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Rigidity and flexibility of isometric extensions

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Abstract. In this paper we consider the rigidity and flexibility of $C^{1,\theta}$ isometric extensions. We show that the Hölder exponent $\theta_0 = \frac{1}{2}$ is critical in the following sense: if $u \in C^{1,\theta}$ is an isometric extension of a smooth isometric embedding of a codimension one submanifold Σ and $\theta > \frac{1}{2}$, then the tangential connection agrees with the Levi-Civita connection along Σ. On the other hand, for any $\theta < \frac{1}{2}$ we can construct $C^{1,\theta}$ isometric extensions via convex integration which violate such property. As a byproduct we get moreover an existence theorem for $C^{1,\theta}$ isometric embeddings, $\theta < \frac{1}{2}$, of compact Riemannian manifolds with C^1 metrics and sharper amount of codimension.

1. Introduction

Let $(\mathcal{M}, g)$ be an n-dimensional compact smooth Riemannian manifold and $m > n$. Recall that an isometric embedding of $(\mathcal{M}, g)$ into $(\mathbb{R}^m, e)$ is an injective C^1 map $u\colon \mathcal{M} \hookrightarrow \mathbb{R}^m$ such that $u^\sharp e = g$. Here, e is the Euclidean metric and $u^\sharp e$ denotes the pullback metric on $\mathcal{M}$. In local coordinates this amounts to the system of partial differential equations

$$\sum_{k=1}^{m} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} = g_{ij} \tag{1.1}$$

for $1 \le i, j \le n$, where $g = g_{ij}\, dx^i dx^j$ using summation over repeated indices.

Classical results in differential geometry indicate that *sufficiently regular* global isometric embeddings into Euclidean space with low co-dimension (i.e., $m - n$ is small) are often rigid (i.e., *unique* up to rigid motions). Most prominent is the rigidity of the Weyl problem: given a metric g on the sphere $\mathbb{S}^2$ with positive Gaussian curvature, isometric embeddings $u\colon (\mathbb{S}^2, g) \hookrightarrow \mathbb{R}^3$ are rigid in the class C^2 (cf. [12, 22]). On the other hand, the celebrated Nash–Kuiper theorem (cf. [29, 31]) implies that such spheres can be isometrically embedded into arbitrarily small balls of $\mathbb{R}^3$ if one only requires the embedding to be C^1. A natural question is whether there exists a threshold regularity which distinguishes these two drastically different behaviors.

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As shown in [1–5,7,32] (see also [14] for a short, modern proof), isometric embeddings $u \in C^{1,\theta}$ of positively curved closed surfaces into $\mathbb{R}^3$ are rigid for $\theta > \frac{2}{3}$. On the other hand, the flexibility of isometric embeddings granted by the Nash–Kuiper theorem also holds for isometric embeddings $u\colon (\mathcal{M}, g) \hookrightarrow \mathbb{R}^{n+1}$ of compact n-dimensional manifolds whenever $u \in C^{1,\theta}$ with $\theta < \frac{1}{1+n+n^2}$ for $n \geq 3$ (cf. [6,11,14]), and with $\theta < \frac{1}{5}$ for $n = 2$ (cf. [11,16]). The threshold exponent has been conjectured to be $\theta = \frac{1}{2}$ (see [18]).

The situation looks different when the co-dimension of the ambient space is sufficiently large. A result by A. Källén (cf. [27]) shows that, in this case, flexibility of isometric embeddings extends to the regularity $C^{1,\theta}$ for any $\theta < 1$, and thus there is no rigidity below C^2.

On the other hand, in [15] the authors find that a weaker form of rigidity is present above the conjectured threshold regularity $C^{1,\frac{1}{2}}$ no matter the codimension: they show that when $u \in C^{1,\theta}(\mathcal{M}, \mathbb{R}^m)$ is an isometric embedding with $\theta > \frac{1}{2}$, a weak notion of tangential connection can be defined on the (irregular) embedded submanifold (see also [1–4] for a similar weak notion of tangential connection) and that it agrees with the Levi-Civita connection. In the case of isometric embeddings $u \in C^{1,\theta}((\bar{D}_1, g), \mathbb{R}^m)$ of the closed unit disc taking fixed (smooth) boundary values it is then shown that this compatibility of the weak tangential connection with the intrinsic metric leads to an angle constraint of the tangent space at points of the embedded boundary curve. In contrast, for every $\theta < \frac{1}{2}$, the authors construct isometric embeddings $u \in C^{1,\theta}$ violating this constraint by extending the (smooth) boundary datum to a $C^{1,\theta}$ isometric embedding of the disc by means of a convex integration process. Thus, for this particular example, the result in [15] gives a geometric illustration of the criticality of the Hölder exponent $\theta = \frac{1}{2}$, at least in the presence of a "boundary condition".

In this paper we study the rigidity and flexibility properties of general isometric extensions. A first observation shows that the angle constraint of [15] is simply a consequence of the compatibility of the weak tangential connection with the intrinsic metric and of the embedding agreeing with a smooth one along a lower-dimensional submanifold. It is therefore also present for general isometric extensions of C^2 isometries which are $C^{1,\theta}$ for $\theta > \frac{1}{2}$. On the flexibility side, we want to construct isometric extensions $u \in C^{1,\theta}$, $\theta < \frac{1}{2}$, violating this constraint.

The problem of extending an isometric map $f\colon \Sigma \to \mathbb{R}^m$, where $\Sigma \subset \mathcal{M}$ is a co-dimension one submanifold, was first considered by Jacobowitz in [26] in the high-regularity and high co-dimension setting. He gave a necessary condition on the second fundamental forms of $\iota\colon \Sigma \hookrightarrow \mathcal{M}$ and $f\colon \Sigma \to \mathbb{R}^m$ for the existence of a C^2 extension $u\colon \mathcal{M} \hookrightarrow \mathbb{R}^m$. He also found a sufficient condition (which turns out to be almost necessary) for such an extension, which can be stated as that the image $f(\Sigma)$

shall be "more curved" than Σ. Besides, as discussed in [10, 26], isometric extension can also be viewed as a Cauchy problem for isometric embeddings and certain non-degeneracy conditions on the curvature are important to prove the existence of a sufficiently smooth solution (for local extensions from a point resp. a curve on 2-dimensional manifolds, see [19,30] resp. [9,17,20,28]). The existence of isometric C^1 extensions in low co-dimension was then investigated in [24]. The authors showed that Jacobowitz' obstruction for C^2 extensions is also an obstruction for C^1 extensions and found a sufficient condition (similar to the one in [26]) for *one-sided* extensions (see [24] or below for a similar definition). Under such a condition they proved an existence theorem for isometric C^1 extensions analogous to the Nash–Kuiper theorem. This was then improved to the $C^{1,\theta}$ category for $\theta < \frac{1}{1+n+n^2}$ in [10], although the one-sided extensions are only defined locally around a point.

In this paper we show that, under the same sufficient condition, we can find one-sided isometric extensions (defined on a full one-sided neighborhood of the submanifold Σ) with regularity $C^{1,\theta}$ for $\theta < \frac{1}{2}$, for which the tangential connection does not agree with the Levi-Civita connection along the submanifold Σ.

To precisely state our results we introduce our setting. We consider a smooth, compact, orientable n-dimensional manifold $\mathcal{M}$ equipped with a C^1 Riemannian metric g and an orientable submanifold $\Sigma \subset \mathcal{M}$ of co-dimension one. Suppose that $f\colon \Sigma \to \mathbb{R}^m$ is a smooth isometric embedding for some $m > n$ and denote by

$$L\colon T\Sigma \times T\Sigma \to C^\infty(\Sigma)$$

the (scalar) second fundamental form of the embedding $\iota\colon \Sigma \hookrightarrow \mathcal{M}$, and by

$$\bar{L}\colon T\Sigma \times T\Sigma \to f^* N f(\Sigma)$$

the second fundamental form of the embedding f. In [24] Hungerbühler–Wasem showed that a sufficient condition for the existence of a C^1 one-sided isometric extension (cf. [24] for the definition) of $f\colon \Sigma \to \mathbb{R}^m$ is that there exists a smooth vector field $\mu\colon \Sigma \to \mathbb{R}^m$ satisfying for every $p \in \Sigma$,

$$\begin{aligned} &\text{(i)} && \mu(p) \in N_{f(p)} f(\Sigma), \\ &\text{(ii)} && |\mu(p)| = 1, \\ &\text{(iii)} && \langle \mu(p), \bar{L}(\cdot,\cdot)\rangle - L(\cdot,\cdot) \text{ is positive definite on } T_p\Sigma. \end{aligned} \tag{1.2}$$

Here, $\langle \cdot,\cdot\rangle$ denotes the Euclidean scalar product. Under this assumption we will be able to extend the isometric embedding f to some neighborhood of Σ which is best described by the exponential map. Let $\nu \in \Gamma(N\Sigma)$ be the unique unit normal vector field respecting the orientation of Σ in $\mathcal{M}$. Since Σ is compact, there exists $\epsilon_0 > 0$ such that $F\colon \Sigma \times]-\epsilon_0, \epsilon_0[\to \mathcal{M}$ given by

$$F(p,t) = \exp_p(t\nu(p)) \tag{1.3}$$

is a diffeomorphism. For $\epsilon \le \epsilon_0$, we then call $\Sigma_\epsilon^+ := F(\Sigma \times [0,\epsilon[)$ a *one-sided neighborhood* of Σ in $\mathcal{M}$. Lastly, we let

$$\mathcal{I}_m^\theta(\Sigma_\epsilon^+) = \{v \in C^{1,\theta} \mid v\colon (\Sigma_\epsilon^+, g) \hookrightarrow \mathbb{R}^m \text{ is an isometric embedding with } v|_\Sigma = f\}.$$

Now we are in a position to state our results. One of our main results concerns the rigidity and flexibility of $C^{1,\theta}$ isometric extensions.

Theorem 1.1. *Let Σ be a codimension one oriented submanifold of the compact Riemannian manifold $(\mathcal{M}, g)$, where $g \in C^1$, and let ν be the unique unit normal vector field respecting the orientation. Suppose moreover that $f\colon \Sigma \to \mathbb{R}^m$ is an isometric embedding satisfying (1.2) and let $X \in \Gamma(T\Sigma)$ be any vector field tangent to Σ. Then the following hold:*

(1) *if $\theta > \frac{1}{2}$, $m \ge n+1$, and $u \in \mathcal{I}_m^\theta(\Sigma_\epsilon^+)$, then*

$$\langle du(\nu), \overline{L}(X,X)\rangle = L(X,X);$$

(2) *for any $\theta < \frac{1}{2}$, $m \ge n + 2n_*$, there is $\epsilon > 0$ and $u \in \mathcal{I}_m^\theta(\Sigma_\epsilon^+)$ such that*

$$\langle du(\nu), \overline{L}(X,X)\rangle > L(X,X)$$

at all points where X does not vanish.

Here, $n_* = \frac{n(n+1)}{2}$ is the number of equations in (1.1). As mentioned above, the proof of part (1) is a simple observation given the result in [15, Proposition 2.2] (see Section 2). The isometries in (2) are constructed via a convex integration process similar to [15]. Roughly speaking, the technical difference of (2) to the corresponding part in [15] is that it is of global instead of local nature. Therefore we adapt the "gluing" method introduced in [11] for the construction of *global* $C^{1,\theta}$ isometric embeddings to our needs. In particular, to get the regularity $C^{1,\theta}$, $\theta < \frac{1}{2}$, we need a more subtle decomposition lemma (see Lemma 3.4) than in [11], it is similar to the one used in [15,27]. This however leads to technical difficulties due to the different type of cut-off functions used in [11] as compared to [15]; they are resolved in Proposition 4.1.

The iteration technique used in the proof of part (2) has its origin in Nash's original construction [31]. The latter inspired the more general framework of convex integration, which remarkably also found application in the question of non-uniqueness of fluid mechanic equations and led for example to the resolution of Onsager's conjecture (see [8, 13, 25]), a striking analogue to the dichotomy of rigidity vs flexibility of isometric embeddings.

One of the main building blocks of the proof of Theorem 1.1 (2) is the iteration Proposition 4.2. With it we can prove our second result, the existence of global isometric embeddings of compact manifolds with C^1 metric.

Theorem 1.2. *Let $(\mathcal{M}, g)$ be an n-dimensional compact manifold, $n \geq 2$, with C^1 metric g. For any $\theta < \frac{1}{2}$, there exist infinitely many $C^{1,\theta}$ isometric embeddings*

$$u\colon (\mathcal{M}, g) \hookrightarrow \mathbb{R}^{n+2n_*}.$$

Such an existence result is not new: it is already contained in [27]. The novelty of Theorem 1.2 is that the target dimension $n + 2n_* = n(n+2)$, is much smaller than the dimension $2n + 3(n+1)(n^2+n+2)$ in [27] for the case where the metric is of class C^1, i.e., $\beta = 1$ in [27].

The remaining part of the paper is organized as follows. We prove Theorem 1.1 (1) in Section 2. The main part of the paper is then devoted to proving Theorem 1.1 (2). We first introduce some notations and useful lemmas in Section 3. We then prove the most important building block, an iteration proposition, in Section 4. With it we are able to show Theorem 1.1 (2) and Theorem 1.2 in Section 5 and Section 6, respectively.

2. The proof of Theorem 1.1 (1): Rigidity part

Recall that for a smooth isometric embedding $u\colon \mathcal{M} \hookrightarrow \mathbb{R}^m$, a curve $\gamma\colon [0,1] \to \mathcal{M}$ and a vector field $X \in \Gamma(T\mathcal{M})$ it holds by Gauss' formula

$$\left(\frac{d}{dt}\bigg|_{t=t_0} du(X)(\gamma) \right)^{\mathsf{T}} = du\big(\nabla^{\mathcal{M}}_{\dot\gamma(t_0)} X\big)$$

for $t_0 \in [0,1]$. Here, $^{\mathsf{T}}$ denotes the orthogonal projection onto $Tu(\mathcal{M}) \subset \mathbb{R}^m$.

Thus, in particular, if $\nu \in \Gamma(T\mathcal{M})$, then

$$\left\langle \frac{d}{dt} du(X)(\gamma), du(\nu)(\gamma) \right\rangle = \big\langle du\big(\nabla^{\mathcal{M}}_{\dot\gamma(t)} X\big), du(\nu)(\gamma)\big\rangle \tag{2.1}$$

holds on $[0,1]$. In [15, Proposition 2.2], the authors show that the left-hand side of the latter equation is well defined as a distribution whenever $u \in C^{1,\theta}$ for $\theta > 1/2$, and that (2.1) still holds.

From this, part (1) of Theorem 1.1 follows easily. Let $X \in \Gamma(T\Sigma)$, $p \in \Sigma$ and $\gamma\colon [0,1] \to \Sigma$ with $\gamma(0) = p$, $\dot\gamma(0) = X_p$. Let, moreover, $\nu \in \Gamma(T\mathcal{M})$ be the unique unit normal to Σ respecting the orientation. Since $u = f$ on Σ and f is smooth, the function $du(X)(\gamma)\colon [0,1] \to \mathbb{R}^m$ is smooth, so that (2.1) holds as a pointwise equality of continuous functions even if u is only $C^{1,\theta}$ with $\theta > \frac{1}{2}$.

With the help of the Gauss formula for the smooth embeddings $f\colon \Sigma \to \mathbb{R}^m$ and $\iota\colon \Sigma \hookrightarrow \mathcal{M}$, we then find

$$\begin{aligned}\langle du_p(\nu), \overline{L}_p(X,X)\rangle &= \left\langle du_p(\nu), \frac{d}{dt}\bigg|_{t=0} du(X)(\gamma)\right\rangle \\ &= \big\langle du_p(\nu), du\big(\nabla^{\mathcal{M}}_{\dot\gamma(0)}X\big)\big\rangle = g\big(\nu_p, \nabla^{\mathcal{M}}_{X_p}X\big) = L_p(X,X)\end{aligned}$$

since u is an isometry.

3. Preliminaries

In this section we introduce some notations, function spaces and basic lemmas which are needed for the proof of the flexibility part of Theorem 1.1.

3.1. Hölder Norms and interpolation

Let $\Omega \subset \mathbb{R}^n$ be an open set. In the following, the maps f can be real valued, vector valued, or tensor valued. In every case, the target is equipped with the Euclidean norm, denoted by $|f(x)|$. The Hölder norms are then defined as follows:

$$\|f\|_0 = \sup_\Omega |f|, \quad \|f\|_m = \sum_{j=0}^m \max_{|\beta|=j} \|\partial^\beta f\|_0,$$

where β denotes a multi-index, and

$$\begin{aligned}[f]_\theta &= \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\theta}, \\ [f]_{m+\theta} &= \max_{|\beta|=m} \sup_{x\neq y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\theta}, \quad 0<\theta\leq 1.\end{aligned}$$

Then the Hölder norms are given as

$$\|f\|_{m+\theta} = \|f\|_m + [f]_{m+\theta}.$$

We recall the standard interpolation inequality

$$[f]_r \leq C\|f\|_0^{1-\frac{r}{s}}[f]_s^{\frac{r}{s}}$$

for $s > r \geq 0$ and the Leibniz rule

$$\|fg\|_r \leq C(r)\big(\|f\|_r\|g\|_0 + \|f\|_0\|g\|_r\big). \tag{3.1}$$

We also collect two classical estimates on Hölder norms of compositions in [27].

Lemma 3.1. *Let* $\Psi\colon \Sigma_1 \to \mathbb{R}$ *be a function and* $u, v\colon \mathbb{R}^n \to \Sigma_1$*. Then for any* $r, s \geq 0$*, it holds*

$$\|\Psi \circ u\|_r \leq C(r)\big(\|\Psi(\cdot)\|_r \|u\|_1^r + \|\Psi(\cdot)\|_1 \|u\|_r + \|\Psi(\cdot)\|_0\big), \quad r \geq 1,$$
$$\|\Psi \circ u\|_r \leq \min\big(\|\Psi(\cdot)\|_r \|u\|_1^r, \|\Psi(\cdot)\|_1 \|u\|_r\big) + \|\Psi(\cdot)\|_0, \qquad 0 \leq r \leq 1.$$

Other properties of the Hölder norm can be found in references such as [14, 16].

3.2. Mollification estimates

We will frequently regularize maps by convolution with a standard mollifier, i.e., a radially symmetric $\varphi_\ell \in C_c^\infty(B_\ell(0))$ with $\int \varphi_\ell = 1$, where $\ell > 0$ denotes the length-scale. Such a regularization of Hölder functions enjoys the following estimates (for a proof, see [14, 16]).

Lemma 3.2. *For any* $r, s \geq 0$ *and* $0 < \theta \leq 1$*, we have*

$$[f * \varphi_\ell]_{r+s} \leq C\ell^{-s}[f]_r,$$
$$\|f - f * \varphi_\ell\|_r \leq C\ell^{1-r}[f]_1, \quad \text{if } 0 \leq r \leq 1,$$
$$\|(fg) * \varphi_\ell - (f * \varphi_\ell)(g * \varphi_\ell)\|_r \leq C\ell^{2\theta - r}\|f\|_\theta \|g\|_\theta,$$

with constant C *depending only on* s, r, θ, φ*.*

In the course of the proof of Theorem 1.1, we will regularize maps $f \in C^k(\overline{\Omega}, \mathbb{R}^m)$ defined on $\overline{\Omega}$. The resulting regularized maps will then have a smaller domain of definition. To counteract this, we first extend f to a map $\overline{f} \in C^k(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\|f\|_{C^k(\mathbb{R}^m)} \leq C\|f\|_{C^k(\overline{\Omega})},$$

where the constant $C > 0$ depends only on k, n and Ω. Such a procedure is given by Whitney's extension theorem; see [34]. We then mollify the resulting extensions at some length-scale $\ell > 0$ to obtain $\tilde{f} = \overline{f} * \varphi_\ell \in C^\infty(\overline{\Omega}, \mathbb{R}^m)$. We will not further specify this.

3.3. Hölder norms and mollification on manifolds

Using a partition of unity, Hölder spaces and mollification can be defined on the compact manifold $\mathcal{M}$ as follows. We fix a finite atlas of $\mathcal{M}$ with charts (Ω_i, ϕ_i), we let $\{\chi_i\}$ be a partition of unity subordinate to $\{\Omega_i\}$ and set

$$\|f\|_k = \sum_i \|(\chi_i f) \circ \phi_i^{-1}\|_k,$$

and

$$f * \varphi_\ell = \sum_i \big((\chi_i f) \circ \phi_i^{-1} * \varphi_\ell\big) \circ \phi_i .$$

One can check that the estimates (3.1) as well as Lemma 3.1 and 3.2 still hold (with constants which may depend on the fixed charts).

3.4. Matrix decomposition

A key step in the construction of isometric embedding, as pioneered by Nash [31] and used in all the subsequent variants of the iteration process, is a suitable decomposition of the metric error. We recall the version used in [14, 16].

Lemma 3.3. *Let $n \geq 2$ and let $\bar{P} \in \mathbb{R}^{n\times n}$ be a positive definite matrix. There exists a constant $r_0 > 0$, vectors $\nu_1, \dots, \nu_{n_*} \in \mathbb{S}^{n-1}$ and smooth functions a_k such that*

$$P = \sum_{k=1}^{n_*} a_k^2(P)\nu_k \otimes \nu_k ,$$

for any positive definite matrix $P \in \mathbb{R}^{n\times n}$ with

$$|P - \bar{P}| < r_0 .$$

For our purposes we need to perturb Lemma 3.3 in two ways: First of all we want to vary the "reference" matrix $\bar{P}$ slightly and allow a matrix field P_0 with suitably small oscillation; this simply follows from a compactness argument. Secondly, we can perturb the coefficients a_k to obtain a slightly subtler decomposition. This is similar to the decomposition used in [27] and can proved with the standard implicit function theorem (compare [15, Proposition 5.4]).

Lemma 3.4. *Let $n \geq 2$ and $\gamma \geq 1$. There exists a constant $\sigma_0(\gamma) > 0$ and vectors $\nu_1, \dots, \nu_{n*} \in \mathbb{S}^{n-1}$ with the following property. If $P_0 \colon \bar{\Omega} \to \mathbb{R}^{n\times n}$ is a matrix field with*

$$\gamma^{-1}\mathrm{Id} \leq P_0 \leq \gamma \mathrm{Id} \quad \text{and} \quad \operatorname{osc}_\Omega P_0 < \sigma_0,$$

and if $P \colon \bar{\Omega} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$, and $\{\Lambda_i\}_{i=1}^{n_*}, \{\Theta_{ij}\}_{i,j=1}^{n_*} \subset C^1(\bar{\Omega}, \mathbb{R}^{n\times n}_{\mathrm{sym}})$ are such that

$$\|P - P_0\|_0 + \sum_{i=1}^{n_*} \|\Lambda_i\|_0 + \sum_{i,j=1}^{n_*} \|\Theta_{ij}\|_0 < \sigma_0,$$

then there exist C^1 functions $a_1, \dots, a_{n_*} \colon \bar{\Omega} \to \mathbb{R}$ with

$$P = \sum_{i=1}^{n_*} a_i^2 \nu_i \otimes \nu_i + \sum_{i=1}^{n_*} a_i \Lambda_i + \sum_{i,j=1}^{n_*} a_i a_j \Theta_{ij} . \tag{3.2}$$

Moreover, a_i are given as

$$a_i(x) = \Phi_i\big(P(x), \{\Lambda_k(x)\}, \{\Theta_{kl}(x)\}\big) \tag{3.3}$$

for C^1 functions Φ_i, and consequently, we have the estimates

$$\|a_i\|_k \leq C_k\Bigg(\|P\|_k + \sum_{j=1}^{n_*}\|\Lambda_j\|_k + \sum_{j,l=1}^{n_*}\|\Theta_{jl}\|_k\Bigg)$$

for $k = 0, 1$ and $1 \leq i \leq n_*$. Here, the constants $C_k \geq 1$ depend only on k, σ_0.

3.5. Existence of normal vector fields

Another key ingredient in the iteration process is the use of suitable normal vector fields to the embedding. The following lemma concerns their existence. It is similar to [15, Proposition 5.3], except that no C^1-closeness to a reference embedding is required. A proof is contained in the appendix.

Lemma 3.5. *Let $N \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be open and simply connected. Assume $v \in C^{N+1}(\bar{\Omega}, \mathbb{R}^m)$ is such that*

$$\gamma^{-1}\mathrm{Id} \leq \nabla v^T \nabla v \leq \gamma \mathrm{Id} \tag{3.4}$$

for some $\gamma > 1$. There exists a family of vector fields $\{\zeta_1, \ldots, \zeta_{m-n}\} \subset C^N(\bar{\Omega}, \mathbb{R}^m)$ such that

$$\begin{aligned} &\langle \zeta_i, \zeta_j \rangle = \delta_{ij}, \quad \nabla v \cdot \zeta_i = 0, \\ &[\zeta_i]_l \leq C(l, \gamma)\big(1 + [v]_{l+1}\big), \end{aligned} \tag{3.5}$$

for all $0 \leq l \leq N$. Here $\delta_{ij} = 1$ when $i = j$, and vanishes else.

4. Iteration proposition

The isometric embedding $u \in \mathcal{I}^{\theta}_m(\Sigma^+_\epsilon)$ in Theorem 1.1 (2) will be constructed by an iteration procedure producing a sequence of embeddings u_q converging to u. The practice, as pioneered by Nash in [31], of decomposing the metric error $g - u_q^\sharp e$ and adding a Nash-twist for each term in the decomposition was improved by A. Källen in [27]. In the latter paper, the author gains extra regularity (at the expense of increased codimension compared to Nash's result) by absorbing the leading error terms into the decomposition. We use a similar decomposition (see (3.2)). However, since we employ the framework of [11] we need to be able to "add' metric pieces which have the form $\rho^2(g + h)$ for compactly supported functions ρ and suitably small $(0, 2)$-tensors h. The missing lower bound on ρ seems however not to be compatible with the decomposition lemma; a technical difficulty which is overcome by introducing an extra cut-off scale (see (4.14) and (4.15)).

In this proposition G is assumed to be the coordinate expression of a given C^1 metric in some chart which is identified with an open bounded subset $\Omega \subset \mathbb{R}^n$. Moreover, the constant σ_0 is given by Lemma 3.4.

Proposition 4.1. *Fix $\gamma \geq 1$ and parameters $0 < \delta < 1$ and $\lambda > 1$. Assume G is a C^1 metric on $\overline{\Omega} \subset \mathbb{R}^n$ with*

$$\gamma^{-1}\mathrm{Id} \leq G \leq \gamma \mathrm{Id}, \quad \|G\|_1 \leq \gamma, \quad \textit{and} \quad \mathrm{osc}_{\Omega} G < \sigma_0,$$

and $u \in C^2(\overline{\Omega}, \mathbb{R}^{n+2n_*})$, $\rho \in C^1(\overline{\Omega})$, $H \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ are such that

$$\begin{aligned} \gamma^{-1}\mathrm{Id} \leq \nabla u^T \nabla u \leq \gamma \mathrm{Id} \quad & \textit{in } \Omega, \\ \|u\|_2 \leq \delta^{1/2}\lambda, & \end{aligned} \tag{4.1}$$

and

$$0 \leq \rho \leq \delta^{1/2}, \qquad \|\rho\|_1 \leq \delta^{1/2}\lambda, \tag{4.2}$$

$$\|H\|_0 \leq \frac{\sigma_0}{2}, \qquad \|H\|_1 \leq \lambda. \tag{4.3}$$

Then for every $\tau > 1$, there exists a constant $\lambda_0(\tau, \gamma, \sigma_0) \geq 1$ such that if

$$\lambda \geq \lambda_0, \tag{4.4}$$

then there exists an embedding $v \in C^2(\overline{\Omega}; \mathbb{R}^{n+2n_*})$ and $\mathcal{E} \in C^1(\overline{\Omega}; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ such that

$$\begin{aligned} \nabla v^T \nabla v = \nabla u^T \nabla u + \rho^2 (G + H) + \mathcal{E} \quad & \textit{in } \Omega, \\ v = u \quad \textit{on } \Omega \setminus (\mathrm{supp}\, \rho + B_{\lambda^{1-2\tau}}(0)) & \end{aligned} \tag{4.5}$$

with estimates

$$\|v - u\|_0 \leq C\delta^{1/2}\lambda^{-\tau}, \tag{4.6}$$

$$\|v - u\|_1 \leq C\delta^{1/2}, \tag{4.7}$$

$$\|v\|_2 \leq C\delta^{1/2}\lambda^{\tau}, \tag{4.8}$$

and

$$\|\mathcal{E}\|_0 \leq C\delta\lambda^{2-2\tau}, \qquad \|\mathcal{E}\|_1 \leq C\delta\lambda. \tag{4.9}$$

Here, $C \geq 1$ is a constant depending only on γ, σ_0.

Remark 4.1 (Constants). As usual, the value of the constants C appearing in the following proof can change from line to line. In addition, all the constants are allowed to depend on γ and σ_0. For the sake of readability, we will suppress this dependence in the notation.

Proof. Fix $\tau > 1$. Regularize u at length scale $\lambda^{-\tau}$ to get $\widetilde{u} \in C^{\infty}(\overline{\Omega})$. Then the smooth embedding $\widetilde{u}$ satisfies

$$\|u - \widetilde{u}\|_1 \leq C\delta^{1/2}\lambda^{1-\tau}, \quad \|\widetilde{u}\|_2 \leq C\delta^{1/2}\lambda, \quad \|\widetilde{u}\|_3 \leq C\delta^{1/2}\lambda^{\tau+1}.$$

Note that

$$\nabla\widetilde{u}^T\nabla\widetilde{u} = \nabla u^T\nabla u - (\nabla u - \nabla\widetilde{u})^T\nabla\widetilde{u} - \nabla u^T(\nabla u - \nabla\widetilde{u}),$$

which then implies

$$(2\gamma)^{-1}\mathrm{Id} \leq \nabla\widetilde{u}^T\nabla\widetilde{u} \leq (2\gamma)\mathrm{Id}$$

provided $\lambda^{1-\tau} \leq C(\gamma)^{-1}$ for some constant $C(\gamma)$, which follows from (4.4) for λ_0 large enough. Then $\widetilde{u}\colon \overline{\Omega} \hookrightarrow \mathbb{R}^{n+2n^*}$ is an embedding of $\overline{\Omega}$. Thus by Lemma 3.5, there exist $2n_*$ unit normal vectors $\{\zeta_k, \eta_k, k = 1, \ldots, n_*\}$ to the surface $\widetilde{u}(\overline{\Omega})$ satisfying the estimates (3.5). Fix moreover the vectors $\nu_1, \ldots, \nu_{n*} \in \mathbb{S}^{n-1}$ provided by Lemma 3.4 for our fixed $\gamma \geq 1$. Similarly to [15], we then define

$$\begin{aligned}
A_k &= \cos(\lambda^\tau\nu_k\cdot x)\zeta_k \otimes \nu_k - \sin(\lambda^\tau\nu_k\cdot x)\eta_k \otimes \nu_k,\\
B_k &= \sin(\lambda^\tau\nu_k\cdot x)\nabla\zeta_k + \cos(\lambda^\tau\nu_k\cdot x)\nabla\eta_k,\\
D_k &= \sin(\lambda^\tau\nu_k\cdot x)\zeta_k + \cos(\lambda^\tau\nu_k\cdot x)\eta_k.
\end{aligned}$$

By (3.1), it is not hard to derive

$$\begin{aligned}
\|A_k\|_0 + \|D_k\|_0 &\leq C\big(1 + \|\nabla\widetilde{u}\|_0\big) \leq C,\\
\|A_k\|_1 + \|D_k\|_1 &\leq C\big(\lambda^\tau\|\nabla\widetilde{u}\|_0 + \|\nabla^2\widetilde{u}\|_0\big) \leq C(\lambda^\tau + \delta^{1/2}\lambda) \leq C\lambda^\tau,\\
\|B_k\|_0 &\leq C\|\nabla^2\widetilde{u}\|_0 \leq C\delta^{1/2}\lambda,\\
\|B_k\|_1 &\leq C\big(\|\nabla^2\widetilde{u}\|_0\lambda^\tau + \|\widetilde{u}\|_3\big) \leq C\delta^{1/2}\lambda^{1+\tau}.
\end{aligned} \tag{4.10}$$

Note that $\nabla\widetilde{u}^T A_k = 0$. Thus, we have

$$\begin{aligned}
\|\nabla u^T A_k\|_0 &= \|(\nabla u - \nabla\widetilde{u})^T A_k\|_0 \leq C\delta^{1/2}\lambda^{1-\tau},\\
\|\nabla u^T A_k\|_1 &\leq C\big(\|\nabla\widetilde{u}\|_1\|A_k\|_0 + \|\nabla u - \nabla\widetilde{u}\|_0\|A_k\|_1\big)\\
&\leq C(\delta^{1/2}\lambda + \delta^{1/2}\lambda^{1-\tau}\lambda^\tau) \leq C\delta^{1/2}\lambda.
\end{aligned} \tag{4.11}$$

Clearly, we have the same estimates for $\nabla u^T D_k$:

$$\|\nabla u^T D_k\|_0 \leq C\delta^{1/2}\lambda^{1-\tau}, \quad \|\nabla u^T D_k\|_1 \leq C\delta^{1/2}\lambda. \tag{4.12}$$

We now set

$$\begin{aligned}
\Lambda_k &= 2\mathrm{sym}(\nabla u^T A_k) + 2\lambda^{-\tau}\mathrm{sym}(\nabla u^T B_k),\\
\Theta_{ij} &= 2\lambda^{-\tau}\mathrm{sym}(A_i^T B_j) + 2\lambda^{-2\tau}\mathrm{sym}(B_i^T B_j).
\end{aligned}$$

With the help of (4.10)–(4.12), we then deduce

$$
\begin{aligned}
\|\Lambda_k\|_0 &\le C\big(\|\nabla u^T A_k\|_0 + \lambda^{-\tau}\|\nabla u\|_0\|B_k\|_0\big) \le C\delta^{1/2}\lambda^{1-\tau},\\
\|\Lambda_k\|_1 &\le C\big(\|\nabla u^T A_k\|_1 + \lambda^{-\tau}\big(\|\nabla u\|_1\|B_k\|_0 + \|\nabla u\|_0\|B_k\|_1\big)\big) \le C\delta^{1/2}\lambda,\\
\|\Theta_{ij}\|_0 &\le C\lambda^{-\tau}\big(\|A_i\|_0\|B_j\|_0 + \lambda^{-\tau}\|B_i\|_0\|B_j\|_0\big) \le C(\gamma)\delta^{1/2}\lambda^{1-\tau},\\
\|\Theta_{ij}\|_1 &\le C\lambda^{-\tau}\big(\|A_i\|_1\|B_j\|_0 + \|A_i\|_0\|B_j\|_1 + \lambda^{-\tau}\big(\|B_i\|_1\|B_j\|_0 + \|B_i\|_0\|B_j\|_1\big)\big)\\
&\le C(\delta^{1/2}\lambda + \delta\lambda^{2-\tau}) \le C\delta^{1/2}\lambda. \qquad (4.13)
\end{aligned}
$$

Fix now a parameter $\epsilon > 0$ defined by

$$
\epsilon^{1/2} = C_0(\gamma,\sigma_0)\delta^{1/2}\lambda^{1-\tau}, \qquad (4.14)
$$

where $C_0(\gamma,\sigma_0) \ge 1$ is a constant to be chosen later. Observe that upon choosing $\lambda_0(\gamma,\sigma_0,\tau)$ large enough we can achieve $\epsilon^{1/2} < \delta^{1/2}$. Next, fix a monotone decreasing function $\psi \in C^\infty([0,\infty[)$ such that

$$
\psi(\rho) = \begin{cases} \frac{1}{\rho} & \text{if } \rho \ge 2\epsilon^{1/2},\\ \epsilon^{-1/2} & \text{if } \rho \le \epsilon^{1/2}. \end{cases} \qquad (4.15)
$$

Clearly,

$$
\|\psi(\rho(\cdot))\|_0 \le C\epsilon^{-1/2}, \quad \|\psi(\rho(\cdot))\|_1 \le C\epsilon^{-1}\delta^{1/2}\lambda,
$$

due to the assumption (4.2). It therefore follows that

$$
\begin{aligned}
\|\psi(\rho)\Lambda_k\|_0 &\le C\epsilon^{-1/2}\delta^{1/2}\lambda^{1-\tau},\\
\|\psi(\rho)\Lambda_k\|_1 &\le C(\epsilon^{-1/2}\delta^{1/2}\lambda + \epsilon^{-1}\delta\lambda^{2-\tau}) \le C\epsilon^{-1/2}\delta^{1/2}\lambda,
\end{aligned}
$$

since $C_0 \ge 1$. Thus, if C_0 in (4.14) is chosen large enough (and afterward, λ_0 in (4.4) is large enough to guarantee $\epsilon < \delta$), we have the following bound

$$
\|H\|_0 + \sum_{k=1}^{n_*}\|\psi(\rho)\Lambda_k\|_0 + \sum_{i,j=1}^{n_*}\|\Theta_{ij}\|_0 \le \frac{\sigma_0}{2} + C\epsilon^{-1/2}\delta^{1/2}\lambda^{1-\tau} < \sigma_0.
$$

A direct application of Lemma 3.4 (with $P_0 = G$, $P = G + H$) then enables us to get n_* functions $\{a_k\} \subset C^1(\overline{\Omega})$ such that

$$
G + H = \sum_{k=1}^{n_*} a_k^2 \nu_k \otimes \nu_k + \sum_{k=1}^{n_*} a_k\psi(\rho)\Lambda_k + \sum_{i,j=1}^{n_*} a_i a_j \Theta_{ij},
$$

i.e.,

$$\rho^2(G+H)=\sum_{k=1}^{n_*}(\rho a_k)^2\nu_k\otimes\nu_k+\sum_{k=1}^{n_*}\rho^2\psi(\rho)a_k\Lambda_k \\ +\sum_{i,j=1}^{n_*}(\rho a_i)(\rho a_j)\Theta_{ij}. \tag{4.16}$$

Notice that $\rho^2\psi(\rho)=\rho$ if $\rho\geq 2\epsilon^{1/2}$, so that, at least in this region, we get a decomposition of the form (3.2) for the degenerate metric piece $\rho^2(G+H)$ with coefficients ρa_k. By Lemma 3.4, for $k=1,\dots,n_*$, we have

$$\begin{aligned} 0\leq a_k &\leq C\Big(\|G\|_0+\|H\|_0+\sum_{k=1}^{n_*}\|\psi(\rho)\Lambda_k\|_0+\sum_{i,j=1}^{n_*}\|\Theta_{ij}\|_0\Big)\leq C,\\ \|a_k\|_1 &\leq C\Big(\|G\|_1+\|H\|_1+\sum_{k=1}^{n_*}\|\psi(\rho)\Lambda_k\|_1+\sum_{i,j=1}^{n_*}\|\Theta_{ij}\|_1\Big)\\ &\leq C\epsilon^{-1/2}\delta^{1/2}\lambda. \end{aligned}$$

However, it follows from the description (3.3) and the estimates (4.13) that the following improved estimate holds

$$\begin{aligned} \|\rho\nabla a_k\|_0 &\leq C\|\rho\nabla(\psi(\rho)\Lambda_k)\|_0\leq C\big(\|\rho\psi'(\rho)\Lambda_k\nabla\rho\|_0+\|\rho\psi(\rho)\nabla\Lambda_k\|_0\big)\\ &\leq C(\epsilon^{-1/2}\delta\lambda^{2-\tau}+\delta^{1/2}\lambda)\leq C\delta^{1/2}\lambda, \end{aligned}$$

since $|\rho\psi'(\rho)|\leq C\epsilon^{-1/2}$ and $|\rho\psi(\rho)|\leq C$. We can then infer that by (3.1),

$$\|\rho a_k\|_1\leq C\big(\|a_k\nabla\rho\|_0+\|\rho\nabla a_k\|_0\big)\leq C\delta^{1/2}\lambda. \tag{4.17}$$

Now we set $b_k:=\rho a_k$ and mollify b_k at length scale $\lambda^{1-2\tau}$ to get $\widetilde{b}_k$. By (4.2), (4.17) and Lemma 3.2, for any $j\in\mathbb{N}$, we have

$$\begin{aligned} \|\widetilde{b}_k\|_0 &\leq C\delta^{1/2},\\ \|\widetilde{b}_k\|_{j+1} &\leq C_j\delta^{1/2}\lambda^{(2\tau-1)j+1},\\ \|\widetilde{b}_k-b_k\|_0 &\leq C_j\delta^{1/2}\lambda^{2-2\tau}. \end{aligned} \tag{4.18}$$

Finally, we define our desired embedding as

$$v=u+\frac{1}{\lambda^\tau}\sum_{k=1}^{n_*}\widetilde{b}_kD_k.$$

From the definition it is clear that $v = u$ on $\Omega \setminus (\operatorname{supp}\rho + B_{\lambda^{1-2\tau}})$, i.e., (4.5) holds. Besides, a straightforward calculation shows

$$\nabla v = \nabla u + \sum_{k=1}^{n_*} \tilde{b}_k A_k + \frac{1}{\lambda^\tau} \sum_{k=1}^{n_*} \tilde{b}_k B_k + \frac{1}{\lambda^\tau} \sum_{k=1}^{n_*} D_k \nabla \tilde{b}_k.$$

Since $A_k^T D_k = 0$ for $k = 1, 2, \dots, n_*$, the induced metric will be

$$\begin{aligned}
\nabla v^T \nabla v = \nabla u^T \nabla u &+ \sum_{k=1}^{n_*} \tilde{b}_k^2 \nu_k \otimes \nu_k + 2 \sum_{k=1}^{n_*} \tilde{b}_k \operatorname{sym}\Big(\nabla u^T A_k + \frac{1}{\lambda^\tau} \nabla u^T B_k\Big) \\
&+ \frac{2}{\lambda^\tau} \sum_{k=1}^{n_*} \operatorname{sym}(\nabla u^T D_k \nabla \tilde{b}_k) + \frac{2}{\lambda^\tau} \sum_{i,j=1}^{n_*} \tilde{b}_i \tilde{b}_j \operatorname{sym}(A_i^T B_j) \\
&+ \frac{2}{\lambda^{2\tau}} \sum_{i,j=1}^{n_*} \tilde{b}_i \tilde{b}_j \operatorname{sym}(B_i^T B_j) + \frac{2}{\lambda^{2\tau}} \sum_{i,j=1}^{n_*} \tilde{b}_i \operatorname{sym}(B_i^T D_j \nabla \tilde{b}_j) \\
&+ \frac{\tilde{\chi}^2}{\lambda^{2\tau}} \sum_{k=1}^{n_*} \nabla \tilde{b}_k^T \nabla \tilde{b}_k.
\end{aligned}$$

Hence by (4.16), we calculate the metric error

$$\nabla v^T \nabla v - (\nabla u^T \nabla u + \rho^2 (G + H)) = \mathcal{E}_1 + \mathcal{E}_2,$$

with

$$\begin{aligned}
\mathcal{E}_1 &= \sum_{k=1}^{n_*} (\tilde{b}_k^2 - b_k^2) \nu_k \otimes \nu_k + \sum_{k=1}^{n_*} (\tilde{b}_k - \rho\psi(\rho) b_k) \Lambda_k + \sum_{i,j=1}^{n_*} (\tilde{b}_i \tilde{b}_j - b_i b_j) \Theta_{ij}, \\
\mathcal{E}_2 &= \frac{2}{\lambda^\tau} \sum_{k=1}^{n_*} \operatorname{sym}(\nabla u^T D_k \nabla \tilde{b}_k) \\
&\quad + \frac{2}{\lambda^{2\tau}} \sum_{i,j=1}^{n_*} \tilde{b}_i \operatorname{sym}(B_i^T D_j \nabla \tilde{b}_j) + \frac{1}{\lambda^{2\tau}} \sum_{k=1}^{n_*} \nabla \tilde{b}_k^T \nabla \tilde{b}_k.
\end{aligned}$$

In the following, we shall bound the above two errors in order to get (4.9). We start with

$$\|\tilde{b}_k^2 - b_k^2\|_0 \le \|\tilde{b}_k + b_k\|_0 \|\tilde{b}_k - b_k\|_0 \le C\delta\lambda^{2-2\tau},$$

where we used (4.18). Similarly, with (3.1) one can then estimate

$$\|\tilde{b}_k^2 - b_k^2\|_1 \le C\delta\lambda.$$

Completely analogously one can estimate the term

$$\|(\tilde{b}_i \tilde{b}_j - b_i b_j)\Theta_{ij}\|_0 \le C\delta^{3/2}\lambda^{3-3\tau} \le C\delta\lambda^{2-2\tau}$$

and

$$\|(\tilde{b}_i\tilde{b}_j - b_i b_j)\Theta_{ij}\|_1 \le C\delta\lambda.$$

For the second term in $\mathcal{E}_1$, we write

$$\tilde{b}_k - \rho\psi(\rho)b_k = \tilde{b}_k - b_k + b_k(1-\rho\psi(\rho))$$

and observe that by definition $1-\rho\psi(\rho) = 0$ for $\rho \ge 2\epsilon^{1/2}$. Thus, remembering that $b_k = \rho a_k$, we have that $|b_k| \le C\epsilon^{1/2}$ on $\mathrm{spt}(1-\rho\psi(\rho))$. Hence,

$$\begin{aligned}\|(\tilde{b}_k - \rho\psi(\rho)b_k)\Lambda_k\|_0 &\le C\delta^{1/2}\lambda^{1-\tau}\big(\|\tilde{b}_k - b_k\|_0 + \|b_k(1-\rho\psi(\rho))\|_0\big)\\ &\le C\delta^{1/2}\lambda^{1-\tau}(\delta^{1/2}\lambda^{2-2\tau} + \epsilon^{1/2}) \le C\delta\lambda^{2-2\tau}\end{aligned}$$

by the definition of ϵ in (4.14). Similarly,

$$\begin{aligned}&\|(\tilde{b}_k - \rho\psi(\rho)b_k)\Lambda_k\|_1\\ &\quad\le C\delta^{3/2}\lambda^{3-2\tau} + C\delta^{1/2}\lambda^{1-\tau}\big(\delta^{1/2}\lambda + \epsilon^{1/2}\|\nabla(1-\rho\psi(\rho))\|_0\big)\\ &\quad\le C\delta\lambda^{2-\tau} \le C\delta\lambda,\end{aligned}$$

since $|\nabla(\rho\psi(\rho))| \le C\epsilon^{-1/2}\delta^{1/2}\lambda + C|\psi(\rho)\nabla\rho| \le C\epsilon^{-1/2}\delta^{1/2}\lambda$.

Combining the previous estimates, we get

$$\|\mathcal{E}_1\|_0 \le C\delta\lambda^{2-2\tau}, \quad \|\mathcal{E}_1\|_1 \le C\delta\lambda.$$

The estimation of $\mathcal{E}_2$ is lengthy but straightforwardly obtained by (3.1), (4.10), (4.12), (4.18) and yields

$$\|\mathcal{E}_2\|_0 \le C\delta\lambda^{2-2\tau}, \quad \|\mathcal{E}_2\|_1 \le C\delta\lambda.$$

This in turn implies (4.9), since

$$\begin{aligned}\|\nabla v^T\nabla v - (\nabla u^T\nabla u + \rho^2(G+H))\|_0 &\le \|\mathcal{E}_1\|_0 + \|\mathcal{E}_2\|_0 \le C\delta\lambda^{2-2\tau},\\ \|\nabla v^T\nabla v - (\nabla u^T\nabla u + \rho^2(G+H))\|_1 &\le \|\mathcal{E}_1\|_1 + \|\mathcal{E}_2\|_1 \le C\delta\lambda.\end{aligned}$$

It remains to show the estimates (4.6)–(4.8). Clearly, by the formulae for v and its derivative and the estimates (4.10), (4.18), we have

$$\|v-u\|_0 \le \lambda^{-\tau}\sum_{k=1}^{n_*}\|\tilde{b}_k\|_0\|D_k\|_0 \le C\delta^{1/2}\lambda^{-\tau},$$

and

$$\begin{aligned}\|v-u\|_1 &\le \sum_{k=1}^{n_*}\|\tilde{b}_k\|_0\|A_k\|_0 + \lambda^{-\tau}\sum_{k=1}^{n_*}\big(\|\tilde{b}_k\|_0\|B_k\|_0 + \|D_k\|_0\|\nabla\tilde{b}_k\|_0\big)\\ &\le C(\delta^{1/2} + \delta^{1/2}\lambda^{1-\tau}) \le C\delta^{1/2}.\end{aligned}$$

Thus, we achieve (4.6)–(4.7). For the second derivatives, we also apply (4.10), (4.18) and (3.1) to obtain

$$\begin{aligned}\|v-u\|_2 &\\ &\leq C\sum_{k=1}^{n_*}\big(\|\widetilde{b}_k\|_0\|A_k+\lambda^{-\tau}B_k\|_1+\|\widetilde{b}_k\|_1\|A_k+\lambda^{-\tau}B_k\|_0+\lambda^{-\tau}\|D_k\nabla\widetilde{b}_k\|_1\big)\\ &\leq C(\delta^{1/2}\lambda^{\tau}+\delta^{1/2}\lambda+\delta^{1/2}\lambda^{\tau})\leq C\delta^{1/2}\lambda^{\tau}.\end{aligned}$$

With (4.1) and the fact $\tau>1$, we arrive at (4.8) and finish the proof. ∎

With Proposition 4.1, we can modify the inductive result [11, Proposition 4.1] to fit our setting, which will help us to construct *adapted short embeddings* iteratively. We recall the definition of adapted short embeddings from [10, 11].

Definition 4.1. Given a closed subset $\Sigma\subset\mathcal{M}$ and $\theta\in]0,1[$, an embedding $u\colon\mathcal{M}\to\mathbb{R}^m$ is called *adapted short embedding with respect to* Σ *with exponent* θ if

(1) $u\in C^{1,\theta}(\mathcal{M})$;

(2) there exists a non-negative function $\rho\in C(\mathcal{M})$ with $\Sigma=\{\rho=0\}$ and a symmetric $(0,2)$-tensor $h\in C(\mathcal{M})$ with $-\frac{1}{2}g\leq h\leq\frac{1}{2}g$ such that

$$g-u^\sharp e=\rho^2(g+h);$$

(3) $u\in C^2(\mathcal{M}\setminus\Sigma)$, $\rho,h\in C^1(\mathcal{M}\setminus\Sigma)$ and there exists a constant $A\geq 1$ such that in any chart Ω_k

$$\begin{aligned}|\nabla^2 u(x)|&\leq A\rho(x)^{1-\frac{1}{\theta}},\\ |\nabla\rho(x)|&\leq A\rho(x)^{1-\frac{1}{\theta}},\\ |\nabla h(x)|&\leq A\rho(x)^{-\frac{1}{\theta}},\end{aligned}$$

for any $x\in\Omega_k\setminus\Sigma$.

Let u be an adapted short embedding with respect to some compact set $\Sigma\subset\mathcal{M}$ with exponent θ (cf. Definition 4.1). In particular,

$$g-u^\sharp e=\rho^2(g+h),$$

with $\Sigma=\{\rho=0\}$. Furthermore, let $S\supset\Sigma$ be another compact subset. Our next goal is to show that, under certain conditions, we can perturb u using Proposition 4.1 to construct another adapted short embedding with respect to the larger compact set S with some exponent $\theta'<\theta$. In particular, we will be able to successively perturb u to make it isometric along the skeleta of a suitable triangulation, eventually ending up with an isometry of a neighborhood of Σ for the flexibility part of Theorem 1.1, respectively, an isometry of $\mathcal{M}$ in Theorem 1.2. We recall from [11] the geometric condition which the two compact sets $\Sigma \subset S$ have to satisfy.

Condition 4.1. *There exists a geometric constant $\overline{r} > 0$ such that for any $\delta > 0$ the set*

$$\{x \in \mathcal{M} : \operatorname{dist}(x, \Sigma) \geq \delta \text{ and } \operatorname{dist}(x, S) \leq \overline{r}\delta\}$$

is contained in a pairwise disjoint union of open sets, each contained in a single chart Ω_k.

Recall that in Proposition 4.1 we impose a smallness-condition on the oscillation of our metric g. We now fix an atlas for $\mathcal{M}$ respecting this condition as follows. Fix an arbitrary atlas of finitely many charts Ω_k. By compactness there exists $\gamma_0 \geq 1$ such that

$$\gamma_0^{-1}\mathrm{Id} \leq G \leq \gamma_0 \mathrm{Id}, \quad \|G\|_{C^1(\Omega_k)} \leq \gamma_0$$

on any Ω_k, where, as above, G is the coordinate expression of g. If necessary, we then subdivide Ω_k to achieve $\operatorname{osc}_{\Omega_k} G < \sigma_0(\gamma_0)$. The charts in Definition 4.1 and Condition 4.1 are assumed to satisfy these assumptions.

With Proposition 4.1 we are now ready to state and prove our inductive proposition, analogous to the iteration in [11, Proposition 4.1]. The main difference in the proof when compared to the one of [11] is the choice of τ (when applying our Proposition 4.1) and corresponding estimates on h.

Proposition 4.2. *Let $0 < \theta < \frac{1}{2}$, $b > 1$, $\sigma < \frac{\sigma_0}{4}$. There exists a constant*

$$A_0 = A_0(\theta, \sigma, b) \geq 1,$$

such that the following holds.

Let $\Sigma \subset S$ be compact subsets of $\mathcal{M}$ satisfying Condition 4.1. Let $u \in C^{1,\theta}(\mathcal{M})$ be an adapted short embedding with respect to Σ such that $g - u^\sharp e = \rho^2(g + h)$ with $\rho \leq 1/4$ in $\mathcal{M}$, $\Sigma = \{\rho = 0\}$, and in any chart Ω_k,

$$\begin{aligned} |\nabla^2 u| &\leq A\rho^{1-\frac{1}{\theta}}, \quad &|\nabla \rho| &\leq A\rho^{1-\frac{1}{\theta}}, \\ |h| &\leq \sigma, &|\nabla h| &\leq A\rho^{-\frac{1}{\theta}}, \end{aligned} \tag{4.19}$$

for some $A \geq A_0$. Then there exists an adapted short embedding $\overline{u} \in C^{1,\theta'}(\mathcal{M})$ with respect to S such that

$$g - \overline{u}^\sharp e = \overline{\rho}^2(g + \overline{h}), \quad \overline{\rho} \leq \rho, \quad \|\overline{u} - u\|_0 \leq A^{-1/2},$$

and $\overline{u} = u$, $d\overline{u} = du$ on Σ.[1]

[1]The equality $du = d\overline{u}$ on Σ is intended as an equality of sections of the bundle $T^*\mathcal{M} \to \Sigma$.

Moreover, in any chart Ω_k,

$$\begin{aligned} |\nabla^2 \overline{u}| &\le A' \overline{\rho}^{1-\frac{1}{\theta'}}, \quad & |\nabla \overline{\rho}| &\le A' \overline{\rho}^{1-\frac{1}{\theta'}}, \\ |\overline{h}| &\le \sigma', \quad & |\nabla \overline{h}| &\le A' \overline{\rho}^{-\frac{1}{\theta'}}, \end{aligned} \tag{4.20}$$

with

$$A' = A^{b^2}, \quad \theta' = \frac{\theta}{b^2}, \quad \sigma' = 4\sigma.$$

Proof. As in [11], the proof is also divided into three steps.

Step 1. Parameters, cut-off functions and error size sequence. This step is the same as in [11]. First, recall that on any chart it holds that

$$\gamma_0{}^{-1}\mathrm{Id} \le G \le \gamma_0 \mathrm{Id}, \quad \|G\|_{C^1(\Omega_k)} \le \gamma_0, \quad \mathrm{osc}_{\Omega_k} G < \sigma_0(\gamma_0),$$

and let $\gamma := 4\gamma_0$. By $\rho < \frac{1}{4}$ and the assumption that u is an adapted embedding, it is easy to get

$$\gamma^{-1}\mathrm{Id} \le \nabla u^T \nabla u \le \gamma \mathrm{Id}.$$

Next, set

$$\delta_1 := \max_{x \in \mathcal{M}} \rho^2,$$

and for $q \ge 1$,

$$\lambda_q = A\delta_q^{-\frac{1}{2\theta}}, \quad \lambda_{q+1} = \lambda_q^b.$$

When A is sufficiently large (depending on θ, σ), we have

$$\delta_{q+1} \le \frac{1}{4}\delta_q, \quad \lambda_{q+1} \ge 2\lambda_q. \tag{4.21}$$

We also decompose $\mathcal{M}$ with respect to Σ and S. Let

$$r_q = A^{-1}\delta_{q+1}^{\frac{1}{2\theta}} = \lambda_{q+1}^{-1},$$

and define for $q = 0, 1, 2, \ldots,$

$$S_q = \{x : \mathrm{dist}(x, S) < r_* r_q\}, \quad \widetilde{S}_q = \{x : \mathrm{dist}(x, S) < \widetilde{r}_* r_q\},$$
$$\Sigma_q = \{x : \mathrm{dist}(x, \Sigma) < r_{**} r_q\},$$

where $r_* < \widetilde{r}_*$ and r_{**} are geometric constants to be chosen in the following order:

(1) Choose $r_{**} > 0$ so that[2]

$$\rho(x) > \frac{3}{2}\delta_{q+2}^{1/2} \quad \text{implies} \quad x \notin \Sigma_{q+1}. \tag{4.22}$$

[2]Such a choice is possible, since (4.19) implies that ρ is Hölder continuous with exponent θ.

(2) Set $\widetilde{r}_* = \overline{r} r_{**}$, where $\overline{r} > 0$ is the geometric constant in Condition 4.1, which for any $q \in \mathbb{N}$ implies that

$$\widetilde{S}_q \setminus \Sigma_q \text{ is contained in a pairwise disjoint union of open sets, each contained in a single chart } \Omega_k. \tag{4.23}$$

(3) Choose $r_* < \widetilde{r}_*$ so that $\frac{1}{2}\widetilde{r}_* < r_* < \widetilde{r}_*$. Then by (4.21), we have

$$\widetilde{S}_{q+1} \subset S_q \subset \widetilde{S}_q \quad \text{for all } q.$$

Next, we fix cut-off functions $\phi, \widetilde{\phi}, \psi, \widetilde{\psi} \in C^\infty(0,\infty)$ with $\phi, \widetilde{\phi}$ monotonic increasing, $\psi, \widetilde{\psi}$ monotonic decreasing such that

$$\phi(s), \widetilde{\phi}(s) = \begin{cases} 1 & s \geq 2, \\ 0 & s \leq \frac{3}{2}, \end{cases} \qquad \psi(s), \widetilde{\psi}(s) = \begin{cases} 1 & s \leq r_*, \\ 0 & s \geq \widetilde{r}_*, \end{cases}$$

and in addition,

$$\widetilde{\phi}(s) = 1 \text{ on } \operatorname{supp} \phi, \qquad \widetilde{\psi}(s) = 1 \text{ on } \operatorname{supp} \psi.$$

As in [11], set

$$\chi_q(x) = \phi\left(\frac{\rho(x)}{\delta_{q+2}^{1/2}}\right)\psi\left(\frac{\operatorname{dist}(x,S)}{r_{q+1}}\right), \quad \widetilde{\chi}_q(x) = \widetilde{\phi}\left(\frac{\rho(x)}{\delta_{q+2}^{1/2}}\right)\widetilde{\psi}\left(\frac{\operatorname{dist}(x,S)}{r_{q+1}}\right).$$

Using (4.19) and the choice of $r_q, r_*, \widetilde{r}_*$ and the cut-off functions, we easily deduce

$$|\nabla \chi_q|, |\nabla \widetilde{\chi}_q| \leq CA\delta_{q+2}^{-\frac{1}{2\theta}} = C\lambda_{q+2}, \tag{4.24}$$

$$\operatorname{dist}(\operatorname{supp} \chi_q, \partial \operatorname{supp} \widetilde{\chi}_q) \geq C^{-1}A^{-1}\delta_{q+2}^{\frac{1}{2\theta}} = C^{-1}\lambda_{q+2}^{-1}. \tag{4.25}$$

for some constant C depending on $r_*, \widetilde{r}_*$, and moreover

$$\begin{aligned} \{x \in S_{q+1} | \rho(x) > 2\delta_{q+2}^{1/2}\} &\subset \{x \in \mathcal{M} : \chi_q(x) = 1\}, \\ \operatorname{supp} \chi_q &\subset \{x \in \mathcal{M} : \widetilde{\chi}_q(x) = 1\}, \\ \operatorname{supp} \widetilde{\chi}_q &\subset \left\{x \in \widetilde{S}_{q+1} : \rho(x) > \frac{3}{2}\delta_{q+2}^{1/2}\right\}. \end{aligned} \tag{4.26}$$

From (4.22) and (4.23), we then deduce that $\operatorname{supp} \widetilde{\chi}_q$ is contained in a pairwise disjoint union of open sets, each contained in a single chart Ω_k.

Finally, we define the sequence of error size $\{\rho_q\}$. Set $\rho_0 = \rho$ and define ρ_q for $q = 1, 2, \ldots$ inductively as

$$\rho_{q+1}^2 = \rho_q^2(1 - \chi_q^2) + \delta_{q+2}\chi_q^2. \tag{4.27}$$

One can prove by induction (cf. [11, Lemma 4.1]) that the thus defined maps ρ_q have the following properties.

Lemma 4.1. *Let $\{\rho_q\}$ be defined in (4.27). Then for any $q = 0, 1, \dots,$*

(i) *on* $\operatorname{supp} \widetilde{\chi}_q$ *it holds that*

$$\frac{3}{2}\delta_{q+2}^{1/2} \le \rho_q \le 2\delta_{q+1}^{1/2};$$

(ii) *for every x, we have $\rho_{q+1}(x) \le \rho_q(x)$;*

(iii) *if $\rho_q(x) \le \delta_{q+1}^{1/2}$, then $x \notin \bigcup_{j=0}^{q-1} \operatorname{supp} \widetilde{\chi}_j$ and consequently $\rho_q(x) = \rho(x)$;*

(iv) *if $\rho_q(x) \ge \delta_{q+1}^{1/2}$, then either $\chi_q(x) = 1$ or $x \notin S_{q+1}$.*

Now we are ready to inductively construct a sequence of adapted short embeddings.

Step 2. Inductive construction. This step is similar to that of [11, Proposition 4.1], but we need to pay more attention to the choice of τ and the estimate of h. We will construct a sequence of smooth adapted short embeddings (u_q, ρ_q, h_q) such that the following hold:

$(1)_q$ for all $\mathcal{M}$, we have

$$g - u_q^\sharp e = \rho_q^2(g + h_q);$$

$(2)_q$ if $x \notin \bigcup_{j=0}^{q-1} \operatorname{supp} \widetilde{\chi}_j$, then $(u_q, \rho_q, h_q) = (u_0, \rho_0, h_0)$ and $du_q = du_0$ along Σ;

$(3)_q$ the following estimates hold in $\mathcal{M}$:

$$|\nabla^2 u_q| \le A^{b^2} \rho_q^{1-\frac{b^2}{\theta}}, \quad |\nabla \rho_q| \le A^{b^2} \rho_q^{1-\frac{b^2}{\theta}}, \tag{4.28}$$

$$|h_q| \le 4\sigma, \qquad |\nabla h_q| \le A^{b^2} \rho_q^{-\frac{b^2}{\theta}}; \tag{4.29}$$

$(4)_q$ on $\{x : \rho_0(x) > \delta_{q+1}^{1/2}\} \cap S_q$, we have the sharper estimates

$$|\nabla^2 u_q| \le A^b \rho_q^{1-\frac{b}{\theta}}, \quad |\nabla \rho_q| \le A^b \rho_q^{1-\frac{b}{\theta}}, \tag{4.30}$$

$$|h_q| \le \sigma, \quad |\nabla h_q| \le A^b \rho_q^{-\frac{b}{\theta}}; \tag{4.31}$$

$(5)_q$ we have the global estimate for $q \ge 1$:

$$\|u_q - u_{q-1}\|_0 \le \bar{C}\delta_q^{1/2}\lambda_q^{-1}, \tag{4.32}$$

$$\|u_q - u_{q-1}\|_1 \le \bar{C}\delta_q^{1/2}, \tag{4.33}$$

where $\bar{C}$ is the constant in the conclusions of Proposition 4.1 in (4.6)–(4.7).

Initial step $q = 0$. Set $(u_0, \rho_0, h_0) = (u, \rho, h)$. Since $b > 1$, it is easy to check $(1)_0$–$(2)_0$ and $(4)_0$ from (4.19).

Inductive step $q \mapsto q+1$. Suppose (u_q, ρ_q, h_q) is an adapted short embedding on $\mathcal{M}$ satisfying $(1)_q$–$(5)_q$. We then construct $(u_{q+1}, \rho_{q+1}, h_{q+1})$. In fact, ρ_{q+1} has already been defined in (4.27). We shall estimate (u_q, ρ_q, h_q) on supp $\widetilde{\chi}_q$. As derived in [11], on supp $\widetilde{\chi}_q$, we have

$$\begin{gathered}\frac{3}{2}\delta_{q+2}^{1/2} \leq \rho_q \leq 2\delta_{q+1}^{1/2},\\ |\nabla \rho_q| \leq \delta_{q+1}^{1/2}\lambda_{q+2}, \quad |\nabla^2 u_q| \leq \delta_{q+1}^{1/2}\lambda_{q+2}, \quad \left|\frac{\nabla \rho_q}{\rho_q}\right| \leq \lambda_{q+2},\\ |h_q| \leq \sigma, \quad |\nabla h_q| \leq \lambda_{q+2}.\end{gathered} \tag{4.34}$$

We then want to apply Proposition 4.1 to construct (u_{q+1}, h_{q+1}). To this end define

$$\widetilde{\rho}_q = \chi_q \sqrt{\rho_q^2 - \delta_{q+2}}, \quad \widetilde{h}_q = \frac{\widetilde{\chi}_q \rho_q^2}{\rho_q^2 - \delta_{q+2}} h_q.$$

From (4.34), on supp $\widetilde{\chi}_q$, one has

$$\frac{5}{4}\delta_{q+2} \leq \rho_q^2 - \delta_{q+2} \leq 4\delta_{q+1},$$

hence $\widetilde{\rho}_q$ and $\widetilde{h}_q$ are well defined. Note that with these definitions, we then have

$$\widetilde{\rho}_q^2(g + \widetilde{h}_q) = \chi_q^2\big((\rho_q^2 - \delta_{q+2})g + \rho_q^2 h_q\big) = \chi_q^2(g - u_q^\sharp e - \delta_{q+2} g)$$

using that $\widetilde{\chi}_q = 1$ on the support of χ_q and the inductive assumption $(1)_q$. Thus, by adding the tensor $\widetilde{\rho}_q^2(g + \widetilde{h}_q)$, we will be able to get a map u_{q+1}, which, up to an error of size δ_{q+2}, is isometric on the support of χ_q.

We therefore want to estimate $\widetilde{\rho}_q$ and $\widetilde{h}_q$ and choose δ, λ in Proposition 4.1 accordingly. We thus set $\Omega = \operatorname{supp} \widetilde{\chi}_q$, and observe

$$\begin{gathered}|\nabla \sqrt{\rho_q^2 - \delta_{q+2}}| \leq C|\nabla \rho_q|,\\ \frac{\rho_q^2}{\rho_q^2 - \delta_{q+2}} = 1 + \frac{\delta_{q+2}}{\rho_q^2 - \delta_{q+2}} \leq 2,\\ \left|\nabla \frac{\rho_q^2}{\rho_q^2 - \delta_{q+2}}\right| = \left|\nabla \frac{\delta_{q+2}}{\rho_q^2 - \delta_{q+2}}\right| \leq C\left|\frac{\nabla \rho_q}{\rho_q}\right|,\end{gathered}$$

where C are geometric constants. Therefore, using (4.24) and (4.34), we can infer

$$\begin{gathered}0 \leq \widetilde{\rho}_q \leq \rho_q \leq 2\delta_{q+1}^{1/2}, \quad |\widetilde{h}_q| \leq 2\sigma \leq \frac{\sigma_0}{2},\\ |\nabla \widetilde{\rho}_q| \leq C\big(|\nabla \chi_q|\rho_q + |\nabla \rho_q|\big) \leq C\delta_{q+1}^{1/2}\lambda_{q+2},\\ |\nabla \widetilde{h}_q| \leq C\left(|\nabla \widetilde{\chi}_q||h_q| + \left|\frac{\nabla \rho_q}{\rho_q}\right||h_q| + |\nabla h_q|\right) \leq C\lambda_{q+2}.\end{gathered}$$

Therefore, $(u_q, \widetilde{\rho}_q, \widetilde{h}_q)$ satisfies all the assumptions in Proposition 4.1 on supp $\widetilde{\chi}_q$ with δ, λ given by $4\delta_{q+1}, C\lambda_{q+2}$, respectively. Setting

$$\tau = 1 + \frac{1-\theta}{b}(b-1) > 1,$$

we only need to make sure that $C\lambda_{q+2} \geq \lambda_0(\gamma, \sigma_0, \tau)$ in (4.4). This however follows by choosing $A \geq A_0(\theta, \sigma, b)$ large enough.

Thus, recalling (4.23) that supp $\widetilde{\chi}_q$ is contained in a pairwise disjoint union of open sets, each contained in a single chart, we may apply Proposition 4.1 in each open set separately in local coordinates to add the term $\widetilde{\rho}_q^2(g + \widetilde{h}_q)$. Overall we obtain u_{q+1} and $\mathcal{E}$ such that

$$g - u_{q+1}^\sharp e = (g - u_q^\sharp e)(1-\chi_q^2) + \delta_{q+2} g \chi_q^2 + \mathcal{E}.$$

with u_{q+1} satisfying

$$|\nabla^2 u_{q+1}| \leq C\delta_{q+1}^{1/2}\lambda_{q+2}^{\tau} = C\delta_{q+1}^{1/2}\lambda_{q+1}^{b+(1-\theta)(b-1)}, \tag{4.35}$$

and $\mathcal{E}$ satisfying

$$|\mathcal{E}| \leq C\delta_{q+1}\lambda_{q+2}^{2-2\tau} = C\delta_{q+2}\lambda_{q+1}^{-2(1-2\theta)(b-1)}, \tag{4.36}$$

$$|\nabla\mathcal{E}| \leq C\delta_{q+1}\lambda_{q+2} = C\delta_{q+2}\lambda_{q+1}^{b+2\theta(b-1)}, \tag{4.37}$$

which are implied by $\delta_{q+1} = \lambda_{q+1}^{2\theta(b-1)}\delta_{q+2}$ and (4.8)–(4.9). From (4.5), one gets

$$\operatorname{supp}(u_{q+1} - u_q), \quad \operatorname{supp}\mathcal{E} \subset \operatorname{supp}\chi_q + B_{\kappa_q}(0),$$

with

$$\kappa_q = (C\lambda_{q+2})^{1-2\tau} \leq \lambda_{q+1}^{-2(1-\theta)(b-1)}\lambda_{q+2}^{-1} \leq C^{-1}\lambda_{q+2}^{-1},$$

where C is the constant in (4.25) and the last inequality holds provided A is sufficiently large. Consequently, $u_{q+1} = u_q$, $du_{q+1} = du_q$ and $\mathcal{E} = 0$ outside supp $\widetilde{\chi}_q$.

Moreover, (4.32) and (4.33) for the case $q+1$ follow immediately from (4.6)–(4.7), hence $(5)_{q+1}$ is verified. We also define

$$h_{q+1} = (1-\chi_q^2)\frac{\rho_q^2}{\rho_{q+1}^2}h_q + \frac{\mathcal{E}}{\rho_{q+1}^2},$$

so that

$$g - u_{q+1}^\sharp e = \rho_{q+1}^2(g + h_{q+1}),$$

verifying $(1)_{q+1}$. Note that on supp $\widetilde{\chi}_q$ using (4.34) one has

$$\begin{aligned} \rho_{q+1}^2 &\leq 4\delta_{q+1}(1-\chi_q^2) + \delta_{q+2}\chi_q^2 \leq 4\delta_{q+1}, \\ \rho_{q+1}^2 &\geq \frac{9}{4}\delta_{q+2}(1-\chi_q^2) + \delta_{q+2}\chi_q^2 \geq \delta_{q+2}. \end{aligned} \tag{4.38}$$

Thus, h_{q+1} is well defined. Besides we can also derive that (ρ_{q+1}, h_{q+1}) agrees with (ρ_q, h_q) outside $\operatorname{supp} \widetilde{\chi}_q$. It remains to verify $(2)_{q+1}$–$(4)_{q+1}$ on $\operatorname{supp} \widetilde{\chi}_q$.

Verification of $(2)_{q+1}$. If $x \notin \bigcup_{j=0}^{q} \operatorname{supp} \widetilde{\chi}_j$, then $\widetilde{\chi}_q(x) = 0$ and therefore

$$(u_{q+1}, \rho_{q+1}, h_{q+1}) = (u_q, \rho_q, h_q) = (u_0, \rho_0, h_0).$$

Verification of $(3)_{q+1}$. On $\operatorname{supp} \widetilde{\chi}_q$, we first calculate

$$\begin{aligned} |\nabla \rho_{q+1}| &= \frac{|\nabla \rho_{q+1}^2|}{2\rho_{q+1}} \leq \frac{C}{\rho_{q+1}}\big(|\rho_q \nabla \rho_q| + |\nabla \chi_q|(\rho_q^2 + \delta_{q+2})\big) \\ &\leq C \frac{\delta_{q+1}\lambda_{q+2}}{\delta_{q+2}^{1/2}} = C A^{b+(b-1)\theta} \delta_{q+1}^{1-\frac{b}{2}(1+\frac{1}{\theta})} \\ &\leq A^{b^2} (2\delta_{q+1}^{1/2})^{1-\frac{b^2}{\theta}} \leq A^{b^2} \rho_{q+1}^{1-\frac{b^2}{\theta}}, \end{aligned} \tag{4.39}$$

where we have used (4.24), (4.34) and (4.38). For the inequality in the last line we have used that

$$1 - \frac{b}{2}\Big(1 + \frac{1}{\theta}\Big) \geq \frac{1}{2}\Big(1 - \frac{b^2}{\theta}\Big), \quad 2(b-1)\theta + b \leq b^2$$

(from $b > 1$ and $2\theta < 1$) and A sufficiently large to absorb geometric constants.

Similarly, using (4.34), (4.35)–(4.36) and (4.38), we obtain

$$\begin{aligned} |h_{q+1}| &\leq |h_q| + \frac{|\mathcal{E}|}{\rho_{q+1}^2} \leq 2\sigma + C\lambda_{q+1}^{-2(1-2\theta)(b-1)} \leq 3\sigma, \\ |\nabla^2 u_{q+1}| &\leq C\delta_{q+1}^{1/2} \lambda_{q+1}^{b+(1-\theta)(b-1)} \leq C\delta_{q+1}^{1/2} \lambda_{q+1}^{b^2-\theta(b-1)} \\ &\leq C A^{b^2-\theta(b-1)} \delta_{q+1}^{\frac{1}{2}(1-\frac{b^2}{\theta})} \leq A^{b^2} \rho_{q+1}^{1-\frac{b^2}{\theta}}, \end{aligned}$$

where we have used

$$(1-\theta)(b-1) + \theta(b-1) \leq b^2 - b$$

(by $b > 1$) and again assumed A sufficiently large to absorb the constants C.

For $|\nabla h_{q+1}|$, we calculate as follows:

$$\begin{aligned} |\nabla h_{q+1}| &\leq |\nabla h_q| + \frac{1}{\rho_{q+1}^2}\big(|\nabla \mathcal{E}| + \delta_{q+2}|\nabla(h_q \chi_q^2)|\big) + \frac{2|\nabla \rho_{q+1}|}{\rho_{q+1}^3}\big(\delta_{q+2}|h_q| + |\mathcal{E}|\big) \\ &\leq C\lambda_{q+2} + C\lambda_{q+1}^{b+2\theta(b-1)} + C\frac{\delta_{q+1}\lambda_{q+2}}{\delta_{q+2}}\big(\sigma + \lambda_{q+1}^{-2(1-\theta)(b-1)}\big) \\ &\leq C\lambda_{q+2} + C\lambda_{q+1}^{b+2\theta(b-1)} + C\frac{\delta_{q+1}}{\delta_{q+2}}\lambda_{q+2} \\ &\leq C\lambda_{q+1}^{b+2\theta(b-1)}, \end{aligned} \tag{4.40}$$

where we have used (4.24), (4.34), (4.36), (4.37) and (4.39). Using again the inequality $b + 2\theta(b-1) < b^2 - (1-2\theta)(b-1)$, we further estimate

$$\begin{aligned} |\nabla h_{q+1}| &\le C\lambda_{q+1}^{b^2-(1-2\theta)(b-1)} \\ &\le CA^{b^2-(1-2\theta)(b-1)}\delta_{q+1}^{-\frac{b^2}{2\theta}} \le A^{b^2}\rho_{q+1}^{-\frac{b^2}{\theta}}, \end{aligned} \tag{4.41}$$

where we have again used that A is sufficiently large. Thus we have shown (4.28) for $q+1$, i.e., $(3)_{q+1}$ is verified.

Verification of $(4)_{q+1}$. Observe that by (4.26),

$$\begin{aligned} \{x \in S_{q+1} : \rho_0(x) > \delta_{q+2}^{1/2}\} = \{\chi_q(x) = 1\} \\ \cup \{x \in S_{q+1} : \delta_{q+2}^{1/2} \le \rho_0(x) \le 2\delta_{q+2}^{1/2}\}. \end{aligned}$$

If $x \in \{\chi_q = 1\}$, then

$$\rho_{q+1} = \delta_{q+2}^{1/2}, \quad h_{q+1} = \frac{\mathcal{E}}{\delta_{q+2}}.$$

Using (4.35),

$$\begin{aligned} |\nabla^2 u_{q+1}| &\le C\delta_{q+1}^{1/2}\lambda_{q+1}^{b+(1-\theta)(b-1)} \\ &= C\delta_{q+1}^{1/2}\lambda_{q+1}^{2b-\theta(b-1)-1} \le CA^{2-\frac{1}{b}-b}A^b\delta_{q+2}^{\frac{1}{2}(1-\frac{b}{\theta})}. \end{aligned} \tag{4.42}$$

where we have used $2 - \frac{1}{b} < b$. By taking A sufficiently large we absorb the geometric constant C and deduce (4.30).

In order to verify (4.31), we calculate using (4.36)–(4.37):

$$\begin{aligned} |h_{q+1}| &\le C\lambda_{q+2}^{-2(1-2\theta)(b-1)} \le \sigma, \\ |\nabla h_{q+1}| &\le C\lambda_{q+1}^{b+2\theta(b-1)} \le \lambda_{q+2}^{b} = A^b\delta_{q+2}^{-\frac{b}{2\theta}}. \end{aligned}$$

using $b + 2\theta(b-1) < b^2$. By choosing A sufficiently large, we can then absorb again the geometric constants and conclude (4.31). Hence, $(4)_{q+1}$ is obtained for this case.

On the other hand, if

$$x \in \{x \in \Sigma_{q+1} : \delta_{q+2}^{1/2} \le \rho_0(x) \le 2\delta_{q+2}^{1/2}\},$$

then

$$(u_q, \rho_q, h_q) = (u_0, \rho_0, h_0)$$

by $(2)_q$ and $\rho_0 \le 2\delta_{q+2}^{1/2}$. Thus,

$$\begin{aligned} \rho_{q+1}^2 &\ge \delta_{q+2}(1-\chi_q^2) + \delta_{q+2}\chi_q^2 \ge \delta_{q+2}, \\ \rho_{q+1}^2 &\le 4\delta_{q+2}(1-\chi_q^2) + \delta_{q+2}\chi_q^2 \le 4\delta_{q+2}. \end{aligned} \tag{4.43}$$

Therefore, choosing again A sufficiently large to absorb geometric constants,

$$\begin{aligned}|h_{q+1}| &\le |h_0| + \left|\frac{\mathcal{E}}{\rho_{q+1}^2}\right| \\ &\le \sigma + C\lambda_{q+1}^{-2(1-2\theta)(b-1)} \le 2\sigma.\end{aligned}$$

Moreover, calculating as in (4.39), but this time using (4.43),

$$\begin{aligned}|\nabla \rho_{q+1}| = \frac{|\nabla \rho_q^2|}{2\rho_{q+1}} &\le \frac{C}{\rho_{q+1}}\big(|\rho_q \nabla \rho_q| + |\nabla \chi_q|(\rho_q^2 + \delta_{q+2})\big) \\ &\le C\delta_{q+2}^{1/2}\lambda_{q+2} = CA\delta_{q+2}^{\frac{1}{2}(1-\frac{1}{\theta})} \le A^b \delta_{q+2}^{\frac{1}{2}(1-\frac{b}{\theta})} \le A^b \rho_{q+1}^{1-\frac{b}{\theta}}.\end{aligned}$$

Similarly, proceeding as in (4.40)–(4.41), we have

$$|\nabla h_{q+1}| \le C\lambda_{q+2}^{1+2\theta(1-\frac{1}{b})} = CA^{1+2\theta(1-\frac{1}{b})}\delta_{q+2}^{-\frac{1}{2\theta}-(1-\frac{1}{b})} \le A^b \rho_{q+1}^{-\frac{b}{\theta}}.$$

Finally, the estimate for $\nabla^2 u_{q+1}$ has already been obtained in (4.42). Therefore, $(4)_{q+1}$ is verified also in this case.

Overall, we have shown that $(u_{q+1}, \rho_{q+1}, h_{q+1})$ satisfies $(1)_{q+1}$–$(5)_{q+1}$.

Step 3. Conclusion. We are now in a position to take the limit as $q \to \infty$. Recalling (4.21), we see that

$$\delta_q^{1/2} \le 2^{-q-1} \quad \text{and} \quad \delta_q^{1/2}\lambda_q^{-1} \le A^{-1}2^{-q-1}.$$

In particular, from $(5)_q$ we see that $\{u_q\}$ is a Cauchy sequence in $C^1(\mathcal{M})$.

From the formula (4.27) and Lemma 4.1, we deduce

$$0 \le \rho_q - \rho_{q+1} \le 2\delta_{q+1}^{1/2},$$

so that $\{\rho_q\}$ is a Cauchy sequence in $C^0(\mathcal{M})$. From $(1)_q$–$(3)_q$, we can also deduce that $\{h_q\}$ is a Cauchy sequence in $C^0(\mathcal{M})$; indeed, this follows from the formula $(1)_q$, the fact that $u_q^\sharp e$ and ρ_q^2 are Cauchy sequences, and (4.29).

Furthermore, since $\operatorname{supp} \tilde{\chi}_q \subset S_q$ and $\bigcap_q S_q = S$, using $(2)_q$ we see that for any $x \in \mathcal{M} \setminus S$ there exists $q_0 = q_0(x)$ such that

$$(u_q, \rho_q, h_q) = (u_{q_0}, \rho_{q_0}, h_{q_0})$$

for all $q \ge q_0(x)$. Similarly, since

$$\operatorname{supp} \tilde{\chi}_q \subset \{\rho > \delta_{q+1}^{1/2}\},$$

(u_q, ρ_q, h_q) agrees with (u, ρ, h) on Σ. Thus, there exist

$$\begin{aligned}
&\overline{u} \in C^1(\mathcal{M}) \cap C^2(\mathcal{M} \setminus S),\\
&\overline{\rho} \in C^0(\mathcal{M}) \cap C^1(\mathcal{M} \setminus S),\\
&\overline{h} \in C^0(\mathcal{M}, \mathbb{R}^{2\times 2}) \cap C^1(\mathcal{M} \setminus S, \mathbb{R}^{2\times 2}),
\end{aligned}$$

such that

$$u_q \to \overline{u}, \quad u_q^\sharp e \to \overline{u}^\sharp e, \quad \rho_q \to \overline{\rho}, \quad h_q \to \overline{h} \text{ uniformly on } \mathcal{M}.$$

The limit $(\overline{u}, \overline{\rho}, \overline{h})$ satisfies

$$g - \overline{u}^\sharp e = \overline{\rho}^2 (g + \overline{h}) \quad \text{on } \mathcal{M}$$

using $(1)_q$. By $(2)_q$, $\overline{u} = u$ and $d\overline{u} = du$ on Σ. Moreover, we have

$$\|\overline{u} - u\|_0 \le \sum_{q=1}^{\infty} \|u_q - u_{q-1}\|_0 \le \overline{C} A^{-1} \sum_{q=1}^{\infty} 2^{-q-1} = \frac{1}{2} \overline{C} A^{-1} \le A^{-1/2}$$

using $(5)_q$ and ensuring A is large enough to absorb the constant $\overline{C}$, and, using $(3)_q$,

$$\begin{aligned}
|\nabla^2 \overline{u}| &\le A^{b^2} \overline{\rho}^{1-\frac{b^2}{\theta}}, \quad & |\nabla \overline{\rho}| &\le A^{b^2} \overline{\rho}^{1-\frac{b^2}{\theta}},\\
|\overline{h}| &\le 4\sigma, & |\nabla \overline{h}| &\le A^{b^2} \overline{\rho}^{-\frac{b^2}{\theta}}.
\end{aligned}$$

Finally, from Lemma 4.1 and (4.26), we see that

$$\rho_q \le 2\delta_{q+1}^{1/2} \quad \text{on } S.$$

Combined with the observation above that for any $x \notin S \supset \Sigma$, we have

$$\overline{\rho}(x) = \rho_q(x) > 0$$

for some q; we deduce $\{\overline{\rho} = 0\} = \Sigma$. This proves that $(\overline{u}, \overline{\rho}, \overline{h})$ is an adapted short embedding with respect to $S \supset \Sigma$ with exponent $\theta' = \frac{\theta}{b^2}$, and satisfying (4.20) as required. The proof of Proposition 4.2 is completed. ∎

5. Proof of Theorem 1.1 (2): Flexibility part

The goal of this section is to show the flexibility part of Theorem 1.1. The proof is divided into three steps.

Step 1. Short extension. In the first step we want to construct an embedding which is isometric on Σ and strictly short on $\Sigma_\epsilon^+ \setminus \Sigma$ for a one-sided neighborhood $\Sigma_\epsilon^+ \subset M$. The construction is analogous to the one in [10] (see also [24]) except that we want to define u not only locally around a point $p \in \Sigma$.

Recall that the one-sided neighborhood is defined as $\Sigma_\epsilon^+ = F(\Sigma \times [0, \epsilon[\,)$ for $F\colon \Sigma \times]-\epsilon_0, \epsilon_0[\to \mathcal{M}$ given by $F(p,t) = \exp_p(t\nu(p))$. We then define our short extension $u\colon \Sigma_\epsilon^+ \to \mathbb{R}^m$ by

$$u(F(p,t)) = f(p) + t\mu(p) - t^2\mu(p).$$

We claim that u is isometric on Σ and strictly short on $\Sigma_\epsilon^+ \setminus \Sigma$ if ϵ is small enough. Indeed, fix a finite atlas $\{(V_i, \psi_i)\}_{i=1}^N$ for the manifold Σ and extend it to Σ_ϵ using F. More precisely, set $U_i = F(V_i,]-\epsilon_0, \epsilon_0[\,)$ and define $\varphi_i\colon U_i \to \mathbb{R}^n$ by

$$\varphi_i(F(p,t)) = (\psi_i(p), t).$$

Clearly in these coordinates it holds that $\Sigma = \{t = 0\}$, and one can check that the metric in each U_i is of the form

$$g = \sum_{i,j=1}^{n-1} g_{ij}\,dx^i dx^j + (dt)^2.$$

Moreover, the scalar second fundamental form of the inclusion $\iota\colon \Sigma \hookrightarrow \mathcal{M}$ is given by

$$L_{ij}(x) = -\frac{1}{2}\partial_t g_{ij}(x,0).$$

By expanding g_{ij} around $t = 0$, we then get

$$g_{ij}(x,t) = g_{ij}(x,0) - 2tL_{ij}(x) + O(t^2).$$

On the other hand, we compute

$$\langle \partial_i u, \partial_j u\rangle = \langle \partial_i f, \partial_j f\rangle + t\big(\langle \partial_i f, \partial_j \mu\rangle + \langle \partial_j f, \partial_i \mu\rangle\big) + O(t^2),$$

and

$$\langle \partial_i u, \partial_t u\rangle = 0, \quad \langle \partial_t u, \partial_t u\rangle = (1-2t)^2$$

thanks to the properties of μ. Since f is an isometry and

$$\langle \partial_i f, \partial_j \mu\rangle = \langle \partial_j f, \partial_i \mu\rangle = -\langle \overline{L}_{ij}, \mu\rangle,$$

we therefore get

$$g - \nabla u^T \nabla u = 2t \begin{pmatrix} \langle \overline{L}_{ij}, \mu\rangle - L_{ij} & 0 \\ 0 & 2 \end{pmatrix} + O(t^2). \tag{5.1}$$

Clearly, $u^\sharp e = g$ on Σ. Moreover, if $\epsilon > 0$ is small enough, assumption (1.2) implies that there exists $C \geq 1$ such that

$$(g - \nabla u^T \nabla u)|_{(x,t)} \geq C^{-1} t \mathrm{Id}$$

on Σ_ϵ^+, showing the strict shortness of u on $\Sigma_\epsilon^+ \setminus \Sigma$.

Lastly, we observe that for $p \in \Sigma$ it holds that

$$du_p(\nu) = \partial_t u(F(p,0)) = \mu(p),$$

and consequently, for any $X \in T_p\Sigma \setminus \{0\}$,

$$\langle du(\nu), \bar{L}(X,X)\rangle = \langle \mu, \bar{L}(X,X)\rangle > L(X,X). \tag{5.2}$$

Step 2. Adapted short extension. Given the short extension u from Step 1, we want to construct an adapted short embedding v with $u = v$ and $du = dv$ on Σ. The step is similar to corresponding construction in [10], the main differences being the choice of frequency parameter below to make our extension of class C^{1,θ_0} for any $\theta_0 < \frac{1}{2}$ and the global nature of the present construction. We use one stage of adding primitive metric errors to construct an adapted short embedding. Choose $\gamma, M > 1$ such that the short extension $u\colon \Sigma_\epsilon^+ \to \mathbb{R}^m$ constructed in Step 1 satisfies $u \in C^2(\Sigma_\epsilon^+)$ with

$$\gamma^{-1}\mathrm{Id} \leq \nabla u^T \nabla u \leq \gamma \mathrm{Id}, \quad \|u\|_{C^2(U_i)} \leq M$$

in every chart U_i. We then define

$$\rho^2(x,t) = \frac{1}{n} \operatorname{tr}(g - \nabla u^T \nabla u).$$

Observe that this is a well-defined function on Σ_ϵ^+ since the trace is invariant under coordinate transformations. By (5.1), we can seek a constant $C \geq 1$ so that, for all $(x,t) \in \Sigma_\epsilon^+$,

$$C^{-1} t^{1/2} \leq \rho(x,t) \leq C t^{1/2}, \quad |\nabla \rho(x,t)| \leq C t^{-1/2}, \quad |\nabla^2 \rho(x,t)| \leq C t^{-3/2}. \tag{5.3}$$

Furthermore, there exists $\alpha > 0$ such that

$$g - \nabla u^T \nabla u \geq C^{-1} \rho^2 \mathrm{Id} \geq 2\alpha \rho^2 g$$

in every chart. We assume without loss of generality that $\alpha\rho^2 \leq \frac{1}{16}$ on Σ_ϵ^+ and $\alpha < 1$. In particular, using [31, Lemma 1] (see also [33, Lemma 1.9]), we obtain the decomposition

$$\frac{g - \nabla u^T \nabla u}{\rho^2} - \alpha g = \sum_{k=1}^{\tilde{N}} \bar{b}_{k,i}^2 \varpi_{k,i} \otimes \varpi_{k,i}$$

in U_i, where $\varpi_{k,i} \in \mathbb{S}^{n-1}$, $\overline{b}_{k,i} \in C^\infty(U_i)$ and $\tilde{N} \in \mathbb{N}$, with estimates of the form

$$\|\overline{b}_{k,i}\|_{C^j(U_i)} \leq C$$

for $j = 0, 1, 2$. Setting $b_k = \overline{b}_k \rho$ we derive

$$g - \nabla u^T \nabla u - \alpha\rho^2 g = \sum_{k=1}^{\tilde{N}} b_{k,i}^2 \varpi_{k,i} \otimes \varpi_{k,i}$$

in U_i, with estimates, for $j = 0, 1, 2$ and $k = 1, \ldots, \tilde{N}$,

$$|\nabla^j b_{k,i}(x,t)| \leq C t^{1/2-j} \quad \text{for } (x,t) \in U_i. \tag{5.4}$$

Now we define a Whitney-decomposition of $\Sigma_\epsilon^+ \setminus \Sigma$ as follows: Set $d_q = 2^{-q}\epsilon$ for $q = 1, 2, \ldots$ and define

$$\Sigma_q^i = F\big(V_i,]d_{q+1}, d_{q-1}[\big) = U_i \cap \big(\Sigma_{d_{q-1}}^+ \setminus \overline{\Sigma_{d_{q+1}}^+}\big).$$

We then let $\{\chi_q^i\}_{q,i}$ be a partition of unity on $\Sigma_\epsilon^+ \setminus \Sigma$ subordinate to the decomposition

$$\Sigma_\epsilon^+ \setminus \Sigma = \bigcup_{q=1}^{\infty} \bigcup_{i=1}^{N} \Sigma_q^i$$

with the following standard properties:

(a) $\operatorname{supp}\chi_q^i \subset \Sigma_q^i$, in particular $\operatorname{supp}\chi_q^i \cap \operatorname{supp}\chi_{q+2}^i = \emptyset$;

(b) $\sum_{i=1}^N \sum_{q=0}^\infty (\chi_q^i)^2 = 1$ in $\Sigma_\epsilon^+ \setminus \Sigma$;

(c) for any q, i and $j = 0, 1, 2$ we have $\|\chi_q^i\|_{C^j(\Sigma_q^i)} \leq C d_q^{-j}$.

Consequently, we can write

$$\begin{aligned} g - u^\sharp e - \alpha\rho^2 g = &\sum_{i=1}^N \sum_{k=1}^{\tilde{N}} \sum_{q \text{ odd}} (\chi_q^i b_{k,i})^2 \varpi_{k,i} \otimes \varpi_{k,i} \\ &+ \sum_{i=1}^N \sum_{k=1}^{\tilde{N}} \sum_{q \text{ even}} (\chi_q^i b_{k,i})^2 \varpi_{k,i} \otimes \varpi_{k,i}. \end{aligned}$$

We now add similar perturbations to the map u as in Proposition 4.1 in order to remove most of the metric error. This can be done as in [10, Proposition 3.1], which we can directly apply since from property (c) and (5.4), we deduce

$$\|\chi_q^i b_{k,i}\|_{C^j(\Sigma_q^i)} \leq C d_q^{1/2-j}.$$

Thus the assumptions of [10, Proposition 3.1] hold in each Σ_q^i with parameters

$$\delta = d_q, \quad \varepsilon = d_q, \quad \theta = d_q^{-1}, \quad \tilde{\theta} = d_q^{-1}.$$

Observe that, using property (a), we may "add" each primitive metric

$$(\chi_q^i b_{k,i})^2 \varpi_{k,i} \otimes \varpi_{k,i}$$

with q odd in parallel, and serially[3] in i and k. We then repeat the same process for q even. Then [10, Proposition 3.1] yields, for any $K \geq C(M, \gamma)$, an embedding $v \in C^2(\Sigma_\epsilon^+, \mathbb{R}^m)$ such that, for all $q \in \mathbb{N}$ and $i = 1, \ldots, N$,

$$\|v - u\|_{C^0(\Sigma_q^i)} \leq C(M, \gamma) d_q^{3/2} K^{-1}, \tag{5.5}$$

$$\|v - u\|_{C^1(\Sigma_q^i)} \leq C(M, \gamma) d_q^{1/2}, \tag{5.6}$$

$$\|v\|_{C^2(\Sigma_q^i)} \leq C(M, \gamma) d_q^{-1/2} K^{2N\tilde{N}}. \tag{5.7}$$

Since the perturbations in each step are compactly supported away from Σ, we have $u = v$ and $du = dv$ along Σ. Moreover,

$$v^\sharp e = u^\sharp e + \sum_{i=1}^{N} \sum_{k=1}^{\tilde{N}} \sum_{q=1}^{\infty} (\chi_q^i b_{k,i})^2 \varpi_{k,i} \otimes \varpi_{k,i} + \mathcal{E}$$

with

$$\|\mathcal{E}\|_{C^0(\Sigma_q^i)} \leq C(M, \gamma) d_q K^{-1},$$

$$\|\mathcal{E}\|_{C^1(\Sigma_q^i)} \leq C(M, \gamma) K^{2N\tilde{N}-1},$$

for every $i = 1, \ldots, N$. Now we are in a position to show that v is our desired adapted short embedding. First of all, observe that for any $\theta_0 < \frac{1}{2}$ and any $i = 1, \ldots, N$, by (5.6)–(5.7),

$$\|v - u\|_{C^{1,\theta_0}(\Sigma_q^i)} \leq \|v - u\|_{C^1(\Sigma_q^i)}^{1-\theta_0} \|v - u\|_{C^2(\Sigma_q^i)}^{\theta_0} \leq C(M, \gamma) d_q^{(1-2\theta_0)/2}$$

is bounded independently of q and i. Consequently, $v \in C^{1,\theta_0}(\overline{\Sigma}_\epsilon^+)$. Besides, for $(x, t) \in \Sigma_q^i$, we have $t \sim d_q \sim \rho^2(x, t)$. Therefore, from (5.3) and (5.7), we get

$$|\nabla \rho(x, t)| \leq C(M, \gamma) \rho(x, t)^{-1} \leq C(M, \gamma) \rho(x, t)^{1 - \frac{1}{\theta_0}},$$

$$|\nabla^2 v(x, t)| \leq C(M, \gamma) \rho(x, t)^{-1} \leq C(M, \gamma) \rho(x, t)^{1 - \frac{1}{\theta_0}},$$

[3]In fact, one could also exploit the fact that the codimension $m - n \geq 2n_*$ to perform the steps in k simultaneously as well. This would lead to an improved bound in (5.7), but this is not needed for our purpose.

Similarly,

$$|\mathcal{E}(x,t)| \le C(M,\gamma)K^{-1}\rho^2(x,t),$$
$$|\nabla\mathcal{E}(x,t)| \le C(M,\gamma)K^{2N\tilde{N}-1}.$$

Let

$$h = -\frac{\mathcal{E}}{\alpha\rho^2},$$

so that

$$g - v^\sharp e = \alpha\rho^2 g - \mathcal{E} = \alpha\rho^2(g+h),$$

and then

$$\begin{aligned}|h(x,t)| &\le C(M,\gamma)(\alpha K)^{-1} < \frac{\sigma_0}{4^{n+1}},\\ |\nabla h(x,t)| &\le C(M,K,\gamma)(\alpha^{1/2}\rho(x,t))^{-2} + C(M,\gamma)(\alpha K)^{-1}\rho(x,t)^{-\frac{1}{\theta_0}}\\ &\le C(M,K,\gamma)(\alpha^{1/2}\rho(x,t))^{-\frac{1}{\theta_0}},\end{aligned}$$

provided K is taken large enough depending on M,γ,α,σ_0.

Step 3. Isometric extension. By the construction of v, we therefore have

$$g - v^\sharp e = \alpha\rho^2(g+h)$$

on Σ_ϵ^+, v is isometric on Σ, and additionally $v = u$, $dv = du$ on Σ. Thus in particular,

$$dv(\nu) = du(\nu) = \mu$$

along Σ, and therefore (5.2) holds with v replacing u. Besides, we have $\alpha^{1/2}\rho \le \frac{1}{4}$ and

$$\begin{aligned}|\nabla(\alpha^{1/2}\rho)| &\le A(\alpha^{1/2}\rho)^{1-\frac{1}{\theta_0}}, \quad & |\nabla^2 v| &\le A(\alpha^{1/2}\rho)^{1-\frac{1}{\theta_0}},\\ |h| &\le \frac{\sigma_0}{4^{n+1}}, & |\nabla h| &\le A(\alpha^{1/2}\rho)^{-\frac{1}{\theta_0}}.\end{aligned}$$

Now fix a triangulation of Σ by $(n-1)$-simplices, such that every simplex is contained in a single chart V_i. Given any $(n-1)$-simplex Δ^{n-1}, we can subdivide the product $\Delta^{n-1} \times [0,\epsilon]$ in a standard way (see, for example, [21]) into a number of n-simplices. We then use the map F from (1.3) to obtain a triangulation $\mathcal{T}$ of $\bar{\Sigma}_\epsilon^+$.

Now set $S = \Sigma \cup \mathcal{V}$, where $\mathcal{V}$ is the vertex set of the triangulation. Then Σ and S satisfy Condition 4.1, and therefore we can apply Proposition 4.2 to obtain a new adapted short embedding with respect to S. Iterating the construction, i.e., setting

$$S_0 = S, \quad S_k = \Sigma \cup T_k,$$

where T_k is the union of the k-faces of the triangulation, and

$$\Sigma_k = S_{k-1}$$

for $k = 1, \dots, n$, we finally end up with a adapted short embedding with respect to $S_n = \bar{\Sigma}_\epsilon^+$, i.e., $\bar{u}\colon \bar{\Sigma}_\epsilon^+ \to \mathbb{R}^m$ is an isometric embedding. It holds $\bar{u} \in C^{1,\theta'}(\bar{\Sigma}_\epsilon^+, \mathbb{R}^m)$ for

$$\theta' = \theta_0 b^{-2n},$$

where $b > 1$ is arbitrary. Since $\theta_0 < \frac{1}{2}$ is arbitrary as well, it follows that we can achieve any regularity $C^{1,\theta}$ for $\theta < \frac{1}{2}$.

Finally, $\bar{u} = v = f$ on Σ, so $\bar{u}$ extends f. Moreover, $d\bar{u} = dv$ on Σ, so that also

$$\langle d\bar{u}(\nu), \bar{L}(X, X)\rangle = \langle dv(\nu), \bar{L}(X, X)\rangle > L(X, X)$$

for any tangent vector X to Σ, finishing the proof.

6. Proof of Theorem 1.2

We will concentrate on the case of immersions. The extension to embeddings is straightforward and follows well-established strategies (see [16, 31, 33]).

With Proposition 4.2 at our disposal, the strategy for proving Theorem 1.2 for immersions is clear: we perform an induction on dimension on the skeleta of a given regular triangulation of $\mathcal{M}$.

As in Section 4, we fix a finite atlas of charts $\{\Omega_k\}$ on $\mathcal{M}$ such that on every chart

$$\gamma^{-1}\mathrm{Id} \le G \le \gamma\mathrm{Id} \quad \text{and} \quad \operatorname{osc}_{\Omega_k} G \le \sigma_0(\gamma)/2$$

for some $\gamma > 1$, where $\sigma_0(\gamma)$ is the constant given in Proposition 4.1. In addition, fix a triangulation $\mathcal{T}$ on $\mathcal{M}$ whose skeleta consist of a finite union of C^1 submanifolds, such that each triangle $T \in \mathcal{T}$ is contained in a single chart.

We first take any C^∞ embedding of $\mathcal{M}$ in $\mathbb{R}^{n+2n_*}$. Then we change a scale of such embedding such that the resulting immersion, which we denote by u, is short. By compactness of $\mathcal{M}$ we may also make u strictly short, i.e.,

$$g - u^\sharp e > 0$$

on $\mathcal{M}$ in the sense of quadratic forms. Next, we will start our inductive construction as in [11]. In the first step, we recall the construction of an adapted short immersion $\tilde{u}$ of $\mathcal{M}$ with respect to $\Sigma = \emptyset$.

Proposition 6.1. *Let $u \in C^2(\mathcal{M};\mathbb{R}^{n+2n_*})$ be a strictly short immersion. There exists $0 < \delta^* \le 1/8$ and $A^* \ge 1$, depending on u and g, such that for any $A \ge A^*$, there exists a strictly short immersion $\widetilde{u}$ and associated $\widetilde{h}$ with*

$$g - \widetilde{u}^\sharp e = \delta^*(g + \widetilde{h})$$

with

$$\frac{1}{2} g \le \widetilde{u}^\sharp e \le g,$$

and such that the following estimates hold:

$$\|\widetilde{u} - u\|_0 \le \delta^* A^{-\alpha^*}, \quad \|\widetilde{u}\|_2 \le A, \tag{6.1}$$

$$\|\widetilde{h}\|_0 \le A^{-\alpha^*}, \qquad \|\widetilde{h}\|_1 \le A. \tag{6.2}$$

The exponent α^* only depends on $\mathcal{M}$.

Next, fix $\theta_0 < 1/2$ and $\epsilon > 0$. Set $u_0 = \widetilde{u}$, $h_0 = \widetilde{h}$ as obtained from Proposition 6.1 with $A = A_0$ sufficiently large (to be determined below), and also $\widetilde{\rho}^2 = \delta^*$. From (6.1), we deduce

$$\|u - u_0\|_0 \le \frac{\varepsilon}{4} \tag{6.3}$$

by assuming A_0 is sufficiently large. From (6.1)–(6.2), we further have

$$\|\nabla^2 u_0\|_0 \le A_0 \le A_0(\delta^*)^{\frac{1}{2} - \frac{1}{2\theta_0}},$$

$$\|h_0\|_0 \le A_0^{-\alpha^*} \le \frac{\sigma_0}{4^{n+1}},$$

$$\|\nabla h_0\|_0 \le A_0^{1-\alpha^*} \le A_0(\delta^*)^{\frac{\alpha_0}{2} - \frac{1}{2\theta_0}},$$

where σ_0 is in Proposition 4.1. Therefore we deduce that u_0 is an adapted short immersion with respect to the empty set $\Sigma_0 = \emptyset$ with exponent θ_0, and furthermore the estimates (4.19) are satisfied by (u_0, ρ_0, h_0) with (A, θ) replaced by (A_0, θ_0).

For any $b > 1$, we can apply Proposition 4.2 to obtain a C^{1,θ_1} adapted short immersion (u_1, ρ_1, h_1) with respect to $\Sigma_1 = \mathcal{V}$, where $\mathcal{V}$ is the vertex set of the triangulation $\mathcal{T}$ and such that (4.19) and (4.20) hold with

$$A_1 = A_0^{b^2}, \quad \theta_1 = \frac{\theta_0}{b^2}.$$

We then continue this process along the skeleta $\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_{n+1} = \mathcal{M}$ and obtain adapted short immersions (u_j, ρ_j, h_j) with respect to Σ_j, $j = 1, 2, \ldots, n+1$, with

$$A_{j+1} = A_j^{b^2}, \quad \theta_{j+1} = \frac{\theta_j}{b^2}.$$

After $n+1$ steps, we finally obtain a global $C^{\theta_{n+1}}$ isometric immersion $v := u_{n+1}$ of $\mathcal{M}$ with

$$\theta_{n+1} = b^{-2n-2}\theta_0.$$

Note that for any fixed θ_0, taking $b \to 1$, we will have $\theta_{n+1} \to \theta_0$. Thus, for any $\theta' < \theta_0$, there exists a choice of $b > 1$ so that $\theta' < \theta_{n+1} < \theta_0$. In this way we can achieve any exponent $\theta < \frac{1}{2}$. Finally, observe that (recalling (6.3))

$$\begin{aligned}\|u - v\|_0 &\le \|u - u_0\|_0 + \sum_{j=0}^{n} \|u_{j+1} - u_j\|_0 \\ &\le \varepsilon/4 + \sum_{j=0}^{n} A_j^{-1/2} \le \varepsilon/4 + (n+1)A_0^{-1/2} \le \varepsilon\end{aligned}$$

by choosing A_0 sufficiently large. This completes the proof of Theorem 1.2. ∎

A. Proof of Lemma 3.5

Step 1. Without loss of generality we assume $\Omega = B_1(0)$. In a first step we construct a family of vector fields $\zeta_1, \dots, \zeta_{m-n}$ which satisfies (3.5) on a small neighborhood of the origin. To do this, pick orthonormal vectors $\xi_1, \dots, \xi_{m-n} \in \mathbb{R}^m \setminus dv_0(T_0\overline{B}_1)$. We then set

$$\nu_i = \xi_i - \sum_{j=1}^{n} r_{ij}\partial_j v,$$

where r_{ij} are chosen to guarantee $\langle \nu_i, \partial_k v\rangle = 0$ for every i and k. This is possible since $\nabla v^T \nabla v \ge \gamma^{-1} Id$. Indeed, denote $b_{ik} = \langle \xi_i, \partial_k v\rangle$ and observe that $\langle \nu_i, \partial_k v\rangle = 0$ for all i, k is equivalent to

$$R \cdot \nabla v^{\mathsf{T}} \nabla v = B,$$

where R and B are the $(m-n) \times n$ matrices with entries $R_{ij} = r_{ij}$ and $B_{ij} = b_{ij}$. We can then simply set

$$R = B \cdot (\nabla v^{\mathsf{T}} \nabla v)^{-1}.$$

We claim that in a neighborhood of the origin the family $\{\nu_i\}_{i=1}^{m-n}$ is linearly independent and therefore constitutes a frame for the normal bundle. A Gram–Schmidt process will then produce the desired vector fields.

To show the claim, we write

$$(\nabla v^{\mathsf{T}} \nabla v)^{-1}_{ij} = (\det \nabla v^{\mathsf{T}} \nabla v)^{-1} P_{ij}(\nabla v),$$

where $P_{ij}(\nabla v)$ is a polynomial in the arguments $\partial_k v^l$. Observe that assumption (3.4) implies $[v]_1 \leq C(\gamma)$. Hence, with Lemma 3.1 and assumption (3.4) we find

$$\|r_{ij}\|_0 \leq C(\gamma)[v]_2\varepsilon,$$

where we used that

$$|b_{ik}| = |\langle \xi_i, \partial_k v(x)\rangle| = |\langle \xi_i, \partial_k v(x) - \partial_k v(0)\rangle| \leq [v]_2\varepsilon$$

for $x \in \overline{B}_\varepsilon$. With this estimate we find

$$\|\langle \nu_i, \nu_j\rangle - \delta_{ij}\|_{C^0(\bar{B}_\varepsilon)} = \|\langle \nu_i, \nu_j\rangle - \langle \xi_i, \xi_j\rangle\|_{C^0(\bar{B}_\varepsilon)} \leq C(n,\gamma)[v]_2\varepsilon.$$

Therefore, if $\varepsilon \equiv \varepsilon(n, \gamma, [v]_2) > 0$ is small enough, the vector fields $\nu_1, \ldots, \nu_{m-n}$ are linearly independent. Before continuing with the Gram–Schmidt process, observe the following estimates for $0 < l \leq N$:

$$[r_{ij}]_l \leq C_l\big(C(\gamma)[v]_{l+1} + C(\gamma)[b_{ij}]_l\big) \leq C_l(\gamma)[v]_{l+1},$$

thanks to the Leibniz rule. Therefore, we have the same estimates for the vector fields

$$[\nu_i]_l \leq C_l(\gamma)[v]_{l+1}. \tag{A.1}$$

Now we set

$$\zeta_1 = \frac{\nu_1}{|\nu_1|},$$

and observe that for small enough $\varepsilon > 0$, we have $|\nu_1| \geq \frac{1}{2}$ so that, thanks to Lemma 3.1 and (A.1), $\zeta_1 \in C^N(\overline{B}_\varepsilon)$ with

$$[\zeta_1]_{C^l(\bar{B}_\varepsilon)} \leq C_l[\nu_1]_{C^l(\bar{B}_\varepsilon)} \leq C_l(\gamma)[v]_{C^l(\bar{B}_\varepsilon)}$$

for all $0 \leq l \leq N$. Moreover, on $\overline{B}_\varepsilon$ we have

$$|\zeta_1 - \xi_1| \leq \frac{2|\nu_1 - \xi_1|}{|\nu_1|} \leq C(\gamma)[v]_2\varepsilon.$$

Now we assume $\zeta_1, \ldots, \zeta_{k-1}$ are already constructed with

$$\begin{aligned}
&\langle \zeta_i, \zeta_j\rangle = \delta_{ij},\\
&\nabla v \cdot \zeta_i = 0,\\
&[\zeta_i]_{l,\beta} \leq C_l(\gamma)[v]_{l+1} \quad \text{for all } 0 \leq l \leq N,
\end{aligned}$$

on $\overline{B}_\varepsilon$, and in addition

$$|\zeta_i - \xi_i| \leq C(\gamma)[v]_2\varepsilon. \tag{A.2}$$

We then set

$$\theta_k = \nu_k - \sum_{j=1}^{k-1} \langle \nu_k, \zeta_j \rangle \zeta_j, \quad \zeta_k = \frac{\theta_k}{|\theta_k|}.$$

It remains to show that ζ_k satisfies (3.5) and (A.2). Observe that

$$\langle \nu_k, \zeta_j \rangle = \langle \nu_k - \xi_k, \zeta_j \rangle + \langle \xi_k, \zeta_j - \xi_j \rangle,$$

so that $|\langle \nu_k, \zeta_j \rangle| \leq C(\gamma)[v]_2 \varepsilon$ on $\overline{B}_\varepsilon$ and by the Leibniz rule also

$$[\langle \nu_k, \zeta_j \rangle]_{C^l(\overline{B}_\varepsilon)} \leq C_l(\gamma)[v]_{C^{l+1}(\overline{B}_\varepsilon)}.$$

In particular, $|\theta_k| \geq \frac{1}{4}$ for ε small enough, and hence, with Lemma 3.1,

$$[\zeta_k]_{C^l(\overline{B}_\varepsilon)} \leq C_l(\gamma)[v]_{C^{l+1}(\overline{B}_\varepsilon)}.$$

Therefore, ζ_k satisfies (3.5). Since moreover

$$|\zeta_k - \xi_k| \leq \frac{2|\theta_k - \xi_k|}{|\theta_k|} \leq C\big(|\theta_k - \nu_k| + |\nu_k - \xi_k|\big) \leq C(\gamma)[v]_2 \varepsilon,$$

the first step is completed.

Step 2. In this step we show that one can continue the vector fields to maps on $\overline{B}_1$ satisfying the same constraints. Consider the set

$$R = \big\{\rho \in [0,1] : \text{there exist } \zeta_1, \ldots, \zeta_{m-n} \in C^{N,\alpha}(\overline{B}_\rho) \text{ satisfying (3.5) on } \overline{B}_\rho\big\}.$$

As we saw in Step 1, R is non-empty. Set $\overline{\rho} = \sup R$. We claim that $\overline{\rho} \in R$. To see this, let $\rho_q \uparrow \overline{\rho}$ and fix the corresponding families of vector fields ζ_i^q. Now assume that there exists $\delta = \delta(\gamma, v) > 0$ such that each ζ_i^q can be extended to a map $\widetilde{\zeta}_i^q \in C^N(\overline{B}_{\sigma_q})$ with

$$[\widetilde{\zeta}_i^q]_{C^l(\overline{B}_{\sigma_q})} \leq C(\gamma)\big(1 + [v]_{l+1}\big), \tag{A.3}$$

where $\sigma_q = \min\{1, \rho_q + \delta\}$. We will prove this fact at the end of this proof in Step 3. With it, we can repeat the procedure of Step 1: We set

$$\nu_i^q = \widetilde{\zeta}_i^q - \sum_{j=1}^{n} r_{ij}^q \partial_j v,$$

where, again, r_{ij}^q are chosen such that every ν_i^q is orthogonal to v. We need to show that, for δ small enough, ν_i^q are linearly independent to perform the Gram–Schmidt process. Set $b_{ik}^q = \langle \widetilde{\zeta}_i^q, \partial_k v \rangle$ and observe that, for $\rho_q < |x| \leq \sigma_q$,

$$b_{ik}^q(x) = \Big\langle \widetilde{\zeta}_i^q(x) - \zeta_i^q\Big(\rho_q \frac{x}{|x|}\Big), \partial_k v(x) \Big\rangle + \Big\langle \zeta_i^q\Big(\rho_q \frac{x}{|x|}\Big), \partial_k v(x) - \partial_k v\Big(\rho_k \frac{x}{|x|}\Big) \Big\rangle.$$

Thus,

$$|b_{ik}^q| \le C(\gamma)[\tilde{\zeta}_i^q]_{C^1(\bar{B}_{\sigma_q})}\delta + [v]_2\delta \le C\big(\gamma, [v]_2\big)\delta$$

thanks to (A.3). Thus, as before it follows

$$|r_{ij}^q| \le C\big(\gamma, [v]_2\big)\delta.$$

Now we write

$$\langle \nu_i^q, \nu_j^q \rangle = \langle \tilde{\zeta}_i^q, \tilde{\zeta}_j^q \rangle + E,$$

where E is an error term with $|E| \le C(\gamma, [v]_2)\delta$ thanks to the estimate on r_{ij}^q. We expand

$$\begin{aligned}\langle \tilde{\zeta}_i^q, \tilde{\zeta}_j^q \rangle &= \Big\langle \tilde{\zeta}_i^q - \zeta_i^q\Big(\rho_q \frac{x}{|x|}\Big), \tilde{\zeta}_j^q \Big\rangle \\ &\quad + \Big\langle \tilde{\zeta}_i^q\Big(\rho_q \frac{x}{|x|}\Big), \tilde{\zeta}_j^q - \zeta_j^q\Big(\rho_q \frac{x}{|x|}\Big)\Big\rangle + \Big\langle \zeta_i^q\Big(\rho_q \frac{x}{|x|}\Big), \zeta_j^q\Big(\rho_q \frac{x}{|x|}\Big)\Big\rangle \\ &= \delta_{ij} + \tilde{E},\end{aligned}$$

where again $|\tilde{E}| \le C(\gamma, [v]_2)\delta$. Hence, for $\delta(\gamma, v)$ small enough, ν_i^q are linearly independent. The estimates (A.1) can be derived in the same way. As in Step 1, we can then apply the Gram–Schmidt process to generate the vector fields $\bar{\zeta}_i^q$ satisfying (3.5) on $\bar{B}_{\sigma_q}$. Consequently, $\sigma_q \in R$. By definition, $\bar{\rho} \ge \sigma_q$ for all q. Letting $q \to \infty$, we find $\rho \ge \min\{1, \rho + \delta\}$, which shows $\sigma_q = 1$ for q large enough. Hence, $1 \in R$, which completes Step 2.

Step 3. In this step we show that there exists a $\delta \equiv \delta(\gamma, v) > 0$ such that any map $\zeta \in C^N(\bar{B}_\rho)$ with

$$[\zeta]_{C^l(\bar{B}_\rho)} \le C_l(\gamma)\big(1 + [v]_{C^{l+1}(\bar{B}_1)}\big) \tag{A.4}$$

can be extended to a map $\tilde{\zeta} \in C^N(\mathbb{R}^n)$ such that

$$[\tilde{\zeta}]_{C^l(\bar{B}_\sigma)} \le C_l(\gamma)\big(1 + [v]_{C^{l+1}(\bar{B}_1)}\big), \tag{A.5}$$

where $\sigma = \min\{1, \rho + \delta\}$ and $C_l(\gamma)$ might differ from the constant in (A.4).

The existence of such an extension is a classical fact, originally due to Whitney [34]. However, we could not find a reference stating the estimates (A.5), which is why we redo the argument in the following.

For $k \in \mathbb{N}$, $y \in \bar{B}_\rho$ and $x \in \mathbb{R}^n$, we denote by $T_y^k \zeta(x)$ the k-th order Taylor polynomial of ζ around y at x, i.e.,

$$T_y^k \zeta(x) = \sum_{|\beta| \le k} \frac{\partial^\beta \zeta(y)}{\beta!}(x - y)^\beta,$$

with the usual conventions concerning the multi-indices β. Let χ_j be a partition of unity subordinate to decomposition of $\mathbb{R}^n \setminus \overline{B}_\rho$ such that no point is in the support of more than $M(n)$ functions χ_j, the diameter of the support of χ_j is at most twice its distance to $\overline{B}_\rho$, and

$$|\partial^\beta \chi_j(x)| \le C_\beta d(x)^{-|\beta|}$$

for $x \in \mathbb{R}^n \setminus \overline{B}_\rho$, where $d(x) = \mathrm{dist}(x, \overline{B}_\rho)$. For a proof we refer to [23, Lemma 2.3.7].

We then set $\tilde{\zeta}(x) = \zeta(x)$ for $x \in \overline{B}_\rho$, and

$$\tilde{\zeta}(x) = \sum_j \chi_j(x) T^N_{y_j}\zeta(x)$$

otherwise, where $y_j \in \partial\overline{B}_\rho$ minimizes the distance to the support of χ_j. In [23, Theorem 2.3.6] it is shown that $\tilde{\zeta} \in C^N$ with $\partial^\beta\tilde{\zeta} = \partial^\beta\zeta$ on $\overline{B}_\rho$ for every $|\beta| \le k$. We want to show that $\tilde{\zeta}$ also satisfies (A.5).

Observe first that if $x \in \mathrm{supp}\,\chi_j$ then

$$|x - y_j| \le \mathrm{diam}(\mathrm{supp}\,\chi_j) + \mathrm{dist}(\mathrm{supp}\,\chi_j, \overline{B}_\rho) \le 3d(x).$$

Hence, for such x, we have

$$\begin{aligned} |\partial^\beta T^N_{y_j}\zeta(x)| &= \left| \sum_{|\mu| \le N - |\beta|} \frac{\partial^{\beta+\mu}\zeta(y_j)}{\mu!}(x - y_j)^\mu \right| \\ &\le [\zeta]_{|\beta|} + \sum_{i=1}^{N-|\beta|} d(x)^i [\zeta]_{|\beta|+i} \\ &\le [\zeta]_{|\beta|} + C_N(\gamma) d(x)\big(1 + \|v\|_{N+1}\big), \end{aligned}$$

for any multi-index β with $0 \le |\beta| \le N$. Consequently, if $\rho < |x| \le \rho + \delta$ we find

$$|\partial^\beta T^N_{y_j}\zeta(x)| \le [\zeta]_{|\beta|} + 1, \tag{A.6}$$

if δ is chosen small enough depending on γ and v. In particular, this shows the estimate (A.5) for $l = 0$ in view of (A.4). Now fix $1 \le l \le N$ and multi-indices α, β with $|\alpha| + |\beta| = l$. We want to show the estimate

$$\left| \sum_j \partial^\alpha \chi_j \partial^\beta T^N_{y_j}\zeta \right| \le C_l(\gamma)\big(1 + [v]_{l+1}\big).$$

If $\alpha = 0$ this follows from the estimate (A.6) together with the assumption (A.4). Therefore, we can assume $|\alpha| \ge 1$. We write

$$\sum_j \partial^\alpha \chi_j(x) \partial^\beta T^N_{y_j}\zeta(x) = \sum_j \partial^\alpha \chi_j(x) \sum_{|\mu| \le N-|\beta|} \frac{\partial^{\beta+\mu}\zeta(y_j)}{\mu!}(x - y_j)^\mu$$

$$
\begin{aligned}
&= \sum_j \partial^\alpha \chi_j(x) \sum_{|\mu|<|\alpha|} \frac{\partial^{\beta+\mu}\zeta(y_j)}{\mu!}(x-y_j)^\mu \\
&\qquad + \sum_j \partial^\alpha \chi_j(x) \sum_{|\alpha|\le|\mu|\le N-|\beta|} \frac{\partial^{\beta+\mu}\zeta(y_j)}{\mu!}(x-y_j)^\mu \\
&=: \mathrm{I}(x) + \mathrm{II}(x)
\end{aligned}
$$

Recall that $|\partial^\alpha \chi_j(x)| \le C d(x)^{-|\alpha|}$. Since $|\alpha| = l - |\beta|$, we can estimate the second sum by

$$
\begin{aligned}
|\mathrm{II}(x)| &\le C d(x)^{-|\alpha|}\left([\zeta]_l d(x)^{|\alpha|} + \sum_{i=1}^{N-|\beta|} d(x)^{|\alpha|+i}[\zeta]_{l+i}\right) \\
&\le C[\zeta]_l + C_N(\gamma) d(x)\big(1 + \|v\|_{N+1}\big) \le C_l(\gamma)\big(1 + [v]_{l+1}\big)
\end{aligned}
$$

if $\rho < |x| \le \rho + \delta$ and δ small enough, i.e., $d(x) \le \delta$, thanks to (A.4). To estimate $\mathrm{I}(x)$ we set $x^* = \rho\frac{x}{|x|}$ and observe that, by Taylor's theorem,

$$
\partial^{\beta+\mu}\zeta(y_j) - T_{x*}^{|\alpha|-|\mu|-1}\partial^{\beta+\mu}\zeta(y_j) = \sum_{|\tilde\mu|=|\alpha|-|\mu|} \frac{\partial^{\tilde\mu+\beta+\mu}\zeta(\xi)}{\tilde\mu!}(y_j - x^*)^{\tilde\mu},
$$

for some $\xi \in [x^*, y_j]$. Now $|x^* - y_j| \le d(x) + |x - y_j| \le 4d(x)$, so that

$$
\left|\partial^{\beta+\mu}\zeta(y_j) - T_{x*}^{|\alpha|-|\mu|-1}\partial^{\beta+\mu}\zeta(y_j)\right| \le C[\zeta]_{|\alpha|+|\beta|} d(x)^{|\alpha|-|\mu|} = C[\zeta]_l d(x)^{|\alpha|-|\mu|}.
$$

since by assumption $|\alpha| + |\beta| = l$. Therefore, it holds that

$$
\left|\sum_j \partial^\alpha \chi_j(x) \sum_{|\mu|<|\alpha|} \frac{(x-y_j)^\mu}{\mu!}\big(\partial^{\beta+\mu}\zeta(y_j) - T_{x^*}^{|\alpha|-|\mu|-1}\partial^{\beta+\mu}\zeta(y_j)\big)\right| \le C[\zeta]_l. \tag{A.7}
$$

To conclude it suffices to observe

$$
\begin{aligned}
&\sum_{|\mu|\le|\alpha|-1} \frac{(x-y_j)^\mu}{\mu!} T_{x^*}^{|\alpha|-|\mu|-1}\partial^{\beta+\mu}\zeta(y_j) \\
&\quad = \sum_{|\mu|\le|\alpha|-1}\left(\sum_{|\tilde\mu|\le|\alpha|-|\mu|-1} \frac{\partial^{\tilde\mu+\beta+\mu}\zeta(x^*)}{\tilde\mu!\mu!}(x-y_j)^\mu (y_j - x^*)^{\tilde\mu}\right) \\
&\quad = \sum_{|\mu|\le|\alpha|-1} \frac{\partial^{\beta+\mu}\zeta(x^*)}{\mu!}\left(\sum_{\tilde\mu\le\mu} \frac{\mu!}{\tilde\mu!(\mu-\tilde\mu)!}(x-y_j)^\mu (y_j - x^*)^{\mu-\tilde\mu}\right) \\
&\quad = \sum_{|\mu|\le|\alpha|-1} \frac{\partial^{\beta+\mu}\zeta(x^*)}{\mu!}(x-x^*)^\mu = T_{x^*}^{|\alpha|-1}\partial^\beta\zeta(x).
\end{aligned} \tag{A.8}
$$

Since $\sum_j \partial^\alpha \chi_j(x) = 0$, we can simply subtract $T_{x^*}^{|\alpha|-1} \partial^\beta \zeta(x)$ to find

$$|\mathrm{I}(x)| = \Big| \sum_j \partial^\alpha \chi_j(x) \Big(\sum_{|\mu| \leq |\alpha|-1} \frac{\partial^{\beta+\mu} \zeta(y_j)}{\mu!} (x - y_j)^\mu - T_{x^*}^{|\alpha|-1} \partial^\beta \zeta(x) \Big) \Big| \leq C [\zeta]_l$$

in view of (A.8) and (A.7), which, thanks to (A.4), finishes the proof.

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