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A theory of cobordism for non-spherical links

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Abstract. We define an equivalence relation, called algebraic cobordism, on the set of bilinear forms over the integers. When $n \geq 3$, we prove that two $2n-1$ dimensional, simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms. As an algebraic link is a simple fibered link, our criterion for cobordism allows us to study isolated singularities of complex hypersurfaces up to cobordism.

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0. Introduction

In this work we present a cobordism theory for links which is motivated by the study of the topology of isolated singularities of complex hypersurfaces. Let us be more precise:

(0.1) Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, be a holomorphic germ with an isolated singular point at the origin. We denote by D^{2k}_δ the compact ball of radius δ centred at 0 in $\mathbb{C}^k$, and by S^{2k-1}_δits boundary. The orientation-preserving homeomorphism class of the pair $(D^{2n+2}_\varepsilon, f^{-1}(0) \cap D^{2n+2}_\varepsilon)$ does not depend on the choice of a sufficiently small ε, by definition it is *the topological type* of f. The orientation preserving diffeomorphism class of the pair $(S^{2n+1}_\varepsilon, K(f))$, where $K(f) = (f^{-1}(0)) \cap S^{2n+1}_\varepsilon$ is the link of f. The Milnor's conic structure theorem (see [M3, 68]) shows that the link $K(f)$ determines the topological type of f. Moreover, J. Milnor has also proved that:

1. $f_{/|f|} : S^{2n+1}_\varepsilon \setminus K(f) \to S^1$ is a differentiable fibration which is trivial on $U \setminus K(f)$, when U is a sufficiently "small" open tubular neighbourhood of $K(f)$.
2. The manifold $K(f)$ is $(n-2)$-connected.
3. The adherence F of a fiber of $f_{/|f|}$ is a compact, oriented, $(n-1)$-connected

smooth submanifold of S_ε^{2n+1} having $K(f)$ as boundary. By definition F is *the Milnor fiber* of $K(f)$.

(0.2) More generally, we will say that a *link* is a $(n-2)$-connected, oriented, smooth, closed, $(2n-1)$ dimensional submanifold of S^{2n+1}. A *knot* is a spherical link (i.e. a link abstractly homeomorphic to S^{2n-1}). It is well-known that, for any link K, there exists a smooth, compact, oriented $2n$-submanifold F of S^{2n+1}, having K as boundary ; such a manifold F is called a *Seifert surface* for K.

(0.3) Following M. Kervaire [K1, 65], we say that two links K_0 and K_1, abstractly diffeomorphic to the same manifold $\mathcal{K}$, are *cobordant* if there exists an embedding Φ, $\Phi : \mathcal{K} \times [0,1] \to S^{2n+1} \times [0,1]$, such that:

$$\Phi(\mathcal{K} \times \{0\}) = K_0 \text{ and } \Phi(\mathcal{K} \times \{1\}) = -K_1,$$

where $-K_1$ is the link K_1 with the orientation reversed.

(0.4) Let F be a $2n$ dimensional oriented smooth manifold of S^{2n+1}, and let G be the quotient of $\mathrm{H}_n(F, \mathbb{Z})$ by its $\mathbb{Z}$-torsion.

The *Seifert form* associated to F is the bilinear form $A : G \times G \to \mathbb{Z}$ defined as follows (see also [K2, 70] p.88 or [L2, 70], p.185): let (x, y) be in $G \times G$, then $A(x, y)$ is the linking number in S^{2n+1} of x and $i_+(y)$, where $i_+(y)$ is the cycle y "pushed" in $(S^{2n+1} \setminus F)$ by the positively oriented vector field normal to F in S^{2n+1}.

By definition a *Seifert form for a link* K is the Seifert form associated to a Seifert surface for K.

When $n \geq 2$, J. Levine ([L1, 69]) and M. Kervaire ([K2, 70]) gave a complete characterization of cobordism classes of knots in terms of Witt-equivalence classes of Seifert forms.

(0.5) A *simple link* is a link which has a $(n-1)$-connected Seifert surface. A link K is a simple fibered link if there exists a differentiable fibration $\varphi : S^{2n+1} \setminus K \to S^1$, φ being trivial on $U \setminus K$, where U is a "small" open tubular neighbourhood of K, and having $(n-1)$-connected fibers, the adherence of which are Seifert surfaces for K. In this paper we define in §1 (see (1.2)) an equivalence relation on integral bilinear forms which is much more sophisticated than "Witt-equivalence" and the theorems 2 and 3, stated in §1, imply:

Theorem A. *If $n \geq 3$, two simple fibered links are cobordant if and only if they have algebraically cobordant Seifert forms.*

(0.6) By definition an *algebraic link* is a link $K(f)$ associated, as described above, to a holomorphic germ f with an isolated singularity. Furthermore, Milnor's theory of singular complex hypersurfaces implies that algebraic links are simple fibered links. So theorem 2' and 3 stated in §1 imply:

Theorem B. *If $n \geq 3$, two algebraic links are cobordant if and only if the Seifert forms associated to their Milnor's fibers are algebraically cobordant.*

In [Lê, 72], D.T. Lê showed that two cobordant algebraic links of plane curves (i.e. when $n = 1$) are isotopic. In [DB-M, 93], P. du Bois and F. Michel found (using the classical cobordism theory for knots of M. Kervaire and J. Levine), for all $n \geq 3$, examples of non isotopic but cobordant algebraic knots. But in general algebraic links are not spherical links. So theorem B gives a cobordism theory for algebraic links.

Furthermore, having algebraically cobordant Seifert forms is also a necessary condition of cobordism for simple fibered links when n is 1 or 2. So we obtain in §5, without any restriction of dimension, a "Fox-Milnor" relation (see [F-M, 66]) for the Alexander polynomials of cobordant simple fibered links which implies:

(0.7) **Corollary.** *Let K_0 and K_1 be two algebraic links having respectively Δ_0 and Δ_1 as characteristic polynomials of monodromy. If K_0 and K_1 are cobordant then the product $\Delta_0.\Delta_1$ is a square in $\mathbb{Z}[X]$.*

(0.8) **Comments.** In [V1, 77] and [V2, 78] R. Vogt gave, when $n \geq 3$, a sufficient, but not necessary, condition of cobordism for simple links having torsion free homology groups. As shown in [DB-M, 93] the sufficient condition of cobordism for algebraic links given in [Sz, 89] by S. Szczepanski, cannot be true. So the problem of finding a criterion for cobordism of simple fibered links was largely open. Our definition of algebraic cobordism for Seifert forms solves the problem.

(0.9) In this paper we use the following **notations**: If X is a differentiable manifold we denote by ∂X its boundary, by $\mathring{X}$ its interior and by $\mathrm{H}_k(X)$ the k^{th}-homology group of X with coefficients in $\mathbb{Z}$. If a is a k-cycle of X we denote by $[a]$ its homology class in $\mathrm{H}_k(X)$. If G is an abelian group let $\mathrm{rk}(G)$ be the rank of G, and $\mathrm{Tors}(G)$ be the torsion subgroup of G.

1. Definitions and statement of results

Let $\mathcal{A}$ be the set of bilinear forms defined on free $\mathbb{Z}$-modules G of finite rank.

Let ε be $+1$ or -1.

(1.1) If A is in $\mathcal{A}$, let us denote by A^T the transpose of A, by S the ε-symmetric form $A + \varepsilon A^T$ associated to A, by $S^* : G \to G^*$ the adjoint of S (G^* being the dual $\mathrm{Hom}_{\mathbb{Z}}(G;\mathbb{Z})$ of G), by $\overline{S} : \overline{G} \times \overline{G} \to \mathbb{Z}$ the ε-symmetric non degenerated form induced by S on $\overline{G} = G_{/\mathrm{Ker}\, S^*}$. A submodule M of G is pure if $G_{/M}$ is torsion free. If M is any submodule of G let us denote by $M^\wedge$ the smallest pure submodule of G which contains M. In fact $M^\wedge$ is equal to $(M \otimes \mathbb{Q}) \cap G$. For a submodule M of G we denote by $\overline{M}$ the image of M in $\overline{G}$.

Definition. *Let $A : G \times G \to \mathbb{Z}$ be a bilinear form in $\mathcal{A}$. The form A is Witt associated to 0 if the rank m of G is even and if there exists a pure submodule M of rank $\frac{m}{2}$ in G such that A vanishes on M ; such a module M is called a* *metabolizer for A.*

(1.2) **Definition.** *Let $A_i : G_i \times G_i \to \mathbb{Z}$, i=0,1, be two bilinear forms in $\mathcal{A}$. Let G be $G_0 \oplus G_1$ and A be $(A_0 \oplus -A_1)$. The form A_0 is algebraically cobordant to A_1 if there exists a metabolizer M for A such that $\overline{M}$ is pure in $\overline{G}$, an isomorphism φ from $\mathrm{Ker}\, S_0^*$ to $\mathrm{Ker}\, S_1^*$ and an isomorphism θ from $\mathrm{Tors}\,(\mathrm{Coker}\, S_0^*)$ to $\mathrm{Tors}\,(\mathrm{Coker}\, S_1^*)$ which satisfy the two following conditions:*

c.1: $M \cap \mathrm{Ker}\, S^* = \{(x, \varphi(x)); x \in \mathrm{Ker}\, S_0^*\}$,

c.2: $d(S^*(M)^\wedge) = \{(x, \theta(x)); x \in \mathrm{Tors}\,(\mathrm{Coker}\, S_0^*)\}$, *where d is the quotient map from G^* to $\mathrm{Coker}\, S^*$.*

In §2 (see (2.3)) we prove:

Theorem 1. *Algebraic cobordism is an equivalence relation on the set $\mathcal{A}$.*

(1.3) From now on, A_0 and A_1 will always be two Seifert forms associated to some $(n-1)$-connected Seifert surfaces F_0 and F_1, of two simple links K_0 and K_1. Let us justify the definition of algebraic cobordism. As a generalization of the Kervaire-Levine theory of knot cobordism we obtain in §3 (see (3.10)):

Proposition. *If K_0 and K_1 are cobordant simple links, then $A = A_0 \oplus -A_1$ has a metabolizer.*

Remark. Let ε be $(-1)^n$, then for i=0,1, $S_i = A_i + \varepsilon A_i^T$ is the intersection form on $\mathrm{H}_n(F_i)$, $\mathrm{Ker}\, S_i^*$ is the image of $\mathrm{H}_n(K_i)$ in $\mathrm{H}_n(F_i)$ and $\mathrm{Coker}\, S_i^*$ is isomorphic to $\tilde{\mathrm{H}}_{n-1}(K_i)$. So for spherical links, both $\mathrm{Ker}\, S_i^*$ and $\mathrm{Coker}\, S_i^*$ are zero, and conditions c.1 and c.2 in definition (1.2) vanish. Then, for spherical links, two Witt associated Seifert forms are algebraically cobordant, and we recover the Kervaire-Levine criterion for cobordism.

In the non-spherical case, the topology of the cobordism implies that the restriction of A_0 on $\mathrm{Ker}\, S_0^*$ is isomorphic (on $\mathbb{Z}$) to the restriction of A_1 on $\mathrm{Ker}\, S_1^*$ (it is easy to check it directly, and it is also implied by the more general proposition (3.10)). This necessary condition for cobordism is not implied by the fact that $A_0 \oplus -A_1$ is Witt associated to 0, but by condition c.1 in definition (1.2). The topology of the cobordism also implies that the linking forms on $\mathrm{Tors}\,(\mathrm{H}_{n-1}(K_i))$ are isomorphic. This necessary condition for cobordism is contained in point c.2 of definition (1.2).

(1.4) The major result of this work is theorem 2 proved in §3 (see (3.10) and (3.13)):

Theorem 2. *Let K_0 and K_1 be two cobordant simple links. If K_0 and K_1 have $(n-1)$-connected Seifert surfaces F_0 and F_1 with unimodular Seifert forms A_0*

and A_1, then A_0 is algebraically cobordant to A_1.

Remark. Let i be 0 or 1. Let us suppose that K_i is a simple fibered link and let F_i be a $(n-1)$-connected fiber of a fibration $\varphi_i : S^{2n+1} \setminus K_i \to S^1$; then, the Seifert form A_i associated to F_i is unimodular. Conversely, if $n \geq 3$ and if A_i is unimodular then K_i is a simple fibered link (see [K-W, 77] chap. V, §3, p.118).

So, theorem 2 implies:

Theorem 2'. *Let K_0 and K_1 be two simple fibered links having F_0 and F_1 as $(n-1)$-connected fibers of differentiable fibrations φ_0 and φ_1. If K_0 is cobordant to K_1, then the Seifert forms A_0 and A_1, associated respectively to F_0 and F_1, are algebraically cobordant.*

(1.5) Using classical methods of surgery, we prove in §4 (see (4.4) and (4.5)):

Theorem 3. *Let n be greater or equal to 3 and let K_0 and K_1 be two $2n-1$ dimensional simple links. If the Seifert forms A_0 and A_1, associated to some $(n-1)$-connected Seifert surfaces F_0 and F_1 of K_0 and K_1, are algebraically cobordant then K_0 is cobordant to K_1.*

(1.6) Proposition (3.10), which does not use (as remarked in (3.12)) any hypothesis on the Seifert forms, gives:

Theorem 4. *Let K_0 and K_1 be two cobordant simple links. If A_0 (resp. A_1) is a Seifert form associated to any $(n-1)$-connected Seifert surface for K_0 (resp. K_1), then $A_0 \oplus -A_1$ has a metaboliser M such that $M \cap \operatorname{Ker} S^* = \{(x, \varphi(x)); x \in \operatorname{Ker} S_0^*\}$, where φ is an isomorphism between $\operatorname{Ker} S_0^*$ and $\operatorname{Ker} S_1^*$.*

2. Algebraic cobordism

(2.0) Let A_0 and A_1 be two algebraically cobordant forms, let A be the form $A_0 \oplus -A_1$ defined on $G = G_0 \oplus G_1$ and S be $A + \varepsilon A^T$. In this section we prove proposition (2.1) which shows that the algebraic cobordism between A_0 and A_1 allows us to describe S ; this characterization of S is fundamental to prove theorem 3 (see §4). Let M, φ and θ be as in (1.2), let m be $\operatorname{rk}(G)$ and r be $\operatorname{rk}(\operatorname{Ker} S_0^*)$. Then definition (1.2) implies that $s = \operatorname{rk}(S^*(M)) = \frac{1}{2}\operatorname{rk}(S^*(G))$ and $\operatorname{rk}(M) = r + s = \frac{m}{2}$.

We use the following notations: if E is any subset of G we denote by $\langle E \rangle$ the submodule of G, generated by E. If L is any submodule of G then:

$$L^{\perp} = \{x \in G \;\; s.t. \;\; S(x, l) = 0 \; \forall l \in L\}$$

$$\mathrm{Hom}_{\mathbb{Z}}(G_{|L}, \mathbb{Z}) = \{f \in G^* \ \ s.t. \ \ f(l) = 0 \ \forall l \in L\}$$

Moreover if L_1 and L_2 are two submodules of G, orthogonal for S, we denote by $L_1 \oplus^{\perp} L_2$ their (orthogonal) direct sum.

Lemma. *We have:* $S^*(G) \cap S^*(M)^\wedge = S^*(M^\perp)$.

Proof. Let r be the rank of $\mathrm{Ker}\, S_0^*$ and s be the rank of $S^*(M)$. As M is a metabolizer for S which fulfills condition c.1 in (1.2) we have:

$\mathrm{rk}(\mathrm{Ker}\, S^*) = 2\,\mathrm{rk}(M \cap \mathrm{Ker}\, S^*) = 2\,\mathrm{r\,k}(\mathrm{Ker}\, S_0^*) = 2\,r$, $\mathrm{rk}(S^*(G)) = 2\,s$ and $\mathrm{rk}(M^\perp) = s + 2\,r$. Hence $M^\perp = (M + \mathrm{Ker}\, S^*)^\wedge$ and $S^*(M^\perp) \subset S^*(G) \cap S^*(M)^\wedge$.

Moreover, $S^*(M)$ is of finite index in $\mathrm{Hom}_{\mathbb{Z}}(G_{|M^\perp}; \mathbb{Z})$. As $\mathrm{Hom}_{\mathbb{Z}}(G_{|M^\perp}; \mathbb{Z})$ is a pure submodule of G^*, we get $S^*(M)^\wedge = \mathrm{Hom}_{\mathbb{Z}}(G_{|M^\perp}; \mathbb{Z})$. So if $S^*(x) \in S^*(M)^\wedge$, then $S^*(x, l) = 0$ for all l in $M^\perp$ and x is in $M^\perp$. □

Since $S^*(M)$ is of finite index in $S^*(M)^\wedge$, one can write $(S^*(M)^\wedge)_{/S^*(M)} \cong \bigoplus_{i=1}^{s} \mathbb{Z}_{/a_i\mathbb{Z}}$ where $a_i \in \mathbb{N} \setminus \{0\}$ and a_i divides a_{i+1} (we do not exclude that there exists an integer l such that $a_i = 1$ for $i = 1, \ldots, l$).

Proposition. *The submodule* $\overline{M}$ *is pure in* $\overline{G}$ *if and only if* $S^*(M^\perp) = S^*(M)$.

Proof. We suppose that $\overline{M}$ is pure in $\overline{G}$. As $M \cap \mathrm{Ker}\, S^* = \Delta(\varphi)$ has rank r, the rank of $M + \mathrm{Ker}\, S^*$ is $s + 2r$. So $M + \mathrm{Ker}\, S^*$ is of finite index in $M^\perp$. Let x be in $M^\perp$; there exists a positive integer k such that $kx = y + m$, where y is in $\mathrm{Ker}\, S^*$, m is in M ; so $\overline{m} = k\overline{x}$. Since $\overline{M}$ is pure in $\overline{G}$ then $\overline{x}$ is in $\overline{M}$, so there exists y' in $\mathrm{Ker}\, S^*$ such that $x + y'$ is in M. Finally $S^*(x) = S^*(x + y') \in S^*(M)$, and $S^*(M^\perp) \subset S^*(M)$. But $M \subset M^\perp$ so $S^*(M^\perp) = S^*(M)$.

We suppose that $S^*(M) = S^*(M^\perp)$. First we prove that $\overline{M^\perp}$ is pure in $\overline{G}$. Let z be in $M^\perp$ with $\overline{z} = k\overline{x}$ where x is in G and k is a positive integer. So there exists y in $\mathrm{Ker}\, S^*$ such that $kx = z + y$. For all m in M we have $S(kx, m) = S(z+y, m) = 0$, so $S(x, m) = 0$ and x is in $M^\perp$. Now we prove that $S^*(M^\perp) = S^*(M)$ implies $\overline{M} = \overline{M^\perp}$. Let z be in $M^\perp$. If $S^*(z) = f$ there exists m in M such that $S^*(m) = f$. So $z - m = y$ is in $\mathrm{Ker}\, S^*$, and $\overline{z} = \overline{m}$ is in $\overline{M}$. Finally, since $\overline{M^\perp}$ is pure in $\overline{G}$ and $\overline{M^\perp} \subset \overline{M}$ we get $\overline{M^\perp} = \overline{M}$ is pure in $\overline{G}$. □

By definition (1.2) $\overline{M}$ is pure in $\overline{G}$, so lemma (2.0) and proposition (2.0), and, conditions c.1 and c.2 in definition (1.2) imply that $\mathrm{Coker}\, S^*$ is isomorphic to $\mathbb{Z}^{2r} \oplus (\bigoplus_{i=1}^{s} \mathbb{Z}_{/a_i\mathbb{Z}})^2$.

(2.1) **Proposition.** *There exists a basis* $\mathcal{B} = \{m_i, m_i^*; {\scriptstyle i=1,\dots,s+r}\}$ *of* G *such that:*

1. $\{m_i; {\scriptstyle i=1,\dots,s+r}\}$ *is a basis of* M*,*
2. $\{m_i, m_i^*; {\scriptstyle i=s+1,\dots,s+r}\}$ *is a basis of* $\operatorname{Ker} S^*$ *and* $\{m_i^*; {\scriptstyle i=s+1,\dots,s+r}\}$ *is a basis of* $\operatorname{Ker} S_0^*$*,*
3. *the submodules* $\langle m_i, m_i^*\rangle, {\scriptstyle i=1,\dots,s+r}$ *; are orthogonal for* S*, i.e.:* $G = \bigoplus_{1\le i\le s+r}^{\perp} \langle m_i, m_i^*\rangle$*,*
3. *when* ${\scriptstyle i=1,\dots,s}, S(m_i, m_i^*) = a_i$*.*

Definition. *Such a basis is called a good basis of* G *associated to* M*.*

The form $S = A + \varepsilon A^T$ is always an even form. Moreover, when the a_i are odd we get the following corollary:

Corollary. *When the* a_i *are odd, the isomorphic class of* S *is given by* $m = \operatorname{rk}(G)$ *and the isomorphic class of* $\operatorname{Coker} S^*$*.*

Proof of proposition (2.1). In (2.0) we have seen that $S^*(M)^\wedge = \operatorname{Hom}_{\mathbb{Z}}(G_{|M^\perp}; \mathbb{Z})$. Let M_0 be any direct summand complement of $(M \cap \operatorname{Ker} S^*)$ in M. There exits a basis $\{m_i; {\scriptstyle i=1,\dots,s}\}$ of M_0 and a basis $\{h_i; {\scriptstyle i=1,\dots,s}\}$ of $\operatorname{Hom}_{\mathbb{Z}}(G_{|M^\perp}; \mathbb{Z})$ such that $S^*(m_i) = a_i\, h_i$ where $a_i \in \mathbb{N} \setminus \{0\}$ and a_i divides a_{i+1}. Let m_1^* be any element in G such that $G = \operatorname{Ker} h_1 \oplus \langle m_1^*\rangle$ and $h_1(m_1^*) = S(m_1, m_1^*).a_1^{-1} = 1$.

Claim. For all x in G, a_1 divides $S(x, m_1^*)$.

If $a_1 = 1$ it is obvious. If $a_1 > 1$, condition c.2 in (1.2) implies that $(S^*(G)^\wedge)_{/S^*(G)}$ is isomorphic to $(S^*(M)^\wedge)_{/S^*(M)})^2 \cong (\bigoplus_{i=1}^{s} \mathbb{Z}_{/a_i\,\mathbb{Z}})^2$ and the rank of $S^*(G)$ is $2\,s$. So a_1 divides $S^*(x)$ for all x in G.

Now, we will construct an orthogonal complement $(M_1 \oplus R_1)$ for $\langle m_1, m_1^*\rangle$ in G such that:

i) $M = \langle m_1\rangle \oplus M_1$,

ii) $\operatorname{Ker} h_1 = M \oplus R_1$.

Let M_1 be the submodule of M generated by $m_i' = m_i - a_1^{-1}\, S(m_i, m_1^*).m_1$, $2 \le i \le s$, and $M \cap \operatorname{Ker} S^*$. By construction M_1 is orthogonal to $\langle m_1, m_1^*\rangle$ and $M = \langle m_1\rangle \oplus M_1$.

By construction $\operatorname{Ker} h_1$ is orthogonal to m_1 and M is in $\operatorname{Ker} h_1$.

If $\{x_i, {\scriptstyle i=2,\dots,s+r}\}$ is a basis of any direct summand complement of M in $\operatorname{Ker} h_1$, let R_1 be the submodule of $\operatorname{Ker} h_1$ generated by x_i' where: $x_i' = x_i - a_1^{-1}\, S(x_i, m_1^*).m_1$. Then $\operatorname{Ker} h_1 = \langle m_1\rangle \oplus M_1 \oplus R_1$ and R_1 is orthogonal to m_1^*.

Now we have an orthogonal decomposition of G in $\langle m_1, m_1^*\rangle \oplus^\perp (M_1 \oplus R_1)$. By induction on s we obtain an orthogonal decomposition:

$$G = (\oplus^{\perp}\langle m_i, m_i^*\rangle) \oplus^{\perp} (M_s \oplus R_s) \text{ where } \operatorname{Ker} S^* = M_s \oplus R_s.$$

Let $\{m_{s+1}, \dots, m_{s+r}\}$ be any basis of $\operatorname{Ker} S^* \cap M$. Thanks to condition c.1, $\operatorname{Ker} S^* \cap M = \{(x, \varphi(x)); x \in \operatorname{Ker} S_0^*\}$. So we can choose any basis $\{m_{s+1}^*, \dots, m_{s+r}^*\}$ of $\operatorname{Ker} S_0^*$ to build up a basis of G which fulfills proposition (2.1). □

(2.2) Now, we use the notations established in §1 and the following convention: if $f : R \to S$ is an isomorphism of $\mathbb{Z}$-modules, $\Delta(f)$ is the submodule $\{(x, f(x)); x \in R\}$ in $R \oplus S$. To prove theorem 1, we need the following proposition which gives an equivalent definition of algebraic cobordism.

Proposition. *Let A_0 and A_1 be in $\mathcal{A}$. Then A_0 is algebraically cobordant to A_1 if and only if there exists a pure submodule H of $G = G_0 \oplus G_1$ on which $A = A_0 \oplus -A_1$ vanishes, an isomorphism φ from $\operatorname{Ker} S_0^*$ to $\operatorname{Ker} S_1^*$ and an isomorphism θ from $\operatorname{Tors}(\operatorname{Coker} S_0^*)$ to $\operatorname{Tors}(\operatorname{Coker} S_1^*)$ such that:*

c.11: $\Delta(\varphi) \subset H$,

c.12: *the image $\overline{H}$ of H in $\overline{G} = G_{/\operatorname{Ker} S^*}$ is a metabolizer for $\overline{S} = \overline{S_0} \oplus -\overline{S_1}$,*

c.2: $d(S^*(H)^\wedge) = \Delta(\theta)$.

Proof. Let M, φ, θ be as in definition (1.2). Then M satisfies c.1 and c.2. The existence of φ shows that $\operatorname{Ker} S_0^*$ and $\operatorname{Ker} S_1^*$ have the same rank, r. So the rank of $\overline{G}$ is $(m_0 + m_1 - 2r)$. By c.1 $M \cap \operatorname{Ker} S^* = \Delta(\varphi)$ and $\operatorname{rk}(M) = \frac{m_0+m_1}{2}$ because M is a metabolizer for A. So $\operatorname{rk}(\overline{M}) = \frac{m_0+m_1}{2} - r$ and $\overline{S}$ vanishes on $\overline{M}$. It implies that $\overline{M}$ is a metabolizer for $\overline{S}$.

Conversely let H, φ and θ be as in the statement of proposition (2.1). As $\Delta(\varphi)$ is pure in H and in $\operatorname{Ker} S^*$, there exists a direct sum decomposition $H \cap \operatorname{Ker} S^* = \Delta(\varphi) \oplus M_0$. As $\operatorname{Ker} S^*$ is pure in G, there exists also a direct sum decomposition $H = M_1 \oplus (H \cap \operatorname{Ker} S^*)$. Let M be $M_1 \oplus \Delta(\varphi)$. By construction A vanishes on M, $M \cap \operatorname{Ker} S^* = \Delta(\varphi)$ and $S^*(M) = S^*(H)$. So M, φ and θ satisfy c.1 and c.2 of definition (1.2). Furthermore, $\overline{H} = \overline{M_1} = \overline{M}$ and by c.12 the rank of $\overline{H}$ is $\frac{m_0+m_1}{2} - r$. But M_1 being isomorphic to $\overline{M_1}$, the rank of M is $\frac{m_0+m_1}{2}$ and M is a metabolizer for A. □

(2.3) *Proof of theorem 1.* The only non trivial property to check is the transitivity of the relation "algebraic cobordism".

(2.4) **Lemma.** *Let $B_i : G_i \times G_i \to \mathbb{Z}$ be in $\mathcal{A}$, $i = 0, 1, 2$. Let m_i be the rank of G_i. If there exists a metabolizer H_{01} (resp. H_{12}) for $B_0 \oplus -B_1$ (resp. $B_1 \oplus -B_2$) and if the B_i are non-degenerate, the form $B_0 \oplus -B_2$ vanishes on $H_{02} = \pi(L)$ and $\operatorname{rk} H_{02} = \frac{1}{2}\operatorname{rk}(G_0 \oplus G_2)$, where: $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$, $H = H_{01} \oplus H_{12}$,*

$\Delta = \{(y,y) \in G_1 \oplus G_1 \ ; \ y \in G_1\}$, $L = H \cap (G_0 \oplus \Delta \oplus G_2)$ *and* π *is the projection of* G *on* $G_0 \oplus G_2$.

Proof. As $B_0 \oplus -B_2$ vanishes on H_{02} by construction, it is sufficient to prove that the rank of H_{02} is $\frac{m_0+m_1}{2}$. The definition of H_{02} gives the following exact sequence:

$$0 \to L \cap \Delta \xrightarrow{i} L \xrightarrow{\pi} H_{02} \to 0.$$

So we get:

$$(*) \quad \mathrm{rk}(L) = \mathrm{rk}(L \cap \Delta) + \mathrm{rk}(H_{02}).$$

If v is in H, there exists unique x in G_0, y_1 and y_2 in G_1 and z in G_2 such that $v = (x, y_1, y_2, z)$. Let $\rho : H \to G_1 \oplus G_1$ be defined by $\rho(v) = (y_1 - y_2, 0)$. Let us denote by L_1 the image $\rho(H)$. By construction L is the kernel of ρ and we get the exact sequence: $0 \to L \xrightarrow{i} H \xrightarrow{\rho} L_1 \to 0$. Both this sequence and $(*)$ show:

$$(**) \quad \frac{m_0 + m_2 + 2\,m_1}{2} - \mathrm{rk}(L_1) = \mathrm{rk}(L \cap \Delta) + \mathrm{rk}(H_{02}).$$

Claim. By $(B_1 \oplus -B_1)$, $\Delta \cap L$ is orthogonal to $L_1 \oplus \Delta$.

Indeed, Δ is self-orthogonal ; if (y,y) is in $\Delta \cap L$, then $(0,y)$ is in H_{01} and $(y,0)$ is in H_{12}. On the other hand, an element of L_1 is of the form $(y_1, -y_2)$ where there exists (x, y_1) in H_{01} and (y_2, z) in H_{12}. So $B_1(y, y_1) = B_1(y_1, y) = 0$ and $-B_1(y, y_2) = -B_1(y_2, y) = 0$.

The rank of $L_1 \oplus \Delta$ is $m_1 + \mathrm{rk}(L_1)$. The claim implies that the rank of the restriction of $B_1 \oplus -B_1$ to $(\Delta \cap L) \times (G_1 \oplus G_1)$ is smaller or equal to $m_1 - \mathrm{rk}(L_1)$. But $B_1 \oplus -B_1$ is non-degenerate by hypothesis, so: $\mathrm{rk}(\Delta \cap L) \leq m_1 - \mathrm{rk}(L_1)$. By $(**)$ it implies: $\frac{m_0+m_2}{2} \leq \mathrm{rk}(H_{02})$.

As B_0 and B_2 are non-degenerate by hypothesis and as $B_0 \oplus -B_2$ vanishes on H_{02}, $\mathrm{rk}(H_{02}) \leq \frac{m_0+m_2}{2}$. It ends the proof of the lemma. □

Let us go back to the proof of theorem 1. Let A_i be algebraically cobordant to A_{i+1}, $i = 0, 1$. Let $M_{i,i+1}$ be a metabolizer for $A_i \oplus -A_{i+1}$ with the isomorphisms φ_i and θ_i fulfilling conditions c.1 and c.2 in definition (1.2).

Let us take the following notations: $G = G_0 \oplus G_1 \oplus G_1 \oplus G_2$, $S_{02} = S_0 \oplus -S_2$, $G_{02} = G_0 \oplus G_2$, $S = S_0 \oplus -S_1 \oplus S_1 \oplus -S_2$, $\Delta = \{(x,x) \ ; \ x \in G_1\} \subset G_1 \oplus G_1$, d be the quotient map from G to $\mathrm{Coker}\, S^*$ and d_{02} the quotient map from G^*_{02} to $\mathrm{Coker}\, S^*_{02}$. Let π (resp. $\tilde{\pi}$) be the obvious projection from G (resp. $\mathrm{Coker}\, S^*$) to $G_0 \oplus G_2$ (resp. $\mathrm{Coker}\, S^*_{02}$). Since $\overline{M}_{i,i+1}$ is pure in $\overline{G}_i \oplus \overline{G}_{i+1}$ we have the following decompositions $M^{\perp}_{i,i+1} = \Delta(\varphi_i) \oplus \mathrm{Ker}\, S^*_i \oplus R_{i,i+1}$ with $M_{i,i+1} = \Delta(\varphi_i) \oplus R_{i,i+1}$, and $\overline{R}_{i,i+1}$ is pure in $\overline{G}_i \oplus \overline{G}_{i+1}$. Let $Q_{i,i+1}$ be any direct summand complement of $M^{\perp}_{i,i+1}$ in $G_i \oplus G_{i+1}$. If $T_{i,i+1} = R_{i,i+1} \oplus Q_{i,i+1}$, then we have the following decomposition $G = \mathrm{Ker}\, S^*_{01} \oplus \mathrm{Ker}\, S^*_{12} \oplus T_{01} \oplus T_{12}$. Let us denote by T_0 (resp. T_1, T'_1, T_2) the projection of T_{01} (resp. T_{01}, T_{12}, T_{12}) to G_0 (resp. G_1, G_1, G_2). We modify R_{12} and Q_{12} by adding to them some elements of $\Delta(\varphi_1)$ in order to have $T_1 = T_1'$. Moreover, we have the following equalities: $G_i = \operatorname{Ker} S_i^* \oplus T_i$ $i = 0,1,2$.

Let T_{02} be $T_{02} = \pi(T_{01} \oplus T_{12}) = T_0 \oplus T_2$. Let R_{02} be the smallest pure submodule of T_{02} which contains the projection of $(R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)$ on T_{02}: $R_{02} = (\pi((R_{01} \oplus R_{12}) \cap (G_0 \oplus \Delta \oplus G_2)))^\wedge$; and let A be $A_0 \oplus -A_2$, φ be $\varphi_1 \circ \varphi_0$ and θ be $-(\theta_1 \circ \theta_0)$.

By proposition (2.2), to prove that A_0 is algebraically cobordant to A_2 it is sufficient to prove that $H = \Delta(\varphi) \oplus R_{02}$ is a metabolizer for $A_0 \oplus -A_2$, and, H fulfill conditions c.11, c.12 and c.2 of (2.2). First we remark that H fulfills c.11 by definition.

(2.5) **Lemma.** *We have the equality* $d_{02}(S_{02}^*(H)^\wedge) = \Delta(-\theta_1 \circ \theta_0)$.

(2.6) **Lemma.** *The submodule* H *is a metabolizer for* A*, and* $\overline{H}$ *is a metabolizer for* $\overline{S_0} \oplus -\overline{S_2}$.

Proof of lemma (2.5). By construction: $d(S^*(G)^\wedge) = \operatorname{Tors}(\operatorname{Coker} S^*)$ and $d_{02}(S_{02}^*(H)^\wedge) = \tilde{\pi}(d(S^*(L)^\wedge))$. But c.2 implies:
$d(S^*(L)^\wedge) = (\Delta(\theta_0) \oplus \Delta(\theta_1)) \cap d(S^*(G_0 \oplus \Delta \oplus G_2)^\wedge)$, so:
$d(S^*(L)^\wedge) = \{(x, \theta_0(x), y, \theta_1(y)); x \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\ ,\ y = -\theta_0(x)\}$.
Finally: $d_{02}(S_{02}^*(H)^\wedge) = \{(x, -\theta_1 \circ \theta_0(x)); x \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\} = \Delta(-\theta_1 \circ \theta_0)$.
□

Proof of lemma (2.6). The restriction $S_{i,i+1|T_{i,i+1}}$ on $T_{i,i+1}$, of the ε-symetric bilinear form $S_{i,i+1}$, is non-degenerate ; and the submodule $R_{i,i+1}$ is a metabolizer for $S_{i,i+1|T_{i,i+1}}$, $i = 0,1$. By construction T_0 (resp. T_1, T_2) is the projection of T_{01} (resp. T_{01}, T_{12}) onto G_0 (resp. G_1, G_2). So we have $S_{i,i+1|T_{i,i+1}} = S_{i|T_i} \oplus - S_{i+1|T_{i+1}}$. We use lemma (2.4) replacing B_i by $S_{i|T_i}$, so $S_{02|T_{02}}$ vanishes on R_{02} and $\operatorname{rk} R_{02} = \frac{1}{2}\operatorname{rk} T_{02}$. Since the pure submodule H of $G_{02} = \operatorname{Ker} S_{02}^* \oplus T_{02}$ is defined by the equality $H = \Delta(\varphi) \oplus R_{02}$ then $\operatorname{rk} H = \frac{1}{2}\operatorname{rk} G_{02}$. Moreover for all h_1, h_2 in H there exist two integers a_1 and a_1 such that for $i = 1,2$ we have: $a_i\, h_i = \pi(m_i)$ and $m_i = (x_i, \varphi_0(x_i), \varphi_0(x_i), \varphi(x_i)) + (m_{0,i}, m_{1,i}, m_{1,i}, m_{2,i})$ is in $M_{01} \oplus M_{12}$. So $A(h_1, h_2) = \frac{1}{a_1 a_2}(A_{01} \oplus -A_{12})(m_1, m_2) = 0$, so A vanishes on the pure submodule H of G_{02}. Finally H is a metabolizer for A. By construction $S_{02|T_{02}}$ is isomorphic to $\overline{S}_{02}$, so as R_{02} is pure in T_{02} then $\overline{R}_{02}$ is a metabolizer for $\overline{S}_{02}$. □

The above properties of H, and, lemmas (2.5) and (2.6) imply conditions c.12 and c.2 of proposition (2.2), and A_0 is algebraically cobordant to A_2. This ends the proof of theorem 1. □

3. The necessary condition to have a cobordism

Let K_0 and K_1 be two cobordant links. Let us denote by $\mathcal{S}$ the product $S^{2n+1} \times [0,1]$ and by Σ its oriented boundary. The definition of cobordism gives a submanifold $C = \Phi(\mathcal{K} \times [0,1])$ of $\mathcal{S}$ such that $\Sigma \cap C = K_0 \coprod (-K_1)$. Let N be $F_0 \cup C \cup (-F_1)$ where F_i is a Seifert surface for K_i. By construction N is a closed, compact, oriented, $2n$-submanifold of $\mathcal{S}$.

(3.1) **Lemma.** *There exists a smooth oriented, compact, submanifold W of $\mathcal{S}$ such that N is the boundary of W.*

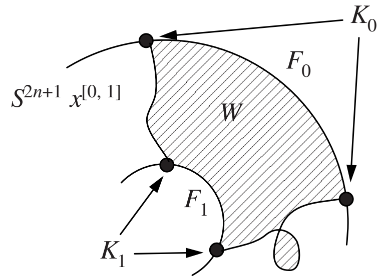
Proof. This lemma is a consequence of classical obstruction theory. If $n \geq 3$ a proof is written in [L2, 70], p. 183. As the existence of W is fundamental to obtain theorem 2, we write a proof which works in any dimension.

Let C_j for $j = 1, \dots, k$ be the k connected components of C. As C has a trivial normal bundle in $\mathcal{S}$, it is possible to choose disjoint, closed, tubular neighbourhoods U_j of C_j and a diffeomorphism $\Psi : C \times D^2 \to U = \coprod_{1 \leq j \leq k} U_j$. Now we have meridians m_j on ∂U_j defined by: $m_j = \Psi(P_j \times S^1)$ where P_j is some point of C_j and m_j is oriented such that the linking number of m_j and C_j (in $\mathcal{S}$) is $+1$. Let X be $\mathcal{S} \setminus \overset{\circ}{U}$, v be the diffeomorphism induced by the inclusion of ∂X in U, e be the excision isomorphism and ∂^i (resp. ∂^i_X) be the connectant homomorphism for the pair $(\mathcal{S}, U)$ (resp. $(X, \partial X)$). Then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
\overset{\partial^0_X}{\to} & H^1(X,\partial X) & \overset{\rho}{\to} & H^1(X) & \overset{\sigma}{\to} & H^1(\partial X) & \overset{\partial^1_X}{\to} & H^2(X,\partial X) & \to \\
 & \cong\uparrow e & & \uparrow & & v\uparrow & & \cong\uparrow e & \\
\overset{\partial^0}{\to} & H^1(\mathcal{S},U) & \to & 0 = H^1(\mathcal{S}) & \to & H^1(U) & \overset{\cong\partial^1}{\to} & H^2(\mathcal{S},U) & \to \quad 0
\end{array}
$$

The commutativity of all the squares of the above diagram implies that the homomorphism ρ is zero so σ is injective and ∂^i_X is surjective for $0 \leq i \leq 2n-1$. We have the following direct sum decomposition: $H^1(\partial X) = \sigma(H^1(X)) \oplus v(H^1(U))$. Any element of $\sigma(H^1(X))$ is represented by a differentiable map from ∂X to S^1, which is, up to homotopy, characterized by its degree on each meridian m_j, and which has a unique extension to X. Let $g : X \to S^1$ be the unique, up to homotopy, differentiable map which has degree $+1$ on each meridian. Thanks to the Thom-Pontriagin construction there exists a differentiable map $f : \Sigma \setminus (K_0 \coprod -K_1) \to S^1$ which has $\overset{\circ}{F_0} \coprod (-\overset{\circ}{F_1})$ as regular fiber and f has degree $+1$ on the meridians of the connected components of $K_0 \coprod (-K_1)$. So f and g have homotopic restrictions on $X \cap \Sigma$ and we can choose g such that its restriction on $X \cap \Sigma$ coincides with f.

Then g has a regular fiber $\overline{W}$ such that $\overline{W} \cap \Sigma = (F_0 \coprod -F_1) \cap X$. The union of $\overline{W}$ with a small collar in U is the manifold W such that $N = \partial W$. □



(3.2) Let us take A_0 (resp. A_1) the Seifert form associated to a $(n-1)$-connected Seifert surface F_0 (resp. F_1) for K_0 (resp. K_1). Let $\tau : K_0 \to K_1$ be the diffeomorphism defined by: $\tau(P) = \Phi(\Phi^{-1}(P) \times \{1\})$ where P is any point of K_0. The diffeomorphism τ induces isomorphisms $\theta_j : \mathrm{H}_j(K_0) \to \mathrm{H}_j(K_1)$ such that for any j-cycle x of K_0, $(x, \theta_j(x))$ is a boundary in $C = \Phi(\mathcal{K} \times [0,1])$. Let $\chi_i : \mathrm{H}_n(K_i) \to \mathrm{H}_n(F_i)$ and $\lambda_i : \mathrm{H}_n(F_i) \to \mathrm{H}_n(N)$, $i = 0, 1$, be the homomorphisms induced by the inclusions $K_i \subset F_i \subset N$. The Mayer-Vietoris exact sequence associated to the decompostion of N in the union of $F_0 \cup C$ and $C \cup (-F_1)$ gives:

$$\to \mathrm{H}_n(K_0) \xrightarrow{\chi} \mathrm{H}_n(F_0) \oplus \mathrm{H}_n(F_1) \xrightarrow{\lambda} \mathrm{H}_n(N) \xrightarrow{\delta} \mathrm{H}_{n-1}(K_0) \to$$

where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ and $\lambda = (\lambda_0, \lambda_1)$

(3.3) **Remark.** Let m_i be $\mathrm{rk}(\mathrm{H}_n(F_i))$, m be $\mathrm{rk}(\mathrm{H}_n(N))$ and r be $\mathrm{rk}(\chi(\mathrm{H}_n(K_0)))$. By Poincaré duality $m = m_0 + m_1$, $r = \mathrm{rk}(\delta(\mathrm{H}_n(N)))$ and $r = \mathrm{rk}(\mathrm{Ker}\, S_i^*)$ where S_i^* is the adjoint of the intersection form S_i on $\mathrm{H}_n(F_i)$.

(3.4) Construction of the isomorphisms $\varphi : \mathrm{Ker}\, S_0^* \to \mathrm{Ker}\, S_1^*$ and $\theta : \mathrm{Tors}(\mathrm{Coker}\, S_0^*) \to \mathrm{Tors}(\mathrm{Coker}\, S_1^*)$.

Let $S_{i*} : \mathrm{H}_n(F_i) \to \mathrm{H}_n(F_i, K_i)$ and $\partial : \mathrm{H}_n(F_i, K_i) \to \mathrm{H}_{n-1}(K_i)$ be the homomorphisms given by the long exact sequence for the pair (F_i, K_i). Let $U : \mathrm{H}^n(F_i) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(F_i); \mathbb{Z})$ be the universal coefficient isomorphism (F_i is $(n-1)$-connected) and let $P : \mathrm{H}_n(F_i, K_i) \to \mathrm{H}^n(F_i)$ be the Poincaré duality isomorphism. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \chi_i(\mathrm{H}_n(K_i)) & \to & \mathrm{H}_n(F_i) & \xrightarrow{S_{i*}} & \mathrm{H}_n(F_i, K_i) & \xrightarrow{\partial} & \partial(\mathrm{H}_n(F_i, K_i)) & \to & 0 \\
 & & \| & & \| & & \cong\downarrow U \circ P & & \downarrow \Delta_i & & \\
0 & \to & \mathrm{Ker}\, S_i^* & \to & \mathrm{H}_n(F_i) & \xrightarrow{S_i^*} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(F_i); \mathbb{Z}) & \xrightarrow{d} & \mathrm{Coker}\, S_i^* & \to & 0
\end{array}$$

By definition $\Delta_i : \partial(\mathrm{H}_n(F_i, K_i)) \to \operatorname{Coker} S_i^*$ is the quotient of the isomorphism $U \circ P$, so Δ_i is an isomorphism.

Let us consider again the isomorphism $\theta_j : \mathrm{H}_j(K_0) \to \mathrm{H}_j(K_1)$, which is defined in (3.2) thanks to the existence of the cobordism. Since F_i is $(n-1)$-connected then $\partial(\mathrm{H}_n(F_i, K_i)) = \widetilde{\mathrm{H}}_{n-1}(K_i)$ and $\theta_n(\operatorname{Ker}\chi_0) = \operatorname{Ker}\chi_1$, so $\theta_{n-1} \circ \partial(\mathrm{H}_n(F_0, K_0))= \partial(\mathrm{H}_n(F_1, K_1))$.

Let θ be the restriction of the isomorphism $\Delta_1 \circ \theta_{n-1} \circ \Delta_0^{-1}$ on the $\mathbb{Z}$-torsion of $\operatorname{Coker} S_0^*$.

Let φ be the restriction of θ_n on $\chi_0(\mathrm{H}_n(K_0))$. As $\chi_i(\mathrm{H}_n(K_i)) = \operatorname{Ker} S_i^*$, so φ is defined on $\operatorname{Ker} S_0^*$.

We denote by $\Delta(\varphi)$ the submodule $\{(x, \varphi(x));\ x \in \operatorname{Ker} S_0^*\}$ of $\mathrm{H}_n(F_0)\oplus\mathrm{H}_n(F_1)$.

(3.5) **Remark.** By construction φ fulfills: $\varphi \circ \chi_0 = \chi_1 \circ \theta_n$ and $\Delta(\varphi) = \chi(\mathrm{H}_n(K_0))$ where $\chi = (\chi_0, \chi_1 \circ \theta_n)$ as in (3.2).

(3.6) To prove theorem 2, we will construct a metabolizer M (in $\mathrm{H}_n(F_0 \coprod -F_1)$) for $A = A_0 \oplus -A_1$. This metabolizer M will fulfill conditions c.1 and c.2 in definition (1.2) of the algebraic cobordism, for the isomorphisms φ and θ defined in (3.4). To do that, we have to choose an oriented submanifold W of $\mathcal{S}$ with $\partial(W) = N$ (thanks to (3.1) such a W exists). Let $j : \mathrm{H}_n(N) \to \mathrm{H}_n(W)$ be the homomorphism induced by the inclusion of N in W.

(3.7) **Lemma.** *The form $A = A_0 \oplus -A_1$ vanishes on $\lambda^{-1}(\operatorname{Ker} j^\wedge)$.*

Proof. It is sufficient to prove that A vanishes on $\lambda^{-1}(\operatorname{Ker} j)$. Let $a = [x]$ and $b = [y]$ be two homology classes in $\lambda^{-1}(\operatorname{Ker} j)$. As λ is induced by the inclusion of $F_0 \coprod -F_1$ in N (see (3.2)), there exists two $(n+1)$-chains α and β in W such that $\partial\alpha = x$ and $\partial\beta = y$. Let i_+ be the positively oriented normal vector field to W in $\mathcal{S}$. The intersection of α and $i_+(\beta)$ is zero. Hence the linking number in Σ of x and $i_+(y)$ is zero. But this linking number is, by definition, equal to $A(a, b)$, so $A(a, b) = 0$ and the lemma is proved. □

(3.8) **Lemma.** *Let m be the rank of $\mathrm{H}_n(N)$. The rank of $\operatorname{Ker} j$ is $\frac{m}{2}$.*

Proof. The long exact sequence for the pair (W, N) gives the exactness of:

$$0 \to \mathrm{H}_{2n+1}(W) \to \mathrm{H}_{2n+1}(W, N) \to \mathrm{H}_{2n}(N) \to \ldots \to \mathrm{H}_{n+1}(W, N) \to \operatorname{Ker} j \to 0$$

The alternating sum of the ranks in this exact sequence together with the Poincaré duality give:

$$\operatorname{rk}(\operatorname{Ker} j) = \frac{\operatorname{rk}(\mathrm{H}_n(N))}{2} = \frac{m}{2}.$$

□

(3.9) **Lemma.** *There exists a direct summand decomposition of $\lambda^{-1}(\operatorname{Ker} j^\wedge)$ in*

$\Delta(\varphi)\oplus R_0\oplus R$ where $\Delta(\varphi)=\{(x,\varphi(x));\ x\in\operatorname{Ker}S_0^*\}$, $R_0=\lambda^{-1}(\operatorname{Ker}j^\wedge)\cap\operatorname{Ker}S_0^*$, and R is any direct summand complement of $\lambda^{-1}(\operatorname{Ker}j^\wedge)\cap\operatorname{Ker}S^*$ in $\lambda^{-1}(\operatorname{Ker}j^\wedge)$.

Proof. As the considered submodules of $\lambda^{-1}(\operatorname{Ker}j^\wedge)$ are pure, the lemma comes from the following equalities:

$\chi(\mathrm{H}_n(K_0))=\operatorname{Ker}\lambda\subset\lambda^{-1}(\operatorname{Ker}j^\wedge)$ (see (3.2)),

$\Delta(\varphi)=\chi(\mathrm{H}_n(K_0))$ (see (3.5)),

$\operatorname{Ker}S^*=\chi(\mathrm{H}_n(K_0))\oplus\operatorname{Ker}S_0^*$. □

(3.10) **Proposition.** *The submodule $M=\Delta(\varphi)\oplus R$ of $\lambda^{-1}(\operatorname{Ker}j^\wedge)$ is a metabolizer for $A=A_0\oplus -A_1$, which fulfills: $M\cap\operatorname{Ker}S^*=\Delta(\varphi)$.*

Proof. By lemma (3.9), $M\cap\operatorname{Ker}S^*=\Delta(\varphi)$. By (3.6), A vanishes on M. So we only have to show that M is of rank $\frac{m}{2}$. As remarked in (3.3), $r=\operatorname{rk}(\delta(\mathrm{H}_n(N)))$, so $\operatorname{rk}(\delta(\operatorname{Ker}j^\wedge))\le r$. Let us consider the following exact sequence induced by (3.2): $0\to\Delta(\varphi)\xrightarrow{\chi}\lambda^{-1}(\operatorname{Ker}j^\wedge)\xrightarrow{\lambda}\operatorname{Ker}j^\wedge\xrightarrow{\delta}\delta(\operatorname{Ker}j^\wedge)\to 0$. This exact sequence together with the equalities: $\operatorname{rk}(\operatorname{Ker}j^\wedge)=\frac{m}{2}$ (see (3.8)), $\operatorname{rk}(\Delta(\varphi))=r$; give $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker}j^\wedge))=r+\frac{m}{2}-\operatorname{rk}(\delta(\operatorname{Ker}j^\wedge))$. So $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker}j^\wedge))\ge\frac{m}{2}$.

We can remark that if A is non degenerated (as supposed in theorem 2) then we have $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker}j^\wedge))\le\frac{1}{2}\operatorname{rk}(\mathrm{H}_n(F_0)\oplus\mathrm{H}_n(F_1))=\frac{m}{2}$, because A vanishes on $\lambda^{-1}(\operatorname{Ker}j^\wedge)$ (see (3.6)). So, if A is non degenerated, $\operatorname{rk}(\lambda^{-1}(\operatorname{Ker}j^\wedge))=\frac{m}{2}$, $\operatorname{rk}(\delta(\operatorname{Ker}j^\wedge))=r$, $\operatorname{rk}(R_0)=0$ and $M=\lambda^{-1}(\operatorname{Ker}j^\wedge)$ is a metabolizer for A.

Come back to the general case. Let r_0 be the rank of R_0. By construction: $\operatorname{rk}(M)=\operatorname{rk}(\lambda^{-1}(\operatorname{Ker}j^\wedge))-r_0=r+\frac{m}{2}-\operatorname{rk}(\delta(\operatorname{Ker}j^\wedge))-r_0$.

(3.11) **Lemma.** *The rank l of $\delta(\mathrm{H}_n(N))_{/\delta(\operatorname{Ker}j^\wedge)}$ is greater or equal to r_0.*

Proof. Let $\{e_j\}$, $j=1,\dots,r_0$ be a basis of R_0. Let $\{e_j^*\}$ be in $\mathrm{H}_n(N)\otimes_{\mathbb{Z}}\mathbb{Q}$ such that $S_N(\lambda(e_j),e_j^*)=\delta_{ij}$ where S_N is the intersection form on $\mathrm{H}_n(N)\otimes_{\mathbb{Z}}\mathbb{Q}$. The e_j^* exists because S_N is unimodular. Let R^* be the submodule of $\mathrm{H}_n(N)\otimes_{\mathbb{Z}}\mathbb{Q}$ generated by $\{e_j^*\}$. Since $R_0\cap\operatorname{Ker}\lambda=\{0\}$, then $\operatorname{rk}(\lambda(R_0))=r_0$. As S vanishes on R_0, then S_N vanishes on $\lambda(R_0)$. It implies that $\operatorname{rk}(R^*)=\operatorname{rk}(R_0)=r_0$, and $\operatorname{Ker}j\cap R^*=\{0\}$. Since $R_0\subset\operatorname{Ker}S_0^*$, we have $S(x,y)=0$ for all x in R_0 and all y in $\mathrm{H}_n(F_0\coprod -F_1)$. So $R^*\cap\lambda(\mathrm{H}_n(F_0\coprod -F_1))=\{0\}$ and $\operatorname{rk}(\delta(\mathrm{H}_n(N))_{/\delta(\operatorname{Ker}j^\wedge)})=l\ge\operatorname{rk}(\delta(R^*))=\operatorname{rk}(R^*)=r_0$. □

In order to end the proof of (3.10), we only have to show that $\operatorname{rk}(R)=\frac{m}{2}-r$. But $\operatorname{rk}(\delta(\operatorname{Ker}j^\wedge))=r-l$; so we already have shown that $\operatorname{rk}(R)=\operatorname{rk}(M)-r=\frac{m}{2}-(r-l)-r_0$.

By lemma (3.11), we have $l-r_0\ge 0$, so $\operatorname{rk}(R)\ge\frac{m}{2}-r$. But $R\cap\operatorname{Ker}S^*=\{0\}$ by construction, and the form $\overline{S}$ induced by S on $\mathrm{H}_n(F_0\coprod -F_1)_{/\operatorname{Ker}S^*}$ is non-degenerate of rank $m-2r$. So $\mathrm{rk}(R) \leq \frac{m}{2} - r$ because $\overline{S}$ vanishes on $\overline{R} = {}^{R}/_{(R\cap \operatorname{Ker} S^*)}$. □

(3.12) **Remark.** We have found a metabolizer $M = \Delta(\varphi) \oplus R$ for A which fulfills condition c.1 of the algebraic cobordism without any condition on A. We already have got theorem 4 (see (1.6)). To prove condition c.2 and $\overline{M}$ is pure in $\overline{G}$, we will have to choose $(n-1)$-connected Seifert surfaces F_i for K_i on which the Seifert forms A_i are unimodular. So the following proposition (3.13) together with proposition (3.10) imply theorem 2 stated in (1.4).

Let θ_{n-1} be the isomorphism betweeen $\mathrm{H}_{n-1}(K_0)$ and $\mathrm{H}_{n-1}(K_1)$ defined in (3.2), and let θ the isomorphism between $\mathrm{Tors}(\operatorname{Coker} S_0^*)$ and $\mathrm{Tors}(\operatorname{Coker} S_1^*)$ defined in (3.4). Using the notation of (2.2), let $\Delta(\theta_{n-1})$ (resp. $\Delta(\theta)$) be the group $\{(x, \theta_{n-1}(x))\ ;\ x \in \mathrm{Tors}(\mathrm{H}_{n-1}(K_0))\}$ (resp. $\{(x, \theta(x))\ ;\ x \in \mathrm{Tors}(\operatorname{Coker} S_0^*)\}$).

(3.13) **Proposition.** *If A_0 and A_1 are unimodular the metabolizer $M = \Delta(\varphi) \oplus R$ of $A = A_0 \oplus -A_1$, fulfills $d(S^*(M)^\wedge) = \Delta(\theta)$; and $\overline{M}$ is pure in ${}^{\mathrm{H}_n(F)}/_{\operatorname{Ker} S^*}$.*

Proof. Let us denote $F_0 \coprod -F_1$ by F, $K_0 \coprod -K_1$ by K, and $S_0^* \oplus -S_1^*$ by S^*. We consider for F the following commutative diagram already constructed for F_i in (3.4):

$$\begin{array}{ccccccccccc}
0 & \to & \operatorname{Ker} S_* & \hookrightarrow & \mathrm{H}_n(F) & \overset{S_*}{\to} & \mathrm{H}_n(F,K) & \overset{\partial}{\to} & \partial(\mathrm{H}_n(F,K)) & \to & 0 \\
 & & \| & & \| & & \cong\downarrow U\circ P & & \cong\downarrow \Delta_0 \oplus \Delta_1 & & \\
0 & \to & \operatorname{Ker} S^* & \hookrightarrow & \mathrm{H}_n(F) & \overset{S^*}{\to} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(F);\mathbb{Z}) & \overset{d}{\to} & \operatorname{Coker} S^* & \to & 0
\end{array}$$

(3.14) **Lemma.** *The equality $d(S^*(M)^\wedge) = \Delta(\theta)$ is equivalent to the equality $\partial(S_*(M)^\wedge) = \Delta(\theta_{n-1})$.*

Proof. The lemma is a consequence of the two following statements:

The restriction of $\Delta_0 \oplus \Delta_1$ on $\Delta(\theta_{n-1})$ is an isomorphism to $\Delta(\theta)$ because $\theta \circ \Delta_0 = \Delta_1 \circ \theta_{n-1}$ by construction (see (3.4)).

The restriction of $\Delta_0 \oplus \Delta_1$ on $\partial(S_*(M)^\wedge)$ is an isomorphism to $d(S^*(M)^\wedge)$ because the commutativity of the above diagram gives $U \circ P(S_*(M)^\wedge) = S^*(M)^\wedge$. □

Let $\kappa : \mathrm{H}_n(N) \to \mathrm{H}_n(N,C)$ be the homomorphism which is defined in the long exact sequence for the pair (N,C) and $\rho : \mathrm{H}_n(N,C) \to \mathrm{N}_n(F,K)$ be the inverse of the excision isomorphism induced by the inclusion of the pair $(F,K) \subset (N,C)$. Let $\xi = \rho \circ \kappa : \mathrm{H}_n(N) \to \mathrm{H}_n(F,K)$ and $\overline{\theta} = (\mathrm{Id}, \theta_{n-1}) : \mathrm{H}_{n-1}(K_0) \to \mathrm{H}_{n-1}(K)$.

With the notations used in (3.2) we have the following commutative diagram:

$$(\star)\qquad \begin{array}{ccccccccccc} \to & \mathrm{H}_n(K_0) & \overset{\chi}{\to} & \mathrm{H}_n(F) & \overset{\lambda}{\to} & \mathrm{H}_n(N) & \overset{\delta}{\to} & \mathrm{H}_{n-1}(K_0) & \to \\ & & & \| & \text{(I)} & \downarrow \xi & \text{(II)} & \downarrow \overline{\theta} & \\ \to & \mathrm{H}_n(K) & \overset{\chi_0 \oplus \chi_1}{\to} & \mathrm{H}_n(F) & \overset{S_*}{\to} & \mathrm{H}_n(F,K) & \overset{\partial}{\to} & \mathrm{H}_{n-1}(K) & \to \end{array}$$

The square (I) is commutative by fonctoriality, and (II) is commutative by definition of ξ and $\overline{\theta}$.

(3.15) **Lemma.** *If A_0 and A_1 are unimodular, then we have $\delta(\operatorname{Ker} j^\wedge) = \tilde{H}_{n-1}(K_0)$.*

We first show that lemma (3.15) implies proposition (3.13).

We show that $\overline{M}$ is pure in $\mathrm{H}_n(F)_{/\operatorname{Ker} S^*}$, which is equivalent to prove that the quotient $\mathrm{H}_n(F)_{/(\operatorname{Ker} S^* + M)}$ is torsion free. Since $A = A_0 \oplus -A_1$ is non-degenerate $M = \lambda^{-1}(\operatorname{Ker} j^\wedge)$. Furthermore by diagram $(\star)$ we get $\lambda(\operatorname{Ker} S^*) = \operatorname{Ker} \xi$. Let pr be the projection of $\mathrm{H}_n(N)$ on $\mathrm{H}_n(N)_{/(\operatorname{Ker} j^\wedge + \operatorname{Ker} \xi)}$, so $\operatorname{Ker}(\mathrm{pr} \circ \lambda) = M + \operatorname{Ker} S^*$. The quotient of $\mathrm{pr} \circ \lambda$ induces an injective map from $\mathrm{H}_n(F)_{/(\operatorname{Ker} S^* + M)}$ into $\mathrm{H}_n(N)_{/(\operatorname{Ker} j^\wedge + \operatorname{Ker} \xi)}$.

Claim. The module $\mathrm{H}_n(N)_{/(\operatorname{Ker} j^\wedge + \operatorname{Ker} \xi)}$ is torsion free.

Proof of the claim. There exists x_i, $i = 1, \dots, r$, in $\operatorname{Ker} j^\wedge$ such that $\tilde{H}_{n-1}(K_0) = \bigoplus_{i=1}^{r} \langle \delta(x_i) \rangle \oplus \operatorname{Tors}(\tilde{H}_{n-1}(K_0))$. Let $(y_i)_{i=1,\dots,r}$ a basis of $\operatorname{Ker} \xi$ such that $S_N(x_i, y_j) = \delta_{ij}$. By induction on r, we can construct these bases such that $\mathrm{H}_n(N) = T \oplus^\perp T^\perp$ where $T = \bigoplus_{i=1}^{r} \langle x_i, y_i \rangle$. If we denote by D the module $D = T^\perp \cap \operatorname{Ker} j^\wedge$ and by D^* any direct summand complement of D in $T^\perp$, then we get:
$\mathrm{H}_n(N)_{/(\operatorname{Ker} \xi + \operatorname{Ker} j^\wedge)} \cong D^*$ which is torsion free. □

Finally $\mathrm{H}_n(F)_{/(\operatorname{Ker} S^* + M)}$ is torsion free and $\overline{M}$ is pure in $\mathrm{H}_n(F)_{/(\operatorname{Ker} S^*)}$.

So if $n = 1$, the links K_0 and K_1 have torsion free homology groups ($\mathcal{K}$ is a one dimensional compact manifold), so $\operatorname{Tors}(\operatorname{Coker} S^*) = \{0\}$ and we have already proved proposition (3.13).

Now let us take $n \geq 2$.

Thanks to lemma (3.14), the equality: $\Delta(\theta_{n-1}) = \partial(S_*(M)^\wedge)$ gives proposition (3.13). The above diagram $(\star)$ and lemma (3.15) imply: $\overline{\theta}(\mathrm{H}_{n-1}(K_0)) =$

$\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$. To show that the inclusion: $\Delta(\theta_{n-1}) \subset \partial(S_*(M)^\wedge)$ is an equality, it is sufficient to take any x in $(\partial(S_*(M)^\wedge) \cap \partial(\mathrm{H}_n(F_0, K_0))$, and to show that such a x is zero.

Let us denote by L (resp. L_i) the linking form on $\mathrm{Tors}\,(\mathrm{H}_{n-1}(K))$ (resp. $\mathrm{Tors}\,(\mathrm{H}_{n-1}(K_i))$). By definition (see remark (3.16)) such a form $L = L_0 \oplus -L_1$ is non degenerated and vanishes on $\partial(S_*(M)^\wedge)$ because $S_0 \oplus -S_1$ vanishes on M. Let $(y, \theta_{n-1}(y))$ be in $\Delta(\theta_{n-1})$. Then $L(x,(y,\theta_{n-1}(y))) = L_0(x,y) = 0$ for all $y \in \mathrm{Tors}\,(\mathrm{H}_{n-1}(K_0))$. The non degeneracy of L_0 implies $x = 0$. This ends the proof of proposition (3.13). □

(3.16) **Remark.** The linking form L is defined as follows (see [L-L, 75] prop. 2.1): Let x, y be in $\mathrm{Tors}\,(\mathrm{H}_{n-1}(K))$ such that p and q are the smallest positive integers with $p.x = q.y = 0$. Let $\overline{x}$ and $\overline{y}$ be in $\mathrm{H}_n(F)$ such that $\partial(S_*(\overline{x}) \otimes \frac{1}{p}) = x$ and $\partial(S_*(\overline{y}) \otimes \frac{1}{q}) = y$. Then: $L(x,y) \equiv \frac{1}{p.q}\, S(\overline{x},\overline{y}) \mod \mathbb{Z}$.

Proof of lemma (3.15). As shown in (3.10), if $A_0 \oplus -A_1$ is non degenerated, $M = \lambda^{-1}(\mathrm{Ker}\, j^\wedge)$ has rank $\frac{m}{2}$ and is the chosen metabolizer. So λ induces a monomorphism $\overline{\lambda}$ on $\mathrm{H}_n(F)_{/M}$ to $\mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge}$ and we get the following exact sequence:

$$0 \to \mathrm{H}_n(F)_{/M} \xrightarrow{\overline{\lambda}} \mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge} \xrightarrow{\overline{\delta}} \tilde{H}_{n-1}(K_0)_{/\delta(\mathrm{Ker}\, j^\wedge)} \to 0.$$

As $\overline{\lambda}$ is injective and M is pure in $\mathrm{H}_n(F)$ there exists two $\mathbb{Z}$-bases $\{\overline{e}_j; {\scriptstyle j=1,\dots,\frac{m}{2}}\}$ of $\mathrm{H}_n(F)_{/M}$ and $\{\overline{k}_j; {\scriptstyle j=1,\dots,\frac{m}{2}}\}$ of $\mathrm{H}_n(N)_{/\mathrm{Ker}\, j^\wedge}$ such that $\overline{\lambda}(\overline{e}_j) = p_j.\overline{k}_j$ with $p_j \in \mathbb{Z} \setminus \{0\}$. Let E (resp. H) be a direct summand complement of M (resp. $\mathrm{Ker}\, j^\wedge$) in $\mathrm{H}_n(F)$ (resp. $\mathrm{H}_n(N)$). Let also $\{e_j; {\scriptstyle j=1,\dots,\frac{m}{2}}\}$ (resp. $\{k_j; {\scriptstyle j=1,\dots,\frac{m}{2}}\}$) be a $\mathbb{Z}$-basis of E (resp. H) such that $e_j \equiv \overline{e}_j \mod M$ (resp. $k_j \equiv \overline{k}_j \mod \mathrm{Ker}\, j^\wedge$). By construction $\lambda(e_j) - p_j.k_j = x \in \mathrm{Ker}\, j^\wedge$. So there exists a $(n+1)$-chain γ in W and a positive integer a such that: $\partial\gamma = a\,\lambda(e_j) - a\, p_j.k_j$. Let ρ be a $(n+1)$-chain of $S^{2n+1} \times [0,1]$ with $\partial\rho = k_j$. So $a\, e_j$ is the boundary of $\gamma + a\, p_j.\rho$ in $S^{2n+1} \times [0,1]$.

Statement: for all m in M, p_j divides $A(e_j, m)$.

Let m be in $M = \lambda^{-1}(\mathrm{Ker} j^\wedge)$ and Δ be a $(n+1)$-chain in $S^{2n+1} \times [0;1]$ such that $\partial\Delta = i_+(m)$. By definition $A(a\, e_j, m)$ is the intersection in $S^{2n+1} \times [0,1]$ of $\gamma + a\, p_j.\rho$ and Δ. But $\lambda(a\, m) \in \mathrm{Ker}\, j$ so there exists a $(n+1)$-chain μ in W such that $\partial\mu = a\, m$. We have $\partial(i_+(\mu)) = a\, i_+(m)$. Since $\partial(a\,\Delta) = a\, i_+(m)$, we get $\gamma \cap (a\,\Delta) = \gamma \cap (i_+(\mu)) = 0$. But $a > 0$, so $a(\gamma \cap \Delta) = 0$ implies $\gamma \cap \Delta = 0$. Finaly $A(a\, e_j, m) = a\, p_j.(\rho \cap \Delta)$ and p_j divides $A(e_j, m)$.

If A is unimodular the statement implies that $p_j = \pm 1$ for all $j = 1, \dots, \frac{m}{2}$. So $\overline{\lambda}$ is an isomorphism and his cokernel is zero. As asked we have got: $\delta(\mathrm{Ker}\, j^\wedge) = \tilde{H}_{n-1}(K_0)$. This ends the proof of lemma (3.15). □

(3.17) **Remark.** As above we can also prove that: for all m in M p_j divides $A(m, e_j)$.

4. The sufficient condition to have a cobordism

(4.1) Let K_0 and K_1 be two $2n-1$ dimensional simple links, with $n \geq 3$. We suppose that there exists $(n-1)$-connected Seifert surfaces F_0 and F_1, for K_0 and K_1, such that the associated Seifert forms A_0 and A_1 are algebraically cobordant. We consider K_0 (resp. $-K_1$) as embedded in the sphere $S^{2n+1} \times \{0\}$ (resp. $S^{2n+1} \times \{1\}$) which are oriented as the boundary of $S^{2n+1} \times [0,1]$.

Let x be in $S^{2n+1} \times \{0\}$ such that $(x \times [0,1]) \cap (F_0 \coprod -F_1)$ is empty, and let U be a "small" open ball around x in $S^{2n+1} \times \{0\}$. The boundary S of the disk $D = (S^{2n+1} \times [0,1]) \setminus (U \times [0,1])$ contains $F_0 \coprod -F_1$. Let G be the closure of the connected sum, in S, of the interiors $\overset{\circ}{F}_0$ and $-\overset{\circ}{F}_1$. By construction $A = A_0 \oplus -A_1$ is the Seifert form of $K_0 \coprod -K_1$, associated to G.

(4.2) *Proof of theorem 3*. In order to prove theorem 3 we will do in D, an embeded surgery on G, the result of which being a manifold $\tilde{G}$ diffeomorphic to $\mathcal{K} \times [0,1]$.

By proposition (2.1) we can choose a good basis $\mathcal{B} = \{(m_i, m_i^*); {\scriptstyle i=1,\dots,s+r}\}$ of $\mathrm{H}_n(G)$. Thanks to J. Milnor ([M1, 61] lemma 6 p. 50), any cycle of G can be represented by the image of an embedding of S^n. Furthermore:

(4.3) **Proposition.** *There exists $s+r$ disjointed embeddings $\psi_i : D^{n+1} \times D^n \to D$ such that for any $i \in \{1, \dots, s+r\}$ we have*
1- $[\psi_i(S^n \times \{0\})] = m_i$,
2- $(\psi_i(D^{n+1} \times D^n)) \cap G = \psi_i(D^{n+1} \times D^n) \cap S = \psi_i(S^n \times D^n)$.

Proof. Let $\overline{\psi_i} : S^n \to G$ be an embedding of S^n which represents m_i. Let i, j with $i \neq j$, be in $\{1, \dots, s+r\}$, then m_i and m_j are in the metabolizer M and we have: $S(m_i, m_j) = A(m_i, m_j) + (-1)^n A(m_j, m_i) = 0$. Since $n \geq 3$, thanks to Whitney's procedure [Wh, 44] we can choose the $\overline{\psi_i}$ such that $\overline{\psi_i}(S^n) \cap \overline{\psi_j}(S^n) = \emptyset$. Since $n \geq 2$, the Whitney obstruction to extend $\overline{\psi_i}$ to disjoint embeddings ψ_i of D^{n+1} in the $(2n+2)$-disk D, is the matrix $A(m_i, m_j)$ which is zero. Furthermore, $A(m_i, m_i) = 0$ is the classical obstruction to extend ψ_i to $\psi_i : D^{n+1} \times D^n \to D$. (see [Br, 72] and for details see [Bl, 94] proposition 5.1.2, p.58). We choose this extension ψ_i such that the restriction to $S^n \times D^n$ is a tubular neighbourhood of $\psi_i(S^n)$ in G. □

So thanks to proposition (4.3) we obtain a submanifold $\tilde{G}$ of D as follows:

$$\tilde{G} = (G \setminus (\coprod_{i=1}^{s+r} \psi_i(S^n \times D^n)) \cup (\coprod_{i=1}^{s+r} \psi_i(D^{n+1} \times S^{n-1})).$$

(4.4) **Proposition.** *The inclusion k_o (resp. k_1) of K_0 (resp. K_1) in $\tilde{G}$, induces isomorphisms $k_{o,j}$ (resp. $k_{1,j}$) from $\mathrm{H}_j(K_0)$ (resp. $\mathrm{H}_j(K_1)$) to $\mathrm{H}_j(\tilde{G})$ for all j.*

(4.5) **Corollary.** *We have* $\mathrm{H}_*(\tilde{G}, K_0) = \mathrm{H}_*(\tilde{G}, K_1) = 0$.

This corollary (4.5) and the h-cobordism theorem imply that $\tilde{G}$ is diffeomorphic to $K_0 \times [0,1]$. More precisely $\dim \tilde{G} = 2n \geq 6$ and:

h-cobordism Theorem [M2, 65]**.** *Let $\mathcal{M}$ be a k-dimensional differentiable compact manifold with $\partial\mathcal{M} = \mathcal{M}_0 \coprod \mathcal{M}_1$ such that $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ are simply connected. If $\mathrm{H}_*(\mathcal{M}, \mathcal{M}_0) = 0$ and $k \geq 6$ then $\mathcal{M}$ is diffeomorphic to $\mathcal{M}_0 \times [0,1]$.*

So to end the proof of theorem 3 we only have to prove proposition (4.4).

Proof of proposition (4.4). According to proposition (2.1), the intersection form on $\mathrm{H}_n(F)$ splits in an orthogonal sum on the submodules $\langle m_i, m_i^* \rangle$, $i = 1, \ldots, s+r$. So the proof of (4.4) when $s + r = 1$ implies the general case.

Let us suppose that $\mathrm{rk}(M) = 1$ and let m be a generator of M, then $\mathrm{H}_n(G) = \langle m, m^* \rangle$. We denote by $\psi : D^{n+1} \times D^n \to D$ an embedding choosen as in proposition (4.3), by $\eta : S^n \to G$ an embedding such that $[\eta(S^n)] = m^*$, and by G_T the manifold $G_T = G \setminus \psi(S^n \times \overset{\circ}{D}{}^n)$.

(4.6) The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $G = G_T \cup \psi(S^n \times D^n)$ gives:

$$0 \to \mathrm{H}_n(\psi(S^n \times S^{n-1})) \to \mathrm{H}_n(G_T) \oplus \mathrm{H}_n(\psi(S^n \times D^n)) \to \mathrm{H}_n(G)$$

$$\xrightarrow{\delta} \mathrm{H}_{n-1}(\psi(S^n \times S^{n-1})) \to \mathrm{H}_{n-1}(G_T) \to 0.$$

where δ is given by the intersection of cycles with m.

(4.7) The Mayer-Vietoris sequence associated to the following decomposition of the manifold: $\tilde{G} = G_T \cup \psi(D^{n+1} \times S^{n-1})$ gives:

$$0 \to \mathrm{H}_n\big(\psi(S^n \times S^{n-1})\big) \xrightarrow{\alpha} \mathrm{H}_n(G_T) \to \mathrm{H}_n(\tilde{G}) \xrightarrow{\gamma} \mathrm{H}_{n-1}(\psi(S^n \times S^{n-1}))$$

$$\xrightarrow{\beta} \mathrm{H}_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus \mathrm{H}_{n-1}(G_T) \to \mathrm{H}_{n-1}(\tilde{G}) \to 0.$$

Remark that the homomorphism β is injective into $\mathrm{H}_{n-1}(\psi(D^{n+1} \times S^{n-1}))$, hence $\gamma = 0$ and the sequence (4.7) splits up into:

(4.8) $0 \to \mathrm{H}_n(\psi(S^n \times S^{n-1})) \xrightarrow{\alpha} \mathrm{H}_n(G_T) \to \mathrm{H}_n(\tilde{G}) \to 0$,

(4.9) $0 \to \mathrm{H}_{n-1}(\psi(S^n \times S^{n-1})) \xrightarrow{\beta} \mathrm{H}_{n-1}(\psi(D^{n+1} \times S^{n-1})) \oplus \mathrm{H}_{n-1}(G_T) \to \mathrm{H}_{n-1}(\tilde{G}) \to 0$.

Since $\mathrm{rk}(M) = 1 = s + r$ we have to consider the two following cases: $s = 0, r = 1$ and $s = 1, r = 0$.

$\star$ 1^{st} case: $s = 0$ and $r = 1$, then $\mathrm{Ker}\, S^* = \langle m, m^* \rangle$.

In sequence (4.6) we have $\mathrm{Ker}\,\delta = \langle m, m^* \rangle$, then $\mathrm{H}_n(G_T) = \langle [\psi(S^n \times \{1\})], [\eta(S^n)] \rangle$ and $\mathrm{H}_{n-1}(G_T) = \langle [\psi(\{1\} \times S^{n-1})] \rangle$. In sequence (4.8) we have $\mathrm{Im}\,\alpha =$

$\langle[\psi(S^n \times \{1\})]\rangle$, so $\mathrm{H}_n(\tilde{G}) = \langle[\eta(S^n)]\rangle$. By construction of the good basis (2.1), $[\eta(S^n)]$ is a generator of $\mathrm{Im}\,(\mathrm{H}_n(K_0) \to \mathrm{H}_n(G))$. So the inclusion of K_0 in $\tilde{G}$ induces the isomorphism: $k_{0,n} : \mathrm{H}_n(K_0) \xrightarrow{\cong} \mathrm{H}_n(\tilde{G})$.

Since $\mathrm{H}_{n-1}(G_T) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$ in sequence (4.9), we have $\mathrm{H}_{n-1}(\tilde{G}) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$. Condition c.1 of the algebraic cobordism gives that there exists a in $\mathrm{Ker}\, S_0^*$ such that $m = (a, \varphi(a))$. If we denote by $\gamma_0 : \mathrm{H}_n(K_0) \to \mathrm{H}_n(G)$ the homomorphism induced by the inclusion, then we can choose b in $\mathrm{H}_{n-1}(K_0)$ such that $\mathrm{H}_{n-1}(K_0) = \langle b \rangle$ and b is the dual of $\gamma_0^{-1}(a)$ for the intersection form of K_0. There exists B in $\mathrm{H}_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. The boundary of the n-chain $(B - (B \cap \psi(S^n \times \mathring{D}^n)))$ is homologous to the $(n-1)$-cycle $b - (\psi(\{1\} \times S^{n-1}))$, hence b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $\mathrm{H}_{n-1}(\tilde{G}) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$. Thus the inclusion of K_0 in $\tilde{G}$ induces the isomorphism: $k_{0,n-1} : \mathrm{H}_{n-1}(K_0) \xrightarrow{\cong} \mathrm{H}_{n-1}(\tilde{G})$.

⋆ 2^{nd} case: $s = 1$ and $r = 0$, then $\mathrm{Ker}\, S^* = \{0\}$ and $\mathrm{H}_n(K_0) = 0$.

In sequence (4.6) we have $\mathrm{Ker}\,\delta = \langle m \rangle$, then $\mathrm{H}_n(G_T) = \langle[\psi(S^n \times \{1\})]\rangle$ and $\mathrm{H}_{n-1}(G_T) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$. In sequence (4.8) we have $\mathrm{Im}\,\alpha = \langle[\psi(S^n \times \{1\}]\rangle$. Since $\mathrm{H}_n(G_T) = \langle[\psi(S^n \times \{1\})]\rangle$ we have $\mathrm{H}_n(\tilde{G}) = 0 = \mathrm{H}_n(K_0)$.

- if $S_*(m)$ is indivisible (i.e. $\mathrm{H}_{n-1}(K_0) = 0$), then δ in (4.6) is surjective. Thus $\mathrm{H}_{n-1}(\tilde{G}) = 0 = \mathrm{H}_{n-1}(K_0)$.
- If $a \neq 1$ is the greatest divisor of $S_*(m)$ (i.e. $\mathrm{H}_{n-1}(K_0) \cong \mathbb{Z}_{/a\mathbb{Z}}$) then condition c.2 of algebraic cobordism together with lemma (3.14) give that there exists c in $\mathrm{H}_{n-1}(K_0)$ such that $\partial(\frac{1}{a} S_*(m)) = (c, \theta_{n-1}(c))$. Let b in $\mathrm{H}_{n-1}(K_0)$ be the dual of c for the linking form of K_0. There exists B in $\mathrm{H}_n(G, K_0)$ such that $\partial B = b$ and the intersection between B and m is $+1$. As before the boundary of the n-chain $B - (B \cap \psi(S^n \times \mathring{D}^n))$ is the n-cycle $b - \psi(\{1\} \times S^{n-1})$, hence b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $\mathrm{H}_{n-1}(G)$. Since $\mathrm{H}_{n-1}(G_T) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$ in sequence (4.9) we have $\mathrm{H}_{n-1}(\tilde{G}) = \langle[\psi(\{1\} \times S^{n-1})]\rangle$. Thus b and $[\psi(\{1\} \times S^{n-1})]$ are homologous in $\mathrm{H}_{n-1}(\tilde{G})$ and the inclusion of K_0 in $\tilde{G}$ induces the isomorphism: $k_{0,n-1} : \mathrm{H}_{n-1}(K_0) \xrightarrow{\cong} \mathrm{H}_{n-1}(\tilde{G})$.

Since $\tilde{G}$ is obtained by surgery on n-cycles, this surgery only modifies homology groups of dimensions n and $n-1$. Hence for $k \neq n, n-1$ we have $\mathrm{H}_k(G) \cong \mathrm{H}_k(K_0) \overset{k_{0,k}}{\cong} \mathrm{H}_k(\tilde{G})$. By symmetry we also have the same results with K_1. Finally $k_{0,j}$ and $k_{1,j}$ are some isomorphisms for all j. This ends the proof of proposition (4.4), and the proof of theorem 3. □

5. Appendix – Alexander polynomials of cobordant links.

Let K be a $2n-1$ dimensional simple link, and $\varepsilon = (-1)^n$. One can associate a polynomial $\Delta \in \mathbb{Z}[X]$ to any Seifert surface F for the link K, defined by: $\Delta(X) =$

$\det(XA + \varepsilon A^T)$, where A is the Seifert form associated to F. Such a polynomial Δ is called a Alexander polynomial for the link K. Changing the Seifert surface to another multiplies Δ by $\pm X^m$ with m in $\mathbb{Z}$.

For a polynomial γ in $\mathbb{Z}[X]$ we define the polynomial γ^* by: $\gamma^*(X) = X^{\deg \gamma}\, \gamma(X^{-1})$.

(5.1) **Proposition.** *Let K_0 and K_1 be two cobordant simple $2n-1$ dimensional links. If Δ_0 and Δ_1 are Alexander polynomials for K_0 and K_1, then there exists γ in $\mathbb{Z}[X]$ such that: $\gamma\,\gamma^* = \pm\Delta_0\,\Delta_1$.*

Remark. If F is the Milnor fiber of an algebraic link K, then the associated Alexander polynomial is the characteristic polynomial of the monodromy. Hence the above proposition and the monodromy theorem imply corollary (0.7).

Proof of proposition (5.1). We denote by F_0 and F_1 two $(n-1)$-connected Seifert surfaces for K_0 and K_1, and by A_0 and A_1 the associated Seifert forms. The links K_0 and K_1 are cobordant so proposition (3.10) implies that the form $A = A_0 \oplus -A_1$ has a metabolizer M. Therefore, there exists a basis for $\mathrm{H}_n(F_0) \oplus \mathrm{H}_n(F_1)$ such that in this basis the matrix for A is $\begin{pmatrix} 0 & B_1 \\ B_2 & B_3 \end{pmatrix}$ where B_i, i=1,2,3 are square matrices. We have $\Delta_0(X).\Delta_1(X) = \det(XA + \varepsilon A^T)$, hence $\Delta_0(X).\Delta_1(X) = \varepsilon.\det(XB_1 + \varepsilon B_2^T).\det(XB_2 + \varepsilon B_1^T)$. Let $\gamma(X)$ be $\det(XB_1 + \varepsilon B_2^T)$, then $\gamma^*(X) = \det(XB_2 + \varepsilon B_1^T)$. Finally we get $\gamma.\gamma^* = \pm\Delta_0.\Delta_1$. □

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References

[Bl, 94] V. Blanlœil, Cobordisme des entrelacs fibrés simples et forme de Seifert. *Thèse de l'Université de Nantes* (1994).

[Bl, 95] V. Blanlœil, Cobordisme des entrelacs fibrés simples et forme de Seifert. *Note aux Comptes Rendus de l'Académie des Sciences de Paris* (1995), t. 320, Série I, p. 985–988.

[Br, 72] W. Browder, Surgery on Simply-connected Manifolds. *Erger. Math.* **65** Springer, 1972.

[DB-M, 93] P. du Bois et F. Michel, Cobordism of Algebraic Knots via Seifert Forms. *Invent. Math.* **111** (1993), 151–169.

[D1, 74] A. Durfee, Fibered Knots and Algebraic Singularities. *Topology* **13** (1974), 47–59.

[D2, 77] A. Durfee, Bilinear and Quadratic Forms on Torsion Modules. *Advances in Mathematics* **25** (1977), 133–164.

[F-M, 66] R. Fox and J. Milnor, Singularities of 2-spheres in 4-spaces and Cobordism of Knots. *Osaka J. Math.* **3** (1966), 257–267.

[K1, 65] M. Kervaire, Les noeuds de dimension supérieure. *Bulletin de la Société Mathématique de France* **93** (1965), 225–271.

[K2, 70] M. Kervaire, Knot Cobordism in Codimension Two. Manifolds Amsterdam 1970, *Lecture Notes* **197**, 83–105.

[K-W, 77] M. Kervaire, C. Weber, A Survey of Multidimensional knots. Knot Theory, Proceedings. Plans-sur-Bex, Switzerland 1977, *Lecture Notes* **685**, 61-134.

[L-L, 75] J. Lannes and F. Latour, Forme quadratique d'enlacement et applications. *Société Mathématique de France Astérisque* **26** (1975).

[L1, 69] J. Levine, Knot Cobordism in Codimension Two. *Comment Math. Helv.* **44** (1969), 229–244.

[L2, 70] J. Levine, An Algebraic Classification of Some Knots of Codimension Two. *Comment. Math. Helv.* **45** (1970), 185–198.

[Lê, 72] D. T. Lê, Sur les noeuds algébriques. *Compos. Math.* **25** (1972), 281–321.

[M1, 61] J. Milnor, A Procedure for Killing Homotopy Groups of Differentiable Manifolds. *Proceeding of Symposia in Pure Math. (A.M.S.)* (1961) t.3, 39–55.

[M2 65] J. Milnor, Lectures on the h-Cobordism Theorem. Princeton Mathematical Notes, Princeton U. Press, 1965.

[M3, 68] J. Milnor, Singular Points of Complex Hypersurfaces. *Annals of Math. Studies* **61** (1968).

[Sa, 74] K. Sakamoto, The Seifert Matrices of Milnor Fiberings defined by Holomorphic Functions. *J. Math. Soc. Japan* **26**(4) (1974), 714–721.

[V1, 77] R. Vogt, Cobordismus von Knoten. *Lect. Notes in Math.* **685** (1977), 218–226.

[V2, 78] R. Vogt, Cobordismus von hochzusammenhängeden Knoten. *Inauguraldissertation zur Erlangung des Doktorgrades*. Bonn 1978.

[W, 70] C.T.C. Wall, *Surgery on Compact Manifolds.* Academic Press, New York 1970.

[Wh, 44] H. Whitney, The Self-Intersection of a Smooth n-Manifold in $2n$-Spaces. *Annals of Math.* **45** (1944), 220–246.

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