Weak commutativity, virtually nilpotent groups, and Dehn functions

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Abstract. The group $\mathfrak{X}(G)$ is obtained from $G * G$ by forcing each element g in the first free factor to commute with the copy of g in the second free factor. We make significant additions to the list of properties that the functor $\mathfrak{X}$ is known to preserve. We also investigate the geometry and complexity of the word problem for $\mathfrak{X}(G)$. Subtle features of $\mathfrak{X}(G)$ are encoded in a normal abelian subgroup $W < \mathfrak{X}(G)$ that is a module over $\mathbb{Z}Q$, where $Q = H_1(G, \mathbb{Z})$. We establish a structural result for this module and illustrate its utility by proving that $\mathfrak{X}$ preserves virtual nilpotence, the Engel condition, and growth type – polynomial, exponential, or intermediate. We also use it to establish isoperimetric inequalities for $\mathfrak{X}(G)$ when G lies in a class that includes Thompson's group F and all non-fibred Kähler groups. The word problem is soluble in $\mathfrak{X}(G)$ if and only if it is soluble in G. The Dehn function of $\mathfrak{X}(G)$ is bounded below by a cubic polynomial if G maps onto a non-abelian free group.

1. Introduction

In [40], Saïd Sidki defined the weak commutativity functor $\mathfrak{X}$, which assigns to a group G the group $\mathfrak{X}(G)$ obtained from the free product $G * G$ by forcing each element g in the first free factor to commute with the copy of g in the second free factor. More precisely, taking a second copy $\overline{G}$ of G and fixing an isomorphism $g \mapsto \overline{g}$, one defines $\mathfrak{X}(G)$ to be the quotient of the free product $G * \overline{G}$ by the normal subgroup $\langle\!\langle [g, \overline{g}] : g \in G \rangle\!\rangle$. At first glance, it seems that if G is infinite then one is likely to require infinitely many relations to define $\mathfrak{X}(G)$, even if G is finitely presented. But in [10], we proved that if G is finitely presented, then so too is $\mathfrak{X}(G)$. Thus, Sidki's construction provides an intriguing new source of finitely presented groups.

Our purpose in this article is two-fold: we investigate the complexity of the word problem for $\mathfrak{X}(G)$, with emphasis on Dehn functions and isoperimetric inequalities,

and we make significant additions to the list of properties that the functor $\mathfrak{X}$ is known to preserve.

Previous investigations have already shown that the functor $\mathfrak{X}$ preserves certain interesting classes of groups. Sidki himself [40, Theorem C] showed that if G lies in any of the following classes then $\mathfrak{X}(G)$ lies in the same class: finite π-groups, where π is a set of primes; finite nilpotent groups; solvable groups. Gupta, Rocco and Sidki [24] proved that the class of finitely generated nilpotent groups is closed under $\mathfrak{X}$, and Lima and Oliveira [32] proved the same for polycyclic-by-finite groups. In [31], Kochloukova and Sidki proved that if G is a soluble group of type FP_∞, then $\mathfrak{X}(G)$ is a soluble group of type FP_∞, but in [10] we proved that when F is a non-abelian free group, $\mathfrak{X}(F)$ is not of type FP_∞.

The emergence of unexpected behaviour in $\mathfrak{X}(G)$ is restricted almost entirely to groups with infinite abelianisation, as the following pair of contrasting results illustrates.

Proposition A. *If G is a word-hyperbolic group that is perfect, then $\mathfrak{X}(G)$ is biautomatic. In particular, $\mathfrak{X}(G)$ is of type* FP_∞ *and satisfies a quadratic isoperimetric inequality.*

Theorem B. *Let G be a finitely presented group. If G maps onto a non-abelian free group, then the Dehn function of $\mathfrak{X}(G)$ is bounded below by a cubic polynomial, and $\mathfrak{X}(G)$ has a subgroup of finite index $X_0 < \mathfrak{X}(G)$ with $H_3(X_0, \mathbb{Q})$ infinite-dimensional.*

There are many hyperbolic groups G that are perfect but have subgroups of finite index $G_0 < G$ such that G_0 maps onto   a non-abelian free group: hyperbolic 3-manifold groups that are integer homology spheres have this property, as do free products of finite perfect groups, and the small-cancellation groups obtained by applying a suitable version of the Rips construction [38] to perfect groups. Thus, Proposition A and Theorem B underscore how radically $\mathfrak{X}(G)$ can change when one replaces G by a subgroup of finite index.

The remainder of our results rely on a structural result concerning the subgroups L and $W = W(G) < \mathfrak{X}(G)$ that we shall now describe. There are natural surjections

$$\mathfrak{X}(G) \to G \times \bar{G} \quad \text{and} \quad \mathfrak{X}(G) \to G,$$

the latter defined by sending both g and $\bar{g}$ to g. The kernels of these maps, D and L, commute, and the second map splits to give $\mathfrak{X}(G) = L \rtimes G$. Sidki [40] identified the crucial role that the abelian group $W := D \cap L$ and the exact sequence

$$1 \to W \to \mathfrak{X}(G) \overset{\rho}{\to} G \times G \times G$$

play in understanding $\mathfrak{X}(G)$, where

$$\rho(g) = (g, g, 1) \quad \text{and} \quad \rho(\overline{g}) = (1, g, g).$$

(We sometimes write $W(G)$ instead of W.) If G is finitely presented and perfect, then $W < \mathfrak{X}(G)$ is finitely generated and central. But for finitely presented groups that are not perfect, W need not be finitely generated and the conjugation action of $\mathfrak{X}(G)$ on W can be difficult to understand. As $DL < \mathfrak{X}(G)$ acts trivially, the action of $\mathfrak{X}(G)$ factors through $Q := \mathfrak{X}(G)/DL = G/G'$. (Throughout, we use the standard notation $G' = [G, G]$.) Thus, W is a $\mathbb{Z}Q$-module.

The action of $\mathfrak{X}(G)$ on L by conjugation induces an action of G on L/L' that factors through Q, and our analysis of W as a $\mathbb{Z}Q$-module begins with the observation that this action is *nilpotent*, i.e., there exists an integer d such that $[\ell, g_1, \dots, g_d] \in L'$ for all $l \in L$ and all $g_i \in G$; see Proposition 4.1.

We then consider the $\mathbb{Z}Q$-module

$$M = ((G'/G'') \otimes_{\mathbb{Z}} (G'/G''))_{Q_0},$$

where the action of Q on the first factor is induced by conjugation in G while the action on the second factor is trivial, and the co-invariants in the definition of M are taken with respect to the action of

$$Q_0 = \{(q, q^{-1}) \mid q \in Q\} \le (G/G') \times (G/G')$$

by conjugation on $(G'/G'') \otimes_{\mathbb{Z}} (G'/G'')$. The apparent asymmetry in the Q-action is an illusion: because we have factored out the action of Q_0, the action of $Q = G/G'$ on M can equally be described as being trivial on the first factor of the tensor product and by conjugation on the second factor.

The following theorem explains why M plays an important role in the understanding of W. Its proof relies heavily on homological methods.

Theorem C. *Let G be a finitely generated group, let Q be the abelianisation of G, and consider W as a $\mathbb{Z}Q$-module in the manner described above. Then there exists a submodule $W_0 < W$ such that*

(1) *W_0 is a subquotient of the $\mathbb{Z}Q$-module $M = ((G'/G'') \otimes_{\mathbb{Z}} (G'/G''))_{Q_0}$;*

(2) *the action of Q on W/W_0 is nilpotent.*

As a first illustration of the utility of these structural results (Proposition 4.1 and Theorem C), we add to the list of properties that $\mathfrak{X}(G)$ is known to preserve. We remind the reader that $\mathfrak{X}(G)$ can change radically when one replaces G by a subgroup of finite index.

Theorem D. *If G is finitely generated and virtually nilpotent, then $\mathfrak{X}(G)$ is virtually nilpotent.*

The virtual nilpotency class of $\mathfrak{X}(G)$ will in general be greater than that of G. This is already the case for virtually abelian groups [24].

Recall that a group G is *n-Engel* if $[a, b, \ldots, b] = 1$ for all $a, b \in G$, where b appears n times in the left-normed commutator. We shall deduce the following result from Theorem C. Our proof relies on Gruenberg's result that the maximal metabelian quotient G/G'' of a finitely generated Engel group is nilpotent [21].

Theorem E. *If G is a finitely generated n-Engel group, then $\mathfrak{X}(G)$ is m-Engel for $m = n + d + s + 3$, where d is the nilpotency class of G/G'' and s is the nilpotency class of the action of G on L/L'.*

We return to results about Dehn functions, reminding the reader that the Dehn function of a finitely presented group $G = \langle A \mid R \rangle$ is the least function $\delta_G(n)$ such that any word w in the kernel of $\text{Free}(A) \to G$, with length $|w| \le n$, can be expressed as a product of at most $\delta_G(n)$ conjugates of the relations $r \in R$ and their inverses. The group G is said to satisfy a polynomial isoperimetric inequality if $\delta_G(n)$ is bounded above by a polynomial function of n. See Section 2.2 for more details, including the sense in which $\delta_G(n)$ is independent of the chosen presentation. A finitely presented group has a soluble word problem if and only if its Dehn function is recursive.

Theorem F. *Let G be a finitely presented group. The word problem in $\mathfrak{X}(G)$ is soluble if and only if the word problem in G is soluble.*

Estimating the complexity of the word problem in $\mathfrak{X}(G)$ by means of isoperimetric inequalities is a more subtle task. The proof of the following result relies heavily on the understanding of W as a $\mathbb{Z}Q$-module that is established through Theorem C, as well as results concerning the Dehn functions of subdirect products of groups [14].

Theorem G. *If G is a finitely presented group whose maximal metabelian quotient G/G'' is virtually nilpotent, then there is a polynomial $p(x)$ such that*

$$\delta_G(n) \preceq \delta_{\mathfrak{X}(G)}(n) \preceq p \circ \delta_G(n);$$

in particular, G satisfies a polynomial isoperimetric inequality if and only if $\mathfrak{X}(G)$ satisfies a polynomial isoperimetric inequality (of different degree, in general).

The degree of the polynomial $p(x)$ depends upon the torsion-free rank of W (which is finite since G'/G'' is finitely generated [31]) and upon the torsion-free rank of $G/[G, G]$; it can be estimated by following the proof of Theorem 6.4.

Theorem G applies to many groups of geometric interest. For example, in [13] Delzant proved that for any compact Kähler manifold M that does not fibre (equi-

valently, $\pi_1 M$ does not map onto a surface group), the maximal metabelian quotient of $\pi_1 M$ is virtually nilpotent. The maximal metabelian quotient of the Torelli subgroup of the mapping class group of a surface of high genus is also virtually nilpotent, as is the corresponding subgroup of the outer automorphism group of a free group [16].

Corollary H. *Let G be the fundamental group of a compact Kähler manifold that does not fibre. If G satisfies a polynomial isoperimetric inequality, then so does $\mathfrak{X}(G)$.*

Theorem G is already interesting for groups whose commutator subgroup is perfect. A much-studied group with this property is Thompson's group

$$F = \langle x_0, x_1, \ldots, x_n, \ldots \mid x_i^{x_j} = x_{i+1} \text{ for } j < i \rangle.$$

It is easy to show that F can be finitely presented. Famously, F' is simple. Brown and Geoghegan [12] proved that F is of type FP_∞, and Guba [22] proved that the Dehn function of F is quadratic, building on his earlier work with Sapir [23].

Corollary I. *The Sidki double $\mathfrak{X}(F)$ of Thompson's group F satisfies a polynomial isoperimetric inequality.*

It is also possible to estimate the Dehn function of $\mathfrak{X}(G)$ in some situations where Theorem G does not apply but where one nevertheless has control of W. This is the case, for example, when G is a metabelian Baumslag–Solitar group $\mathrm{BS}(1, m)$ with $m > 1$; Kochloukova and Sidki [31] prove that $W = 1$ in this case, so it follows from Theorem 6.1 that $\mathfrak{X}(G)$ has an exponential Dehn function, since G does.

Related to $\mathfrak{X}(G)$, one has the group $\mathcal{E}(G) = (G, \bar{G} \mid [\mathcal{D}, \mathcal{L}] = 1)$ studied in [33], where $\mathcal{D} = [G, \bar{G}]$ and $\mathcal{L} = \langle\{g^{-1}\bar{g} \mid g \in G\}\rangle$ are subgroups of $G * \bar{G}$. The circumstances under which $\mathcal{E}(G)$ is finitely presented are much more restrictive than for $\mathfrak{X}(G)$; it is necessary but not sufficient that G be finitely presented; see [30]. Correspondingly, the proof of the following result is much more straightforward than that of Theorem G.

Proposition J. *Whenever $\mathcal{E}(G)$ is finitely presented, $\delta_{\mathcal{E}(G)}(n) \preceq \max\{n^4, \delta_G(n)^2\}$.*

In the final section of this paper, we return to the search for classes of groups that are closed under passage from G to $\mathfrak{X}(G)$. Our focus now is on the growth of groups. We remind the reader that the growth function of a group G with finite generating set S counts the number of elements in balls of increasing radius in the word metric d_S. The growth type is polynomial, exponential, or intermediate (i.e., subexponential but faster than any polynomial).

We shall see that the main arguments in the proof of Theorem D can be abstracted to provide the following criterion for closure under $\mathfrak{X}$; the key point is the control on the $\mathbb{Z}Q$-module W provided by Theorem C.

Theorem K. *Let $\mathcal{P}$ be a class of finitely generated groups such that*

- *every metabelian group in $\mathcal{P}$ is virtually nilpotent;*

- *$\mathcal{P}$ is closed under the taking of finitely generated subgroups, quotients, central extensions, extensions by and of finite groups, and finite direct products.*

Then $\mathfrak{X}(G) \in \mathcal{P}$ if and only if $G \in \mathcal{P}$.

If one drops the requirement that $\mathcal{P}$ is closed under direct products, then the conclusion has to be modified: $\mathfrak{X}(G) \in \mathcal{P}$ if and only if $\mathrm{im}(\rho) \in \mathcal{P}$, where $\rho\colon \mathfrak{X}(G) \to G \times G \times G$ is as described above. Both versions of the theorem remain true if the only central extensions under which $\mathcal{P}$ is assumed closed are those with finitely generated kernel. And in either version, it follows from [31] that $W(G)$ is finitely generated as an abelian group for all $G \in \mathcal{P}$.

Gromov [20] proved that groups of polynomial growth are virtually nilpotent, and the converse is straightforward. Thus the first part of the following corollary is covered by Theorem D. The closure properties required in the other cases are discussed in Section 10.

Corollary L. *Let G be a finitely generated group. Then*

(1) *$\mathfrak{X}(G)$ has polynomial growth if and only if G has polynomial growth;*

(2) *$\mathfrak{X}(G)$ has exponential growth if and only if G has exponential growth;*

(3) *$\mathfrak{X}(G)$ has intermediate growth if and only if G has intermediate growth.*

In the context of Theorem K, we shall also discuss the class of finitely generated groups that have the fixed-point property for cones, as studied by Monod in [35] (see Section 10).

This paper is organised as follows. After gathering necessary preliminaries, in Section 3 we prove Proposition A and Theorem B, and in Section 4 we prove Theorem D. In Section 5 we prove Theorem C and a related structural result that is needed in later sections, Corollary 5.2. Sections 6 and 9 contain our results concerning isoperimetric inequalities and word problems. Engel groups are discussed in Section 7 and Proposition J is proved in Section 8. In the final section, we extract a proof of Theorem K from the arguments in earlier sections, discuss the growth of groups, and prove Corollary L.

2. Preliminaries

Throughout, our notation for conjugation is $x^y := y^{-1}xy$, our commutator convention is $[x, y] = x^{-1}y^{-1}xy$, and iterated commutators are left-normed $[a_1, \ldots, a_n] := [[a_1, \ldots, a_{n-1}], a_n]$.

We assume that the reader is familiar with the use of homological methods in group theory as described in [3] and [11].

2.1. On the structure of the group $\mathfrak{X}(G)$

We recall some subgroups and decompositions of $\mathfrak{X}(G)$ defined by Sidki [40]. We follow the notations from [40] except that we write $\bar{G}$ and $\bar{g}$ where Sidki writes G^{ψ} and g^{ψ}. In the introduction we defined

$$D = D(G) := [G, \bar{G}] \text{ and } L = L(G) := \langle\!\langle \{g^{-1}\bar{g} \mid g \in G\} \rangle\!\rangle.$$

Note that D is the kernel of the natural map $\mathfrak{X}(G) \to G \times \bar{G}$ and that L is the kernel of the map $\mathfrak{X}(G) \to G$ that sends both g and $\bar{g}$ to g. This last map has an obvious splitting

$$\mathfrak{X}(G) = L \rtimes G.$$

We adopt the notation $\ell_g = g^{-1}\bar{g} \in L$, where $g \in G$. Following [40], one sees that L is actually generated (not just normally generated) by the set of ℓ_g, because

$$\ell_u^x = x^{-1}u^{-1}\bar{u}x = (ux)^{-1}\bar{u}(\overline{x}\overline{x}^{-1})x = (ux)^{-1}(\overline{ux})(\overline{x}^{-1}x) = \ell_{ux}\ell_x^{-1} \quad (2.1)$$

for all $x, u \in G$, and $\ell_u^{\bar{x}} = \ell_u^{x\ell_x} = \ell_x^{-1}\ell_u^x\ell_x = \ell_x^{-1}\ell_{ux}$.

By taking the direct product of the maps with kernels L and D (and re-ordering the factors), we obtain a map

$$\rho: \mathfrak{X}(G) \to G \times G \times \bar{G} \cong G \times G \times G$$

with

$$\rho(g) = (g, g, 1), \quad \rho(\bar{g}) = (1, g, g)$$

for all $g \in G$. The kernel of ρ is

$$W = W(G) := D \cap L.$$

Sidki showed in [40, Lemma 4.1.6 (ii)] that D commutes with L, and therefore W is central in DL; in particular, W is abelian.

The image of ρ is

$$\{(g_1, g_2, g_3) \mid g_1 g_2^{-1} g_3 \in [G, G]\}.$$

Note that $\mathrm{im}(\rho)$ contains the commutator subgroup of $G \times G \times G$; it is the kernel of the epimorphism

$$G \times G \times G \to G/[G, G],$$

whose restriction to the first and third coordinates is $g \mapsto g[G, G]$ and whose restriction to the second coordinate is $g \mapsto g^{-1}[G, G]$. The projection of $\mathrm{im}(\rho)$ to each pair of coordinates in $G \times G \times G$ is onto, so by the Virtual Surjection to Pairs theorem [8], if G is finitely presented then $\mathrm{im}(\rho)$ is finitely presented.

For the remainder of the paper, we fix the notation $Q := G/[G, G]$. The action of $G < \mathfrak{X}(G)$ by conjugation on W and on L/L' (hence, on the homology of L/W) factors through Q.

Lemma 2.1. *As $\mathbb{Z}Q$-modules, $W \cap L'$ is a quotient of $H_2(L/W, \mathbb{Z})$.*

Proof. The 5-term exact sequence in homology associated to the short exact sequence

$$1 \to W \to L \to L/W \to 1$$

(where W is central) gives an exact sequence of $\mathbb{Z}Q$-modules

$$H_2(L/W, \mathbb{Z}) \to H_0(L/W, W) \to H_1(L, \mathbb{Z}) \to H_1(L/W, \mathbb{Z}) \to 0$$

and the kernel of $H_0(L/W, W) = W \to H_1(L, \mathbb{Z})$ is $W \cap L'$. ∎

We shall need later the following result of Kochloukova and Sidki [31].

Lemma 2.2. *Suppose that G is of type* FP_2. *If G'/G'' is finitely generated, then so is $W(G)$.*

2.2. Dehn functions and isoperimetric inequalities

We recall some standard facts and terminology concerning the geometry of the word problem in finitely presented groups. The reader unfamiliar with this material may wish to consult [4] for a more thorough introduction.

Let $\mathcal{P} \equiv \langle X \mid R \rangle$ be a finite presentation of a group G. By definition, a word w in the free group $F(X)$ represents the identity in G if and only if it lies in the normal subgroup generated by the set R, which means that there is an equality in $F(X)$ of the form

$$w = \prod_{i=1}^{M} \theta_i^{-1} r_i \theta_i \tag{2.2}$$

with $r_i \in R \cup R^{-1}$ and $\theta_i \in F(X)$. The difficulty of solving the word problem in G by means of a naïve attack seeking equalities of this form depends on the size of M in a minimal such expression, as well as the length $|\theta_i|$ of the conjugating elements θ_i.

Van Kampen's lemma establishes a correspondence between equalities in the free group of this form (more formally, sequences $(\theta_i, r_i)_{i=1}^{M}$) and a class of contractible, labelled, planar diagrams; see, e.g., [4]. This correspondence enables one to invoke

geometric arguments in pursuit of understanding the complexity of the word problem in G. It also explains the following terminology: M, the number of factors in the product on the right-hand side of the above equality, is defined to be the *area* of the product and $\max\{|\theta_i| : 1 \le i \le m\}$ is defined to be its *radius*.

For each word $w \in F(X)$ that represents the identity in G, one defines Area(w) to be the least area of any expression of this form for w, and the *Dehn function* of $\mathcal{P}$ is the function $\delta \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta(n) = \max\{\text{Area}(w) \mid w =_G 1 \text{ and } |w| \le n\}.$$

A pair of functions (α, ρ), with $\alpha, \rho \colon \mathbb{N} \to \mathbb{N}$, is called an *area-radius pair* for P if for each word $w \in F(X)$ that represents the identity in G if $|w| \le n$, then there is an equality of the form (2.2) with area at most $\alpha(n)$ and radius at most $\rho(n)$. (It is important to note that these bounds hold simultaneously.)

Different finite presentations of a group G will give rise to different Dehn functions, but they will be equivalent in the following sense. For functions $f, g \colon \mathbb{N} \to \mathbb{N}$, define $f \preceq g$ if there exists a positive integer C such that

$$f(n) \le C g(Cn + C) + Cn + C$$

for all $n \in \mathbb{N}$, and define $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

Likewise, if (α, ρ) is an area-radius pair for a finite presentation of G and $\mathcal{P}'$ is any other finite presentation of G, then there will exist functions $\alpha', \rho' \colon \mathbb{N} \to \mathbb{N}$ with $\alpha \simeq \alpha'$ and $\rho \simeq \rho'$ such that (α', ρ') is an area-radius pair for $\mathcal{P}'$; see, e.g., [14, Proposition 3.15].

Following common practice, we shall write δ_G to denote "the" Dehn function of a finitely presented group, with the understanding that this is only well defined up to $\simeq$ equivalence. Likewise, we shall refer to (α, ρ) as an area-radius pair for G if α and ρ are $\simeq$ equivalent to the functions in an area-radius pair for a finite presentation of G.

It is easy to see that for constants $p, q \ge 1$, the functions $n \mapsto n^p$ and $n \mapsto n^q$ are $\simeq$ equivalent if and only if $p = q$.

A function $\beta \colon \mathbb{N} \to \mathbb{N}$ is called an *isoperimetric function* for G if $\delta_G \preceq \beta$. One says that G *satisfies a polynomial isoperimetric inequality* if there is a constant d such that $\delta_G(n) \preceq n^d$. And an area-radius pair (α, ρ) for G is said to be polynomial if both α and ρ are bounded above by polynomial functions.

A simple cancellation argument (or diagrammatic argument) establishes the following lemma; see, e.g., [17, Lemma 2.2] or [4].

Lemma 2.3. *Let G be a finitely presented group. If β is an isoperimetric function for G, then (up to $\simeq$ equivalence) (β, β) is an area-radius pair for G.*

Corollary 2.4. *If a finitely presented group satisfies a polynomial isoperimetric inequality, then it has a polynomial area-radius pair.*

We need two other easy and well-known facts about isoperimetric functions, the second of which is a special case of the quasi-isometric invariance of Dehn functions.

Lemma 2.5. *If H is a retract of the finitely presented group G, then (up to $\simeq$ equivalence) every area-radius pair for G is also an area-radius pair for H.*

Lemma 2.6. *Let $\phi\colon H \to G$ be a homomorphism of finitely presented groups. If the kernel of ϕ is finite and $\phi(H)$ has finite index in G, then the Dehn function of H is $\simeq$ equivalent to the Dehn function of G.*

2.3. Biautomatic groups

The theory of automatic groups grew out of investigations into the algorithmic structure of Kleinian groups by Cannon and Thurston, and it was developed thoroughly in the book by Epstein et al. [15]. Let G be a group with finite generating set $A = A^{-1}$ and let A^* be the free monoid on A (i.e., the set of all finite words in the alphabet A). An *automatic structure* for G is determined by a normal form

$$\mathcal{A}_{\mathcal{G}} = \{\sigma_g \mid g \in G\} \subseteq A^*$$

such that $\sigma_g = g$ in G. This normal form is required to satisfy two conditions. First, $\mathcal{A}_{\mathcal{G}} \subset A^*$ must be a *regular language*, i.e., the accepted language of a finite state automaton; and second, the edge-paths in the Cayley graph $\mathcal{C}(G, A)$ that begin at $1 \in G$ and are labelled by the words σ_g must satisfy the following *fellow-traveller condition*: there is a constant $K \geq 0$ such that for all $g, h \in G$ and all integers $t \leq \max\{|\sigma_g|, |\sigma_h|\}$,

$$d_A(\sigma_g(t), \sigma_h(t)) \leq K\, d_A(g, h),$$

where d_A is the path metric on $\mathcal{C}(G, A)$ in which each edge has length 1, and $\sigma_g(t)$ is the image in G of the initial subword of length t in σ_g.

A group is said to be *automatic* if it admits an automatic structure. Should G admit an automatic structure with the additional property that for all integers $t \leq \max\{|\sigma_g|, |\sigma_h|\}$,

$$d_A(a.\sigma_g(t), \sigma_h(t)) \leq K\, d_A(ag, h),$$

for all $g, h \in G$ and $a \in A$, then G is said to be *biautomatic*. Biautomatic groups were first studied by Gersten and Short [18]. Automatic and biautomatic groups form two of the most important classes studied in connection with notions of non-positive curvature in group theory; see [5] for a recent survey. From the point of view of this article, the most salient properties of automatic groups are the following.

Proposition 2.7. *If a finitely generated group G is automatic, then it is finitely presented and of type FP_∞. Moreover, $(\alpha(n), \rho(n)) = (n^2, n)$ is an area-radius pair for G.*

Biautomatic groups behave better than automatic groups with respect to central quotients [36] and this is needed in our proof of Proposition A.

3. Proof of Proposition A and Theorem B

3.1. Proof of Proposition A

We remind the reader that every perfect group Γ admits a universal central extension

$$1 \to H_2(\Gamma, \mathbb{Z}) \to \widetilde{\Gamma} \overset{p}{\to} \Gamma \to 1,$$

meaning that for all perfect groups E and surjections $\pi \colon E \to \Gamma$ with central kernel, there is a surjection (central quotient) $s \colon \widetilde{\Gamma} \to E$ with $\pi \circ s = p$.

Lemma 3.1. *If G is perfect, then*

$$1 \to W \to \mathfrak{X}(G) \overset{\rho}{\to} G \times G \times G \to 1$$

is exact, W is central in $\mathfrak{X}(G)$, and $\mathfrak{X}(G)$ is a central quotient of the universal central extension of $G \times G \times G$.

Proof. Let $\ell_x = x^{-1}\overline{x}$. Note that $\rho(\ell_a \ell_b \ell_{ab}^{-1}) = ([a, b], 1, 1)$, from which it follows that the image of DL under ρ contains the commutator subgroup of $P := G \times G \times G$, which is the whole of P, and W is central in DL. Thus, W is central. And since $\mathfrak{X}(G)$ is perfect, the last assertion is an instance of the universal property recalled above. ∎

Proof of Proposition A. Assume now that the perfect group G is hyperbolic. Neumann and Reeves [37] proved that every central extension of a hyperbolic group with finitely generated kernel is biautomatic, so in particular this is true of the universal central extension $\widetilde{G}$. Let $P = G \times G \times G$. By the Künneth formula,

$$H_2(P, \mathbb{Z}) = H_2(G, \mathbb{Z}) \times H_2(G, \mathbb{Z}) \times H_2(G, \mathbb{Z}),$$

so the universal central extension of P is $\widetilde{G} \times \widetilde{G} \times \widetilde{G}$. A direct product of biautomatic groups is biautomatic and Mosher [36] proved that a central quotient of a biautomatic group is again biautomatic, so Lemma 3.1 completes the proof of Proposition A. ∎

Remark 3.2. The preceding argument establishes the following more general fact: let $\mathcal{C}$ be any class of finitely presented groups that is closed under the formation of finite direct sums, central extensions by finitely generated abelian groups, and central quotients; if a finitely presented perfect group G belongs to $\mathcal{C}$, then so does $\mathfrak{X}(G)$.

3.2. Proof of Theorem B

Throughout this section, F will denote the free group of rank 2 with basis $\{a, b\}$. Theorem B reduces easily to the special case $G = F$: one can split the surjection $G \to F$ to regard F as a retract of G; any retraction $A \to B$ extends to a retraction $\mathfrak{X}(A) \to \mathfrak{X}(B)$; and if $C \to D$ is a retraction of finitely presented groups then it is easy to see that the Dehn function of C is bounded below by that of D. Thus, we concentrate on showing that the Dehn function of $\mathfrak{X}(F)$ is bounded below by a cubic polynomial.

Definition 3.3. Let $H < G$ be a pair of finitely generated groups and let d_H and d_G be the word metrics associated to a choice of finite generating sets. The *distortion* of H in G is the function

$$\text{dist}_H^G(n) = \max\{d_H(1, h) \mid h \in H \text{ with } d_G(1, h) \le n\}.$$

One checks easily that, up to Lipschitz equivalence, this function does not depend on the choice of word metrics.

Lemma 3.4. *Let $A < B < C$ be finitely generated groups. If B is a retract of C, then* $\text{dist}_A^B(n) \simeq \text{dist}_A^C(n)$.

Proof. If $r: C \to B$ is a retraction, then we obtain a finite generating set S for C by taking a finite generating set S' for B and adding finitely many generators from the kernel of r. With respect to the associated word metrics, $d_B(1, b) = d_C(1, b)$ for all $b \in B$, in particular for $b = a \in A$. ∎

We need the following connection between Dehn functions and distortion. We remind the reader that the *trivial HNN extension* $B *_H$ of a group B with associated subgroup $H < B$ is the quotient of $B * \langle t \rangle$ by the normal subgroup generated by $\{[t, h] \mid h \in H\}$.

The following lemma is a slight variant on an inequality in [7, Theorem III.Γ.6.20]; see also [1].

Lemma 3.5. *Let $H < B < G_0 < G$ be groups with $G_0 = B *_H$ a trivial HNN extension. If G is finitely presented and $G_0 < G$ has finite index, then the Dehn function of G satisfies*

$$n \, \text{dist}_H^G(n) \preceq \delta_G(n).$$

Proof. Dehn functions and distortion are unaffected (up to $\simeq$ equivalence) by passage to subgroups of finite index, so we may assume that $G = G_0$. Killing the stable letter of the HNN extension retracts $G = G_0$ onto B, so

$$\text{dist}_H^G(n) \simeq \text{dist}_H^B(n),$$

by Lemma 3.4. Note that since G is finitely presented, B is finitely presented and H is finitely generated. We fix a finite presentation $B = \langle X \mid R \rangle$ where X contains a finite generating set Y for H. Then

$$G = \langle X, t \mid R, [t, y] \, (y \in Y) \rangle.$$

For each positive integer n, we choose a word w_n in the free group on X that represents an element $h_n \in H$ with

$$d_H(1, h_n) = \operatorname{dist}_H^B(n) \quad \text{and} \quad d_B(1, h_n) \le n.$$

A standard argument using the t-corridors introduced in [6] shows that any van Kampen diagram for the word $W_n = t^n w_n t^{-n} w_n^{-1}$ must contain at least $n \ \operatorname{dist}_H^B(n)$ 2-cells; see [7, pp. 506–508]. And since $|W_n| \le 4n$, this establishes the desired lower bound on $\delta_G(n)$. ∎

Lemma 3.5 is relevant to our situation because of the following observation noted in [10].

Lemma 3.6. *There is a finite-index subgroup $G_0 < \mathfrak{X}(F) = L \rtimes F$ that contains L as the associated subgroup of a trivial HNN extension $G_0 = B*_L$.*

Proof. Let $\pi \colon \mathfrak{X}(F) = L \rtimes F \to F$ be the retraction with kernel L and recall that $[L, D] = 1$. For a fixed $t \in D \setminus L$, consider $\pi(t) \in F$. Marshall Hall's theorem [26] provides a subgroup $F_0 < F$ of finite index such that $\pi(t)$ is a primitive element of F_0. Let $G_0 = \pi^{-1}(F_0) = L \rtimes F_0$. By choosing a different section of π if necessary, we may assume that $t \in \tilde{F}_0 = 1 \times F_0$. We fix a free splitting $\tilde{F}_0 = C * F_1$, where C is the cyclic subgroup generated by t, and set $B = L \rtimes \tilde{F}_1$. Then $G_0 = B*_L$. ∎

In order to apply Lemma 3.5, we need a lower bound on $\operatorname{dist}_L^{\mathfrak{X}(F)}(n)$. (The bound that we establish is sharp, but we do not require this so omit the proof.)

Lemma 3.7. *The distortion of L in $\mathfrak{X}(F)$ satisfies $\operatorname{dist}_L^{\mathfrak{X}(F)}(n) \succeq n^2$.*

Proof. We maintain the notation $\ell_x = x^{-1}\overline{x}$ for elements of L, where $x \in F$. We proved in [10] that L is generated by $\{\ell_a, \ell_b, \ell_{ab}\}$. In the current setting, it is convenient to replace ℓ_{ab} by $\lambda = \ell_a \ell_b \ell_{ab}^{-1}$. Thus, we work with the finite generating set $\{\ell_a, \ell_b, \lambda\}$ for L and $\{\ell_a, \ell_b, \lambda, a, b\}$ for $\mathfrak{X}(F) = L \rtimes F$.

Consider the elements $c_n := \ell_{a^n} \ell_{b^n} \ell_{a^n b^n}^{-1} \in L$. To see why we focus on these elements, note that $\rho(c_n) = ([a^n, b^n], 1, 1) \in F \times F \times F$.

Note that c_n can be expressed as a word of length $6n$ in the generators $\{a, b, \ell_a, \ell_b\}$, because $\ell_{a^n} = \ell_a^n$ and $\ell_{b^n} = \ell_b^n$, while by (2.1), we have

$$\ell_{a^n b^n} = \ell_{a^n}^{b^n} \ell_{b^n} = b^{-n} \ell_a^n b^n \ell_b^n.$$

We claim that $d_L(1, c_n) \geq n^2$. To prove this, we analyse the structure of words w in the free group $\mathcal{F}$ with basis $\{\ell_a, \ell_b, \lambda\}$ that equal c_n in L. As always, we write u^v as shorthand for $v^{-1}uv$. By making repeated use of the identity $vu^v = uv$, we can express w in $\mathcal{F}$ as a product of the form

$$w = V \prod_{i=1}^{N} \lambda^{\pm\theta_i} \tag{3.1}$$

with $V = V(\ell_a, \ell_b)$ and θ_i words in the letters $\ell_a^{\pm 1}, \ell_b^{\pm 1}$, and $N \leq |w|$. Consider the composition $\bar{p}$ of $\mathcal{F} \to L$ with the map $\bar{\pi} \colon \mathfrak{X}(F) \to \bar{F}$ that kills F and is the identity on $\bar{F}$. Note that

$$\bar{p}(\ell_a) = \bar{a}, \quad \bar{p}(\ell_b) = \bar{b}, \quad \bar{p}(w) = \bar{\pi}(c_n) = 1.$$

By taking the image of the free equality (3.1) under $\bar{p}$, we conclude that $V(\bar{a}, \bar{b}) = 1$ in $\bar{F}$. Hence, V is the empty word and we have an equality

$$w = \prod_{i=1}^{N} \lambda^{\pm\theta_i}$$

in $\mathcal{F}$.

Next we consider the composition p of $\mathcal{F} \to L$ with the map $\pi \colon \mathfrak{X}(F) \to F$ that kills $\bar{F}$ and is the identity on F. This map sends λ to $[a, b]$ and w to $[a^n, b^n]$, while sending $\theta_i(\ell_a, \ell_b)$ to $\theta_i' = \theta_i(a^{-1}, b^{-1})$. Thus, we obtain an equality

$$[a^n, b^n] = \prod_{i=1}^{N} [a, b]^{\pm\theta_i'}$$

in the free group $F = F(a, b)$. A standard exercise (often used to motivate van Kampen's lemma) shows that $[a^n, b^n]$ cannot be expressed in F as a product of fewer than n^2 conjugates of $[a, b]$. Thus, $n^2 \leq N \leq d_L(1, c_n)$, as claimed. ∎

Proof of Theorem B. We have a retraction

$$r \colon \mathfrak{X}(G) \to \mathfrak{X}(F).$$

In [10], we proved that $\mathfrak{X}(F)$ has a subgroup Γ of finite index with $H_3(\Gamma, \mathbb{Q})$ infinite-dimensional, and this injects into $H_3(r^{-1}(\Gamma), \mathbb{Q})$. The Dehn function of $\mathfrak{X}(G)$ is bounded below by the Dehn function of its retract $\mathfrak{X}(F)$. From Lemmas 3.5 and 3.6, we have

$$\delta_{\mathfrak{X}(F)}(n) \succeq n \; \mathrm{dist}_L^{\mathfrak{X}(F)}(n).$$

And in Lemma 3.7, we proved that $n^2 \preceq \mathrm{dist}_L^{\mathfrak{X}(F)}(n)$. ∎

4. $\mathfrak{X}$ preserves virtual nilpotence

Throughout this section we assume that G is finitely generated. We shall prove that if G is virtually nilpotent then $\mathfrak{X}(G)$ is virtually nilpotent. In [24], Gupta, Rocco and Sidki used commutator calculations to prove that if G is nilpotent then so is $\mathfrak{X}(G)$, and gave a bound on the nilpotency class. In this nilpotent case, our proof is shorter and more homological, but it does not give as good a bound as theirs on the nilpotency class of $\mathfrak{X}(G)$.

4.1. Nilpotent actions

If A is an abelian group and B is a group acting on A (on the right), then one can form the semidirect product $A \rtimes B$ in which the action a^b is transformed into conjugation. In multiplicative notation,

$$[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b.$$

The action is said to be *nilpotent* if there is an integer d such that, for all $a \in A$ and $b_1, \ldots, b_d \in B$,

$$[a, b_1, \ldots, b_d] = 1.$$

If B is nilpotent, then $A \rtimes B$ will be nilpotent if and only if the action of B on A is nilpotent.

If we write the group operation in A additively and regard A as a $\mathbb{Z}B$ module, writing the action a^b as $a \circ b$, then $[a, b] = a \circ (b - 1)$ and the vanishing of the above commutator becomes

$$a \circ (b_1 - 1) \cdots (b_d - 1) = 0.$$

Thus A is a nilpotent module over $\mathrm{Aug}(\mathbb{Z}B)$, the augmentation ideal of $\mathbb{Z}B$.

We retain the notation $L = L(G)$ for the normal subgroup of $\mathfrak{X}(G)$ generated by the elements $\ell_g = g^{-1}\overline{g}$. We proved in [10] that when G is finitely generated, L is finitely generated. Lima and Oliveira [32] had proved earlier that L/L' is finitely generated. The action of $\mathfrak{X}(G) = L \rtimes G$ by conjugation on L/L' factors through G (and even $Q = G/G'$).

Proposition 4.1. *For all finitely generated groups G, the action of G on L/L' is nilpotent.*

Proof. Following [40], we consider the set-map

$$L \to \mathrm{Aug}(\mathbb{Z}G)$$

that sends ℓ_g to $(g - 1)$. In order to make this a group homomorphism, we must take a quotient of $\mathrm{Aug}(\mathbb{Z}G)$ to force the image of ℓ_{g^n} to coincide with that of ℓ_g^n, that is,

$$g^n - 1 = n(g - 1),$$

the group operation in $\mathrm{Aug}(\mathbb{Z}G)$ being addition. A simple induction shows that it is enough to factor out by the $\mathbb{Z}G$-ideal $I_2 = \langle (g - 1)^2 : g \in G \rangle$. From [40] and [32], we have that the resulting map $L/L' \to \mathrm{Aug}(\mathbb{Z}G)/I_2$ is an isomorphism of abelian groups.

Moreover, with the action on the target coming from multiplication in the ring $\mathbb{Z}G$, this map is an isomorphism of right $\mathbb{Z}G$-modules, because the image of $\ell_g^x = \ell_{gx}\ell_x^{-1}$ is

$$(g - 1)x = (gx - 1) - (x - 1).$$

(In the light of this we omit $\circ$ from our notation.)

Thus the task of showing that the action of G on L/L' is nilpotent is translated into showing that $V(\mathrm{Aug}(\mathbb{Z}G))^d = 0$ for some $d \in \mathbb{N}$, where $V = \mathrm{Aug}(\mathbb{Z}G)/I_2 \cong L/L'$; equivalently, that $(\mathrm{Aug}(\mathbb{Z}G))^{d+1} \subset I_2$.

Moreover, because $G' = [G, G]$ acts trivially (via conjugation) on L/L', we can analyse the action of the *commutative* quotient ring $\mathbb{Z}Q/I_0$ rather than $\mathbb{Z}G/I_2$, where I_0 is the ideal generated by $\{(q - 1)^2 \mid q \in Q\}$. Taking advantage of this commutativity, for $q_1, q_2 \in Q$, in $\mathbb{Z}Q/I_0$, we have

$$2q_1q_2 - 1 = (q_1q_2)^2 = q_1^2q_2^2 = (2q_1 - 1)(2q_2 - 1) = 4q_1q_2 - 2q_1 - 2q_2 + 1,$$

hence

$$2(q_1 - 1)(q_2 - 1) = 2q_1q_2 - 2q_1 - 2q_2 + 2 = 0.$$

It follows that

$$2(\mathrm{Aug}(\mathbb{Z}G))^2 \subseteq \ker(\mathbb{Z}G/I_2 \to \mathbb{Z}Q/I_0).$$

In other words,

$$2V(\mathrm{Aug}(\mathbb{Z}G))^2 = 0. \tag{4.1}$$

Because $V \cong L/L'$ is finitely generated as an abelian group, $\overline{V} = V/2V$ is finite, say with k elements. The action of $(g - 1)$ on $\overline{V}$ by right multiplication depends only on the image of $(g - 1)$ in $\overline{V}$, so the action of any product $\prod_{i=1}^{k+1}(g_i - 1)$ is the same as that of a product with a repeated factor. And since the action factors through the commutative ring $\mathbb{Z}Q$, the order of the factors does not matter. As $(g_i - 1)^2 = 0$ in $\mathbb{Z}G/I_2$, we conclude that $(\mathrm{Aug}(\mathbb{Z}G))^{k+1}$ annihilates $\overline{V}$. In other words,

$$V(g_1 - 1)\ldots(g_{k+1} - 1) \subseteq 2V. \tag{4.2}$$

By (4.1) and (4.2),

$$V(\mathrm{Aug}(\mathbb{Z}G))^{k+3} \subseteq 2V(\mathrm{Aug}(\mathbb{Z}G))^2 = 0.$$

This completes the proof. ∎

Remark 4.2. The proof given above shows that for every finitely generated group G, the semidirect product $(L/L') \rtimes Q$ contains a subgroup of finite index, namely

$$2(L/L') \rtimes Q,$$

that is nilpotent of class at most 2.

Proof of Theorem D. Consider the sequence of subgroups

$$W \cap L' \subseteq W \subseteq L \subseteq \mathfrak{X}(G),$$

and note that

$$W/(W \cap L') \cong WL'/L' \subseteq L/L' \subseteq \mathfrak{X}(G)/L'.$$

Let $V = L/L'$. From the decomposition $\mathfrak{X}(G) = L \rtimes G$, we have $\mathfrak{X}(G)/L' \cong V \rtimes G$, and Proposition 4.1 tells us that the action of G on V is nilpotent. Thus, if G_0 is a nilpotent subgroup of finite index in G then $V \rtimes G_0$ is nilpotent and $\mathfrak{X}(G)/L'$ is virtually nilpotent.

Let T denote $L \rtimes G_0$, the preimage of $V \rtimes G_0$ in $\mathfrak{X}(G)$; note that $[\mathfrak{X}(G) : T] < \infty$.

The nilpotent action of G on V restricts to a nilpotent action on $W/(W \cap L') \cong WL'/L'$, so for some natural number m, we have

$$W(\mathrm{Aug}(\mathbb{Z}T))^m = [W, T, \ldots, T] \subseteq L' \cap W, \tag{4.3}$$

where we view W as a right $\mathbb{Z}\mathfrak{X}(G)$-module via conjugation. Since $\mathfrak{X}(G)/W \cong \mathrm{im}(\rho)$ is a subgroup of $G \times G \times G$, it is virtually nilpotent. Hence, there is a normal subgroup $T_0 < T$ that contains W and is such that T_0/W is nilpotent. Thus, we have a short exact sequence of groups

$$1 \to W/(W \cap L') \to T_0/(W \cap L') \to T_0/W \to 1,$$

where the nilpotent group T_0/W acts nilpotently on the abelian group $W/(W \cap L')$ by (4.3). Hence,

$$T_0/(W \cap L') \text{ is nilpotent.} \tag{4.4}$$

We will be done if we can show that T_0 is nilpotent.

As T_0 is normal, $[L/W, T_0/W] \subseteq T_0/W$. And as T_0/W is nilpotent, for a sufficiently long commutator

$$[L/W, T_0/W, \ldots, T_0/W] \subseteq [T_0/W, \ldots, T_0/W] = 1;$$

in other words, T_0/W acts nilpotently on L/W. Therefore,

$$T_0/W \text{ acts nilpotently on } H_2(L/W, \mathbb{Z}). \tag{4.5}$$

The result now follows from Lemma 2.1, but we give more details. The central extension

$$1 \to W \cap L' \to L \to L/(W \cap L') \to 1$$

is stem (i.e., the central subgroup $W \cap L'$ is contained in the commutator of the middle group L), so by the general theory of central extensions,

$$W \cap L' \text{ is a quotient of } H_2(L/W, \mathbb{Z}). \tag{4.6}$$

The quotient map, which is given by Hopf's formula, is equivariant with respect to the action of T_0/W, so from (4.5) and (4.6), we have that T_0/W acts nilpotently on $W \cap L'$. Thus,

$$\text{the action by conjugation of } T_0 \text{ on } W \cap L' \text{ is nilpotent.} \tag{4.7}$$

Together, (4.4) and (4.7) imply that T_0 is nilpotent. ∎

5. The structure of $W(G)$ as a $\mathbb{Z}Q$-module

In this section we prove Theorem C, which describes the structure of $W = W(G)$ as a Q-module, where the action of $Q = G/G'$ on W is induced by the action of $G < \mathfrak{X}(G)$ on W by conjugation. We shall make heavy use of the notation and structure established in Section 2.1.

The first insight into the action of Q on W comes from Proposition 4.1, which tells us that the action of Q on the image of W in L/L' is nilpotent. As in Lemma 2.1, we have an exact sequence of $\mathbb{Z}Q$-modules

$$H_2(L/W, \mathbb{Z}) \to W \to L/L' \to H_1(L/W, \mathbb{Z}) \to 0,$$

so our main task now is to understand the structure of $H_2(L/W, \mathbb{Z})$ as a $\mathbb{Z}Q$-module.

We observe that L/W is the image of L under $\rho \colon \mathfrak{X}(G) \to G \times G \times G$, which is

$$\langle (g, 1, g^{-1}) \mid g \in G \rangle = \{(g_1, 1, g_2) \mid g_1 g_2 \in G'\}.$$

Note that L/W is normal in $G \times 1 \times G \cong G \times G$ with quotient $(Q \times Q)/Q_0$, where

$$Q_0 = \{(q, q^{-1}) \mid q \in Q\} < Q \times Q.$$

To lighten the notation, we drop the second coordinate and identify L/W with

$$S := \{(g_1, g_2) \mid g_1 g_2 \in G'\} < G \times G.$$

The G-action on L/W coincides with the action of the first factor $G \times 1 < G \times G$ by conjugation on S, so it is the induced Q-action on the homology of S that we must understand. (Note that this is not the same as the action of Q_0.)

Proposition 5.1. *Let G be a finitely generated group, let Q be the abelianisation of G, let $S = \langle \{(g, g^{-1}) \mid g \in G\} \rangle < G \times G$ and consider the action of Q on the homology of S that is induced by the conjugation action of $G \times 1$. Then, $J := H_2(S, \mathbb{Z})$ has a filtration by $\mathbb{Z}Q$-submodules*

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq J$$

such that

(1) J_1 *is a $\mathbb{Z}Q$-subquotient of the module $M = ((G'/G'') \otimes_{\mathbb{Z}} (G'/G''))_{Q_0}$, where $Q = G/G'$ is acting by conjugation on the first factor and acting trivially on the second factor, while the co-invariants are taken with respect to the standard conjugation action of*

$$Q_0 = \{(q, q^{-1}) \mid q \in Q\} < (G/G') \times (G/G');$$

(2) Q *acts trivially on J/J_3, J_3/J_2 and J_2/J_1.*

Proof. We shall write $\bar{G}$ for the second direct factor of $G \times G$ in order to simplify the notation. Thus, $S < G \times \bar{G}$.

We analyse the LHS spectral sequence associated to the short exact sequence

$$1 \to G' \times \bar{G}' \to S \to Q_0 \to 1.$$

Thus, we consider

$$E^2_{p,q} = H_p(Q_0, H_q(G' \times \bar{G}', \mathbb{Z}))$$

converging to $H_{p+q}(S, \mathbb{Z})$. The terms $E^\infty_{i,j}$ with $i + j = 2$ give a filtration of $H_2(S, \mathbb{Z})$, so our focus is on the terms $E^*_{i,j}$ with $i + j = 2$.

The action of $G = G \times 1$ by conjugation (which determines the Q action) is compatible with all of the maps and decompositions that we consider. The action of G on $E^2_{2,0} = H_2(Q_0, \mathbb{Z})$ factors through the conjugation action of $Q \times 1$ on Q_0, which is trivial since $Q \times Q$ is abelian. Thus,

$$\text{the action of } Q \text{ on } E^\infty_{2,0} \text{ is trivial.} \tag{5.1}$$

Next we consider

$$E_{1,1}^2 = H_1(Q_0, H_1(G' \times \bar{G}', \mathbb{Z})) = H_1(Q_0, (G'/G'') \times (\bar{G}'/\bar{G}''))$$
$$= H_1(Q_0, G'/G'') \oplus H_1(Q_0, \bar{G}'/\bar{G}''). \tag{5.2}$$

The action of $G = G \times 1$ on Q_0 and $\bar{G}'/\bar{G}''$ is trivial, hence G acts trivially on the second factor of (5.2), i.e., on $H_1(Q_0, \bar{G}'/\bar{G}'')$. The action of $(h, 1) \in G \times 1$ on the coefficient module G'/G'' is the same as the action of $(h, \bar{h}^{-1}) \in S$ induced by conjugation in $G \times \bar{G}$. But S acts trivially on $H_1(Q_0, G'/G'')$, because the S-action factors through Q_0. Thus, the action of G on the first factor of (5.2) is also trivial, and

$$Q \text{ acts trivially on } E_{1,1}^\infty. \tag{5.3}$$

Finally, we consider $E_{0,2}^2$. From the Künneth formula, we have the Q-invariant decomposition of Q_0-modules

$$H_2(G' \times \bar{G}', \mathbb{Z}) \cong H_2(G', \mathbb{Z}) \oplus H_2(\bar{G}', \mathbb{Z}) \oplus H_1(G', \mathbb{Z}) \otimes H_1(\bar{G}', \mathbb{Z}).$$

The term $E_{0,2}^2$ is obtained by taking Q_0 co-invariants $H_0(Q_0, -)$ of these modules. The argument used in our analysis of $E_{1,1}^2$ shows that Q acts trivially on

$$H_0(Q_0, H_2(G', \mathbb{Z})) \oplus H_0(Q_0, H_2(\bar{G}', \mathbb{Z})),$$

so

$$E_{0,2}^2 \operatorname{Aug}(\mathbb{Z}Q) \subseteq H_0(Q_0, H_1(G', \mathbb{Z}) \otimes H_1(G', \mathbb{Z})). \tag{5.4}$$

And by definition,

$$H_0(Q_0, H_1(G', \mathbb{Z}) \otimes H_1(G', \mathbb{Z})) = ((G'/G'') \otimes (\bar{G}'/\bar{G}''))_{Q_0} \tag{5.5}$$

is the quotient of $(G'/G'') \otimes (\bar{G}'/\bar{G}'')$ by the conjugation action of

$$Q_0 < (G/G') \times (\bar{G}'/\bar{G}'').$$

This is the module called M in the statement of Theorem C.

The spectral sequence converges to $H_*(S, \mathbb{Z})$, so there is a filtration $\{F^j\}_j$ of $H_2(S, \mathbb{Z})$ such that

$$F^{-1} = 0 \subseteq F^0 \subseteq F^1 \subseteq F^2 = H_2(S, \mathbb{Z}),$$

with $F^i/F^{i-1} \cong E_{i,2-i}^\infty$. We saw in (5.1) and (5.3) that Q acts trivially on F^2/F^1 and F^1/F^0. We set

$$J_2 := E_{0,2}^\infty = F^0, \quad J_1 := J_2 \operatorname{Aug}(\mathbb{Z}Q), \quad J_3 := F^1.$$

By (5.5) and (5.4), $E_{0,2}^2 \operatorname{Aug}(\mathbb{Z}Q)$ is isomorphic to a $\mathbb{Z}Q$-submodule of

$$M = ((G'/G'') \otimes (\bar{G}'/\bar{G}''))_{Q_0}.$$

Thus, J_1 is a $\mathbb{Z}Q$-subquotient of M. ∎

5.1. Proof of Theorem C

We return to consideration of the following exact sequence of $\mathbb{Z}Q$-modules,

$$H_2(L/W, \mathbb{Z}) \xrightarrow{\mu} W \to L/L' \to H_1(L/W, \mathbb{Z}) \to 0.$$

The first map has image $W \cap L'$, so from the filtration $J_1 \subseteq J_2 \subseteq J_3 \subseteq J$ of Proposition 5.1 we obtain a filtration

$$\mu(J_1) \subseteq \mu(J_2) \subseteq \mu(J_3) \subseteq W_1 := W \cap L'$$

by $\mathbb{Z}Q$-submodules, such that $\mu(J_1)$ is a $\mathbb{Z}Q$-subquotient of

$$M = ((G'/G'') \otimes_{\mathbb{Z}} (G'/G''))_{Q_0},$$

where the structure of M as a $\mathbb{Z}Q$-module is as described in the statement of Proposition 5.1, and Q acts trivially on each of $W_1/\mu(J_3)$, $\mu(J_3)/\mu(J_2)$ and $\mu(J_2)/\mu(J_1)$. In particular, defining $W_0 := \mu(J_1)$, we have that Q acts nilpotently on W_1/W_0, indeed

$$W_1.(\text{Aug}(\mathbb{Z}Q))^3 \subseteq W_0.$$

We proved in Proposition 4.1 that Q acts nilpotently on L/L', which contains W/W_1. Thus, Q acts nilpotently on W/W_0 and the theorem is proved. ∎

The following consequence of Theorem C will play a vital role in the proof of Theorems G and K.

Corollary 5.2. *Let G be a finitely generated group such that G/G'' is virtually nilpotent. Then there is a subgroup Q_1 of finite index in $Q = \mathfrak{X}(G)/DL \cong G/G'$ and a filtration of $W = W(G)$ by $\mathbb{Z}Q_1$-submodules such that Q_1 acts trivially on each quotient of the filtration that is infinite.*

Proof. Maintaining the notation of the preceding proof, we have

$$N := W.(\text{Aug}(\mathbb{Z}Q))^{3+s} \subseteq W_0,$$

where s is the nilpotency class of the action of Q on L/L', i.e.,

$$(L/L')(\text{Aug}(\mathbb{Z}Q))^s = 0,$$

but $(L/L')(\text{Aug}(\mathbb{Z}Q))^{s-1} \neq 0$. Write A for G'/G''.

Let T be a subgroup of finite index in G such that T/G'' is nilpotent. Set $A_1 = (T/G'') \cap A$ and $Q_1 = TG'/G'$. Then Q_1 has finite index in Q, A_1 has finite index in A, and Q_1 acts nilpotently on A_1. Let us say that a filtration of $\mathbb{Z}Q_1$-submodules is "*good*" if Q_1 acts trivially on every infinite quotient of the filtration. Note that the

image M_1 of $A_1 \otimes_{\mathbb{Z}} A_1$ in $M = (A \otimes_{\mathbb{Z}} A)_{Q_0}$ has finite index and M_1 is a $\mathbb{Z}Q_1$-submodule of M. Moreover, since Q_1 acts nilpotently on A_1 and M_1 has finite index in M, the filtration $0 \subseteq M_1 \subseteq M$ can be refined to a good filtration of M. Since N is a $\mathbb{Z}Q$-subquotient of M, it is a $\mathbb{Z}Q_1$-subquotient of M, and so N has a good filtration by $\mathbb{Z}Q_1$-submodules. Finally, since Q_1-acts nilpotently on W/N, there is a good filtration of $\mathbb{Z}Q_1$-submodules of W/N. These two good filtrations yield a good filtration of W by $\mathbb{Z}Q_1$-submodules.   ∎

We also highlight the special case of Theorem C in which $G'/G'' = 0$.

Corollary 5.3. *If G is finitely generated with perfect commutator G', then Q acts nilpotently on $W = W(G)$.*

6. Isoperimetric functions for $\mathfrak{X}(G)$

In this section we prove Theorem G. The proof relies on the understanding of W as a $\mathbb{Z}Q$-module that was developed in the previous section as well as on previous results on the isoperimetric functions of subdirect products and central extensions of groups, and on the basic facts about isoperimetric functions gathered in Section 2.2.

6.1. Subdirect products that are co-abelian

Let H be a subgroup of a group G. One says that H is *coabelian in G* if $G' \leq H$. If there exists a subgroup of finite index $G_0 < G$ with $[G_0, G_0] \leq H$, then H is said to be *virtually-coabelian in G*. The *corank* of H in G is defined to be

$$\dim_{\mathbb{Q}}(G_0/(G_0 \cap H) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Note that this is independent of the finite-index subgroup G_0 chosen.

Suppose now that H is a subgroup of a direct product $D = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$. If $\Gamma_i H = D$ for every $1 \leq i \leq n$, we say that H *fills D*. If $[D : \Gamma_i H] < \infty$ for every $1 \leq i \leq n$, we say that H is *virtually filling* in D. Note that these definitions depend on the choice of direct-product decomposition of D.

Our proof of Theorem G relies on the following result from Dison's thesis [14].

Theorem 6.1 ([14, Theorem A]). *Let H be a virtually-filling subgroup of a direct product $D = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$, and suppose that H is virtually-coabelian of co-rank r. Suppose each Γ_i is finitely presented and $n \geq 3$. For each i, let (α_i, ρ_i) be an area-radius pair for some finite presentation of Γ_i. Define*

$$\alpha(l) = \max\big(\{l^2\} \cup \{\alpha_i(l) : 1 \leq i \leq n\}\big)$$

and

$$\rho(l) = \max\big(\{l\} \cup \{\rho_i(l) : 1 \leq i \leq n\}\big).$$

Then $\rho^{2r}\alpha$ is an isoperimetric function for H.

6.2. Isoperimetric functions for central extensions

Our proof of Theorem G also relies on the following special case of the Bryant Park lemma from [2].

Lemma 6.2. *Let G be a finitely presented group and let C be a finitely generated central subgroup. The Dehn functions of G and G/C satisfy the inequality*

$$\delta_G(n) \preceq \delta_{G/C}(n)^2.$$

In particular, if G/C satisfies a polynomial isoperimetric inequality, then so does G.

Proof. In the light of Lemma 2.6, we may assume that C is free abelian, of rank k say.

Let $\langle b_1, \ldots, b_m \mid r_1, \ldots, r_s \rangle$ be a finite presentation of G/C. We fix a basis $\{a_1, \ldots, a_k\}$ for C and choose words $\sigma_1, \ldots, \sigma_s$ in the letters $a_j^{\pm 1}$ such that $r_i \sigma_i = 1$ in G for $i = 1, \ldots, s$. Let $\mu = \max_i |\sigma_i|$. We work with the following presentation of G:

$$\langle a_1, \ldots, a_k, b_1, \ldots, b_m \mid r_i \sigma_i = 1, \ [a_t, a_j] = 1 = [a_t, b_d]$$
$$\text{for } 1 \leq i \leq s, \ 1 \leq j, t \leq k, \ 1 \leq d \leq m \rangle.$$

Let F be the free group on $\{a_1, \ldots, a_k, b_1, \ldots, b_m\}$. For words $v_1, v_2 \in F$, we write $v_1 =_c v_2$ if v_1 and v_2 represent the same element in G and we can transform v_1 into v_2 "at cost at most c," i.e., by applying at most c relations from the above presentation of G; in other words

$$\text{Area}(v_1 v_2^{-1}) \leq c.$$

Let $w \in F$ be a word of length $|w| = n$ and suppose $w = 1$ in G. Using the relations $[a_t, b_d] = 1$, we have

$$w =_{n^2} w_1(\underline{b}) w_2(\underline{a}),$$

where $w_2(\underline{a})$ is a word in the letters $\{a_1^{\pm 1}, \ldots, a_k^{\pm 1}\}$ and $w_1(\underline{b})$ is a word in the letters $\{b_1^{\pm 1}, \ldots, b_m^{\pm 1}\}$; both words have length at most n, and $w_1 = 1$ in G/C. Let $N = \text{Area}(w_1)$, which is bounded by $\delta(n)$, where $\delta = \delta_{G/C}$ is the Dehn function of G/C. Then we have the following equalities, for some choice of relators

$r_{j(i)} \in \{r_1^{\pm 1}, \ldots, r_s^{\pm 1}\}$, where the conjugating elements θ_i can be taken to have length $|\theta_i| \leq \delta(n)$, as in Lemma 2.3:

$$w_1(\underline{b}) = \prod_{1 \leq i \leq N} \theta_i^{-1} r_{j(i)} \theta_i = \prod_{1 \leq i \leq N} \theta_i^{-1} (r_{j(i)} \sigma_{j(i)}) \sigma_{j(i)}^{-1} \theta_i$$

$$=_c \prod_{1 \leq i \leq N} \theta_i^{-1} (r_{j(i)} \sigma_{j(i)}) \theta_i \sigma_{j(i)}^{-1} =_N \prod_{1 \leq i \leq N} \sigma_{j(i)}^{-1} =: w_3(\underline{a}).$$

The first and second equalities are in the free group F, and the last is a definition. The cost c of the third equality counts the number of commutation relations applied to move each $\sigma_{j(i)}^{-1}$ past θ_i. Clearly,

$$c \leq \sum_{1 \leq i \leq N} |\sigma_{j(i)}| . |\theta_i| \leq \mu N \, \delta(n) \leq \mu \delta(n)^2.$$

The cost of the fourth equality comes from applying the relations $r_j \sigma_j = 1$ (followed by free reduction). Note that w_3 has length $|w_3| \leq N\mu$, and hence $w_0 := w_3 w_2$ has length at most $n + N\mu$. And $N \leq \delta(n)$.

At this stage, we have

$$w =_{n^2 + c + N} w_0(\underline{a}),$$

where w_0 is a word that represents the identity in

$$C = \langle a_1, \ldots, a_k \mid [a_i, a_j] = 1, i, j = 1, \ldots, k \rangle$$

and $|w_0| \leq n + \mu \delta(n)$. The Dehn function of C is $q(m) = m^2$, so $w_0 =_M 1$, where

$$M = (n + \mu \delta(n))^2.$$

Thus we have transformed w, which has length $|w| = n$, into the empty word at a total cost of at most

$$n^2 + c + N + (n + \mu \delta(n))^2 \leq n^2 + \mu \delta(n)^2 + \delta(n) + (n + \mu \delta(n))^2,$$

and the last term is $\simeq \delta(n)^2$. ∎

The following corollary bends Lemma 6.2 to the needs of Theorem G.

Corollary 6.3. *Let Γ be a finitely presented group and let $A < \Gamma$ be a normal subgroup that is finitely generated and abelian. Regard A as a $\mathbb{Z}\Gamma$-module via conjugation. Suppose that there is a filtration of A by $\mathbb{Z}\Gamma$-submodules*

$$0 = I_0 < I_1 < \cdots < I_k = A$$

such that Γ acts trivially on each of the quotients I_j/I_{j-1} that is infinite. Then the Dehn functions of Γ and Γ/A satisfy

$$\delta_\Gamma(n) \preceq \delta_{\Gamma/A}(n)^{2^k}.$$

In particular, if Γ/A satisfies a polynomial isoperimetric inequality then so does Γ.

Proof. We proceed by induction on k, the length of the filtration. If $k = 1$ then Lemma 6.2 applies. In the inductive step we apply the case $k - 1$ to Γ/I_1, with A/I_1 filtered by $(I_j/I_1)_j$, noting that $\Gamma/A \cong (\Gamma/I_1)/(A/I_1)$.

If I_1 is finite, then the Dehn function of Γ/I_1 is equivalent to that of Γ, by Lemma 2.6, and we are done. If I_1 is infinite, then our hypothesis on the action of Γ implies that $I_1 < \Gamma$ is central. From the induction we know that

$$\delta_{\Gamma/I_1}(n) \preceq \delta_{\Gamma/A}(n)^{2^{k-1}},$$

and Lemma 6.2 implies that $\delta_\Gamma(n) \preceq \delta_{\Gamma/I_1}(n)^2$. This completes the induction. ∎

6.3. Proof of Theorem G

For the convenience of the reader, we repeat the statement of Theorem G.

Theorem 6.4. *If G is a finitely presented group whose maximal metabelian quotient G/G'' is virtually nilpotent, then there is a polynomial $p(x)$ such that*

$$\delta_G(n) \preceq \delta_{\mathfrak{X}(G)}(n) \preceq p \circ \delta_G(n).$$

In particular, G satisfies a polynomial isoperimetric inequality if and only if $\mathfrak{X}(G)$ satisfies a polynomial isoperimetric inequality (of different degree, in general).

Proof. The group G is a retract of $\mathfrak{X}(G)$, whence the leftmost inequality. For the converse, we appeal to Corollary 5.2, which provides us with a subgroup Q_1 of finite index in Q such that $W(G)$ has a filtration by $\mathbb{Z}Q_1$-submodules

$$0 = I_0 < I_1 < I_2 < \cdots < I_k = W$$

such that Q_1 acts trivially on each infinite quotient I_j/I_{j-1} of this filtration. Let $\pi: \mathfrak{X}(G) \to \mathfrak{X}(G)/DL = Q$ be the canonical epimorphism. Then $T = \pi^{-1}(Q_1)$ has finite index in $\mathfrak{X}(G)$, so their Dehn functions are $\simeq$ equivalent.

As G/G'' is finitely generated and virtually nilpotent, its subgroup G'/G'' is finitely generated, and therefore W is finitely generated, by Lemma 2.2. Thus we can appeal to Corollary 6.3 with W in the role of A and T in the role of Γ. From this we deduce that T (hence, $\mathfrak{X}(G)$) will satisfy an isoperimetric inequality of the required form if T/W does. But T/W has finite index in $\mathfrak{X}(G)/W$, and W is the

kernel of the canonical map $\rho \colon \mathfrak{X}(G) \to G \times G \times G$, whose image is a filling subdirect product that is normal with quotient G/G'. Dison's theorem 6.1 tells us that the Dehn function of this subdirect product is bounded above by a polynomial function of the Dehn function of G. This completes the proof. ∎

Corollary 6.5. *Let G be a finitely presented group whose commutator subgroup is perfect. Then $\mathfrak{X}(G)$ satisfies a polynomial isoperimetric inequality if and only if G does.*

Corollary 6.6. *The group $\mathfrak{X}(F)$ has a polynomial isoperimetric function, where F is Thompson's group.*

Proof. The commutator subgroup of F is simple, and Guba [22] proved that the Dehn function of F is quadratic. ∎

Example 6.7. The Dehn functions of semidirect products $G = \mathbb{Z}^k \rtimes_A \mathbb{Z}$ are completely understood. Bridson and Gersten [6] proved that $\delta_G(n)$ is determined by how the entries of the matrix $A \in \mathrm{GL}(k, \mathbb{Z})$ grow under iteration: $\delta_G(n) \simeq n^2 \|A^n\|$. Thus, $\delta_G(n)$ is polynomial if A is virtually unipotent (i.e., all of its eigenvalues are roots of unity) and is exponential otherwise. According to Theorem G, $\mathfrak{X}(G)$ will satisfy a polynomial isoperimetric inequality if G does, and in the other case the Dehn function of $\mathfrak{X}(G)$ will be exponential. In the polynomial case, $\mathfrak{X}(G)$ is virtually nilpotent, by Theorem D. In all cases, $\mathfrak{X}(G)$ is polycyclic [32].

7. n-Engel groups and $\mathfrak{X}(G)$

Recall that a group G is *n-Engel* if $[a, b, \ldots, b] = 1$ for all $a, b \in G$, where b appears n times in the left-normed commutator.

Theorem 7.1. *If G is a finitely generated n-Engel group, then $\mathfrak{X}(G)$ is m-Engel for $m = n + d + s + 3$, where d is the nilpotency class of G/G'' and s is the nilpotency class of the action of G on L/L'.*

Proof. The n-Engel words are defined inductively by

$$\gamma_1(x, y) = [x, y] \quad \text{and} \quad \gamma_{n+1}(x, y) = [\gamma_n(x, y), y].$$

Let $a, b \in \mathfrak{X}(G)$. Since $\mathfrak{X}(G)/W(G)$ is a subgroup of $G \times G \times G$, it is n-Engel, so

$$\gamma_n(a, b) \in W(G).$$

We again consider $W = W(G)$ as a $\mathbb{Z}Q$-module via conjugation, where $Q = G/G'$. In the proof of Theorem C, we constructed a $\mathbb{Z}Q$-submodule W_0 of W such that

$$W(\text{Aug } \mathbb{Z}Q)^{3+s} \subseteq W_0,$$

where Aug $\mathbb{Z}Q$ is the augmentation ideal of $\mathbb{Z}Q$ and W_0 is a $\mathbb{Z}Q$-subquotient of $M = ((G'/G'') \otimes (G'/G''))_{Q_0}$, where $Q = G/G'$ acts by conjugation on the first factor of the tensor product and trivially on the second factor. Thus,

$$\gamma_{n+s+3}(a, b) \in [[[W, b], \ldots], b] \in W(\text{Aug } \mathbb{Z}Q)^{s+3} \subseteq W_0.$$

Note that W_0 is a $\mathbb{Z}Q$-subquotient of M and M depends only on the metabelian group G/G''. Karl Gruenberg [21] proved that every finitely generated soluble n-Engel group is nilpotent, in particular G/G'' is nilpotent, say of class d. It follows that Q acts nilpotently on $A = G'/G''$, more specifically $A(\text{Aug } \mathbb{Z}Q)^d = 0$. From this it follows that Q acts nilpotently on $M = (A \otimes A)_{Q_0}$, indeed writing $\overline{a_1 \otimes a_2}$ for the image of $a_1 \otimes a_2 \in A \otimes A$ in M, for all $q_1, \ldots, q_d \in Q$ and $a_1, a_2 \in A$, we see that

$$\overline{a_1 \otimes a_2}\, (q_1 - 1)(q_2 - 1) \ldots (q_d - 1) = \overline{a_1(q_1 - 1)(q_2 - 1) \ldots (q_d - 1) \otimes a_2}$$

belongs to the image of $A(\text{Aug } \mathbb{Z}Q)^d \otimes A$ in M, which is trivial. Thus,

$$M(\text{Aug } \mathbb{Z}Q)^d = 0,$$

whence $W_0(\text{Aug } \mathbb{Z}Q)^d = 0$ and

$$\gamma_{n+3+s+d}(a, b) \in W_0(\text{Aug } \mathbb{Z}Q)^d = 0. \qquad \blacksquare$$

8. On the group $\mathcal{E}(G)$

Following Lima and Sidki [33], define

$$\mathcal{E}(G) = \langle G, \bar{G} \mid [\mathcal{D}, \mathcal{L}] = 1 \rangle,$$

where

$$\mathcal{D} = \mathcal{D}(G) = [G, \bar{G}],$$
$$\mathcal{L} = \mathcal{L}(G) = \langle \{ g^{-1}\bar{g} \mid g \in G \} \rangle, \quad \mathcal{W} = \mathcal{W}(G) = \mathcal{D} \cap \mathcal{L}.$$

Each of these groups is normal in $\mathcal{E}(G)$; see [33]. Consider the natural epimorphism

$$\theta \colon \mathcal{E}(G) \to \mathcal{X}(G),$$

which restricts to the identity on $G \cup \bar{G}$. Then

$$D = D(G) = \theta(\mathcal{D}) \quad \text{and} \quad L = L(G) = \theta(\mathcal{L}),$$

where D and L are as defined in introduction. Note that $\mathcal{W} = \theta^{-1}(W(G))$ is central in $\mathcal{D}\mathcal{L}$, and

$$\mathcal{E}(G)/\mathcal{D}\mathcal{L} \cong \mathcal{X}(G)/DL \cong G/G'.$$

Kochloukova [30] proved that the circumstances under which $\mathcal{E}(G)$ is finitely presented are much more restricted than for $\mathcal{X}(G)$.

Theorem 8.1 ([30]). *The group $\mathcal{E}(G)$ is finitely presented if and only if G is finitely presented and G/G' is finite. In this case $\mathcal{W}$ and $\mathcal{L}/\mathcal{L}'$ are finitely generated.*

Proof of Proposition J. By Theorem 8.1, if $\mathcal{E}(G)$ is finitely presented then G/G' is finite, so $H := \mathcal{D}\mathcal{L}$ has finite index in $\mathcal{E}(G)$; in particular, H is finitely presented and its Dehn function is $\simeq$ equivalent to that of $\mathcal{E}(G)$. Moreover, $\mathcal{W}$ is central in H and, by Theorem 8.1, $\mathcal{W}$ is finitely generated. Lemma 6.2 tells us that

$$\delta_H(n) \preceq \delta_{H/\mathcal{W}}(n)^2.$$

But $H/\mathcal{W} \cong DL/W$ is isomorphic to the image of $DL < \mathcal{X}(G)$ under $\rho \colon \mathcal{X}(G) \to G \times G \times G$, which is normal with abelian quotient. As G/G' is finite, the image of ρ has finite index in $G \times G \times G$, and hence its Dehn function is $\simeq$ equivalent to $\max\{n^2, \delta_G(n)\}$. ∎

Remark 8.2. If G is infinite, hyperbolic and perfect, then by arguing as in the proof of Proposition A, one can improve the bound in Proposition J and show that in this case $\delta_{\mathcal{E}(G)}(n) \simeq n^2$.

9. Solubility of the word problem in $\mathcal{X}(G)$

We have seen that bounding the complexity of the word problem in $\mathcal{X}(G)$ by means of reasonably efficient isoperimetric inequalities is a subtle challenge; in particular it depends on more than just the Dehn function of G. In contrast, the following theorem shows that the mere existence of a solution to the word problem depends only on the existence of such an algorithm in G. The proof of this result shows, roughly speaking, that the complexity of solving the word problem in $\mathcal{X}(G)$ is bounded by the greater of the complexity of the word problem in G and the complexity of the word problem in the metabelian group $W(G) \rtimes Q$, where the action of $Q = G/G'$ is induced by the conjugation action of G in $\mathcal{X}(G)$.

Theorem 9.1. *Let G be a finitely presented group. The word problem in $\mathfrak{X}(G)$ is soluble if and only if the word problem in G is soluble.*

Proof. Recall from [10], as in Section 2, that if G is finitely generated then L is finitely generated and if G is finitely presented then $\mathfrak{X}(G)$ is finitely presented. Furthermore, if G is finitely presented then by [8], $\text{im}(\rho) < G \times G \times G$ is finitely presented, so in particular W is finitely generated as a normal subgroup of $\mathfrak{X}(G)$.

Any retract of a group with a soluble word problem has a soluble word problem, so the real content of the theorem is the "if" implication: we assume that G has a soluble word problem and must prove that $\mathfrak{X}(G)$ does. Note that if G has a soluble word problem, then so does $G \times G \times G$.

Once again we focus our attention on the subgroups $W < L < \mathfrak{X}(G)$ and the decomposition $\mathfrak{X}(G) = L \rtimes G$ discussed in Section 2.1, as well as the exact sequence

$$1 \to W \to \mathfrak{X}(G) \xrightarrow{\rho} G \times G \times G.$$

We fix a finite generating set $X = A \cup B \cup C$ for $\mathfrak{X}(G)$ with $A \subset W$, $B \subset L$ and $C \subset G$, where A generates W as a normal subgroup, $A \cup B$ generates L, and $C \subset G$ generates G. We then fix a finite presentation $\langle X \mid R \rangle$ for $\mathfrak{X}(G)$. We also fix a finite presentation $\langle Y \mid S \rangle$ for $G \times G \times G$, where $X \subset Y$ generates the image of ρ and S includes both R and $\{a : a \in A\}$.

Given a word w in the free group on X, we determine whether or not it equals the identity in $\mathfrak{X}(G)$ by employing the following algorithm. First, working in the free group $F = F(X)$, making repeated use of the identity $ua = au^a$, we move all occurrences of letters $a_i \in A$ to the left, thus expressing $w \in F$ as a product $w = w_1 w_2$, where w_1 is a word in the letters A and w_2 is a product of conjugates of letters from $B \cup C$.

We can decide whether or not $w_2 = 1$ in $\langle Y \mid S \rangle$ using the hypothesised solution to the word problem in $G \times G \times G$. If $w_2 \neq 1$, then $w \neq 1$ in $\mathfrak{X}(G)$ and we are done. If $w_2 = 1$ in $\langle Y \mid S \rangle$, then the element of $\mathfrak{X}(G) = \langle X \mid R \rangle$ represented by w_2 lies in the kernel of ρ, which is W, and therefore $w \in W$. We chose A so that it generates W as a normal subgroup, so by searching naïvely through equalities in the free group F, we will eventually find a product of conjugates of the letters $a \in A$ that equals w_2 in $\mathfrak{X}(G) = \langle X \mid R \rangle$; in other words, we find an equality in F of the form

$$w_2 = \prod_{i=1}^{N} a_i^{u_i} \prod_{j=1}^{M} r_j^{\theta_j},$$

where $a_i \in A^{\pm 1}$ and $r_j \in R^{\pm 1}$. Note that since $[L, W] = 1$, we may assume the conjugators u_i are words in the letters $C^{\pm 1}$ alone (since conjugation by $A \cup B$ has

no effect in W). Define w_2' to be the first of the two products in this decomposition of w_2.

At this stage, we have transformed w into a word $w' := w_1 w_2'$ that represents the same element of $W < \mathfrak{X}(G)$ and is expressed in the free group $F(A \cup C)$ as a product

$$w' = \prod_{i=1}^{N'} a_i^{v_i},$$

where $a_i \in A^{\pm 1}$ and the v_i are words in the letters $C^{\pm 1}$. We must decide whether or not $w' = 1$ in W.

The final key point to observe is that because the action of G' by conjugation on $W < L$ in $\mathfrak{X}(G) = L \rtimes G$ is trivial, the natural map

$$W \rtimes G \to W \rtimes (G/G')$$

restricts to an injection on W. It follows that $w' = 1$ in $W < \mathfrak{X}(G)$ if and only if the image of w' under the natural map

$$F(A \cup C) \twoheadrightarrow W \rtimes (G/G')$$

is trivial. And we can decide if this last image is trivial because the word problem is soluble in any finitely generated metabelian group; this follows from classical work of Philip Hall [27, 28], who proved that finitely generated metabelian groups are residually finite and finitely presented in the variety of metabelian groups. In fact, since every finitely generated metabelian group can be embedded in a finitely presented metabelian group with polynomial Dehn function [9], such word problems lie in the complexity class NP. ∎

10. $\mathfrak{X}$-closure and growth

In this section we explain how the structure that emerged in the proof of Theorem C leads to the closure criterion isolated in Theorem K, and we use this criterion to show that $\mathfrak{X}$ preserves growth type (Corollary L).

10.1. Proof of Theorem K

For any group G, we have surjections $\mathfrak{X}(G) \twoheadrightarrow \mathrm{im}(\rho_G) \twoheadrightarrow G$; so if $\mathcal{P}$ is closed under quotients, then

$$\mathfrak{X}(G) \in \mathcal{P} \Rightarrow \mathrm{im}(\rho_G) \in \mathcal{P} \quad \text{and} \quad \mathrm{im}(\rho_G) \in \mathcal{P} \Rightarrow G \in \mathcal{P}.$$

We also have $\operatorname{im}(\rho_G) \leq G \times G \times G$; so if $\mathscr{P}$ is closed under subgroups and finite direct products, then

$$G \in \mathscr{P} \Rightarrow \operatorname{im}(\rho_G) \in \mathscr{P}.$$

With these observations in hand, Theorem K and the variations on it stated in the introduction are immediate consequences of the following result.

Proposition 10.1. *Let G be a finitely generated group. If G/G'' is virtually nilpotent, then there is a subgroup of finite index $H \leq \mathfrak{X}(G)$ that can be obtained from a finite-index subgroup of $\operatorname{im}(\rho_G) \leq G \times G \times G$ by a finite sequence of extensions each of which has finite-abelian or finitely-generated-central kernel.*

Proof. As G/G'' is finitely generated and virtually nilpotent, the results in Section 5 apply. In particular, Corollary 5.2 provides us with a subgroup Q_1 of finite index in $Q = \mathfrak{X}(G)/DL \cong G/G'$ and a finite filtration of W by $\mathbb{Z}Q_1$-submodules

$$1 = W_0 \subseteq W_2 \subseteq \cdots \subseteq W_k = W$$

such that Q_1 acts trivially on each of the quotients W_i/W_{i-1} that is infinite. Moreover, since $G'/G'' \leq G/G''$ is finitely generated, we know from [31] that each W_i is finitely generated as an abelian group.

Let $H \leq \mathfrak{X}(G)$ be the preimage of Q_1. This has finite index, so $H/W_k = H/W$ has finite index in $\operatorname{im}(\rho_G)$. Consider the short exact sequence of groups

$$1 \to W_k/W_{k-1} \to H/W_{k-1} \to H/W_k \to 1.$$

There are two options: either W_k/W_{k-1} is finite or it is central in H/W_{k-1}. Repeating this argument with $k-1, \ldots, 0$ in place of k completes the proof. ■

Remark 10.2. It is straightforward to verify that the class of finitely generated virtually nilpotent groups satisfies the version of Theorem K in which only central extensions by finitely generated kernels are allowed. Thus we obtain a second proof of Theorem D at the expense of using a corollary of Theorem C in the new proof.

10.2. Growth: Proof of Corollary L

Let G be a group, let S be a finite generating set for G and let d_S be the associated word metric. The *growth function* $\operatorname{vol}_{G,S}$ counts the number of elements in balls about the identity in G:

$$\operatorname{vol}_{G,S}(n) := |B_{G,S}(n)|,$$

where $B_{G,S}(n) = \{g \in G \mid d_S(1, g) \leq n\}$.

If there exist constants $C, \delta > 0$ such that

$$\mathrm{vol}_{G,S}(n) \leq C n^{\delta}$$

for all $n > 0$, then G is said to have *polynomial growth*. If $\lim_n \mathrm{vol}_{G,S}(n)^{1/n} > 1$, then G has *exponential growth*, and if $\lim_n \mathrm{vol}_{G,S}(n)^{1/n} = 1$ then G has *sub-exponential growth*. If the growth of G is sub-exponential but not polynomial, then G is said to have *intermediate growth*. It is easy to check that these growth types are independent of the chosen generating set S.

Our main interest in Theorem K lies in the following application, which is a restatement of Corollary L. Note that item (2) of this proposition allows one to construct new groups of intermediate growth $\mathfrak{X}_n(G) = \mathfrak{X}(\mathfrak{X}_{n-1}(G))$ starting from well-known examples such as the Grigorchuk group [19], or the Gupta–Sidki groups [25].

Proposition 10.3. *Let G be a finitely generated group. Then,*

- $\mathfrak{X}(G)$ *has polynomial growth if and only if G has polynomial growth;*
- $\mathfrak{X}(G)$ *has subexponential growth if and only if G has subexponential growth.*

Proof. Gromov [20] proved that groups of polynomial growth are virtually nilpotent, and the converse is straightforward. Thus the first part of this result is covered by Theorem D (alternatively, remark 10.2).

Let $\mathcal{P}$ be the class of finitely generated groups of subexponential growth. It is easy to see that $\mathcal{P}$ is closed under quotients, subgroups, extensions by and of finite groups, and finite direct products. And it is well known that if a soluble group is not virtually nilpotent, then it has exponential growth [34, 41]; so the metabelian groups in $\mathcal{P}$ are virtually nilpotent. Thus, before we can apply Theorem K, it only remains to check the closure of $\mathcal{P}$ under central extensions, which is the content of the lemma that follows. ∎

Lemma 10.4. *Let G be a finitely generated group and let $Z < G$ be a finitely generated central subgroup. If G/Z has subexponential growth, then so does G.*

Proof. We fix a generating set $T = \{z_1, \ldots, z_m\}$ for Z and extend this to a generating set $S = \{z_1, \ldots, z_m, y_1, \ldots, y_r\}$ for G. Let $Y = \{y_1, \ldots, y_r\}$, let $H = \langle Y \rangle$, and note that $HZ = G$. We write $\bar{h}$ for the image of $h \in H$ in $\bar{H} := G/Z$. Note that $\bar{H}$ is generated by $\bar{Y} = \{\bar{y}_1, \ldots, \bar{y}_r\}$.

As $d_S(1, hz) \leq d_Y(1, h) + d_T(1, z)$ for all $h \in H$, $z \in Z$,

$$B_{G,S}(n) \subseteq B_{H,Y}(n) B_{Z,T}(n).$$

As Z is abelian, $|B_{Z,T}(n)| \leq C n^{\delta}$ for some constant C, where δ is the torsion-free rank of Z. Thus it suffices to prove that H has sub-exponential growth.

We define a set-theoretic section of $H \to \bar{H}$ by choosing coset representatives $\sigma(\bar{h}) \in H$ for Z in HZ. As $\bar{H}$ has sub-exponential growth, for every $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$\mathrm{vol}_{\bar{H}, \bar{Y}}(n) \le e^{\varepsilon n}$$

for all $n \ge N_\varepsilon$. Let $\zeta_h := h\sigma(\bar{h})^{-1} \in Z$ and let

$$\lambda_\varepsilon = \max\{d_T(1, \zeta_h) \mid h \in H, \ d_Y(1, h) \le N_\varepsilon\}.$$

For all integers $n > 0$ and all $h \in H$ with $d_Y(1, h) \le n$, if $k = \lceil n/N_\varepsilon \rceil$ then $h = h_1 \dots h_k$ for some $h_i \in B_{H,Y}(N_\varepsilon)$, therefore

$$h = \prod_{i=1}^{k} \zeta_{h_i} \sigma(\bar{h}_i) = \prod_{i=1}^{k} \zeta_{h_i} \prod_{i=1}^{k} \sigma(\bar{h}_i).$$

There are at most $|B_{\bar{H}, \bar{Y}}(N_\varepsilon)|^k$ possible values for the product of the $\sigma(\bar{h}_i)$ and at most $C(k\lambda_\varepsilon)^\delta$ values for the product of the $\zeta_{h_i} \in Z$. And

$$|B_{\bar{H}, \bar{Y}}(N_\varepsilon)|^k = \mathrm{vol}_{\bar{H}, \bar{Y}}(N_\varepsilon)^k \le e^{\varepsilon k N_\varepsilon}.$$

Therefore,

$$|B_{H,Y}(n)| \le e^{\varepsilon k N_\varepsilon} \, C(k\lambda_\varepsilon)^\delta.$$

Noting that $kN_\varepsilon < n + N_\varepsilon$, we deduce that

$$\lim_{n \to \infty} \frac{1}{n} \log |B_{H,Y}(n)| \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_n (\mathrm{vol}_{H,Y}(n))^{1/n} = 1$, as required.   ∎

Remark 10.5. Tianyi Zheng [42, Lemma 1.1] proved that the preceding lemma remains true even if Z is not finitely generated.

10.3. The fixed point property for cones

We close with a brief discussion of two classes of groups related to sub-exponential growth and amenability. We shall be concerned only with finitely generated groups, although the theory is more general. Monod [35] defines a group to have *the fixed point property for cones* if actions of G on convex cones in topological vector spaces must, under specified mild hypotheses, have fixed points. He proves that it is equivalent to require that for every non-zero bounded function f on G and all $t_i \in \mathbb{R}$, $g_i \in G$, if $f \ge 0$, then

$$\sum_{i=1}^{n} t_i g_i f \ge 0 \implies \sum_{i=1}^{n} t_i \ge 0.$$

Following Rosenblatt [39], one says that a group G is *supramenable* if for every non-empty $A \subset G$ there is an invariant, finitely additive measure μ on G such that $\mu(A) = 1$. Kellerhals, Monod and Rördam, proved that G is supramenable if and only if the rooted binary tree cannot be Lipschitz-embedded in G; see [29, Proposition 3.4]. Rosenblatt [39], had earlier proved that a supramenable group cannot contain a non-abelian free semigroup.

Groups of sub-exponential growth have the fixed point property for cones, and if a group has the fixed point property for cones then it is supramenable [35]. It is unknown if these implications can be reversed for finitely generated groups. It is also unknown whether either of the latter classes is closed under the formation of finite direct products.

Proposition 10.6. *Let G be a finitely generated group. $\mathfrak{X}(G)$ has the fixed point property for cones if and only if*

$$\mathrm{im}(\rho_G) = \{(g_1, g_2, g_3) \mid g_1 g_2^{-1} g_3 \in [G, G]\} < G \times G \times G$$

has the same property.

Proof. It suffices to prove that the class of groups with the fixed point property for cones satisfies each of the criteria in Theorem K except closure under direct products. Closure under the operations of forming subgroups, quotients and central extensions, as well as extensions by and of finite groups is proved by Monod in [35]. Thus it only remains to show that a finitely generated metabelian group H with the fixed-point property for cones is virtually nilpotent. As discussed above, H is supramenable, so it cannot contain a non-abelian free semigroup. Rosenblatt [39] proved that a finitely generated solvable group without such a semigroup must have polynomial growth, and it is therefore virtually nilpotent [34, 41]. ∎

Remark 10.7. If the class $\mathcal{P}$ of finitely generated groups with the fixed-point property for cones were closed under the taking of finite direct products, we would be able to reverse the implication $\mathrm{im}(\rho_G) \in \mathcal{P} \implies G \in \mathcal{P}$.

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