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Central quotients of biautomatic groups

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Abstract. The quotient of a biautomatic group by a subgroup of the center is shown to be biautomatic. The main tool used is the Neumann–Shapiro triangulation of S^{n-1}, associated to a biautomatic structure on $\boldsymbol{Z}^n$. Among other applications, a question of Gersten and Short is settled by showing that direct factors of biautomatic groups are biautomatic.

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Biautomatic groups form a wide class of finitely presented groups with interesting geometric and computational properties. These groups include all word hyperbolic groups, all fundamental groups of finite volume Euclidean and hyperbolic orbifolds, all braid groups [ECH+92], and all central extensions of word hyperbolic groups [NRa]. A biautomatic group satisfies a quadratic isoperimetric inequality, has a word problem solvable in quadratic time, and has a solvable conjugacy problem. The class of biautomatic groups has several interesting closure properties. For instance, the centralizer of a finite subset of a biautomatic group is biautomatic [GS91]. Also, biautomatic groups are closed under direct products [ECH+92]. The theory of biautomatic groups is briefly reviewed below.

We present a technique for putting biautomatic structures on central quotients of biautomatic groups:

Theorem A. *Let G be a biautomatic group, and let C be a subgroup of $\mathcal{Z}\,G$, the center. Then G/C is biautomatic.*

Theorem A has several applications. Our first application answers a questions posed by Gersten and Short ([GS91], cf. proposition 4.7):

Theorem B. *Direct factors of biautomatic groups are biautomatic.*

Proof. Suppose $G \times H$ is biautomatic. The centralizer of H is $G \times \mathcal{Z}\,H$, and this

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is a biautomatic group by [GS91] corollary 4.4. Then $\mathcal{Z}\,H$ is a subgroup of the center of $G \times \mathcal{Z}\,H$, so by theorem A, $G \times \mathcal{Z}\,H/\mathcal{Z}\,H = G$ is biautomatic. $\square$

Several recent discoveries have pointed to the useful concept of *poison subgroups*: certain classes of groups cannot occur as subgroups of word hyperbolic groups. For instance, the group $\boldsymbol{Z}^2$ is poison to word hyperbolicity. More generally, any group which has an infinite index central $\boldsymbol{Z}$ subgroup is poison to word hyperbolicity ([CDP90], corollaire 7.2). Our next theorem says that among biautomatic groups, the latter class of poison subgroups occurs no more generally than $\boldsymbol{Z}^2$:

Theorem C. *If the biautomatic group G contains a subgroup with an infinite index central $\boldsymbol{Z}$ subgroup, then G contains a $\boldsymbol{Z}^2$ subgroup.*

Proof. The hypothesis says that G has an infinite cyclic subgroup Z of infinite index in its centralizer $\mathcal{C}_Z$. The group $\mathcal{C}_Z$ is biautomatic by [GS91] corollary 4.4. Since Z is central in $\mathcal{C}_Z$, then by theorem A the group $\mathcal{C}_Z/Z$ is biautomatic. This group is infinite, so by [ECH$^+$92] example 2.5.12, it has an element of infinite order. Any infinite cyclic subgroup of $\mathcal{C}_Z/Z$ pulls back to a $\boldsymbol{Z}^2$ subgroup of $\mathcal{C}_Z < G$. $\square$

Theorem C raises the stakes on the question of whether biautomatic groups satisfy an analogue of Thurston's hyperbolization conjecture: is it true that every biautomatic group either is word hyperbolic or has a $\boldsymbol{Z}^2$ subgroup? This question can now be restated as follows: if every infinite cyclic subgroup in a biautomatic group has finite index in its centralizer, is the group word hyperbolic?

Gersten and Short ask whether a biautomatic group can have an infinitely generated abelian subgroup ([GS91], p. 154). We can reduce this problem as follows:

Theorem D. *Suppose there is a biautomatic group with an infinitely generated abelian subgroup. Then either there is a biautomatic group with an infinite rank abelian subgroup, or there is a biautomatic group with an infinite abelian torsion subgroup.*

Proof. Suppose the biautomatic group G has an abelian subgroup H, infinitely generated and of finite rank $n \geq 0$. If $n \geq 1$, choose an element $h \in H$ of infinite order. By [GS91] corollary 4.4, the centralizer $\mathcal{C}_h$ of h in G is biautomatic. Let $\mathcal{Z}\,\mathcal{C}_h$ be the center of $\mathcal{C}_h$. By theorem A, $\mathcal{C}_h/H \cap \mathcal{Z}\,\mathcal{C}_h$ is biautomatic. Note that $h \in H \cap \mathcal{Z}\,\mathcal{C}_h < H < \mathcal{C}_h$, so the image of H in $\mathcal{C}_h/H \cap \mathcal{Z}\,\mathcal{C}_h$ is an infinitely generated, abelian subgroup of rank $\leq n-1$. By induction, we obtain a biautomatic group Γ with an infinitely generated abelian subgroup of rank 0, i.e. an infinite abelian torsion subgroup. $\square$

Remark. If a group G has an infinite abelian torsion subgroup A then G is not virtually torsion free, for if $K < G$ were a torsion free subgroup of finite index, then there would be two elements $a \neq b \in A$ such that $1 \neq b^{-1}a \in K$, hence $b^{-1}a$ has infinite order; but $b^{-1}a \in A$ has finite order.

Dani Wise has produced biautomatic groups which are not virtually torsion free [Wis95].

Remark. Theorem A has been sharpened in recent work of Neumann and Reeves [NRb], who show that if C is a central subgroup of a biautomatic group G then the central extension $1 \to C \to G \to G/C \to 1$ is defined by a "regular cocycle" of G/C. They have also proved a converse: if H is biautomatic, and if a central extension $1 \to C \to G \to H \to 1$ is defined by a regular cocycle of H, then G is biautomatic. See also [NRa] where the converse is used to prove that every central extension of a word hyperbolic group is biautomatic.

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Proof of theorem A

First we reduce theorem A to a special case:

Theorem E. *If G is a biautomatic group and $Z < G$ is an infinite cyclic central subgroup, then G/Z is biautomatic.*

Proof of theorem A. Let G be a biautomatic group, and let C be a subgroup of the center $\mathcal{Z}\,G$. Since $\mathcal{Z}\,G$ is biautomatic it is finitely generated. Hence C is a finite rank central subgroup, say of rank $k \geq 0$. Now peel off factors of $\boldsymbol{Z}$ one at a time, as in the proof of [GS91] proposition 4.7. If $k \geq 1$ let Z be any infinite cyclic subgroup of C. Applying theorem E it follows that G/Z is biautomatic, and C/Z is a central subgroup of rank $k-1$. Repeating this argument k times, we see that there is a finite index free abelian subgroup $C' < C$ such that C/C' is a central finite subgroup in the biautomatic group G/C'. But the quotient of any biautomatic group by any finite normal subgroup is easily seen to be biautomatic; projecting the biautomatic structure from the total group to the quotient group gives a biautomatic structure on the quotient. Hence, $G/C = (G/C')/(C/C')$ is biautomatic. $\square$

The remainder of the paper is devoted to proving theorem E.

Review of biautomatic groups

An *alphabet* is a finite set. A *word* over an alphabet $\mathcal{A}$ is a finite sequence of elements in $\mathcal{A}$. The empty word is sometimes denoted ϵ. The set of all words in $\mathcal{A}$ is denoted $\mathcal{A}^*$, and this forms a monoid under the operation of concatenation, with ϵ as the identity. A *language over* A is a subset of $\mathcal{A}^*$.

The length of a word w is denoted ℓ_w. If w is written in the form $w = w_1w_2$ then w_1 is called a *prefix* subword and w_2 is a *suffix* subword. If w is written as $w = w_1w_2w_3$ then w_2 is called an *infix* subword, with associated prefix w_1 and suffix w_3. For any integer $t \geq 0$, $w(t)$ denotes the prefix subword of w of length t if $t \leq \ell_w$, and $w(t) = w$ otherwise.

We adopt the graph theoretic notion of a *finite state automaton* over an alphabet $\mathcal{A}$. This is a finite directed graph M whose vertices are called *states*, together with a labelling of each edge by a letter of $\mathcal{A}$, a specified state s_0 called the *start state*, and a specified subset of states called the *accept states*, such that each state has exactly one outgoing edge labelled with each letter of $\mathcal{A}$. A *failure state* is any state which is not an accept state. A *path* in M is always a directed path. Concatenation of paths is denoted by juxtaposition. If $\ell = \ell_\pi$ is the length of π, then the states of M visited by π are denoted $\pi[0], \ldots, \pi[\ell]$, and the subpath from $\pi[s]$ to $\pi[t]$ is denoted $\pi[s,t]$. Reading off the letters on the edges of π in succession yields a word $w_\pi = (a_1 \cdots a_\ell)$ where a_i is the label on the edge $\pi[i-1,i]$. For any word w and any state s, there is a unique directed path π starting at s such that $w = w_\pi$; if $s = s_0$ then we denote this path by π_w. When circumstances require, we shall also denote w_π by $w(\pi)$ and π_w by $\pi(w)$. The set of all words w such that π_w ends at an accept state forms a language over $\mathcal{A}$ denoted $L(M)$.

A language L over $\mathcal{A}$ is called *regular* if there exists a finite state automaton M over $\mathcal{A}$ such that $L = L(M)$. We say that M is a *word acceptor* for L.

Given a finite state automaton M, an *accepted path* is any path from the start state to an accept state. A *live state* is any state lying on an accepted path. A *dead end state* is any state such that all arrows pointing out of that state point directly back into it; note that a dead end state may be a live state. Any path which begins and ends at live states is called a *live path*; note that all interior states of a live path are live states. A *loop* is a path which begins and ends at the same vertex, so a live loop is a loop passing over live states only. Given a loop π, if we write $\pi = \pi_1\pi_2$ then $\pi_2\pi_1$ is also a loop, called a *cyclic permutation* of π.

The basic definition of biautomatic groups involves 2-variable languages (see for example [ECH$^+$92] p. 24, or [GS91] p. 135). For our purposes the equivalent geometric definition of biautomatic groups will suffice ([ECH$^+$92] lemma 2.5.5), so we shall not use 2-variable languages.

Consider a group G, an alphabet $\mathcal{A}$, and a map $\mathcal{A} \to G$. This induces a monoid homomorphism $\mathcal{A}^* \to G$ denoted $w \to \overline{w}$. Given $a \in \mathcal{A}$, we often omit the overline and consider a as an element of G, even if $\mathcal{A} \to G$ is not injective. Thus, abusing terminology, $\mathcal{A}$ is called a *generating set* for G if $\mathcal{A}^* \to G$ is onto. Also, any language $L \subset \mathcal{A}^*$ that maps onto G is called a set of *normal forms* for G. If M is a finite deterministic automaton over $\mathcal{A}$, then for any path π in M the group element $\overline{w}_\pi$ is also denoted $\overline{\pi}$.

Given a generating set $\mathcal{A}$ for G, for each $g \in G$ we define the *word length* of g to be $|g| = \mathrm{Min}\{\ell_w \mid \overline{w} = g\}$, and we define the *word metric* on G by $d(g,h) = |g^{-1}h|$.

A *biautomatic structure* for G consists of a generating set $\mathcal{A}$ for G, and a set of normal forms $L \subset \mathcal{A}^*$ for G, with the following properties:

- L is a regular language
- There exists a constant $K \geq 0$ such that for each $v, w \in L$ and each $a \in \mathcal{A} \cup \{\epsilon\}$, if $\overline{v} = \overline{w}a$ then for all $t \geq 0$,

$$d(\overline{v}(t), \overline{w}(t)) \leq K$$

and if $a\overline{v} = \overline{w}$ then for all $t \geq 0$,

$$d(a\overline{v}(t), \overline{w}(t)) \leq K.$$

The constant K is called a *two-way fellow traveller constant* for the biautomatic structure L (to contrast with an automatic structure, in which only the first inequality is required). As a consequence, for each $v, w \in L$ and any words $\mu, \nu \in \mathcal{A}^*$, if $\overline{\mu}\,\overline{v} = \overline{w}\,\overline{\nu}$, then

$$d(\overline{\mu}\,\overline{v}(t), \overline{w}(t)) \leq K(|\mu| + |\nu|)$$

for all $t \geq 0$.

We shall need the result of [ECH$^+$92] theorem 2.5.1, that any biautomatic structure on a group G has a sublanguage which is a biautomatic structure with uniqueness, meaning that each element of G has a unique normal form.

Now we review several results of [GS91] concerning subgroups of biautomatic groups; these results will be used without comment in what follows. Let L be a biautomatic structure on a group G. A subgroup $H < G$ is called *rational* if the language $\{w \in L \mid \overline{w} \in H\}$ is regular. If this is so, then H is a biautomatic group ([GS91], theorems 3.1 and 2.2). The centralizer of a subset $S \subset G$ is denoted $\mathcal{C}_S$; and $\mathcal{C}_G$, the center of G, is specially denoted $\mathcal{Z}\,G$. If G is biautomatic and S is a finite set or a finitely generated subgroup, then $\mathcal{C}_S$ is rational ([GS91] proposition 4.3); thus, the subgroups $\mathcal{C}_S$, $\mathcal{Z}\,G$ and $\mathcal{Z}\,\mathcal{C}_S$ are rational, and it follows that all these subgroups are biautomatic.

For the remainder of the paper, fix a central extension

$$1 \to Z \to G \to H \to 1$$

where $Z = \langle z \rangle$ is an infinite cyclic, central subgroup of G. Also fix a generating set $\mathcal{A}$. Note that $\mathcal{A}$ projects to a generating set for H as well, and the projection map $G \to H$ does not increase the word metric. Let L be a biautomatic structure with uniqueness for G over $\mathcal{A}$. Let M be the word acceptor automaton for L.

A biautomatic structure for $G/Z = H$

Define a *central loop* in the automaton M to be any live loop representing an element of the center $\mathcal{Z}\,G$. We consider two central loops to be the same if they are cyclic permutations of each other.

Simplicity Lemma (cf. [NS92], lemma 3.1)**.** *Let γ be a central loop in M. Then γ is an iterate of a simple loop in M, and every other simple loop in M is disjoint from γ.*

Proof. If γ is not an iterate of a simple loop, then after cyclically permuting γ, there exist loops μ, ν with the same initial state as γ, such that $\gamma = \mu\nu \neq \nu\mu$. Since γ is central, it follows that $\overline{\nu}\,\overline{\mu} = \overline{\mu}^{-1}\overline{\gamma}\,\overline{\mu} = \overline{\gamma} = \overline{\mu}\,\overline{\nu}$. Choose an accepted path $\pi = \pi_1\pi_2$ concatenated at the common initial state of the loops μ, ν, γ. Then $\pi_1\mu\nu\pi_2$ and $\pi_1\nu\mu\pi_2$ are distinct accepted paths representing $\overline{\pi}\,\overline{\gamma}$, violating uniqueness of L.

If there is another simple loop γ' in M intersecting γ, then after cyclic permutations we may assume that γ and γ' have the same base vertex. Since γ is central then $\overline{\gamma}\,\overline{\gamma}' = \overline{\gamma}'\,\overline{\gamma}$, but $\gamma\gamma' \neq \gamma'\gamma$. Now proceed as above. □

A central loop is *primitive* if it is not an iterate of a shorter central loop. Note that a primitive central loop does not have to be a simple loop in M. A path π in M is said to be *compatible with* a set of primitive central loops $\{\gamma_1, \dots, \gamma_I\}$ if π intersects each γ_i. The set $\{\gamma_1, \dots, \gamma_I\}$ is *live* if it is compatible with an accepted path. Define a *central cycle* in M to be any formal linear combination with positive integer coefficients of a live set of central loops, $c = n_1\gamma_1 + \cdots + n_I\gamma_I$. The element $\overline{c}$ is defined to be $\overline{\gamma}_1^{n_1} \cdots \overline{\gamma}_I^{n_I}$.

If a path π is compatible with a central cycle $c = n_1\gamma_1 + \cdots + n_I\gamma_I$, then we may combine π and c into a well-defined path as follows. Choose $t_1, \dots, t_I$ so that t_i is the minimal integer with $\pi(t_i) \in \gamma_i$. Since these numbers are distinct by the *Simplicity Lemma*, we may reindex so that $t_1 < t_2 < \cdots < t_I$. Now take a cyclic permutation of γ_i so that it is based at the point $\pi[t_i]$; this gives a well-defined loop, since γ_i is an iterate of a simple loop. Then the path

$$\pi{*}{*}c = \pi[0, t_1]{*}\gamma_1^{n_1}{*}\pi[t_1, t_2]{*}\cdots{*}\pi[t_{I-1}, t_I]{*}\gamma_I^{n_I}{*}\pi[t_I, \ell_\pi]$$

is well-defined. If π is an accepted path then $\pi{*}{*}c$ is an accepted path representing $\overline{\pi}\,\overline{c}$. We say that a path q *contains* the central cycle c if there exists a path π compatible with c such that $q = \pi{*}{*}c$.

A subset of an abelian group is *linearly independent* if the identity cannot be expressed as a non-trivial integer linear combination of elements in the set. Note that a linearly independent set cannot contain torsion elements.

Independence Lemma (cf. [NS92], p. 451)**.** *If $\{\gamma_1, \dots, \gamma_I\}$ is a live set of central loops, then $\{\overline{\gamma}_1, \dots, \overline{\gamma}_I\}$ is a linearly independent subset of $\mathcal{Z}\,G$.*

Proof. Let π be any accepted path compatible with $\{\gamma_1, \ldots, \gamma_I\}$. If the lemma is false, there is an equation with positive integer exponents of the form

$$\overline{\gamma}_{i_1}^{m_{i_1}} \cdots \overline{\gamma}_{i_A}^{m_{i_A}} = \overline{\gamma}_{j_1}^{n_{j_1}} \cdots \overline{\gamma}_{j_B}^{n_{j_B}}$$

where $\gamma_{i_a} \neq \gamma_{j_b}$ for $1 \leq a \leq A$, $1 \leq b \leq B$. Let c_1, c_2 be the central cycles given by the two sides of this equation, e.g. $c_1 = m_{i_1}\gamma_{i_1} + \cdots + m_{i_A}\gamma_{i_A}$. Then $\pi{*}{*}c_1$ and $\pi{*}{*}c_2$ are distinct accepted paths representing the same element of G, contradicting the uniqueness property for L. $\square$

Now define a *Z-cycle* to be a central cycle representing an element of Z. A *primitive Z-cycle* is a Z-cycle which is not a positive multiple of any other Z-cycle except for itself. Note that a primitive Z-cycle may be a positive multiple of some central cycle which is not a Z-cycle. Recalling that $Z = \langle z \rangle$, if c is a Z-cycle with $\overline{c} = z^n$ where $n \geq 1$ then c is said to be a *positive* Z-cycle.

Uniqueness Corollary. *There are only finitely many primitive Z-cycles, and an accepted path can be compatible with at most one of them.*

Proof. There are only finitely many live sets of central loops, and by the *Independence lemma* each one has at most one positive linear combination which is a primitive Z-cycle. If an accepted path p is compatible with two distinct primitive Z-cycles, then those two cycles taken together give a live set of central loops which forms a linearly dependent subset of $\mathcal{Z}\,G$, contradicting the *Independence lemma.* $\square$

Define a sublanguage $L_H \subset L$ to consist of all words $w \in L$ such that w is compatible with some positive Z-cycle but w contains no Z-cycle. We will prove that L_H projects to a biautomatic structure on H, but first here is an example.

Consider the group $G = \boldsymbol{Z} \oplus \boldsymbol{Z} = \langle a, b \mid [a, b] = 1\rangle$ with generating set $\mathcal{A} = \{a, b, A = a^{-1}, B = b^{-1}\}$, and with the biautomatic structure.

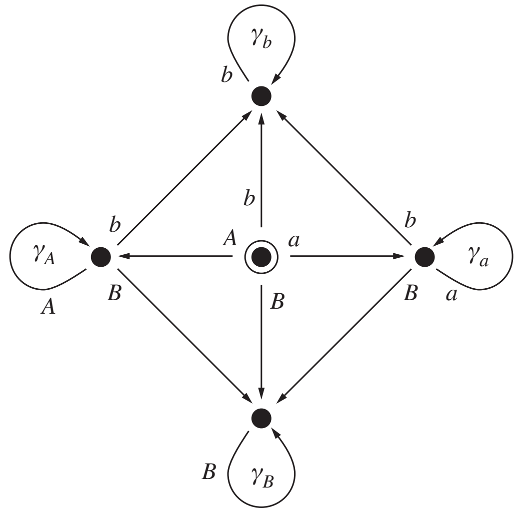
$$L = \{a^m b^n, a^m B^n, A^m b^n, A^m B^n \mid m, n \geq 0\}.$$

A word acceptor for L is shown in the figure below. The primitive central loops in this example are γ_a, γ_b, γ_A, γ_B. The live sets of primitive central loops are

$$\{\gamma_a\}, \{\gamma_b\}, \{\gamma_A\}, \{\gamma_B\}, \{\gamma_a, \gamma_b\}, \{\gamma_a, \gamma_B\}, \{\gamma_A, \gamma_b\}, \{\gamma_A, \gamma_B\}.$$

We now consider several examples of infinite cyclic subgroups $Z \subset \boldsymbol{Z} \oplus \boldsymbol{Z}$, and in each case we describe L_H where $H = \boldsymbol{Z} \oplus \boldsymbol{Z}/Z$.

If $Z = \langle a \rangle$ then the only positive, primitive Z-cycle is γ_a, and in this case $L_H = \{ab^n, aB^n \mid n \geq 0\}$. If $Z = \langle a^k \rangle$ then the only positive, primitive Z-cycle is $k\gamma_a$, and $L_H = \{a^i b^n, a^i B^n \mid 1 \leq i \leq k, n \geq 0\}$.



The central state is the start state. Any missing directed edges lead to a dead end failure state.

If $Z = \langle ab \rangle$, then the only positive, primitive Z-cycle is $\gamma_a + \gamma_b$, and $L_H = \{ab^n, a^n b \mid n \geq 1\}$. If $Z = \langle a^k b^l \rangle$ with $l, k \geq 1$ then the only positive, primitive Z-cycle is $k\gamma_a + l\gamma_b$, and in this case

$$L_H = \{a^i b^n \mid 1 \leq i \leq k, n \geq 1\} \cup \{a^n b^j \mid n \geq 1, 1 \leq j \leq l\}.$$

Returning to the general setting, the proof that L_H projects to a biautomatic structure on H proceeds in three steps. Step 1 proves that L_H is a regular language. Step 2 proves that each coset of Z is represented by some element of L_H. Step 3 is the two-way fellow traveller property.

Step 1: Regularity of L_H

In one special case the proof of regularity is particularly simple. Namely, suppose that each primitive Z-cycle is actually a simple loop of length 1 in the automaton M. It is easy to construct a new automaton which accepts exactly those accepted words of M that touch some Z-loop but do not go around it. First take three separate copies of M denoted M_n for "not touched", M_t for "touched", and M_a for "around"; we imagine these stacked one atop the other. For each edge E of M_n that starts outside a Z-loop and ends on a Z-loop, detach the forward end of E from M_n and attach it to the corresponding state in M_t. For each Z-loop in M_t, detach its forward end from M_t and attach it to the corresponding state in M_a. The result is an automaton M'. For each accept state in M, the corresponding state of M_t is also an accept state of M'; all other states of M' are failure states. The automaton M' is a word acceptor for L_H.

In general, a Z-cycle may be a linear combination of non-simple loops. We show that L_H is regular by reformulating the definition of L_H as a regular predicate, and then applying [ECH$^+$92] theorem 1.4.6. For any central loop γ in M, the language $L_\gamma = \{w \mid \pi_w \cap \gamma \neq \emptyset\}$ is regular, because we may alter M by turning each state lying on γ into a dead end accept state, and the new automaton recognizes L_γ. Also, the set of words $L^+_\gamma = \{w \mid \pi_w \text{ contains } \gamma\}$ is regular, because we may modify M by keeping track not only of the state in M visited by $\pi_w(t)$, but also of the longest subpath of a cyclic permutation of γ traversed by $\pi_w(t)$; this can clearly be done with a finite state automaton.

For each primitive Z-cycle $c = n_1\gamma_1 + \cdots + n_P\gamma_P$, the language $L_c = \{w \mid \pi_w$ is compatible with $c\}$ is the same as $L_{\gamma_1} \cap \cdots \cap L_{\gamma_P}$, hence is regular. Similarly, the language $L^+_c = \{w \mid \pi_w \text{ contains } c\}$ is the same as $L^+_{\gamma_1^{n_1}} \cap \cdots \cap L^+_{\gamma_P^{n_P}}$ hence is regular.

Finally, let $c_1, \ldots, c_N$ be the finite list of all primitive, positive Z-cycles. By the *Uniqueness corollary*, if $w \in L_{c_1}$ then the only possible Z-cycle that π_w may contain is c_1. Thus,

$$L_H = L \cap \left[(L_{c_1} \cap \neg L^+_{c_1}) \cup \cdots \cup (L_{c_N} \cap \neg L^+_{c_N})\right]$$

so L_H is regular.

Step 2: L_H represents each coset of Z in G

For this argument, fix an element $g \in G$. We must show that the coset gZ is represented by some word in L_H. The proof will depend on the properties of the Neumann–Shapiro triangulation of the boundary of an automatic structure on an abelian group.

First we make a reduction: it suffices to construct a word $w \in L$ representing gZ such that π_w contains some positive Z-cycle. For then we may write $\pi_w = \pi_v {**} c^m$ where c is a positive, primitive Z-cycle and m is as large as possible. Then π_v does not contain c, and yet π_v is compatible with c, so by the *Uniqueness corollary* π_v does not contain any other Z-cycle. Hence $v \in L_H$ and v represents gZ.

Let $C = \mathcal{Z}\,\mathcal{C}_g$, and note that C contains both $\mathcal{Z}\,G$ and gZ. We review the biautomatic structure on C induced by that on G. Let $L_C \subset L$ be the regular sublanguage of words w with $\overline{w} \in C$. From the proof of [GS91] theorem 3.1, it follows that there is a generating set $\mathcal{B}$ for C, a biautomatic structure L' for C over $\mathcal{B}$, and a map $\phi\colon \mathcal{B} \to \mathcal{A}^*$, such that the induced map $\mathcal{B}^* \to \mathcal{A}^*$, also denoted ϕ, restricts to a surjection from L' to L_C. By [ECH$^+$92] theorem 2.5.1, we may replace L' by a sublanguage which is a biautomatic structure with uniqueness for C, and hence the map $\phi\colon L' \to L_C$ is a bijection. Let M' be a word acceptor for L' over $\mathcal{B}$. Then we may speak about Z-cycles in M'.

Now we make another reduction. We shall prove that the coset gZ is represented by an accepted path π in M', so that π is compatible with some positive Z-cycle $c' = n_1\gamma_1 + \cdots + n_I\gamma_I$ in M'.

Accepting this for the moment, we use it to complete step 2. Write $\pi = \pi_0\pi_1\cdots\pi_I$ so that for each $k \geq 0$, the path

$$\pi{*}{*}c'^k = \pi_0\gamma_1^{kn_1}\pi_1\cdots\pi_{I-1}\gamma_I^{kn_I}\pi_I$$

is an accepted path representing gZ. Under the mapping $\phi\colon \mathcal{B}^* \to \mathcal{A}^*$, the word $w(\pi{*}{*}c'^k) \in L'$ goes to a word ρ^k in the language $L_C \subset L$. For each k the word ρ^k represents gZ. For each k and for $1 \leq i \leq I$, let t_i^k be the moment of time at which $\pi{*}{*}c'^k$ completes the loop $\gamma_i^{kn_i}$. Note that the image of $w\big((\pi{*}{*}c'^k)[0,t_k^k]\big)$ under ϕ is a prefix of ρ^k, denoted ρ_i^k. Let s_i^k be the state of M at which the path $\pi(\rho_i^k)$ ends. Thus, for each k we obtain an I-tuple of states in M denoted $Q^k = (s_1^k,\ldots,s_I^k)$. There must be two values $k_1 < k_2$ with $Q^{k_1} = Q^{k_2}$. It follows that $\pi(\rho^{k_2}) = \pi(\rho^{k_1}){*}{*}c$ for some positive Z-cycle c in M representing $\overline{c}'^{k_2-k_1} \in Z$. Thus, $\pi(\rho^{k_2})$ is an accepted path in M representing gZ and containing a positive Z-cycle, as required to complete step 2.

Now we review the result of Neumann–Shapiro, [NS92] theorem 1.1, which associates to each automatic structure on the abelian group C, a simplicial decomposition of the boundary. While their result is only stated when C is free abelian, we note that their construction is valid more generally when C is abelian.

Fix an identification $C = \boldsymbol{Z}^k \oplus F$ for some finite abelian group F. We shall sometimes confuse an element of C with its projection onto $\boldsymbol{Z}^k$. Each non-torsion $c \in C$ determines a ray in $\boldsymbol{Z}^k$ whose direction is denoted $[c] \in S^{k-1}$. Neumann and Shapiro associate, to a biautomatic structure L' on C, a rational linear ordered simplicial subdivision Σ of S^{k-1}, as follows. Each state of the word acceptor M' lies on at most one simple loop of M' (see [NS92] lemma 3.1, or the *Simplicity lemma*). Let π be a simple live path in M' initiating at the start state s_0. Let s_1 be the first state on π which lies on a simple loop, and let γ_1 be that loop. Inductively, let s_i be the first state of π after s_{i-1} which lies on a simple loop distinct from γ_{i-1}, and let γ_i be that loop. This induction ends with s_l, and let s_{l+1} be the final state of π. Note that $\{\overline{\gamma}_1,\ldots,\overline{\gamma}_l\}$ is a linearly independent set in C (see [NS92] p. 451, or the *Independence lemma*). We may now define a rational linear ordered $(l-1)$-simplex in S^{k-1}, namely $\sigma_\pi = \langle[\overline{\gamma}_1],\ldots,[\overline{\gamma}_l]\rangle$. Neumann and Shapiro prove that as π varies over all simple paths in M', the collection $\Sigma = \{\sigma_\pi\}$ is an ordered simplicial subdivision of S^{k-1}.

Since a group element determines a ray, we need to know the relation between that element and the simplex at infinity which the ray hits. Let π be as in the previous paragraph, and let $\pi_i = \pi[s_{i-1}, s_i]$, so $\pi = \pi_1\pi_2\cdots\pi_{l+1}$. Define $\pi(n_1,\ldots,n_l) = \pi_1\gamma_1^{n_1}\pi_2\cdots\pi_l\gamma_l^{n_l}\pi_{l+1}$. Note that each element of C is uniquely represented by a path of the form $\pi(n_1,\ldots,n_l)$, for some simple accepted path π and some $n_1,\ldots,n_l \geq 0$. Fix the "visual" metric on S^{k-1}, where the distance between two rays is equal to the angle they subtend. Although we cannot guarantee that the ray $[\overline{\pi}(n_1,\ldots,n_l)]$ hits the simplex σ_π, the following lemma says that it comes visually close:

Visual Lemma. *For each $\epsilon > 0$ there exists a ball $B \subset C$ around the origin such that if $\overline{\pi}(n_1, \dots, n_l) \notin B$ then the visual distance between $[\overline{\pi}(n_1, \dots, n_l)]$ and the point $[\overline{\gamma_l^{n_1}} \cdots \overline{\gamma_l^{n_l}}] \in \sigma_\pi$ is smaller than ϵ.*

Proof. This follows from a geometric principle: as a person walks away from you at the beach, they appear to get smaller and smaller. This principle applies in $\boldsymbol{Z}^n$, and so also in C which is quasi-isometric to $\boldsymbol{Z}^n$. More precisely, for all $\epsilon > 0$ and all $\delta \geq 0$ there exists a ball $B \subset C$ around the origin, such that if $X \subset C$ has diameter at most δ, and if X is not contained in B, then X has visual diameter less than ϵ.

Let δ be the length of the longest simple path in M'. Since

$$\overline{\pi}(n_1, \dots, n_0)^{-1}\big(\overline{\gamma_1^{n_1}} \cdots \overline{\gamma_l^{n_l}}\big) = \overline{\pi}^{-1}$$

it follows that $d\big(\overline{\pi}(n_1, \dots, n_l), \overline{\gamma_1^{n_1}} \cdots \overline{\gamma_l^{n_l}}\big) \leq \delta$. Now choose B according to the above principle, and take $X = \big\{\overline{\pi}(n_1, \dots, n_0), \overline{\gamma_1^{n_1}} \cdots \overline{\gamma_l^{n_l}}\big\}$ to finish the proof. $\square$

To complete step 2, recall that $Z = \langle z \rangle$, and let Star$[z]$ be the union of those simplices of Σ that contain $[z]$. Noting that $[z] \in \operatorname{int}(\operatorname{Star}[z])$, choose ϵ so small that every point of S^{k-1} within visual distance 2ϵ of $[z]$ is contained in $\operatorname{int}(\operatorname{Star}[z])$. Choose a positive integer m so large that $[gz^m]$ is within visual distance ϵ of $[z]$, and so that gz^m lies outside the ball B given by the *Visual lemma*. Now gz^m is represented by a path in M' of the form $\pi(n_1, \dots, n_l)$, for some simple accepted path π and some $n_1, \dots, n_l \geq 0$ as above. By the *Visual lemma* it follows that $\big[\overline{\gamma_1^{n_1}} \cdots \overline{\gamma_l^{n_l}}\big] \in \operatorname{int}(\operatorname{Star}[z])$, hence $[z] \in \sigma_\pi$. Therefore there is a positive Z-cycle c obtained as a linear combination of $\gamma_1, \dots, \gamma_l$. This shows that gz^m is represented by an accepted path $\pi(n_1, \dots, n_l)$ in M' compatible with the positive Z-cycle c, finishing the proof that the coset gZ is represented by the language L_H.

Step 3: The two-way fellow traveller property for L_H

To prove this, consider $v, w \in L_H$ and any $a, b \in \mathcal{A} \cup \{\epsilon\}$, and assume that $a\overline{v}$ and $\overline{w}b$ are congruent modulo Z. Then $a\overline{v} = \overline{w}bz^\beta$ for some β. We shall give a bound $|\beta| \leq B$, where the constant B depends only on Z and on the biautomatic structure on G. If K is a two-way fellow traveller constant for L, it follows that $d(a\overline{v}(t), \overline{w}(t)) \leq K' = (B|z| + 2)K$ for all $t \geq 0$. Thus, K' is a two-way fellow traveller constant for L_H in G, and so also in H. Henceforth, we can and shall assume $\beta \geq 0$.

To give the idea of the proof, we first sketch the case where Z is a rational subgroup of G. Then each Z-cycle in G is a loop, for by [NS92] theorem 3.4, the two ends of Z in the sphere at infinity of $\mathcal{Z}\,G$ are vertices of the Neumann–Shapiro triangulation; and if there were a Z-cycle which was not a loop then that Z-cycle would determine a simplex σ of the triangulation such that the interior of σ

contains an end of Z, contradiction. Thus, we can write $\pi_w = \pi_1\pi_2$, concatenated at a vertex that lies on a primitive Z-loop γ. For simplicity, suppose that γ represents z itself, not a power (at worst, γ represents a bounded power of z). Then $\overline{w}z^\beta$ is represented by the accepted path $\pi' = \pi_1\gamma^\beta\pi_2$. Let $w' = w_{\pi'} = w_{\pi_1}w_\gamma^\beta w_{\pi_2}$. Let k be the length of γ, which is bounded independent of γ. Since $a\overline{v} = \overline{w}'b$ and $v, w' \in L$, then av and $w'b$ are fellow travellers. Assuming by contradiction that β is very large, it follows that v has a long subword v' that fellow travels the subword w_γ^β of w'. Travelling along v' and w_γ^β, at every k^{th} vertex we keep track of two pieces of data: the state of M visited by v', and the word difference between v' and w_γ^β. This data takes values in a finite set, so if β is large enough the data is repeated at two different spots on v'. The subword between these two spots traces out a loop in M, because the states are repeated; and this subword represents an element of Z, because the word difference with powers of $w(\gamma)$ is repeated. Thus, we have shown that v contains a Z-loop, contradicting the fact that $v \in L_H$. This contradiction shows that β cannot be too large, completing the sketch in the case that Z is rational.

In the general case, the path π_w can be written in the form $\pi_1\pi_2\ldots\pi_p\pi_{p+1}$, with π_j and π_{j+1} concatenated at a vertex V_j, so that there is a primitive central loop γ_j based at V_j, and there is a primitive Z-cycle $c = n_1\gamma_1 + \cdots + n_p\gamma_p$ representing z^α, where $0 < \alpha \leq A$ for some constant A depending only on the biautomatic structure on G. Now write $\beta = q\alpha + r$ for some integers $q, r \geq 0$ with $r < \alpha$, so $r < A$. Then there is a word $w' \in L$ such that

$$\pi' = \pi_{w'} = \pi_w{**}(q\cdot c) = \pi_1\gamma_1^{qn_1}\pi_2\cdots\pi_p\gamma_p^{qn_p}\pi_{p+1}$$

is an accepted path representing $\overline{w}' = \overline{w}z^{q\alpha}$. It follows that $a\overline{v} = \overline{w}'bz^r$, so $d(a\overline{v}, \overline{w}'b) \leq r|z| < A|z|$. Thus, the words av and w' are fellow travellers with a constant independent of all choices:

$$d(a\overline{v}(t), \overline{w}'(t)) \leq K_1 = (A|z|+2)K.$$

Let U be the ball of radius K_1 around the origin of G.

Since the automaton M has only finitely many primitive central loops, for each such loop γ there are only finitely many primitive central loops having a power representing some power of $\overline{\gamma}$; let G_γ be this set of loops. There is a positive integer m_γ such that each loop in G_γ has a power representing $\overline{\gamma^{m_\gamma}}$.

Recalling the primitive Z-cycle $c = n_1\gamma_1 + \cdots + n_p\gamma_p$, choose the least positive integral multiple ρ_j of each m_{γ_j} so that $\rho_1\gamma_1 + \cdots + \rho_p\gamma_p$ is a Z-cycle. Note that ρ_j depends only on the primitive Z-cycle c and on j. In particular, there is a global bound $\rho_j \leq R$ independent of c and j.

Fix $j = 1, \ldots, p$ for the moment. We show that if β is sufficiently large, then v has an infix subword traversing a loop of M that represents $\overline{\gamma_j^{\rho_j}}$. Let L_j be the length of $\gamma_j^{\rho_j}$. Factor π' as $\pi_1'\gamma_j^{qn_j}\pi_2'$, and let the corresponding factorization of w' be $w'_1 w^+ w'_2$ with $w^+ = w(\gamma_j^{qn_j})$. We may factor $v = v'_1 v^+ v'_2$, so that for $0 \le t \le qn_j/\rho_j$,

$$d(a\overline{v}'_1\overline{v}^+(tL_j), \overline{w}'_1\overline{w}^+(tL_j)) \le K_1.$$

Let d_t be this word difference, so $d_t \in U$. Let s_t be the state of M at which the word $v'_1 v^+(tL_j)$ terminates.

Noting that $q \ge (\beta - A)/A$, then if

$$\beta \ge \frac{A\rho_j(|U| \cdot |M| + 1)}{n_j} + A$$

it follows that

$$q \ge \frac{\rho_j(|U| \cdot |M| + 1)}{n_j}$$

so

$$\left\lfloor \frac{qn_j}{\rho_j} \right\rfloor \ge |U| \cdot |M|.$$

In this case there are integers $0 \le t_1 < t_2 \le qn_j/\rho_j$ so that $d_{t_1} = d_{t_2}$ and $s_{t_1} = s_{t_2}$. It follows that

$$\overline{v}^+(t_1)^{-1}\overline{v}^+(t_2) = \overline{\gamma_j^{(t_2-t_1)\rho_j}}$$

and that this element is represented by a loop contained in π_v. This loop must be an iterate of some simple loop $\gamma' \in G_{\gamma_j}$, and there must be a lower iterate of γ' representing $\overline{\gamma_j^{\rho_j}}$, since m_{γ_j} divides ρ_j. Hence, π_v contains a loop representing $\overline{\gamma_j^{\rho_j}}$.

Therefore, if $\beta \ge AR(|U| \cdot |M| + 1) + A$ then π_v contains a Z-cycle representing $\overline{\gamma_1^{\rho_1}} + \cdots \overline{\gamma_p^{\rho_p}} \in Z$, contradicting the fact that $v \in L_H$. This finishes the proof that L_H is a biautomatic structure for H.

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