we define a projection $q_\lambda = \sum_{p \in \lambda} p \otimes p$. The net $(q_\lambda \mid \lambda \in \Lambda)$ is evidently decreasing, and it therefore has an ultraweak limit q_∞, also a projection in $M \mathbin{\bar{\otimes}} M$. By our assumption on φ, we have that $\varphi(q_\lambda) = 1$ for each partition λ, and therefore $\varphi(q_\infty) = 1$. We conclude that $\varphi((p_1 \otimes p_2)q_\infty) = 1$.

We now obtain a contradiction by showing that $(p_1 \otimes p_2)q_\infty = 0$ as follows:

$$\begin{aligned}(\varphi_1 \mathbin{\bar{\otimes}} \varphi_2)((p_1 \otimes p_2)q_\infty) &= (\varphi_1 \mathbin{\bar{\otimes}} \varphi_2)(\lim_\lambda (p_1 \otimes p_2)q_\lambda) \\ &= \lim_\lambda (\varphi_1 \mathbin{\bar{\otimes}} \varphi_2)((p_1 \otimes p_2)q_\lambda) \\ &= \lim_\lambda \sum_{p \in \lambda} \varphi_1(p_1 p)\varphi_2(p_2 p) = 0.\end{aligned}$$

The final equality is a consequence of the fact that, for both $i \in \{1, 2\}$, we can partition p_i into projections p that are arbitrarily small in the sense that each satisfies $\varphi_i(p) \leq \varepsilon$ for arbitrarily small $\varepsilon > 0$. Indeed, p_i is the identity of the von Neumann algebra $p_i M$, which has no atoms and on which φ_i is a faithful normal state. Since $\varphi_1 \mathbin{\bar{\otimes}} \varphi_2$ is faithful, we conclude that $(p_1 \otimes p_2)q_\infty = 0$, as claimed. Having obtained a contradiction, we infer that our opening supposition is false. ∎

Proposition A.1.2. *Let M be any von Neumann algebra. Let δ be the largest projection in the spatial tensor product $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$ such that $(p \otimes (1 - p))\delta = 0$ for all projections p in M. Then, $(p_1 \otimes p_1)\delta \neq 0$ for all nonzero projections p_1 in M if and only if M is hereditarily atomic.*

Proof. Let H be the Hilbert space on which M is canonically represented. The von Neumann algebra M^{op} is canonically represented on the conjugate Hilbert space $\bar{H}$, whose vectors are the same as those of H but written with a conjugation symbol so that $\overline{\alpha h} = \bar{\alpha}\bar{h}$ for all $\alpha \in \mathbb{C}$ and all $h \in H$. The inner product on $\bar{H}$ is defined by $\langle \bar{h}_1 | \bar{h}_2 \rangle = \overline{\langle h_1 | h_2 \rangle}$. For each $a \in M^{\mathrm{op}}$ and each $\bar{h} \in \bar{H}$, we define $a\bar{h} = \overline{a^\dagger h}$, where $a^\dagger$ is the Hermitian adjoint of a. It is routine to verify that this defines a faithful representation of M^{op} on $\bar{H}$.

Like any von Neumann algebra, M is the direct sum of a hereditarily atomic von Neumann algebra M_0 and a von Neumann algebra M_1 that has no finite type I factors as a direct summand. Let p_0 and p_1 be the central projections in M corresponding to M_0 and M_1, respectively. Hence, $M_0 = p_0 M$ and $M_1 = p_1 M$.

Assume that M is not hereditarily atomic or, in other words, that p_1 is nonzero. By our assumption on δ, we have that $(p_0 \otimes p_1)\delta = 0$ and $(p_1 \otimes p_0)\delta = 0$, and therefore, $\delta = (p_0 \otimes p_0)\delta + (p_1 \otimes p_1)\delta$. The projection $\delta_1 := (p_1 \otimes p_1)\delta$ is in $M_1 \mathbin{\bar{\otimes}} M_1^{\mathrm{op}}$, and it satisfies $(p \otimes (p_1 - p))\delta_1 = 0$ for all projections p in M_1.

Assume for contradiction that δ_1 is nonzero. It follows that there is a vector w in the Hilbert space $(p_1 H) \otimes (p_1 \bar{H})$ such that $(p \otimes (p_1 - p))w = 0$ for all projections p in M_1. Thus, we have a state φ on $M_1 \mathbin{\bar{\otimes}} M_1^{\mathrm{op}}$ such that $\varphi(p \otimes (p_1 - p)) = 0$ for all projections p in M_1.

The algebra M_1 need not be commutative, but it contains a unital ultraweakly closed $*$-subalgebra that is both commutative and *diffuse* in the sense that it contains no minimal projections. Indeed, the center of M_1 is the direct sum of an atomic von Neumann algebra and a diffuse von Neumann algebra. It follows that M_1 is a direct sum of von Neumann algebras, each of which is either a factor that is not finite type I or a von Neumann algebra with diffuse center. Each such direct summand has a diffuse commutative unital ultraweakly closed $*$-subalgebra, and hence, so does M_1.

Let N be any diffuse commutative unital ultraweakly closed $*$-subalgebra of M_1. The normal state φ restricts to a normal state on $N \mathbin{\bar{\otimes}} N^{\mathrm{op}}$ satisfying the equation $\varphi(p \otimes (p_1 - p)) = 0$ for all projections p in N, and p_1 is the multiplicative unit of N because N is a unital $*$-subalgebra of M_1. Furthermore, since N is commutative, $N^{\mathrm{op}} = N$. Therefore, we may apply Lemma A.1.1 to obtain a contradiction. We conclude that $\delta_1 = 0$, that is, $(p_1 \otimes p_1)\delta = 0$. Therefore, if M is not hereditarily atomic, then there does exist a nonzero projection p_1 such that $(p_1 \otimes p_1)\delta = 0$.

Assume now that M is a hereditarily atomic von Neumann algebra, and let p_1 be a nonzero projection in M. By [30, Prop. 5.4], there exists a set A of finite-dimensional Hilbert spaces such that M is isomorphic to the ℓ^∞-direct sum of the operator algebras $L(X)$, for $X \in A$. Without loss of generality, we may assume that M is equal to such an ℓ^∞-direct sum. For each $X \in A$, let $[X]$ be the corresponding minimal central projection in M.

Let X_1 be such that $p_1[X_1] \neq 0$. Choose an orthonormal basis $x_1, \dots, x_n$ for X_1, and let

$$w = \sum_{i=1}^{n} x_i \otimes \bar{x}_i \in X_1 \otimes \bar{X}_1 \leq H \otimes \bar{H}.$$

Let $[\mathbb{C}w]$ be the corresponding projection in $M \mathbin{\bar{\otimes}} M^{\mathrm{op}}$. Using standard linear algebra, we may directly compute $\langle w|(p_1 \otimes p_1)w\rangle$ to show that $(p_1 \otimes p_1)w \neq 0$; thus,

$$(p_1 \otimes p_1)[\mathbb{C}w] \neq 0.$$

Similarly, for each projection $p \in M$, we may directly compute $\langle w|(p \otimes (1-p))w\rangle$ to show that $(p \otimes (1-p))w = 0$; thus, $[\mathbb{C}w] \leq \delta$. Altogether, we find that $(p_1 \otimes p_1)\delta = 0$ for any nonzero projection p_1 in M. ∎

A.2. Weaver's quantum relations

We substantiate the observation [30] that binary relations between quantum sets are essentially just Weaver's quantum relations [57]. A quantum relation from a von Neumann algebra $M \leq L(H)$ to a von Neumann algebra $N \leq L(K)$ is defined to be an ultraweakly closed subspace $V \leq L(H, K)$ such that $N' \cdot V \cdot M' \leq V$.

For each atom X of a quantum set $\mathcal{X}$, we write $\mathrm{inc}_X \in L(X, \bigoplus \mathrm{At}(\mathcal{X}))$ for the corresponding inclusion isometry.

Proposition A.2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets. Let the von Neumann algebras $\ell^\infty(\mathcal{X})$ and $\ell^\infty(\mathcal{Y})$ be canonically represented on the Hilbert spaces $\bigoplus \mathrm{At}(\mathcal{X})$ and $\bigoplus \mathrm{At}(\mathcal{Y})$, respectively. Quantum relations V from $\ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{Y})$ are in one-to-one correspondence with binary relations R from $\mathcal{X}$ to $\mathcal{Y}$. The correspondence is given by $R(X, Y) = \mathrm{inc}_Y^\dagger \cdot V \cdot \mathrm{inc}_X$, for $X \in \mathrm{At}(\mathcal{X})$ and $Y \in \mathrm{At}(\mathcal{Y})$.*

Proof. For each quantum relation V from $\ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{Y})$, let R_V be the binary relation from $\mathcal{X}$ to $\mathcal{Y}$ defined by $R_V(X,Y) = \mathrm{inc}_Y^\dagger \cdot V \cdot \mathrm{inc}_X$, for $X \in \mathrm{At}(\mathcal{X})$ and $Y \in \mathrm{At}(\mathcal{Y})$. For each binary relation R from $\mathcal{X}$ to $\mathcal{Y}$, let $V_R \leq L(\bigoplus \mathrm{At}(\mathcal{X}), \bigoplus \mathrm{At}(\mathcal{Y}))$ be defined by

$$V_R = \overline{\sum \{\mathrm{inc}_Y \cdot R(X,Y) \cdot \mathrm{inc}_X^\dagger \mid X \in \mathrm{At}(\mathcal{X}),\ Y \in \mathrm{At}(\mathcal{Y})\}},$$

where the symbol $\sum$ denotes the algebraic span of the union and the line indicates closure with respect to the ultraweak topology. This is a quantum relation from $\ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{Y})$ because $\ell^\infty(\mathcal{X})'$ is the closed span of the minimal central projections $[X] := \mathrm{inc}_X \cdot \mathrm{inc}_X^\dagger$ for $X \in \mathrm{At}(\mathcal{X})$, and it is likewise for $\ell^\infty(\mathcal{Y})'$. Indeed, for all atoms $X_0 \in \mathrm{At}(\mathcal{X})$ and $Y_0 \in \mathrm{At}(\mathcal{Y})$, we calculate that

$$[Y_0] \cdot V_R \cdot [X_0] \leq \overline{\mathrm{inc}_{Y_0} \cdot R(X_0,Y_0) \cdot \mathrm{inc}_{X_0}^\dagger} = \mathrm{inc}_{Y_0} \cdot R(X_0,Y_0) \cdot \mathrm{inc}_{X_0}^\dagger \leq V_R,$$

and therefore $\ell^\infty(\mathcal{Y})' \cdot V_R \cdot \ell^\infty(\mathcal{X})' \leq V_R$.

We show that the two constructions invert each other by direct calculation. For each binary relation R from $\mathcal{X}$ to $\mathcal{Y}$, and all atoms $X_0 \in \mathrm{At}(\mathcal{X})$ and $Y_0 \in \mathrm{At}(Y)$, we calculate that

$$\begin{aligned} R_{V_R}(X_0,Y_0) &= \mathrm{inc}_{Y_0}^\dagger \cdot V_R \cdot \mathrm{inc}_{X_0} \leq \overline{\mathrm{inc}_{Y_0}^\dagger \cdot \mathrm{inc}_{Y_0} \cdot R(X_0,Y_0) \cdot \mathrm{inc}_{X_0}^\dagger \cdot \mathrm{inc}_{X_0}} \\ &= R(X_0,Y_0) \\ &= \mathrm{inc}_{Y_0}^\dagger \cdot \mathrm{inc}_{Y_0} \cdot R(X_0,Y_0) \cdot \mathrm{inc}_{X_0}^\dagger \cdot \mathrm{inc}_{X_0} \leq \mathrm{inc}_{Y_0}^\dagger \cdot V_R \cdot \mathrm{inc}_{X_0} \\ &= R_{V_R}(X_0,Y_0). \end{aligned}$$

Similarly, for each quantum relation V from $\ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{Y})$, we calculate that

$$\begin{aligned} V_{R_V} &= \overline{\sum \left\{\mathrm{inc}_Y \cdot R_V(X,Y) \cdot \mathrm{inc}_X^\dagger \mid X \in \mathrm{At}(\mathcal{X}),\ Y \in \mathrm{At}(\mathcal{Y})\right\}} \\ &= \overline{\sum \left\{\mathrm{inc}_Y \cdot \mathrm{inc}_Y^\dagger \cdot V \cdot \mathrm{inc}_X \cdot \mathrm{inc}_X^\dagger \mid X \in \mathrm{At}(\mathcal{X}),\ Y \in \mathrm{At}(\mathcal{Y})\right\}} \\ &= \overline{\sum \left\{[Y] \cdot V \cdot [X] \mid X \in \mathrm{At}(\mathcal{X}),\ Y \in \mathrm{At}(\mathcal{Y})\right\}} = V. \end{aligned}$$

The last equality can be proved by establishing both inclusions. The inclusion of the left side into the right side holds because V is a quantum relation. The inclusion of the right side into the left side holds because the projections $[X]$ for $X \in \mathrm{At}(\mathcal{X})$ sum to the identity, as do the projections $[Y]$ for $Y \in \mathrm{At}(\mathcal{Y})$. Therefore, the constructions $R \mapsto V_R$ and $V \mapsto R_V$ invert each other. ∎

Proposition A.2.2. *The one-to-one correspondence of Proposition A.2.1 is functorial.*

Proof. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets, let V be a quantum relation from $\ell^\infty(\mathcal{X})$ to $\ell^\infty(\mathcal{Y})$, and let W be a quantum relation from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{Z})$. In the notation of the proof of Proposition A.2.1, we are to show that $R_{\overline{W \cdot V}} = R_W \circ R_V$. For all atoms $X \in \mathrm{At}(\mathcal{X})$ and $Z \in \mathrm{At}(\mathcal{Z})$, we calculate that

$$\begin{aligned}
(R_W \circ R_V)(X, Z) &= \sum_{Y \in \mathrm{At}(\mathcal{Y})} R_W(Y, Z) \cdot R_V(X, Y) \\
&= \sum_{Y \in \mathrm{At}(\mathcal{Y})} \mathrm{inc}_Z^\dagger \cdot W \cdot \mathrm{inc}_Y \cdot \mathrm{inc}_Y^\dagger \cdot V \cdot \mathrm{inc}_X \leq \mathrm{inc}_Z^\dagger \cdot W \cdot V \cdot \mathrm{inc}_X \\
&\leq \mathrm{inc}_Z^\dagger \cdot \overline{W \cdot V} \cdot \mathrm{inc}_X = R_{\overline{W \cdot V}}(X, Z) \leq \overline{\mathrm{inc}_Z^\dagger \cdot W \cdot V \cdot \mathrm{inc}_X} \\
&= \overline{\mathrm{inc}_Z^\dagger \cdot W \cdot 1_{\bigoplus \mathrm{At}(Y)} \cdot V \cdot \mathrm{inc}_X} \\
&\leq \overline{\sum_{Y \in \mathrm{At}(\mathcal{Y})} \mathrm{inc}_Z^\dagger \cdot W \cdot \mathrm{inc}_Y \cdot \mathrm{inc}_Y^\dagger \cdot V \cdot \mathrm{inc}_X} \\
&= \overline{(R_W \circ R_V)(X, Z)} = (R_W \circ R_V)(X, Z).
\end{aligned}$$
■

It is immediate from the definition of this one-to-one correspondence in Proposition A.2.1 that it preserves the partial order relation and the adjoint operation. Thus, we obtain an enriched equivalence of dagger categories from the category of quantum sets and binary relations to the category of hereditarily atomic von Neumann algebras and quantum relations.

A.3. Permutation equivariance

We prove Proposition 2.3.3.

Lemma A.3.1. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets.*

(1) *For every predicate P on $\mathcal{X}$, we have $\neg(P \times \top_{\mathcal{Y}}) = (\neg P) \times \top_{\mathcal{Y}}$.*

(2) *For all predicates P_1 and P_2 on $\mathcal{X}$ and all predicates Q_1 and Q_2 on $\mathcal{Y}$, we have*

$$(P_1 \times Q_1) \wedge (P_2 \times Q_2) = (P_1 \wedge P_2) \times (Q_1 \wedge Q_2).$$

(3) *Let R be a predicate on $\mathcal{X} \times \mathcal{Y}$, let Q be the largest predicate on $\mathcal{Y}$ such that $\top_{\mathcal{X}} \times Q \leq R$, and let S be the largest predicate on $\mathcal{Y} \times \mathcal{Z}$ such that $\top_{\mathcal{X}} \times S \leq R \times \top_{\mathcal{Z}}$. Then, $S = Q \times \top_{\mathcal{Z}}$.*

Proof. All three claims are established most easily using the bijective correspondence between the predicates on any quantum set $\mathcal{W}$ and the projections in the corresponding von Neumann algebra $\ell^\infty(\mathcal{W})$. Expressed in terms of projections, claims (1) and (2) are elementary. To prove claim (3), let Q, R, and S correspond to projections $q \in \ell^\infty(\mathcal{Y})$, $r \in \ell^\infty(\mathcal{X}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Y})$, and $s \in \ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Z})$, respectively. The von Neumann algebra $\ell^\infty(\mathcal{Z})$ is canonically represented on a Hilbert space H, the ℓ^2-direct sum of the atoms of $\mathcal{Z}$. If $\mathcal{Z}$ has no atoms, then claim (3) holds trivially, so we may assume that H is nonzero. We are essentially given that

$$q = \sup\{p \in \mathrm{Proj}(\ell^\infty(\mathcal{Y})) \mid 1 \otimes p \leq r\},$$

and $s = \sup\{p \in \mathrm{Proj}(\ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Z})) \mid 1 \otimes p \leq r \otimes 1\}$. In particular, $1 \otimes q \leq r$, so $1 \otimes q \otimes 1 \leq r \otimes 1$, giving $q \otimes 1 \leq s$.

For the opposite inequality, we consider the projection $\tilde{s}$, defined to be the supremum of all projections p in $\ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} L(H)$ satisfying the inequality $1 \otimes p \leq r \otimes 1$, where $L(H)$ is the von Neumann algebra of all bounded operators on H. If a projection p satisfies the inequality $1 \otimes p \leq r \otimes 1$, then so does the projection $(1 \otimes u^\dagger) \cdot p \cdot (1 \otimes u)$ for every unitary operator $u \in L(H)$. It follows that

$$(1 \otimes u^\dagger) \cdot \tilde{s} \cdot (1 \otimes u) = \tilde{s}$$

for all unitaries $u \in L(H)$, so $\tilde{s}$ is in the commutant $(\mathbb{C} \mathbin{\bar{\otimes}} L(H))'$. Since $\tilde{s}$ is also in $\ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} L(H)$, we conclude that $\tilde{s}$ is in $\ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} \mathbb{C}$, a von Neumann subalgebra of $\ell^\infty(\mathcal{Y}) \mathbin{\bar{\otimes}} \ell^\infty(\mathcal{Z})$. Therefore, $s = \tilde{s} = p_1 \otimes 1$, for some projection p_1 in $\ell^\infty(\mathcal{Y})$. We now calculate that

$$1 \otimes p_1 \otimes 1 = 1 \otimes s \leq r \otimes 1,$$

which implies that $1 \otimes p_1 \leq r$, giving us $p_1 \leq q$, by the definition of q as a supremum of projections satisfying this inequality. Finally, we obtain $s = p_1 \otimes 1 \leq q \otimes 1$. ∎

Proposition A.3.2. *Let $\mathcal{X}_1, \dots, \mathcal{X}_p$ be quantum sets, and let $x_1, \dots, x_p$ be distinct variables of sorts $\mathcal{X}_1, \dots, \mathcal{X}_p$, respectively. For each permutation σ of $\{1, \dots, p\}$ and each primitive formula $\phi(x_1, \dots, x_n)$ with $n \leq p$, we have*

$$\begin{aligned}
&[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \cdots \times \mathcal{X}_{\sigma(p)} \mid \phi(x_1, \dots, x_n)]\!] \\
&\quad = (\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid \phi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p}).
\end{aligned}$$

Proof. The proof proceeds by structural induction. To clarify the calculations, we introduce the notations $\mathcal{Y}_i = \mathcal{X}_{\sigma(i)}$ and $y_i = x_{\sigma(i)}$ for $i \in \{1, \dots, p\}$.

Suppose that $\phi(x_1, \dots, x_n)$ is atomic. In that case, $\phi(x_1, \dots, x_n)$ is necessarily of the form $R(x_{\pi(1)}, \dots, x_{\pi(m)})$, for some natural $m \leq n$, for some permutation π of the set $\{1, \dots, n\}$, and for some relation R of arity $(\mathcal{X}_{\pi(1)}, \dots, \mathcal{X}_{\pi(m)})$. We may extend π to a permutation $\tilde{\pi}$ of the set $\{1, \dots, p\}$ by defining $\tilde{\pi}(k) = k$ for all k in $\{n+1, \dots, p\}$.

We then calculate that

$$\begin{aligned}
&[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \cdots \times \mathcal{X}_{\sigma(p)} \mid R(x_{\pi(1)}, \dots, x_{\pi(m)})]\!] \\
&= [\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \cdots \times \mathcal{X}_{\sigma(p)} \mid R(x_{\tilde{\pi}(1)}, \dots, x_{\tilde{\pi}(m)})]\!] \\
&= [\![(y_1, \dots, y_p) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_p \mid R(y_{(\sigma^{-1} \circ \tilde{\pi})(1)}, \dots, y_{(\sigma^{-1} \circ \tilde{\pi})(m)})]\!] \\
&= (\sigma^{-1} \circ \tilde{\pi})_\#(R \times \top_{\mathcal{Y}_{(\sigma^{-1} \circ \tilde{\pi})(m+1)}} \times \cdots \times \top_{\mathcal{Y}_{(\sigma^{-1} \circ \tilde{\pi})(p)}}) \\
&= (\sigma^{-1})_\#(\tilde{\pi}_\#(R \times \top_{\mathcal{X}_{\tilde{\pi}(m+1)}} \times \cdots \times \top_{\mathcal{X}_{\tilde{\pi}(p)}})) \\
&= (\sigma^{-1})_\#(\tilde{\pi}_\#(R \times \top_{\mathcal{X}_{\pi(m+1)}} \times \cdots \times \top_{\mathcal{X}_{\pi(n)}} \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p})) \\
&= (\sigma^{-1})_\#(\pi_\#(R \times \top_{\mathcal{X}_{\pi(m+1)}} \times \cdots \times \top_{\mathcal{X}_{\pi(n)}}) \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p}) \\
&= (\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid R(x_{\pi(1)}, \dots, x_{\pi(m)})]\!] \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p}).
\end{aligned}$$

Suppose that $\phi(x_1, \dots, x_n)$ is of the form $\neg\psi(x_1, \dots, x_n)$ for some nonduplicating formula $\psi(x_1, \dots, x_n)$. Then,

$$
\begin{aligned}
&[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \dots \times \mathcal{X}_{\sigma(p)} \mid \neg\psi(x_1, \dots, x_n)]\!] \\
&= [\![(y_1, \dots, y_p) \in \mathcal{Y}_1 \times \dots \times \mathcal{Y}_p \mid \neg\psi(y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(n)})]\!] \\
&= \neg[\![(y_1, \dots, y_p) \in \mathcal{Y}_1 \times \dots \times \mathcal{Y}_p \mid \psi(y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(n)})]\!] \\
&= \neg[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \dots \times \mathcal{X}_{\sigma(p)} \mid \psi(x_1, \dots, x_n)]\!] \\
&= \neg(\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}) \\
&= (\sigma^{-1})_\#(\neg([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p})) \\
&= (\sigma^{-1})_\#(\neg[\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \psi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}) \\
&= (\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \dots \times \mathcal{X}_n \mid \neg\psi(x_1, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}).
\end{aligned}
$$

We apply Lemma A.3.1 (1) in the second-to-last equality. The case in which $\phi(x_1, \dots, x_n)$ is of the form $\psi_1(x_1, \dots, x_n) \wedge \psi_2(x_1, \dots, x_n)$ is entirely similar; there, we apply Lemma A.3.1 (2).

Suppose that $\phi(x_1, \dots, x_n)$ is of the form $(\forall x_0 \in \mathcal{X}_0)\, \psi(x_0, \dots, x_n)$ for some quantum set $\mathcal{X}_0$ and some variable x_0 that is distinct from the variables $x_1, \dots, x_p$, and that has sort $\mathcal{X}_0$. We extend the permutation σ to a permutation $\widetilde{\sigma}$ of $\{0, \dots, p\}$ by setting $\widetilde{\sigma}(0) = 0$, and we write $\mathcal{Y}_0 = \mathcal{X}_0$ and $y_0 = x_0$. We then calculate that

$$
\begin{aligned}
&[\![(x_{\sigma(1)}, \dots, x_{\sigma(p)}) \in \mathcal{X}_{\sigma(1)} \times \dots \times \mathcal{X}_{\sigma(p)} \mid (\forall x_0 \in \mathcal{X}_0)\, \psi(x_0, \dots, x_n)]\!] \\
&= [\![(y_1, \dots, y_p) \in \mathcal{Y}_1 \times \dots \times \mathcal{Y}_p \mid (\forall y_0 \in \mathcal{Y}_0)\, \psi(y_0, y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(n)})]\!] \\
&= \sup\{R \in \mathrm{Rel}\{\mathcal{Y}_1, \dots, \mathcal{Y}_p\} \mid \top_{\mathcal{Y}_0} \times R \\
&\qquad \leq [\![(y_0, \dots, y_p) \in \mathcal{Y}_0 \times \dots \times \mathcal{Y}_p \mid \psi(y_0, y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(n)})]\!]\} \\
&= \sup\{R \in \mathrm{Rel}\{\mathcal{X}_{\sigma(1)}, \dots, \mathcal{X}_{\sigma(p)}\} \mid \top_{\mathcal{X}_0} \times R \\
&\qquad \leq [\![(x_{\widetilde{\sigma}(0)}, x_{\widetilde{\sigma}(1)}, \dots, x_{\widetilde{\sigma}(p)}) \in \mathcal{X}_{\widetilde{\sigma}(0)} \times \mathcal{X}_{\widetilde{\sigma}(1)} \dots \times \mathcal{X}_{\widetilde{\sigma}(p)} \mid \psi(x_0, x_1, \dots, x_n)]\!]\} \\
&= \sup\{R \in \mathrm{Rel}\{\mathcal{X}_{\sigma(1)}, \dots, \mathcal{X}_{\sigma(p)}\} \mid \top_{\mathcal{X}_0} \times R \\
&\qquad \leq (\widetilde{\sigma}^{-1})_\#([\![(x_0, \dots, x_n) \in \mathcal{X}_0 \times \dots \times \mathcal{X}_n \mid \psi(x_0, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p})\} \\
&= \sup\{R \in \mathrm{Rel}\{\mathcal{X}_{\sigma(1)}, \dots, \mathcal{X}_{\sigma(p)}\} \mid (\widetilde{\sigma})_\#(\top_{\mathcal{X}_0} \times R) \\
&\qquad \leq [\![(x_0, \dots, x_n) \in \mathcal{X}_0 \times \dots \times \mathcal{X}_n \mid \psi(x_0, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}\} \\
&= \sup\{R \in \mathrm{Rel}\{\mathcal{X}_{\sigma(1)}, \dots, \mathcal{X}_{\sigma(p)}\} \mid \top_{\mathcal{X}_0} \times \sigma_\#(R) \\
&\qquad \leq [\![(x_0, \dots, x_n) \in \mathcal{X}_0 \times \dots \times \mathcal{X}_n \mid \psi(x_0, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}\} \\
&= (\sigma^{-1})_\#(\sup\{R' \in \mathrm{Rel}\{\mathcal{X}_1, \dots, \mathcal{X}_p\} \mid \top_{\mathcal{X}_0} \times R' \\
&\qquad \leq [\![(x_0, \dots, x_n) \in \mathcal{X}_0 \times \dots \times \mathcal{X}_n \mid \psi(x_0, \dots, x_n)]\!] \times \top_{\mathcal{X}_{n+1}} \times \dots \times \top_{\mathcal{X}_p}\})
\end{aligned}
$$

$$= (\sigma^{-1})_\#(\sup\{R'' \in \mathrm{Rel}\{\mathcal{X}_1, \dots, \mathcal{X}_n\} \mid \top_{\mathcal{X}_0} \times R''$$
$$\leq [\![(x_0, \dots, x_n) \in \mathcal{X}_0 \times \cdots \times \mathcal{X}_n \mid \psi(x_0, \dots, x_n)]\!]\} \times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p})$$
$$= (\sigma^{-1})_\#([\![(x_1, \dots, x_n) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \mid (\forall x_0 \in \mathcal{X}_0)\, \psi(x_0, \dots, x_n)]\!]$$
$$\times \top_{\mathcal{X}_{n+1}} \times \cdots \times \top_{\mathcal{X}_p}).$$

We apply Lemma A.3.1 (3) in the second-to-last equality. ∎

A.4. Relating $E_{\mathcal{X}}$ and δ_M

Let $\mathcal{X}$ be a quantum set. The von Neumann algebra $\ell^\infty(\mathcal{X}) \mathbin{\overline{\otimes}} \ell^\infty(\mathcal{X})^{\mathrm{op}}$ is canonically isomorphic to $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$ via the ultraweakly continuous map ϕ that is defined by $\phi(a_1 \otimes a_2)(X_1 \otimes X_2^*) = a_1(X_1) \otimes a_2(X_2)^*$, for $a_1, a_2 \in \ell^\infty(\mathcal{X})$ and $X_1, X_2 \in \mathrm{At}(\mathcal{X})$. The map ϕ exists and is a unital normal $*$-homomorphism because the spatial tensor product of hereditarily atomic von Neumann algebras is also their categorical tensor product [17]. Clearly, ϕ maps no minimal central projections to 0, so it is injective on its entire domain. It is also clearly surjective onto each factor of $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$, so it is surjective onto its entire codomain. Thus, ϕ is an isomorphism.

Hence, we regard the projection $\delta_{\ell^\infty(\mathcal{X})}$ that was defined in Section 1.1 as an element of $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$. We show that this projection corresponds to the predicate $E_{\mathcal{X}}$ in the sense that $E_{\mathcal{X}}(X_1 \otimes X_2^*) = L(X_1 \otimes X_2^*, \mathbb{C}) \cdot \delta_{\ell^\infty(\mathcal{X})}(X_1 \otimes X_2^*)$ for all $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.

Lemma A.4.1. *Let X be a finite-dimensional Hilbert space, and let $\zeta \colon X \otimes X^* \to \mathbb{C}$ be a functional. Then, $\zeta \in \mathbb{C}\varepsilon_X$ if and only if $\zeta \cdot (p \otimes (1-p)^*) = 0$ for all projections $p \in L(X)$.*

Proof. Assume that $\zeta = \alpha\varepsilon_X$ for some $\alpha \in \mathbb{C}$, and let $p \in L(X)$ be a projection. Let $x_1 \in pX$ and let $x_2^* \in (1-p)^*X^*$. Then, $\zeta(x_1 \otimes x_2^*) = \alpha\langle x_2 | x_1\rangle = 0$. We conclude that ζ vanishes on $pX \otimes (1-p)^*X^* = (p \otimes (1-p)^*)(X \otimes X^*)$, so $\zeta \cdot (p \otimes (1-p)^*) = 0$.

Assume that $\zeta \cdot (p \otimes (1-p)^*) = 0$ for all projections $p \in L(X)$. Define $a \in L(X)$ by $\langle x_2 | a x_1\rangle = \zeta(x_1 \otimes x_2^*)$ for all $x_1, x_2 \in X$. If x_1 and x_2 are orthogonal, then there is a projection $p \in L(X)$ such that $px_1 = x_1$ and $px_2 = 0$, so $\langle x_2 | a x_1\rangle = \zeta(x_1 \otimes x_2^*) = (\zeta \cdot (p \otimes (1-p)^*))(x_1 \otimes x_2^*) = 0$. It follows that $a \in \mathbb{C}1_X$ and therefore that $\zeta \in \mathbb{C}\varepsilon_X$. ∎

Lemma A.4.2. *Let $\mathcal{X}$ be a quantum set, let r be a projection in $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$, and let R be the corresponding relation of arity $(\mathcal{X}, \mathcal{X}^*)$, i.e., the relation defined by $R(X_1 \otimes X_2^*) = L(X_1 \otimes X_2^*, \mathbb{C}) \cdot r(X_1 \otimes X_2^*)$, for $X_1, X_2 \in \mathrm{At}(\mathcal{X})$. Then, r is orthogonal to $p \otimes (1-p)^*$ for every projection $p \in \ell^\infty(\mathcal{X})$ if and only if $R(X \otimes X^*) \leq \mathbb{C}\varepsilon_X$ for all $X \in \mathrm{At}(\mathcal{X})$ and $R(X_1 \otimes X_2^*) = 0$ for all distinct $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.*

Proof. Fix $X_1, X_2 \in \mathrm{At}(\mathcal{X})$. For all projections p in $\ell^\infty(\mathcal{X})$, the condition

$$r(X_1 \otimes X_2^*) \cdot (p \otimes (1-p)^*)(X_1 \otimes X_2^*) = 0$$

is clearly equivalent to $R(X_1 \otimes X_2^*) \cdot (p(X_1) \otimes (1 - p(X_2)^*)) = 0$. If X_1 and X_2 are distinct, then the latter condition holds for all projections p if and only if $R(X_1 \otimes X_2^*) = 0$

because there is a projection p such that $p(X_1) = 1$ and $p(X_2) = 0$. If X_1 and X_2 are identical, then by Lemma A.4.1, the latter condition holds for all projections p if and only if the elements of $R(X_1 \otimes X_2^*)$ are all scalar multiples of ε_{X_1}. We vary $X_1, X_2 \in \mathrm{At}(\mathcal{X})$ to conclude the statement of the proposition. ∎

Proposition A.4.3. *Let $\mathcal{X}$ be a quantum set. Then, $E_{\mathcal{X}}(X_1 \otimes X_2) = L(X_1 \otimes X_2, \mathbb{C}) \cdot \delta_{\ell^\infty(\mathcal{X})}(X_1 \otimes X_2)$ for all $X_1, X_2 \in \mathrm{At}(\mathcal{X})$.*

Proof. The canonical one-to-one correspondence between projections r in $\ell^\infty(\mathcal{X} \times \mathcal{X}^*)$ and relations R of arity $(\mathcal{X}, \mathcal{X}^*)$ is an isomorphism of orthomodular lattices. Thus, by Lemma A.4.2, the largest projection r that is orthogonal to $p \otimes (1 - p)^*$ for all projections $p \in \ell^\infty(\mathcal{X})$, namely, $r = \delta_{\ell^\infty(\mathcal{X})}$, corresponds to the largest relation R that is less than or equal to $E_{\mathcal{X}}$, namely, $R = E_{\mathcal{X}}$. In other words, $E_{\mathcal{X}}(X_1 \otimes X_2) = L(X_1 \otimes X_2, \mathbb{C}) \cdot \delta_{\ell^\infty(\mathcal{X})}(X_1 \otimes X_2)$ for all $X_1, X_2 \in \mathrm{At}(\mathcal{X})$, as claimed. ∎

A.5. Canonical isomorphisms

We prove Proposition 3.1.1, which concerns Definition 2.2.4.

Proposition A.5.1. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be quantum sets, and let π be a permutation of $\{1, \ldots, n\}$. Let U_π be the canonical isomorphism* [35, Thm. XI.1.1] *from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ to $\mathcal{X}_{\pi(1)} \times \cdots \times \mathcal{X}_{\pi(n)}$ in the symmetric monoidal category of quantum sets and binary relations* [30, Sec. 3]. *Then, $\pi_\#(R) = R \circ U_\pi$ for all relations R of arity $(\mathcal{X}_{\pi(1)}, \ldots, \mathcal{X}_{\pi(n)})$.*

Proof. We first consider the special case that π exchanges $m, m+1 \in \{1, \ldots, n\}$, leaving the other elements fixed. Then,

$$U_\pi = I_{\mathcal{X}_1} \times \cdots \times I_{\mathcal{X}_{m-1}} \times B_{\mathcal{X}_m, \mathcal{X}_{m+1}} \times I_{\mathcal{X}_{m+2}} \times \cdots \times I_{\mathcal{X}_n},$$

where $B_{\mathcal{X}_m, \mathcal{X}_{m+1}}$ is the braiding from $\mathcal{X}_m \times \mathcal{X}_{m+1}$ to $\mathcal{X}_{m+1} \times \mathcal{X}_m$. By the definition of this braiding,

$$B_{\mathcal{X}_m, \mathcal{X}_{m+1}}(X_m \otimes X_{m+1}, X_{m+1} \otimes X_m) = \mathbb{C} b_{X_m, X_{m+1}}$$

for all $X_m \in \mathrm{At}(\mathcal{X}_m)$ and $X_{m+1} \in \mathrm{At}(\mathcal{X}_{m+1})$, with the other components of $B_{\mathcal{X}_m, \mathcal{X}_{m+1}}$ vanishing. Here, $b_{X_m, X_{m+1}}$ is the braiding from $X_m \otimes X_{m+1}$ to $X_{m+1} \otimes X_m$ in the symmetric monoidal category of finite-dimensional Hilbert spaces and linear operators. In other words, $b_{X_m, X_{m+1}} = u_\pi$ in the notation of Definition 2.2.4.

Let $\mathcal{X}_\pi = \mathcal{X}_{\pi(1)} \times \cdots \times \mathcal{X}_{\pi(n)}$. We compute that for all $X_1 \in \mathrm{At}(\mathcal{X}_1)$, $\mathcal{X}_2 \in \mathrm{At}(\mathcal{X}_2)$, etc.,

$$\begin{aligned}(R \circ U_\pi)(X_1 \otimes \cdots \otimes X_n, \mathbb{C}) &= \sum_{X_\pi \in \mathrm{At}(\mathcal{X}_\pi)} R(X_\pi, \mathbb{C}) \cdot U_\pi(X_1 \otimes \cdots \otimes X_n, X_\pi) \\ &= R(X_{\pi(1)} \otimes \cdots \otimes X_{\pi(n)}, \mathbb{C}) \cdot \mathbb{C} u_\pi \\ &= \pi_\#(R)(X_1 \otimes \cdots \otimes X_n).\end{aligned}$$

Therefore, $\pi_\#(R) = R \circ U_\pi$ whenever π exchanges two consecutive elements of $\{1, \dots, n\}$, leaving the other elements fixed.

The general case follows from the fact that any permutation of $\{0, \dots, n\}$ is a product of such 2-cycles. Let **qRel** be the category of quantum sets and binary relations, and let **FdHilb** be the category of finite-dimensional Hilbert spaces and linear operators. For each permutation π of $\{1, \dots, n\}$, we may regard U_π as a natural transformation of functors $\mathbf{qRel}^n \to \mathbf{qRel}$, and similarly, we may regard u_π as a natural transformation of functors $\mathbf{FdHilb}^n \to \mathbf{FdHilb}$. By category theory [35, Thm. XI.1.1], we have that $U_{\pi_2 \circ \pi_1} = U_{\pi_1} \circ U_{\pi_2}$ and $u_{\pi_2 \circ \pi_1} = u_{\pi_1} \circ u_{\pi_2}$ for all permutations π_1 and π_2 of $\{1, \dots, n\}$, where the composition symbol denotes the "horizontal" composition of natural transformations [35, Sec. II.5]. It follows that for all relations R of arity $(\mathcal{X}_{(\pi_2 \circ \pi_1)(1)}, \dots, \mathcal{X}_{(\pi_2 \circ \pi_1)(n)})$, we have that $R \circ U_{\pi_2 \circ \pi_1} = R \circ U_{\pi_1} \circ U_{\pi_2}$ and similarly that $(\pi_2 \circ \pi_1)_\#(R) = \pi_{2\#}(\pi_{1\#}(R))$.

Overall, we find that the set of all permutations π of $\{1, \dots, n\}$ such that $\pi_\#(R) = R \circ U_\pi$ for all n-ary relations R contains all the permutations that exchange two consecutive elements of $\{1, \dots, n\}$ and is closed under composition. We conclude that this set consists of all the permutations of $\{1, \dots, n\}$, and the proposition is proved. ∎

A.6. Quantifying over ordinary sets

Let A be a set. We show that existential quantification over ${}`A$ reduces to disjunction in the expected way. As usual, we view each element $a \in A$ also as a function $\{*\} \to A$, and we identify ${}`\{*\}$ with the monoidal unit $\mathbf{1}$ of the dagger compact category of quantum sets and binary relations.

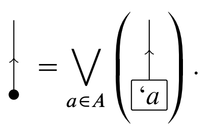
Lemma A.6.1. *Let A be a set, and let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be quantum sets. If $\phi(x, y_1, \dots, y_n)$ is a nonduplicating formula with x being a variable of sort ${}`A$ and $y_1, \dots, y_n$ being variables of sorts $\mathcal{Y}_1, \dots, \mathcal{Y}_n$, respectively, then*

$$\begin{aligned} &[\![(y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\exists x \in {}`A)\, \phi(x, y_1, \dots, y_n) ]\!] \\ &\quad = \bigvee_{a \in A} [\![ (y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid \phi({}`a, y_1, \dots, y_n)]\!]. \end{aligned}$$

Similarly, if $\psi(x_*, y_1, \dots, y_n)$ is a nonduplicating formula with x_* being a variable of sort ${}`A^*$ and $y_1, \dots, y_n$ being variables of sorts $\mathcal{Y}_1, \dots, \mathcal{Y}_n$, respectively, then

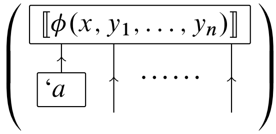
$$\begin{aligned} &[\![(y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\exists x_* \in {}`A^*)\, \phi(x_*, y_1, \dots, y_n) ]\!] \\ &\quad = \bigvee_{a \in A} [\![ (y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid \phi({}`a_*, y_1, \dots, y_n)]\!]. \end{aligned}$$

Proof. This lemma follows from Proposition 3.2.3. The binary relation $\top_{`A}^\dagger$ can be written as a disjunction:



Therefore,

$$[\![(\exists x \in {}`A)\,\phi(x, y_1, \dots, y_n)]\!] = \bigvee_{a\in A}$$



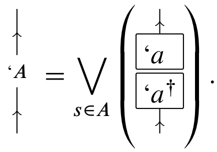
$$= \bigvee_{a\in A} [\![\phi({}`a, y_1, \dots, y_n)]\!].$$

In the second case, the proof is entirely similar because $\top_{`A^*}^{\dagger} = \bigvee_{a\in A} {}`a_*$. ■

Lemma A.6.2. *Let A be a set, and let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be quantum sets. Let $\phi(x, x_*, y_1, \dots, y_n)$ be a nonduplicating formula with x being a variable of sort ‘A, x_* being a variable of sort ‘A^*, and $y_1, \dots, y_n$ being variables of sorts $\mathcal{Y}_1, \dots, \mathcal{Y}_n$, respectively. Then,*

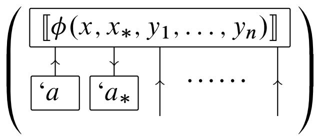
$$[\![(y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid (\exists (x = x_*) \in {}`A \times {}`A^*)\,\phi(x, x_*, y_1, \dots, y_n)]\!]$$
$$= \bigvee_{s\in A} [\![(y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \mid \phi({}`a, {}`a_*, y_1, \dots, y_n)]\!].$$

Proof. This lemma follows from Theorem 3.3.2. The identity on ‘A can be written as a disjunction:



Therefore,

$$[\![(\exists (x = x_*) \in {}`A \times {}`A^*)\,\phi(x, x_*, y_1, \dots, y_n)]\!] = \bigvee_{a\in A}$$



$$= \bigvee_{a\in A} [\![\phi({}`a, {}`a_*, y_1, \dots, y_n)]\!].$$ ■

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Andre Kornell
Department of Computer Science, Tulane University, New Orleans, LA 70118, USA;
akornell@tulane.edu