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Gabriel filters in real closed rings

Niels Schwartz

Abstract. Real closed rings arise in semi-algebraic geometry and topology as well as in the investigation of partially ordered rings. It is shown that localizations of real closed rings with respect to Gabriel filters, or more generally: multiplicative filters, are again real closed. Thus, real closedness is preserved under a large number of important ring theoretic constructions. For a few particularly simple cases the multiplicative filters are classified and the localizations are determined.

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Gabriel topologies provide a very general method of localization which is even applicable in noncommutative situations (cf. [14]; [17]; [46]; [47]). In commutative algebra a large number of important constructions are special cases of Gabriel localizations. Among these are classical rings of quotients, complete rings of quotients ([29], Chapter 2) and sections of quasi–coherent modules over open quasi–compact subsets of affine schemes (Deligne's formula, cf. [48], Proposition 5.16). The present paper deals with these techniques in a context arising in real algebra.

Real closed rings ([38]; [39]; [40]) were first introduced in order to extend semi–algebraic geometry as developed by Delfs and Knebusch (cf. [11]; [27]) to cover the geometry of arbitrary real spectra. But there are other contexts where these rings appear naturally. Arbitrary rings of continuous functions into the real numbers are real closed ([44]). In real algebra a systematic investigation of monoreflectors of the category of reduced partially ordered rings shows that real closed rings play a very distinguished rôle in this category ([34]). When working with real closed rings in various applications it is frequently necessary to know that certain constructions when applied to real closed rings will yield real closed rings. The main results of this paper show that Gabriel localizations have such a preservation property.

Throughout most of the paper localizations are discussed with respect to multiplicative filters (section 2) instead of the more special Gabriel filters. This extends the scope of the applications considerably without any additional effort. For example, it is possible to include results about Nagata's ideal transforms ([2]; [9]; [25], p.30) although they are not Gabriel localizations. Therefore, section 2 contains some basic material about multiplicative filters. In section 3 it is shown how Gabriel localizations can be used to describe global rings of sections of certain subschemes of affine schemes. These results are related to Deligne's formula as presented in [48], Proposition 5.16. They show that many rings arising in semi–algebraic geometry are Gabriel localizations of real closed rings. When dealing with multiplicative filters in real closed rings it is possible to a large extent to restrict attention to l–ideals. (Note that real closed rings are always f–rings.) It is shown in section 1 that every ideal in a real closed ring can be closely approximated by an l–ideal. Therefore, every multiplicative filter has a basis consisting of l–ideals. The advantage of dealing with l–ideals compared with arbitrary ideals is that the relationship between different l–ideals is much easier to understand. This becomes particularly evident in sections 4, 5 and 7. In section 4 the multiplicative filters of finite type and their localizations are studied. Similarly, section 5 deals with multiplicative filters in real closed domains. Section 7 relates the multiplicative filters of a real closed ring to the multiplicative filters of its residue domains. The situation becomes particularly simple when the real closed ring has only finitely many minimal prime ideals. The main results of the paper are contained in section 6. It is shown that every localization of a real closed ring with respect to a mulitplicative filter is real closed. Just to mention a few applications, this implies that complete rings of quotients of real closed rings are real closed, that ideal transforms of real closed rings are real closed, and that many rings of sections over subsets of real closed schemes are real closed.

Notation and terminology All rings are commutative and have a unit. If A is a ring then A^* is its group of units. If $A \subseteq B$ is an extension of rings, if $x \in B$ and C is an intermediate ring then $(A : x)_C = \{y \in C; yx \in A\}$. The reference to the ring C will be dropped if this does not lead to ambiguities. The Zariski spectrum is $Spec(A)$. The subspaces of maximal or minimal ideals are $Max(A)$ and $Min(A)$. If $p \in Spec(A)$ and $a \in A$ then the canonical image of a in A/p is frequently denoted by $a(p)$. If $X \subseteq A$ is any subset then

$$D(X) = \{p \in Spec(A); X \not\subseteq p\},$$
$$V(X) = \{p \in Spec(A); X \subseteq p\}.$$

The *real spectrum* of A is denoted by $Sper(A)$. Basic information about this space can be looked up in [6]; [10]; [28]. Both $Spec(A)$ and $Sper(A)$ are spectral spaces in the sense of [24]. If x and y are points of a spectral space then x is a *specialization* of y and y is *generalization* of x if $x \in \overline{\{y\}}$. If X is a subset of a spectral space then $Gen(X)$ denotes the set of all generalizations of elements of X. The set X is said to be *generically closed* if $X = Gen(X)$. A subset of a spectral space is *constructible* if it belongs to the Boolean algebra of subsets generated by the open quasi–compact sets. The constructible sets are the basis of a topology of the space which is called the *constructible topology.* A subset of the spectral space is *proconstructible* if it is closed with respect to the constructible topology. General references for schemes are [18] and [20]. The terminology concerning lattice–ordered groups (l–groups) and f–rings is the same as in [5]. In this paper all f–rings are reduced. If x belongs to some l–group then $x^+ = \sup(x, 0)$, $x^- = \sup(-x, 0)$, and $|x| = x^+ + x^-$.

1. l–ideals in real closed rings

Any real closed ring A is an f–ring by [40], Corollary I 3.4. Therefore the set $Id(A)$ of all ideals of A contains the set $LId(A)$ of l–ideals as a subset. It will be shown in this section that the set $LId(A)$ is quite dense in $Id(A)$ and that it has very favorable arithmetic properties.

Proposition 1.1. *Let A be an f–ring with bounded inversion (i.e., if $1 \leq a \in A$ then $a \in A^*$). If $G \subseteq A$ is a convex l–subgroup of the additive l–group of A then the ideal GA generated by G is an l–ideal.*

Proof. It is only necessary to show that $0 \leq |x| \leq |y|$ with $y \in GA$ implies $x \in GA$. Because of $0 \leq x^+, x^- \leq |x|$ and $x = x^+ - x^-$ one may assume that $0 \leq x$. If $y = \sum g_i a_i$, $g_i \in G$, $a_i \in A$, define $g = \sum |g_i| \in G$ and $a = \sum |a_i| \in A$. Then it follows that $0 \leq x \leq ga$. Writing

$$a = \sup(a, 1) \inf(a, 1)$$

and using bounded inversion one gets

$$0 \leq x \sup(a, 1)^{-1} \leq g \inf(a, 1) \leq g.$$

By convexity of G, $x \sup(a, 1)^{-1} \in G$, hence $x \in GA$. $\square$

Corollary 1.2. *If A is an f–ring with bounded inversion then every ideal $I \subset A$ contains a largest l–ideal.*

Proof. In the additive l–group of A there is a largest convex l–subgroup G which is contained in I ([5], (2.2.6)). By Proposition 1.1, GA is an l–ideal. Since $GA \subseteq I$ one concludes that $G = GA$ and that this is the desired l–ideal. $\square$

The largest l–ideal contained in the ideal I is denoted by $L(I)$. Changing slightly the definition of [30], Introduction, an f–ring is said to have the 2^{nd} *convexity property* if $0 \leq x \leq y^2$ implies that $x \in (y)$. It was pointed out to me by Warren McGovern that the 2^{nd} convexity property implies bounded inversion. For, if $0 \leq 1 \leq x$ then also $0 \leq 1 \leq x^2$, i.e., $1 \in (x)$, hence $x \in A^*$. For rings having the 2^{nd} convexity property the connection between I and $L(I)$ is particularly close:

Proposition 1.3. *Suppose that A is an f–ring with the 2^{nd} convexity property. Then $I^2 \subseteq L(I)$ for each ideal $I \subseteq A$.*

Proof. Consider the following set:

$$J = \{x \in A; \exists a \in I^2 : |x| \leq |a|\}.$$

It is claimed that J is an l–ideal. Clearly $0 \leq |x| \leq |y|$ with $x \in A$ and $y \in J$ implies $x \in J$. Therefore it suffices to prove that J is an ideal. Pick $x, y \in J$, say $|x| \leq |a|$, $|y| \leq |b|$ with $a, b \in I^2$. Writing $a = \sum \alpha_i \beta_i$, $\alpha_i, \beta_i \in I$ one has $|a| \leq \sum \alpha_i^2 + \beta_i^2$, i.e., there is some $0 \leq c \in I^2$ such that $|a| \leq c$. Similarly, $|b| \leq d$ for some $d \in I^2$. But then

$$|x + y| \leq |x| + |y| \leq c + d = |c + d|$$

with $c + d \in I^2$. Thus, J is additively closed. If $x \in J$, $|x| \leq |a|$ as before, and if $c \in A$ then $|cx| \leq |ca|$, $ca \in I^2$ implies that $cx \in J$. This completes the proof that J is an ideal.

Since $I^2 \subseteq J$ is trivial it suffices to prove $J \subseteq I$. For, then $I^2 \subseteq J \subseteq L(I)$. Pick $x \in J$, $|x| \leq |a|$ with $a \in I^2$. Since $x = x^+ - x^-$ and $0 \leq x^+, x^- \leq |x|$ one may assume that $0 \leq x$. Writing $a = \sum \alpha_i \beta_i$, $\alpha_i, \beta_i \in I$, one has $0 \leq x \leq \sum \alpha_i^2 + \beta_i^2$. By the Theorem of Riesz ([5], Corollaire 1.2.17) there are $y_i, z_i \in A$ with $0 \leq y_i \leq \alpha_i^2$, $0 \leq z_i \leq \beta_i^2$ and $x = \sum y_i + z_i$. The 2^{nd} convexity property now implies that $y_i \in (\alpha_i) \subseteq I$ and $z_i \in (\beta_i) \subseteq I$. Altogether one concludes that $x \in I$. $\square$

The set of ideals of any ring is a complete lattice with intersection as meet and sum as join. Quite clearly, in an f–ring A, intersections of l–ideals are l–ideals. By [5], Proposition 2.1.12, sums of l–ideals are also l–ideals. Thus, $LId(A) \subseteq Id(A)$ is a complete sublattice. If A has bounded inversion then $L : Id(A) \to LId(A)$ preserves arbitrary intersections, but it does not preserve joins, in general. In any f–ring, a trivial computation shows that $(I : J) = \{a \in A; aJ \subseteq I\} \in LId(A)$ whenever $I \in LId(A)$. If A has the 2^{nd} convexity property then finite products of l–ideals are l–ideals ([30], Theorem 4.4(2)) and radical ideals are l–ideals ([30], Theorem 4.1(2)). With the additional condition that every nonnegative element of A has a nonnegative square root it can be shown that the idempotent ideals are exactly the radical ideals. For, if $I = I^2$ then I is radical by [30], Theorem 4.3. Conversely, if $I = \sqrt{I}$ then I is an l–ideal (as noted above), hence I is square dominated (cf. [31] or [32], p. 3111). Now [30], Theorem 4.3, applies to show that $I = I^2$.

In any f–ring the *irreducible l–ideals* ([5], Definition 8.4.2) are of particular importance. These are exactly the l–ideals I for which A/I is totally ordered ([5], Théorème 9.1.5). Since all f–rings are reduced in this paper this is also equivalent to I containing some prime ideal ([5], Théorème 9.3.2). Every minimal prime ideal is an l–ideal ([5], Théorème 9.3.2); every prime ideal which is an l–ideal is irreducible. Any l–ideal is an intersection of irreducible l–ideals ([5], Proposition 8.4.6); for example, $I \in LId(A)$ can be written as $\cap\{I+p; p \in LId(A) \cap Spec(A)\}$. Following [26], p. 212, the set of irreducible prime l–ideals is called the *Keimel spectrum* of A and is denoted by $SpeK(A)$. The sets

$$S(a) = \{I \in SpeK(A); a \notin I\}$$

form a basis for a topology of $SpeK(A)$. It follows from [5], section 10.1, that $SpeK(A)$ is a spectral space in the sense of [24]. For any homomorphism $f : A \to B$ in the category of reduced f–rings the map $SpeK(f) : SpeK(B) \to SpeK(A) : J \to f^{-1}(J)$ is a morphism of spectral spaces. In this way $SpeK$ is a functor from the category of reduced f–rings to the category of spectral spaces.

It was shown above that in an f–ring A with the 2^{nd} convexity property any ideal I is very closely approximated by the l–ideal $L(I)$. It is an obvious question for which f–rings one actually has $Id(A) = LId(A)$. For rings of continuous functions an answer has been known for a long time: Given a completely regular space X, let $C(X)$ be the ring of continuous functions into $\mathbb{R}$. Then every ideal of $C(X)$ is an l–ideal if and only if X is an F–space, if and only if every prime ideal of $C(X)$ contains a unique minimal prime ideal ([15], Theorem 14.25). If this is the case then $C(X)/p$ is a convex subring of the real closed field $qf(C(X)/p)$ (which follows from [15], Theorem 4.7 and Theorem 14.24), i.e., $C(X)$ is an SV–ring in the terminology of [22].

Proposition 1.4. *Let A be an f–ring. In A every ideal is an l–ideal if and only if the following two conditions hold:*
(i) If $p \subseteq A$ is a prime ideal then A/p is totally ordered and convex in its quotient field;
(ii) every prime ideal of A contains a unique minimal one.

Proof. First suppose that every ideal is an l–ideal. If $p \subseteq A$ is a prime ideal then p is an irreducible l–ideal, hence the domain A/p is totally ordered and every ideal of A/p is convex. It is well known (or easy to check) that then A/p is a convex subring of $qf(A/p)$. Next, pick two minimal prime ideals $p, q \subseteq A$, $p \neq q$. It is claimed that $p + q = A$. As p and q are incomparable there is some $a \in A$ such that $a(p) > 0$ in A/p and $a(q) < 0$ in A/q. Since (a) is an l–ideal one has $|a| = c \cdot a$ for some $c \in A$. This implies $c \equiv 1(mod\ p)$ and $c \equiv -1(mod\ q)$, i.e., $2 = (1-c) + (1+c) \in p+q$. Since $(2) \in LId(A)$ and $0 \leq 1 \leq 2$ it follows that $1 \in p+q$.

For the converse it suffices to pick $a, b \in A$ with $|a| \leq |b|$ and to show that then $a \in (b)$. To start with, let $p \subseteq A$ be any prime ideal, $q \subseteq p$ the minimal prime ideal contained in p. Then q is an l–ideal and A/q is a totally ordered domain which is convex in its quotient field. Thus, p/q is a convex ideal of A/q. This implies that $p \subseteq A$ is an l–ideal as well, i.e., the set of prime l–ideals is all of $Spec(A)$. Then every radical ideal is an l–ideal as well. If $I \subseteq A$ is a radical ideal then it is clear that A/I satisfies the conditions (i) and (ii) as well. Now let $I = \cap D(b)$ and let $\pi : A \to B = A/I$ be the canonical homomorphism. The canonical map $Spec(\pi)$ is a homeomorphism of $Spec(B)$ onto $\overline{D(b)} = \{r \in Spec(A); \exists p \in D(b) : p \subseteq r\}$. Suppose that there is some $c \in B$ with $\pi(a) = c\pi(b)$. Then picking $x \in A$ such that $c = \pi(x)$ one has $a = xb$. For, if $a \neq xb$ then there is some $p \in Spec(A)$ with $a - xb \notin p$. If $p \notin D(b)$ then $|a| \leq |b|$ implies that $a, b \in p$, hence $a - xb \in p$, a contradiction. But if $p \in D(b)$ then there is a unique $q \in Spec(B)$ with $p = \pi^{-1}(q)$ and

$$(a - xb)(p) = (\pi(a) - c\pi(b))(q) = 0,$$

once again a contradiction. So, it suffices to show that $c \in B$ exists with $\pi(a) = c\pi(b)$. Therefore, one may assume that $D(b)$ is dense in A, i.e., that $D(b)$ contains every minimal prime ideal.

If $q \in Max(A)$ let $\mu(q) \in Min(A)$ be the unique minimal prime ideal contained in q. Define

$$x(q) = \frac{a(\mu(q))}{b(\mu(q))} \in qf(A/\mu(q)).$$

From $|a| \leq |b|$ it follows that $|x(q)| \leq 1$ in $qf(A/\mu(q))$, by convexity of $A/\mu(q)$ in its quotient field one gets $x(q) \in A/\mu(q)$. Pick $x_q \in A$ with image $x(q)$ in $A/\mu(q)$. The subsets $\overline{\{\mu(q)\}}$ and

$$U_q = \{r \in Spec(A); a(r) \neq x_q(r)b(r)\}$$

of $Spec(A)$ are proconstructible, closed under generalization and are disjoint. Therefore there is some open constructible neighborhood V_q of $\overline{\{\mu(q)\}}$ with $V_q \cap U_q = \emptyset$, say $V_q = D(s_q)$, $s_q \in A$. The canonical image of x_q in A_{s_q} is denoted by c_q. If $q, q' \in Max(A)$ are given then it is easy to check that the images of c_q and $c_{q'}$ in $A_{s_q s_{q'}}$ agree. Thus, considering the scheme $Spec(A)$ one has an open cover formed by the V_q and a section c_q over each V_q such that the sections are compatible. By glueing these local sections together one gets a global section $c \in A$. Since $a = cb$ locally it follows that the same holds globally. $\square$

It was pointed out before the proposition that the conditions (i) and (ii) are not independent for rings of continuous functions. In fact, (ii) implies (i) for these rings, but the reverse implication is false. This follows from [23], Theorem 2.8.

For arbitrary f–rings the implication (ii) $\Rightarrow$ (i) is also false. In fact, it is false even for real closed rings. A counterexample is provided by any real closed domain A which is not a valuation ring. For, in a real closed domain the prime ideals always form a chain (since the support function $supp : Sper(A) \to Spec(A)$ is a homeomorphism – cf. [40], Proposition I 3.8). Such domains can be obtained through the $D + M$–construction of [16], Appendix 2: Let $V \subseteq R$ be a convex subring in a real closed field, let $\overline{R}$ be the real closed residue field, $M \subseteq V$ the maximal ideal. If $R_0 \subseteq V$ is any maximal subfield then $R_0 \subseteq V \to \overline{R}$ is an isomorphism ([37], p. 89, Satz 6; [28], p. 66, Satz 3), hence $V = R_0 + M$. If

$R_1 \subseteq R_0$ is any real closed subfield then $R_1 + M$ is a real closed domain ([36], p. 18, Korollar; [41], Example 13). Whenever $R_1 \subset R_0$ is a proper subfield the ring $R_1 + M$ is not a valuation ring.

Since every ring of continuous functions is a real closed ring ([44], Theorem 1.2) the conditions (i) and (ii) of Proposition 1.4 are independent for real closed rings.

All the results proved in this section apply to real closed rings. It was mentioned already that real closed rings are f–rings. They have bounded inversion by [40], Proposition I 3.1, and the 2^{nd} convexity property by [40], Proof of Proposition 3.8. Nonnegative elements have nonnegative square roots by [40], Proposition I 3.3. Real closed rings have a large number of special properties in addition to these: all prime ideals are convex ([40], Propositon I 3.8); residue fields at prime ideals are real closed ([40], Corollary I 3.26); reduced factor rings are real closed ([43], Lemma 3.7), just to mention a few.

According to [12], Introduction, or [3], Definition 1, a domain is called *divided* if every prime ideal is comparable with every principal ideal.

Proposition 1.5. *Real closed domains are divided domains.*

Proof. If I is any ideal in the real closed domain A and if $p \subseteq A$ is any prime ideal then both $L(I)$ and p are convex in A, hence they are comparable. If $I \not\subseteq p$ then $I^2 \not\subseteq p$, hence $L(I) \not\subseteq p$ (Propositon 1.3). But then $p \subseteq L(I) \subseteq I$. □

The prime ideals in a real closed domain A form a chain, hence A is local. According to [38], Proposition 9, real closed domains are integrally closed. By [35], Corollary 11, real closed domains are going down domains.

It is clear from Proposition 1.4 that most real closed rings have ideals which are not l–ideals. On the other hand, arbitrary ideals can be approximated very well by l–ideals (Proposition 1.3). Therefore, in the investigation of localizations of real closed rings it is frequently sufficient to deal with l–ideals.

A *Gabriel filter* is a set $\mathcal{F} \subseteq Id(A)$ having the following properties:

(a) $\mathcal{F}$ is a filter; and

(b) if $I \in \mathcal{F}$ and $J \in Id(A)$ and $(J : x) \in \mathcal{F}$ for each $x \in I$ then $J \in \mathcal{F}$

(cf. [7], Chapitre 2, Exercises, p. 157 ff; [46], section 1.3; [47], section VI. 5). More generally, call $\mathcal{F}$ a *multiplicative filter* of ideals if $\mathcal{F}$ has property (a) and:

(c) if $I, J \in \mathcal{F}$ then $IJ \in \mathcal{F}$

(cf. [19], p. 601). Note that every Gabriel filter is a multiplicative filter, but not vice versa. Now suppose that A is a real closed ring. If $\mathcal{F}$ is a multiplicative filter then $\mathcal{F} \cap LId(A)$ is a filter in $LId(A)$ having properties (a) and (c). Thus, one may speak of a multiplicative filter of l–ideals. Moreover, Proposition 1.3 shows that $\mathcal{F} \cap LId(A)$ is a filter basis of $\mathcal{F}$. Therefore, there is a bijective correspondence between multiplicative filters in $Id(A)$ and in $LId(A)$.

2. Multiplicative filters in arbitrary rings

If A is any ring then usually the set of multiplicative filters of A lies in between the set of *topologizing filters* (cf. [7], Chapitre 2, p. 157, Exercise 16; these are called *pretopologies* in [46], p. 13) and the set of Gabriel filters. Therefore every multiplicative filter determines a left exact preradical $t_{\mathcal{F}}$ of $\mathbf{A}-\mathbf{Mod}$ ([7], Chapitre 2, p. 157/158, Exercise 17; [46], Proposition 3.3; [47], Proposition VI. 4.2). Also, $\mathcal{F}$ yields another left exact functor $l_{\mathcal{F}} : \mathbf{A}-\mathbf{Mod} \to \mathbf{A}-\mathbf{Mod}$ which is defined by $l_{\mathcal{F}}(M) = \varinjlim_{I\in\mathcal{F}} Hom_A(I,M)$ on the objects ([7], Chapitre 2, p. 158, Exercise 17 c)). The canonical maps $\nu_{\mathcal{F},M} : M \to l_{\mathcal{F}}(M)$ define a natural transformation $\nu_{\mathcal{F}} : id_{\mathbf{A–Mod}} \to l_{\mathcal{F}}$. Going beyond what is possible to do with topologizing filters, a multiplicative filter allows the definition of a bilinear map

$$\psi_M : l_{\mathcal{F}}(A) \times l_{\mathcal{F}}(M) \to l_{\mathcal{F}}(M).$$

On the level of representatives the map is defined exactly as for Gabriel filters ([7], Chapitre 2, p. 159, Exercise 19; [46], §7). Suppose that $a \in l_{\mathcal{F}}(A)$ and $x \in l_{\mathcal{F}}(M)$ are represented by $\alpha : I \to A$, $\xi : J \to M$ with $I, J \in \mathcal{F}$. Then $\alpha^{-1}(J)$ contains $IJ \in \mathcal{F}$, hence belongs to $\mathcal{F}$. Now $\psi_M(a,x)$ is defined to be the canonical image of $\alpha^{-1}(J) \xrightarrow{\alpha} J \xrightarrow{\xi} M$ in $l_{\mathcal{F}}(M)$. With ψ_A as multiplication, $l_{\mathcal{F}}(A)$ is a commutative ring with 1. Using ψ_M as multiplication by scalars, $l_{\mathcal{F}}(M)$ acquires the structure of an $l_{\mathcal{F}}(A)$–module. The canonical map $\nu_{\mathcal{F},A} : A \to l_{\mathcal{F}}(A)$ is a ring homomorphism. The iteration $l_{\mathcal{F}}l_{\mathcal{F}}$ of the functor $l_{\mathcal{F}}$ is denoted by $L_{\mathcal{F}}$. This construction will be considered only if $\mathcal{F}$ is a Gabriel filter. The principle properties of $L_{\mathcal{F}}$ may be found in [7], Chapitre II, p. 157 ff., or [46], §7.

If $\varphi : A \to B$ is a homomorphism between rings then there are several canonical maps between the multiplicative filters of A and B, resp.

Lemma 2.1.

(a) If $\mathcal{F}$ is a multiplicative filters of A then

$$\varphi_*\mathcal{F} = \{J \subseteq B; \varphi^{-1}(J) \in \mathcal{F}\}$$

is a multiplicative filter of B (cf. [7], Chapitre 2, p. 160, Exercise 21 c) [7], p. 96).

(b) If $\mathcal{G}$ is a multiplicative filter of B then

$$\varphi^*\mathcal{G} = \{I \subseteq A; \varphi(I)B \in \mathcal{G}\}$$

is a multiplicative filter of A.

(c) If the given filter in (a) or (b) is a Gabriel filter then so is the new filter.

(d) $\varphi^*\varphi_*\mathcal{F} \supseteq \mathcal{F}$, $\varphi_*\varphi^*\mathcal{G} \subseteq \mathcal{G}$.

(e) If $\psi : B \to C$ *is another homomorphism and* $\mathcal{H}$ *is a multiplicative filter on* C *then* $\psi_*\varphi_*\mathcal{F} = (\psi\varphi)_*\mathcal{F}$ *and* $\varphi^*\psi^*\mathcal{H} = (\psi\varphi)^*\mathcal{H}$. □

Now let $\varphi : A \to l_{\mathcal{F}}(A)$ be the canonical map $\nu_{\mathcal{F},A}$ and define

$$D(\mathcal{F}) = \{p \in Spec(A); p \notin \mathcal{F}\},$$
$$D(\varphi_*\mathcal{F}) = \{q \in Spec(l_{\mathcal{F}}(A)); q \notin \varphi_*\mathcal{F}\}.$$

It will be necessary to study the relationship between these two sets. With $\overline{A} = A/t_{\mathcal{F}}(A)$ the map φ factors into $\psi : A \to \overline{A}$ and $\omega : \overline{A} \to l_{\mathcal{F}}(A)$. Set $\mathcal{G} = \psi_*\mathcal{F}$, $\mathcal{H} = \varphi_*\mathcal{F} = \omega_*\mathcal{G}$.

Lemma 2.2. *If* π *is the functorial map* $Spec(\psi) : Spec(\overline{A}) \to Spec(A)$ *then* π *restricts to a homeomorphism* $D(\mathcal{G}) \to D(\mathcal{F})$.

Proof. If $q \in D(\mathcal{G})$ then $\psi^{-1}(q) \notin \mathcal{F}$, i.e., $\pi(q) \in D(\mathcal{F})$. Thus, π restricts to a well–defined map $D(\mathcal{G}) \to D(\mathcal{F})$. It is also clear that $D(\mathcal{G}) = \pi^{-1}(D(\mathcal{F}))$. (Note for later use that no special property of ψ has been used to get this map. Therefore, if ψ is any ring homomorphism, this map is always well–defined.) Now suppose that $p \notin \mathcal{F}$. Then $t_{\mathcal{F}}(A) \subseteq p$: Pick $a \in t_{\mathcal{F}}(A)$ and choose $I \in \mathcal{F}$ such that $aI = \{0\}$. There exists some $b \in I \backslash p$. Since $ab = 0 \in p$ one sees that $a \in p$. Because of $t_{\mathcal{F}}(A) \subseteq p$ there is a unique prime ideal $q \subseteq \overline{A}$ with $\pi(q) = p$. It remains to show that $q \in D(\mathcal{G})$. For, then the homeomorphism $Spec(\overline{A}) \to V(t_{\mathcal{F}}(A))$ obtained from π by restriction of the codomain restricts further to the bijection $D(\mathcal{G}) \to D(\mathcal{F})$; this is a homeomorphism as well. If one assumes that $q \notin D(\mathcal{G})$ then $q \in \mathcal{G}$, i.e., $p = \pi(q) = \psi^{-1}(q) \in \mathcal{F}$, contradicting the choice of p. □

Because of Lemma 2.2 it is the same thing to study the relationship between $D(\mathcal{F})$ and $D(\mathcal{H})$ or between $D(\mathcal{G})$ and $D(\mathcal{H})$. If $a \in l_{\mathcal{F}}(A)$ has a representative α: $I \to A$ with $I \in \mathcal{F}$ then set $J = \psi(I) \in \mathcal{G}$. Since $a\varphi(x) = \varphi(\alpha(x)) \in \omega(\overline{A})$ ([46], Lemma 7.4) one sees that for every $a \in l_{\mathcal{F}}(A)$ there is some ideal $J \in \mathcal{G}$ with $a\omega(J) \subseteq \omega(\overline{A})$. Therefore, the extension $\omega : \overline{A} \to l_{\mathcal{F}}(A)$ is a special case of the following situation: Consider an extension $\varphi : A \to B$ and a multiplicative filter $\mathcal{F}$ in A such that for every $b \in B$ there is some ideal $I \in \mathcal{F}$ with $b\varphi(I) \subseteq \varphi(A)$.

Again, define $\mathcal{G} = \varphi_*\mathcal{F}$. The relationship between $D(\mathcal{F})$ and $D(\mathcal{G})$ will be studied in this situation. By the proof of Lemma 2.2 the functorial map $\pi = Spec(\varphi)$ restricts to a well–defined map $\pi' : D(\mathcal{G}) \to D(\mathcal{F})$. The following considerations serve to define a map in the opposite direction. Let $p \in D(\mathcal{F})$ and set $S = A \backslash p$. A subset $q \subseteq B$ is defined by

$$q = \{b \in B; \exists s \in S : b\varphi(s) \in \varphi(p)\}.$$

One checks easily that q is a prime ideal. Next it is claimed that $\pi(q) = p$. Obviously, $\varphi(p) \subseteq q$. Now let $a \in A$ and suppose that $\varphi(a) \in q$, say $\varphi(a)\varphi(s) \in \varphi(p)$. This implies $as \in p$, hence $a \in p$ since $s \notin p$.

Altogether, a map $\lambda : D(\mathcal{F}) \to D(\mathcal{G})$ has been defined such that $\pi'\lambda = id$. It is claimed that λ is surjective: Suppose that $q \in D(\mathcal{G})$, set $p = \pi(q)$ and $S = A\backslash p$. Then $p \in D(\mathcal{F})$ and $I \cap S \neq \emptyset$ for each $I \in \mathcal{F}$. If $b \in q$ there is some $s \in S$ such that $b\varphi(s) = \varphi(a)$ for some $a \in A$. Since $b \in q$, it follows that $a \in p$, hence that $b \in \lambda(p)$. Thus, $q \subseteq \lambda(p)$. On the other hand, if $b \in \lambda(p)$, say $b\varphi(s) \in \varphi(p) \subseteq q$ then $b \in q$ because $\varphi(s) \notin q$. Thus, $q = \lambda(p)$. This proves most of

Proposition 2.3. (cf. [7], Chapitre 2, p. 161/162, Exercise 21 b)) *Suppose $\varphi : A \to B$ is injective and $\mathcal{F}$ is a multiplicative filter in A such that for each $b \in B$ there is some $I \in \mathcal{F}$ with $b\varphi(I) \subseteq \varphi(A)$. Then the functorial map $\pi = Spec(\varphi)$ restricts to a homeomorphism $\pi' : D(\varphi_*\mathcal{F}) = \pi^{-1}(D(\mathcal{F})) \to D(\mathcal{F})$.*

Proof. It remains to show that π' is open. A basis of open sets of $D(\mathcal{G})$ is formed by the sets $D(\mathcal{G}) \cap D(b)$ with $b \in B$. Let $p = \pi(q) \in \pi(D(\mathcal{G}) \cap D(b))$ and set $S = A\backslash p$. Again, there is some $s \in S$ such that $b\varphi(s) = \varphi(a)$. It suffices to show that $D(\mathcal{F}) \cap D(a)$ contains p and is contained in $\pi(D(\mathcal{G}) \cap D(b))$. Assume that $a \in p$. Then $b\varphi(s) \in \varphi(p) \subseteq q$ yields $b \in q$ because of $\varphi(s) \notin q$. This is a contradiction, hence $p \in D(\mathcal{F}) \cap D(a)$. Now suppose that $p' \in D(\mathcal{F}) \cap D(a)$. Since π' is bijective, there is a unique $q' \in D(\mathcal{G})$ such that $\pi'(q') = p'$. If $q' \notin D(b)$ then $b \in q'$, hence $\varphi(a) = b\varphi(s) \in q'$.But then $a \in \pi(q') = p'$, a contradiction. □

On the basis of Proposition 2.3 it is an obvious question whether the close relationship between the prime spectra of A and B extends to the local rings of A and B at corresponding prime ideals. The next result gives an answer:

Proposition 2.4. *In the situation of Proposition 2.3, let $p \in D(\mathcal{F})$ and $q \in D(\varphi_*\mathcal{F})$ such that $\pi(q) = p$. Then the canonical homomorphism $\varphi_p : A_p \to B_q$ is an isomorphism.*

Proof. First pick $\frac{a}{s} \in A_p$ such that $\varphi_p(\frac{a}{s}) = 0$, i.e., there is some $t \in B\backslash q$ with $t\varphi(a) = 0$. Let $I \in \mathcal{F}$ such that $t\varphi(I) \subseteq \varphi(A)$. Choosing $r \in I\backslash p$ one gets $t\varphi(r) = \varphi(c)$, hence $\varphi(ac) = t\varphi(a)\varphi(r) = 0$. Since $t \notin q$ and $r \notin p$ it follows that $t\varphi(r) \notin q$, thus $c \notin p$. Now $ac = 0$ implies that $\frac{a}{1} = 0$ in A_p. This proves injectivity. For surjectivity, suppose that $\frac{b}{t} \in B_q$. There is some ideal $I \in \mathcal{F}$ such that $b\varphi(I) \subseteq \varphi(A)$ and $t\varphi(I) \subseteq \varphi(A)$. Because of $p \in D(\mathcal{F}) \subseteq D(I)$ one finds some $s \in I\backslash p$. Then $b\varphi(s) = \varphi(a)$ and $t\varphi(s) = \varphi(r)$ with $r \notin p$. Now $\varphi_p(\frac{a}{r}) = \frac{\varphi(a)}{\varphi(r)} = \frac{b}{t}$, and the proof is complete. □

Corollary 2.5. *Keeping the notation and the hypotheses of Proposition 2.3, sup-*

pose that $D(a) \subseteq D(\mathcal{F})$ for some $a \in A$. Then the canonical homomorphism $\varphi_a : A_a \to B_{\varphi(a)}$ is an isomorphism. □

An immediate consequence of the last couple of results is

Theorem 2.6. *Let $\varphi : A \to B$ be a monomorphism of rings and let $\mathcal{F} \subseteq Id(A)$ be a multiplicative filter such that $D(\mathcal{F}) \subseteq Spec(A)$ is open. Assume that for each $b \in B$ there is some $I \in \mathcal{F}$ with $b\varphi(I) \subseteq \varphi(A)$. Then $D(\varphi_*\mathcal{F}) \subseteq Spec(B)$ is open. Considering both $D(\mathcal{F}) \subseteq Spec(A)$ and $D(\varphi_*\mathcal{F}) \subseteq Spec(B)$ as open subschemes, the functorial morphism $Spec(\varphi) : Spec(B) \to Spec(A)$ of schemes restricts to an isomorphism $D(\varphi_*\mathcal{F}) \to D(\mathcal{F})$.* □

Returning to the original situation, namely the canonical homomorphism $\varphi : A \to l_{\mathcal{F}}(A)$ where $\mathcal{F}$ is any multiplicative filter in A, the preceding results yield

Corollary 2.7. *The functorial map $\pi = Spec(\varphi)$ restricts to a homeomorphism $\pi' : D(\varphi_*\mathcal{F}) = \pi^{-1}(D(\mathcal{F})) \to D(\mathcal{F})$. For every $p \in D(\mathcal{F})$, the homomorphism $\varphi_p : A_p \to l_{\mathcal{F}}(A)_{\pi^{-1}(p)}$ is an isomorphism. If $a \in A$ has the property that $D(a) \subseteq D(\mathcal{F})$ then $\varphi_a : A_a \to l_{\mathcal{F}}(A)_{\varphi(a)}$ is an isomorphism. If $D(\mathcal{F}) \subseteq Spec(A)$ is open then π' is an isomorphism between schemes.* □

Example 2.8. Let $I \subseteq A$ be an ideal.The set

$$\mathcal{F}_I = \{J \subseteq A; \exists n \in \mathbb{N} : I^n \subseteq J\}$$

is a multiplicative filter of A. If I is finitely generated then $\mathcal{F}_I$ is even a Gabriel filter ([48], p. 72). The ring $l_{\mathcal{F}_I}(A)$ will be denoted by A_I, the canonical homomorphism is $\nu_I : A \to A_I$. The direct image of $\mathcal{F}_I$ is $\mathcal{F}_{\nu_I(I)A_I}$. The sets $D(\mathcal{F}) = D(I)$ and $D(\nu_{I^*}\mathcal{F}_I) = D(\nu_I(I)A_I)$ are open subschemes of $Spec(A)$ and $Spec(B)$. The restriction $D(\nu_I(I)A_I) \to D(I)$ of $Spec(\nu_I)$ is an isomorphism of schemes. □

If $\mathcal{F}$ is any multiplicative filter then the isomorphism

$$l_{\mathcal{F}}(A) = \varinjlim_{I \in \mathcal{F}} Hom_A(I, A) \xrightarrow{\cong} \varinjlim_{I \in \mathcal{F}} \varinjlim_{n \in \mathbb{N}} Hom_A(I^n, A) = \varinjlim_{I \in \mathcal{F}} A_I$$

suggests that the rings A_I are particularly useful for the investigation of arbitrary localizations with respect to multiplicative filters. They will be used for the proof that $l_{\mathcal{F}}(A)$ is real closed whenever A is real closed.

Given a homomorphism $\varphi : A \to B$, two canonical maps between the sets of multiplicative filters of A and B were introduced in Lemma 2.1. For a special case there is yet another canonical map:

Lemma 2.9. *Suppose that $\varphi : A \to B$ is a surjective homomorphism of reduced rings and that for every $a \in A$ there exist some $2 \leq n \in \mathbb{N}$ and some $b \in A$ with*

$a = b^n$. If $\mathcal{G}$ is a multiplicative filter of B then

$$\varphi^{-1}\mathcal{G} = \{I \subseteq A; \exists J \in \mathcal{G} : I = \varphi^{-1}(J)\}$$

is a multiplicative filter of A. If $\mathcal{G}$ is a Gabriel filter then so is $\varphi^{-1}\mathcal{G}$.

Proof. It is obvious that $\varphi^{-1}\mathcal{G}$ is a filter. To show that $\varphi^{-1}\mathcal{G}$ is multiplicative pick $I = \varphi^{-1}(J), I' = \varphi^{-1}(J') \in \varphi^{-1}\mathcal{G}$. Then $II' = \varphi^{-1}(J)\varphi^{-1}(J') \subseteq \varphi^{-1}(JJ')$ holds trivially. In fact, the ideals are equal: If $x \in \varphi^{-1}(JJ')$ then $\varphi(x) = \sum b_i b_i'$ with $b_i \in J$, $b_i' \in J'$. By surjectivity of φ there are $a_i \in I$, $a_i' \in I'$ such that $\varphi(a_i) = b_i$, $\varphi(a_i') = b_i'$, hence $y = x - \sum a_i a_i' \in ker(\varphi)$. Writing $y = z^n$ for $2 \leq n \in \mathbb{N}$ one notes that $z \in ker(\varphi)$ (since B is reduced) and $z \in I \cap I'$. Therefore, $x = \sum a_i a_i' + zz^{n-1} \in II'$.

Now assume that $\mathcal{G}$ is a Gabriel filter. Suppose that $I = \varphi^{-1}(J) \in \varphi^{-1}\mathcal{G}$ and that $K \subseteq A$ is an ideal with $(K : x) \in \varphi^{-1}\mathcal{G}$ for all $x \in I$. To start with, pick $x \in ker(\varphi) \subseteq I$ and write $x = y^n$, $2 \leq n \in \mathbb{N}$. Then $y \in ker(\varphi) \subseteq I$ and $y^{n-1} \in ker(\varphi) \subseteq (K : y)$ imply that $x = y^{n-1}y \in K$, i.e., $ker(\varphi) \subseteq K$. It is easy to check that $(\varphi(K) : \varphi(x)) = \varphi(K : x)$ for all $x \in I$. Since $\varphi(K : x) \in \mathcal{G}$ one concludes that $\varphi(K) \in \mathcal{G}$, hence $K = \varphi^{-1}(\varphi(K)) \in \varphi^{-1}\mathcal{G}$. □

In arbitrary rings there is a type of Gabriel filters that is particularly easy to construct (cf. [48], (5.7)): Let $Y \subseteq Spec(A)$ be any subset, let $\mathcal{F}(Y) = \{I \subseteq A; Y \subseteq D(I)\}$. Then $\mathcal{F}(Y)$ is a Gabriel filter. It is obvious that $D(\mathcal{F}(Y)) = Gen(Y)$. Evidently, $\mathcal{F}(Y)$ is the largest Gabriel filter $\mathcal{F}$ with $D(\mathcal{F}) = Gen(Y)$. If Y is open and constructible then there is a finitely generated ideal I with $Y = D(I)$ and in this case $\mathcal{F}(Y) = \mathcal{F}_I$ is the only Gabriel filter $\mathcal{F}$ of finite type with $D(\mathcal{F}) = Y$ (cf. [48], p. 79). More generally, if Y is proconstructible then $\mathcal{F}(Y)$ is of finite type ([48], p. 75) and again this is the only Gabriel filter $\mathcal{F}$ of finite type with $D(\mathcal{F}) = Y$. In this way there is a bijective correspondence between Gabriel filters of finite type and generically closed proconstructible subsets of $Spec(A)$. Explicit examples of Gabriel filters which are not of the form $\mathcal{F}(Y)$ can be obtained from [8], Theorem 3.3: Suppose that V is a nontrivial valuation ring in an algebraically closed field or a proper convex subring in a real closed field. Let $M \subseteq V$ be the maximal ideal. In either case, $M = M^2$. Therefore $\mathcal{F} = \{M, V\}$ is a Gabriel filter. If V is of finite rank (more generally: if there is a largest prime ideal properly contained in M) then $\mathcal{F} \neq \mathcal{F}(Y)$ for any set Y.

In connection with Gabriel filters of the type $\mathcal{F}(Y)$ it is an obvious question what the maps φ_* and φ^* associated with a homomorphism $\varphi : A \to B$ do with such filters. The functorial map $Spec(\varphi) : Spec(B) \to Spec(A)$ is denoted by π.

Lemma 2.10. *Suppose that φ is surjective.*
(a) If $Y \subseteq im(\pi)$ then $\varphi_*\mathcal{F}(Y) = \mathcal{F}(\pi^{-1}(Y))$.
(b) If $Y \subseteq Spec(B)$ then $\varphi^*\mathcal{F}(Y) = \mathcal{F}(\pi(Y))$.

Proof. (a) Suppose that $J \in \varphi_*\mathcal{F}(Y)$. If $q \in \pi^{-1}(Y)$ then $\varphi^{-1}(J) \not\subseteq \pi(q) =$

$\varphi^{-1}(q)$, hence $J \not\subseteq q$. This proves one inclusion. For the other one, pick $J \in \mathcal{F}(\pi^{-1}(Y))$. It is claimed that $\varphi^{-1}(J) \not\subseteq p$ for every $p \in Y$. Given such p, there is $q \in \pi^{-1}(Y)$ with $\pi(q) = p$. Since $J \not\subseteq q$ one finds $x \in \varphi^{-1}(J)$ with $\varphi(x) \in J \backslash q$, hence $x \in \varphi^{-1}(J) \backslash p$. This finishes the proof of (a). – **(b)** If $I \in \varphi^* \mathcal{F}(Y)$ and $p = \pi(q) \in \pi(Y)$ then $\varphi(I) \not\subseteq q$, hence $I \not\subseteq \varphi^{-1}(q) = p$. Thus, $I \in \mathcal{F}(\pi(Y))$. Conversely, if $I \in \mathcal{F}(\pi(Y))$ and $q \in Y$ then $\pi(q) \in \pi(Y)$, hence $I \not\subseteq \pi(q)$. This implies that $\varphi(I) \not\subseteq q$, hence $\varphi(I) \in \mathcal{F}(Y)$, i.e., $I \in \varphi^* \mathcal{F}(Y)$. □

3. Deligne's formula

Generalizing a formula of Deligne, Gabriel localizations can be used to describe the sections of a quasi–coherent sheaf of modules over an open quasi–compact subset of an affine scheme ([48], Proposition 5.16; see also [33], section 4). If one asks only for the global ring of sections of the structure sheaf of some restriction of an affine scheme then the same formula holds in a far more general situation.

Suppose that A is a reduced ring. The structure sheaf of the affine scheme $Spec(A)$ is denoted by $\mathcal{O} = \mathcal{O}_A$, the restriction to any subspace $X \subseteq Spec(A)$ is $\mathcal{O}|_X$. Let $Y \subseteq Spec(A)$ be a subset satisfying the following conditions:

(A) Y is generically closed;
(B) there is an open cover $Y = \bigcup\limits_{\kappa \in K} Y_\kappa$ such that each Y_κ is an intersection of a family $D(s_{\kappa\lambda})$, $\lambda \in \Lambda_\kappa$, of basic open subsets of $Spec(A)$.

For each κ let S_κ be the multiplicative subset of A generated by $\{s_{\kappa\lambda}; \lambda\}$. The locally ringed space $(Y_\kappa, \mathcal{O}|_{Y_\kappa})$ is canonically isomorphic to the affine scheme $Spec(A_{S_\kappa})$. In particular, $(Y, \mathcal{O}|_Y)$ is a scheme. Here are two situations in which these hypotheses are satisfied.

Example 3.1. Any open subset $Y \subseteq Spec(A)$ has properties (A) and (B). □

Example 3.2. Suppose that A is a real closed ring and that $Y \subseteq Spec(A)$ is a generically closed subspace ([40], Definition II 2.1). By definition, Y has an open cover $\cup Y_\kappa$ such that each Y_κ is proconstructible in $Spec(A)$. As Y is generically closed it is clear that so is each Y_κ. Then Y_κ is an intersection of quasi–compact open subsets of $Spec(A)$. It suffices to realize that every quasi-compact open subset is actually a basic open subset of $Spec(A)$. This follows from

$$D(a_1) \cup \ldots \cup D(a_r) = D(a_1^2) \cup \ldots \cup D(a_r^2) = D(a_1^2 + \ldots + a_r^2).$$

□

With Y one associates the Gabriel filter $\mathcal{F}(Y)$ (section 2). It will be shown eventually that there is a canonical isomorphism $L_{\mathcal{F}(Y)}(A) \to \Gamma(Y) = \Gamma(\mathcal{O}|_Y)$. For preparation a few auxiliary results are needed.

Lemma 3.3. *If A is a reduced ring and $\mathcal{F}$ is a multiplicative filter then the torsion ideal $t_{\mathcal{F}}(A) \subseteq A$ is radical.* $\square$

In particular, the ring $B = A/t_{\mathcal{F}(Y)}(A)$ is reduced. Let $\varphi : A \to B$ be the canonical homomorphism; let $\pi = Spec(\varphi)$ be the functorial map of the associated affine schemes.

Lemma 3.4. *The subset $Z = \pi^{-1}(Y) \subseteq Spec(B)$ is dense and has properties (A) and (B). The restriction $Z \to Y$ is a homeomorphism and the morphism $\pi' : (Z, \mathcal{O}_B|_Z) \to (Y, \mathcal{O}_A|_Y)$ obtained from π by restriction is an isomorphism of locally ringed spaces.* $\square$

Because of Lemma 3.4 there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ \Gamma(Y) & \xrightarrow[\varphi']{\cong} & \Gamma(Z) \end{array}$$

At a later point this diagram will be used to reduce the problem to the case that $Y \subseteq Spec(A)$ is dense.

Lemma 3.5. *There is a unique ring homomorphism $\sigma : l_{\mathcal{F}(Y)}(A) \to \Gamma(Y)$ such that $\rho = \sigma\nu$ (where $\rho : A \to \Gamma(Y)$ is the restriction homomorphism and $\nu = \nu_{\mathcal{F}(Y),A}$).*

Proof. To start with, let $I \subseteq A$ be any ideal with $Y \subseteq D(I)$. A homomorphism $\sigma'_I : Hom_A(I, A) \to \Gamma(D(I))$ is defined as follows: If $\alpha \in Hom_A(I, A)$ and $x \in I$ then $\frac{\alpha(x)}{x} \in A_x$ is a section of $Spec(A)$ over $D(x)$. Because of

$$\frac{\alpha(x)}{x} = \frac{y\alpha(x)}{yx} = \frac{\alpha(yx)}{yx} = \frac{x\alpha(y)}{xy} = \frac{\alpha(y)}{y}$$

in A_{xy}, these sections are compatible and can be glued together to yield a section $\sigma'_I(\alpha) \in \Gamma(D(I))$. It is obvious that σ'_I is a homomorphism of A–modules. Then also σ_I, the composition of σ'_I with the restriction $\Gamma(D(I)) \to \Gamma(Y)$, is A–linear. If $J \subseteq I$ then the diagram

$$\begin{array}{ccc} Hom_A(I, A) & \longrightarrow & Hom_A(J, A) \\ & \sigma_I \searrow \quad \swarrow \sigma_J & \\ & \Gamma(Y) & \end{array}$$

is commutative. Going to the limit one obtains a homomorphism $\sigma : l_{\mathcal{F}(Y)}(A) \to \Gamma(Y)$ of A–modules. The explicit definition of the multiplication in $l_{\mathcal{F}(Y)}(A)$ shows that σ is even a ring homomorphism. It is obvious that $\rho = \sigma\nu$. This proves the existence of σ. For uniqueness, suppose that $\sigma' : l_{\mathcal{F}(Y)}(A) \to \Gamma(Y)$ is another homomorphism with $\rho = \sigma'\nu$. Pick some $a \in l_{\mathcal{F}(Y)}(A)$ and a representative $\alpha : I \to A$ of a (where $I \in \mathcal{F}(Y)$). Note that $Ann_{\Gamma(Y)}(\rho(I)) = (0)$, i.e., $\rho(I) \subset \Gamma(Y)$ is a dense ideal. If $x \in I$ then $\nu(\alpha(x)) = a\nu(x)$ ([46], Lemma 7.4), therefore

$$\sigma(a)\rho(x) = \sigma(a\nu(x)) = \rho(\alpha(x)) = \sigma'(a\nu(x)) = \sigma'(a)\rho(x).$$

It follows that $\sigma(a) - \sigma'(a) \in Ann_{\Gamma(Y)}(\rho(I))$, hence $\sigma(a) = \sigma'(a)$. □

Lemma 3.6. *Let A be a reduced ring, $I \subseteq A$ an ideal. If $\alpha \in Hom_A(I, A)$ and $x \in I$ then $D(\alpha(x)) \subseteq \overline{D(x)}$ in $Spec(A)$.*

Proof. Assume by way of contradiction that there is some $p \in D(\alpha(x)) \backslash \overline{D(x)}$. Since $D(x)$ is quasi–compact there exists $a \notin p$ with $a \in \cap D(x)$. As A is reduced this implies that $ax = 0$, hence also $a\alpha(x) = \alpha(ax) = 0$. But $a(p)\alpha(x)(p) \neq 0$ yields a contradiction. □

Lemma 2.10 (a) shows that the Gabriel filters $\mathcal{F}(Z)$ and $\varphi_*\mathcal{F}(Y)$ of B both agree. Note that B and $l_{\mathcal{F}(Y)}(A)$ are $\mathcal{F}(Y)$–torsion free. As $Z \subseteq Spec(B)$ is dense, B is also $\mathcal{F}(Z)$–torsion free. The canonical homomorphisms

$$\begin{aligned} l_{\mathcal{F}(Z)}(B) &\longrightarrow L_{\mathcal{F}(Z)}(B), \\ l_{\mathcal{F}(Y)}(B) &\longrightarrow L_{\mathcal{F}(Y)}(B), \\ L_{\mathcal{F}(Y)}(A) &\longrightarrow L_{\mathcal{F}(Y)}(l_{\mathcal{F}(Y)}(A)) \end{aligned}$$

are therefore all isomorphims ([46], Lemma 7.6). By [7], Chapitre 2, p. 159, Exercise 19 c), and there is a unique homomorphism $l_{\mathcal{F}(Z)}(B) \to l_{\mathcal{F}(Y)}(B)$ making the diagram

$$\begin{tikzcd} B \arrow[r, "="] \arrow[d, "\nu_{\mathcal{F}(Z),B}"'] & B \arrow[d, "\nu_{\mathcal{F}(Y),B}"] \\ l_{\mathcal{F}(Z)}(B) \arrow[r] & l_{\mathcal{F}(Y)}(B) \end{tikzcd}$$

commutative. In fact, this homomorphism is an isomorphism ([7], Chapitre 2, p. 162, Exercise 21 c)). Similarly, there is a unique homomorphism $l_{\mathcal{F}(Y)}(B) \to L_{\mathcal{F}(Y)}(A)$ making the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d] & l_{\mathcal{F}(Y)}(A) \arrow[d] \\ l_{\mathcal{F}(Y)}(B) \arrow[r] & l_{\mathcal{F}(Y)}(A) \end{tikzcd}$$

commutative. According to [46], Lemma 7.6, this homomorphism is an isomorphism. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[rrrrr, "\varphi"] \arrow[dd, "\rho_A"'] \arrow[dr, "\nu_A"] & & & & & B \arrow[dllll] \arrow[dll] \arrow[dl, "\nu_B"'] \arrow[dd, "\rho_B"] \\
& l_{\mathcal{F}(Y)}(A) \arrow[r] \arrow[dl, "\sigma_A"] & L_{\mathcal{F}(Y)}(A) & l_{\mathcal{F}(Y)}(B) \arrow[l, "\cong"'] & l_{\mathcal{F}(Z)}(B) \arrow[l, "\cong"'] \arrow[dr, "\sigma_B"'] & \\
\Gamma(Y) \arrow[rrrrr, "\cong", "\varphi'"'] & & & & & \Gamma(Z)
\end{tikzcd}$$

By the uniqueness of the various isomorphisms and the uniqueness of σ_A and σ_B there exist unique homomorphisms $\sigma'_B : l_{\mathcal{F}(Y)}(B) \to \Gamma(Z)$, $\sigma'_A : L_{\mathcal{F}(Y)}(A) \to \Gamma(Y)$ making the entire diagram commutative.

Theorem 3.7. *The canonical homomorphism $\sigma'_A : L_{\mathcal{F}(Y)}(A) \to \Gamma(Y)$ is an isomorphism.*

Proof. By the definition of σ'_A it suffices to show that σ'_B is an isomorphism. This, in turn, is equivalent to σ_B being an isomorphism. Therefore it remains to show that σ is an isomorphism if $Y \subseteq Spec(A)$ is dense. First injectivity: Pick $a \in l_{\mathcal{F}(Y)}(A)$ such that $\sigma(a) = 0$. Choose a homomorphism $\alpha : I \to A$ representing a. Then $\sigma(a)$ is determined by the family of sections $\frac{\alpha(x)}{x} \in A_x$ over $D(x)$ for all $x \in I$. Since $Y \cap D(x)$ is dense in $D(x)$ the fact that $\frac{\alpha(x)}{x}|_{Y \cap D(x)} = 0$ implies $\frac{\alpha(x)}{x} = 0$. Since A is reduced this yields $x\alpha(x) = 0$ in A. If $\alpha(x) \neq 0$ then there is a minimal prime ideal p with $p \in D(\alpha(x))$. Lemma 3.6 shows that $p \in D(x)$, hence $x(p)\alpha(x)(p) \neq 0$, a contradiction. Thus, $\alpha(x) = 0$ for every $x \in I$. This implies $a = 0$ in $l_{\mathcal{F}(Y)}(A)$, and σ is injective.

To prove surjectivity, suppose that $\gamma \in \Gamma(Y)$. Since $Y \subseteq Spec(A)$ is dense the restriction $\rho : A \to \Gamma(Y)$ is injective. It is claimed that the ideal

$$I = (A : \gamma)_A = \{x \in A; \gamma\rho(x) \in \rho(A)\} \subseteq A$$

belongs to $\mathcal{F}(Y)$, i.e., that $Y \subseteq D(I)$. For any $p \in Y$ there is a neighborhood $Y' \subseteq Y$ of p such that $Y' = \bigcap_{s \in S} D(s)$ for some multiplicative set $S \subseteq A$. The canonical homomorphism $\rho_S : A_S \to \Gamma(Y)_{\rho(S)} \cong \Gamma(Y')$ is an isomorphism. Hence there exist $x \in A$ and $s \in S$ such that $\rho_S(\frac{x}{s}) = \frac{\gamma}{1}$ in $\Gamma(Y)_{\rho(S)}$. This means that $\rho(tx) = \rho(ts)\gamma$ in $\Gamma(Y)$ for some $t \in S$. But then $\gamma\rho(ts) \in \rho(A)$, i.e., $ts \in I$. Since $p \in D(ts)$ it follows that $p \in D(I)$. This proves that $I \in \mathcal{F}(Y)$, hence $Hom_A(I, A)$

contributes to $l_{\mathcal{F}(Y)}(A)$. In particular, consider the A–linear map

$$\begin{aligned} \alpha : I \longrightarrow \rho(A) &\xrightarrow{\rho^{-1}} A \\ x \longrightarrow \gamma\rho(x). & \end{aligned}$$

If $a \in l_{\mathcal{F}(Y)}(A)$ is the class of α then it is claimed that $\sigma(a) = \gamma$ in $\Gamma(Y)$. It suffices to compare the canonical images of $\sigma(a)$ and γ in each local ring A_p with $p \in Y$. But it is clear from the definitions of α and σ that these germs agree. This finishes the proof. □

4. Filters of finite type in real closed rings

A Gabriel filter is said to be of *finite type* if it has a filter basis consisting of finitely generated ideals ([48], p. 75). This definition can be extended immediately to multiplicative filters. However, it is easy to see that every multiplicative filter of finite type is, in fact, a Gabriel filter (cf. [48], p. 72/73). It was pointed out in section 2 that there is a bijective correspondence between Gabriel filters of finite type and generically closed proconstructible subsets of the prime spectrum. The purpose of this section is to give a complete description of the Gabriel filters of finite type in a real closed ring.

It turns out that these filters are trivial in the sense that the corresponding localizations are classical rings of quotients.

Proposition 4.1. *Let A be a real closed ring, let $\mathcal{F}$ be a Gabriel filter of A. Then the following statements are equivalent:*

(a) $\mathcal{F}$ is of finite type.

(b) $\mathcal{F}$ has a basis consisting of principal ideals (i.e., $\mathcal{F}$ is a 1-topology, cf. [47], p. 148).

(c) $D(\mathcal{F}) \subseteq Spec(A)$ is proconstructible.

(d) The localization functor $L_{\mathcal{F}}$ has property (T) (cf. [17], p. 28; [48], p. 93).

If this is the case then $L_{\mathcal{F}}(A) = A_S$ with $S = \{s \in A; (s) \in \mathcal{F}\}$. In particular, $A \to L_{\mathcal{F}}(A)$ is a flat epimorphism.

Proof. The equivalence of (a) and (c) is already clear. The implication $(b) \Rightarrow (a)$ is trivial. To prove $(a) \Rightarrow (b)$, pick some finitely generated ideal $I \in \mathcal{F}$, say $I = (a_1, \dots, a_k)$. Define $a = a_1^2 + \dots + a_k^2$ and consider the principal ideal (a). It is clear that $(a) \subseteq I^2 \subseteq I$. On the other hand, $I^4 \subseteq L((a))$ and $L((a)) \subseteq (a)$ (section 1), hence $(a) \in \mathcal{F}$. – $(b) \Rightarrow (d)$ If (b) holds then $L_{\mathcal{F}}(A) = A_S$ with $S = \{s \in A | (s) \in \mathcal{F}\}$ and $L_{\mathcal{F}}(M) \cong M \otimes_A A_S$ ([48], p. 78). This is exactly property (T). – Finally, $(d) \Rightarrow (a)$ is shown in [48], p. 95. □

Because of [33], Proposition 2.5, the proposition has the following immediate consequence:

Corollary 4.2. *If A is a real closed ring then the following three sets are in bijective correspondence with each other:*
– the set of isomorphism classes of flat epimorphisms over A;
– the set of Gabriel filters of finite type;
– the set of generically closed proconstructible subsets of $Spec(A)$. □

By [40], Theorem I 3.29 and Theorem I 4.9, the results of this section imply that $L_{\mathcal{F}}(A)$ is a real closed ring whenever $\mathcal{F}$ is a Gabriel filter of finite type in a real closed ring.

5. Filters in real closed domains

In this section the Gabriel filters in real closed domains and in some cases also their localizations are determined. The results are reminiscent of the description of the Gabriel filters of a valuation domain in [8], section 3. Let A be a real closed domain. The set $LId(A)$ of l–ideals is the totally ordered set of convex ideals. Every proper l–ideal is irreducible, hence $LId(A) = SpeK(A) \cup \{A\}$. Every multiplicative filter $\mathcal{F}$ is completely determined by $\mathcal{F} \cap SpeK(A)$ (section 1). Hence, the set of multiplicative filters is totally ordered. The first result shows that there is no need to distinguish between multiplicative filters and Gabriel filters.

Proposition 5.1. *If A is a real closed domain and $\mathcal{F}$ is a multiplicative filter then $\mathcal{F}$ is a Gabriel filter.*

Proof. Suppose that $I \in \mathcal{F}$ and that $J \in LId(A)$ with $(J : x) \in \mathcal{F}$ for each $x \in I$. Because of section 1 one may assume that I is also convex. It is claimed that $J \in \mathcal{F}$. If $I^n \subseteq J$ for some n then there is nothing to prove. So suppose that $J \subset I^n$ for every n. There is some $0 < x \in I$ with $x^3 \notin J$. Then $x^2 \notin (J : x)$ and $(J : x)$ is an l–ideal. Therefore, $0 \leq y \leq x^2$ or $0 \leq -y \leq x^2$ for every $y \in (J : x)$, and the 2^{nd} convexity property implies that $(J : x) \subseteq (x)$. But then $(J : x)^2 \subseteq x(J : x) \subseteq J$ shows that $J \in \mathcal{F}$ because $(J : x)^2 \in \mathcal{F}$. □

The filters of finite type are easy to recognize: In $Spec(A)$ a set is generically closed and proconstructible if and only if it consists of the generalizations of a single point. Thus, $\mathcal{F}$ is of finite type if and only if there is a prime ideal $p \subseteq A$ with $\mathcal{F} = \{I \subseteq A; p \subset I\}$. The following is just another way of phrasing this condition:

Proposition 5.2. *The Gabriel filter $\mathcal{F}$ in the real closed domain A is of finite type if and only if there is a largest ideal not belonging to $\mathcal{F}$. If this is the case then this largest ideal is prime.*

Proof. One implication is already clear; for the other one suppose that p is the largest ideal not in $\mathcal{F}$. Since A is a divided domain (Proposition 1.5) it suffices to show that p is a prime ideal. (Note that then every ideal is comparable with p, hence $\mathcal{F} = \{I; p \subset I\}$.) First let $\overline{p}$ be the convex hull of p. From $p^2 \subseteq L(p)$ it follows that $\overline{p}^2 \subseteq L(p) \subseteq p$. If $p \neq \overline{p}$ then $\overline{p} \in \mathcal{F}$, hence $\overline{p}^2 \in \mathcal{F}$. This implies that $p \in \mathcal{F}$, a contradiction. Now assume that p is not prime. There exists some $x \in A$ such that $x^2 \notin p$, but $x^4 \in p$. Being convex, p and $L((x))$ are comparable. Since $x^2 \in L((x))\backslash p$ one knows that $p \subset L((x))$, hence $L((x)) \in \mathcal{F}$. But then also $(x) \in \mathcal{F}$ and $(x^4) \in \mathcal{F}$. Now $(x^4) \subseteq p$ shows that $p \in \mathcal{F}$, a contradiction. □

Now assume that there is no largest ideal not belonging to the filter $\mathcal{F}$. Let $p = \bigcap \mathcal{F}$. Then p is convex since every ideal in $\mathcal{F}$ contains a convex ideal belonging to $\mathcal{F}$. Moreover, p is a prime ideal: Suppose that $x, y \notin p$. There are convex ideals $I, J \in \mathcal{F}$ with $x \notin I$, $y \notin J$. Since I and J are comparable one may assume that $x, y \notin I$. This implies that $0 \leq |z| < |x|, |y|$ for all $z \in I$. Assume that $xy \in I^2$. Then one finds some $0 \leq z \in I$ such that $0 \leq |x|\ |y| \leq z^2$, a contradiction. Thus, $xy \notin I^2$, hence $xy \notin p$ (since $I^2 \in \mathcal{F}$). This shows that p is prime. It is claimed that $p \in \mathcal{F}$, i.e., that p is the smallest element of $\mathcal{F}$. Assume by way of contradiction that this is false. Then $p \notin \mathcal{F}$. Since there is no largest ideal not belonging to $\mathcal{F}$, there must be some $I \notin \mathcal{F}$ such that $p \subset I$. Then also $p \subset I^2 \subseteq L(I) \subseteq I$, hence $L(I) \notin \mathcal{F}$. Pick any $x \in L(I)\backslash p$. There must be some $J \in \mathcal{F}$ with $x \notin J$. Then also $L(J) \in \mathcal{F}$ and $L(J)$ is comparable with $L(I)$. Since $x \in L(I)\backslash L(J)$ it follows that $L(J) \subseteq L(I)$. This is impossible since $L(J) \in \mathcal{F}$ and $L(I) \notin \mathcal{F}$. Altogether this finishes the proof of the following result:

Proposition 5.3. *Let $\mathcal{F}$ be a Gabriel filter in the real closed domain A. If $\mathcal{F}$ is not of finite type then there is some prime ideal p such that $\mathcal{F} = \{I \subseteq A; p \subseteq I\}$.*□

The last two propositions determine the set of all Gabriel filters of the real closed domain A. The rest of the section is devoted to computing localizations with respect to Gabriel filters. If $\mathcal{F} = Id(A)$ then $L_{\mathcal{F}}(A) = 0$. This exceptional and trivial case is excluded in the further considerations, i.e., it is assumed that $(0) \notin \mathcal{F}$. Note that A is then $\mathcal{F}$–torsionfree, hence $L_{\mathcal{F}}(A) = l_{\mathcal{F}}(A)$ ([46], Lemma 7.6). Moreover, the localization is a subring of the real closed field $qf(A)$. It can be described in the following ways:

$$\begin{aligned} L_{\mathcal{F}}(A) &= \{x \in qf(A); (A : x)_A \in \mathcal{F}\} \\ &= \{x \in qf(A); \exists I \in \mathcal{F} : xI \subseteq A\} \end{aligned}$$

(cf. [13], p. 4522).

If $\mathcal{F}$ is of finite type then let p be the largest ideal not belonging to $\mathcal{F}$. It is clear from section 4 that $L_{\mathcal{F}}(A) = A_p$. Note that $pA_p = p$ since A is a divided ring ([12], Introduction). Now suppose that $\mathcal{F}$ is not of finite type. Let $p = \bigcap \mathcal{F}$. Then $\{p\}$ is a basis of the filter $\mathcal{F}$, hence

$$L_{\mathcal{F}}(A) = Hom_A(p, A) = (A : p)_{qf(A)} = (p : p)_{qf(A)}.$$

The first equality is due to the fact that $p = p^2 = \dots$ for any prime ideal in a real closed ring. For the last equality, suppose that $a \in qf(A)$ and that $ap \subseteq A$. It is claimed that even $ap \subseteq p$. Suppose that $0 \leq x \in p$. Then $\sqrt{x}$ exists in p, hence $a\sqrt{x} \in A$. This implies $ax = (a\sqrt{x})\sqrt{x} \in p$.

The set $\mathcal{F}' = \mathcal{F}\backslash\{p\}$ is a Gabriel filter of A as well. Since it is of finite type its localization is known to be $L_{\mathcal{F}'}(A) = A_p$. Let $\varphi : A \to A_p$ be the canonical homomorphism. The direct image $\varphi_*\mathcal{F}$ agrees with the Gabriel filter $\mathcal{F}_{A\backslash p}$ of A_p described in [8], Proposition 1.2. One sees immediately that $\varphi_*\mathcal{F} = \{A_p, pA_p\}$. By [7], Chapitre 2, p. 162, Exercise 21 c), the rings $L_{\mathcal{F}}(A)$ and $L_{\varphi_*\mathcal{F}}(A_p)$ can be identified canonically. Thus, to determine the localization $L_{\mathcal{F}}(A)$ it suffices to deal with the case that $\mathcal{F} = \{A, M\}$, M the maximal ideal of A.

Theorem 5.4. *$L_{\mathcal{F}}(A)$ is a real closed domain with maximal ideal M.*

Proof. Let $V \subseteq qf(A)$ be the largest convex valuation ring with center M in A, i.e., $M = A \cap N$ where N is the maximal ideal of V. Then $L_{\mathcal{F}}(A) \subseteq V$: If $0 < a \in qf(A)\backslash V$ then $a^{-1} \in N$ and there is some $0 < b \in M$ with $0 < a^{-1} < b$. Then also $0 < a^{-\frac{1}{2}} < b^{\frac{1}{2}} \in M$. This implies that $a^{\frac{1}{2}} = a\ a^{-\frac{1}{2}} < a\ b^{\frac{1}{2}}$. Since $a^{\frac{1}{2}} \notin V$ and V is convex one concludes that $ab^{\frac{1}{2}} \notin V$, hence $ab^{\frac{1}{2}} \notin A$. Thus, $a \notin (A : M)_{qf(A)} = L_{\mathcal{F}}(A)$.

Next it is claimed that $L_{\mathcal{F}}(A) \cap N = M$. One inclusion holds trivially, for the other one pick $a \in L_{\mathcal{F}}(A) \cap N$. For any $x \in M$ one has $ax \in A$. By the choice of V there is some $x \in M$ with $0 \leq |a| < x$. Then $|ax| < x^2$ implies that $ax \in (x)$ in A ([42], Satz 1). Writing $ax = cx$ with $c \in A$ one sees that $a = c \in A \cap N = M$.

It is now clear that $M \subseteq L_{\mathcal{F}}(A)$ is a prime ideal. It is even the unique maximal ideal of $L_{\mathcal{F}}(A)$, i.e., $L_{\mathcal{F}}(A)$ is local with maximal ideal M. Suppose that $a \in L_{\mathcal{F}}(A)\backslash M$. It is claimed that also $a^{-1} \in L_{\mathcal{F}}(A)$. Assume by way of contradiction that $a^{-1}M \not\subseteq A$, say $a^{-1}x \notin A$ with $0 < x \in M$. If $|ax| \leq x^r$ for some $1 < r \in \mathbb{Q}$ then $ax \in (x)$ ([30], Satz 1), hence $ax = cx$ with $c \in A$. But then $a = c \in A\backslash M = A^*$, i.e., $a^{-1} \in A$ and $a^{-1}M \subseteq A$, a contradiction. Therefore, $x^r < |ax|$ for all $1 < r \in \mathbb{Q}$. By [42], Satz 1, this implies $x^2 \in (ax)$, hence $a^{-1}x = \frac{x^2}{ax} \in A$, a contradiction. This finishes the proof that $M = L_{\mathcal{F}}(A)\backslash L_{\mathcal{F}}(A)^*$.

To show that $L_{\mathcal{F}}(A)$ is a real closed domain the criterion of [42], Satz 1, will be used. It is clear that $L_{\mathcal{F}}(A)$ is a domain with quotient field $qf(A)$. Since A is real closed this is a real closed field. To show that $L_{\mathcal{F}}(A)$ is integrally closed consider an equation

$$a^n + a_{n-1}a^{n-1} + \dots + a_0 = 0$$

with $a \in qf(A)$, $a_i \in L_{\mathcal{F}}(A)$. For any $x \in M$ one gets

$$(ax)^n + (a_{n-1}x)(ax)^{n-1} + \ldots + a_0 x^n = 0.$$

Note that $a_i x^{n-i} \in A$ for each $i = 0, \ldots, n-1$. Since A is integrally closed one concludes that $ax \in A$. Thus, $a \in (A : M)_{qf(A)} = L_{\mathcal{F}}(A)$. Finally, suppose that $0 \leq a \leq b$ in $L_{\mathcal{F}}(A)$. It is claimed that $a^2 \in (b)$: If $b \in L_{\mathcal{F}}(A)^*$ then this is trivial. Otherwise $b \in L_{\mathcal{F}}(A) \backslash L_{\mathcal{F}}(A)^* = M$. From $0 \leq a \leq b$ it follows that $a \in M$ as well. But then $a^2 \in (b)$ in A, hence also in $L_{\mathcal{F}}(A)$. $\square$

By Theorem 5.4 the maximal ideal of $L_{\mathcal{F}}(A)$ is known. The structure of the ring $L_{\mathcal{F}}(A)$ is particularly simple because of the following general result about real closed domains:

Theorem 5.5. *Let A be a real closed domain. Then there is a field $R \subseteq A$ such that the homomorphism $R \subseteq A \longrightarrow A/M$ is an isomorphism. Thus, $A = R + M$.*

Proof. (cf. [28], p. 66, Satz 3; [37], p. 89, Satz 6) The field of real algebraic numbers is contained in A. Zorn's Lemma shows that there is a maximal subfield $R \subseteq A$. Since A is integrally closed in the real closed field $qf(A)$, R is algebraically closed in $qf(A)$, hence is real closed. Let $\overline{R} \subseteq R/M$ be the image of R. If $\overline{x} \in A/M$ is transcendental over $\overline{R}$ then $R[x] \subseteq A$ is a polynomial ring with $R[x] \cap M = (0)$. But then $R(x) \subseteq A$ and $R \subset R(x)$ is a proper extension. This contradicts the maximality of R. Thus $\overline{R} \subseteq A/M$ is an algebraic extension. Since both fields are real closed they must agree. $\square$

By [1], Theorem 3.10, $A \subseteq L_{\mathcal{F}}(A)$ is a pair of rings having the same prime ideals. In fact, [1], Proposition 3.3, shows that $L_{\mathcal{F}}(A)$ is the largest extension of A having this property. Therefore, and in view of Theorem 5.5, the computation of $L_{\mathcal{F}}(A)$ boils down to determining the residue field. A lower bound for $L_{\mathcal{F}}(A)/M$ is provided by A/M. A first upper bound is obtained as follows: Suppose that $V \subseteq qf(A)$ is the largest convex subring such that $M = A \cap N$, where $N \subseteq V$ is the maximal ideal. Then $L_{\mathcal{F}}(A) \subseteq V$ and $M = L_{\mathcal{F}}(A) \cap N$, hence $L_{\mathcal{F}}(A)/M \subseteq V/N$. Suppose A is a *pseudo valuation domain* (cf. [21]), i.e., for all $x, y \in qf(A)$ and all $p \in Spec(A)$ it follows that $x \in p$ or $y \in p$ whenever $xy \in p$. According to [21], Theorem 2.7, in this case $V = (A : M)_{qf(A)} = L_{\mathcal{F}}(A)$. In particular, one has $L_{\mathcal{F}}(A)/M = V/N$. This proves

Propositon 5.6. *If the real closed ring A is a pseudo valuation domain then $L_{\mathcal{F}}(A)$ is the largest convex subring $V \subseteq qf(A)$ which dominates A.* $\square$

Note that every real closed ring of dimension 1 is a pseudo valuation domain ([38], Lemma 8). Arbitrary real closed pseudo valuation domains are constructed as in [1], Proposition 2.6, where V is a convex valuation ring in a real closed field and $k \subseteq K$ is a real closed subfield of the residue field. It follows from [36], p. 18, Korollar, that the rings so constructed are real closed.

If A is any real closed domain and $p \subseteq M \subseteq A$ is a prime ideal let $\varphi : A \to A/p$ be canonical. The maximal ideal of A/p is M/p and $\overline{\mathcal{F}} = \varphi_*\mathcal{F} = \{A/p, M/p\}$. By [7], Chapitre 2, p. 160, Exercise 19 d) and p. 162, Exercise 21 c), there is a canonical homomorphism

$$L_{\mathcal{F}}(A) \longrightarrow L_{\mathcal{F}}(A/p) \cong L_{\overline{\mathcal{F}}}(A/p)$$

which is a local homomorphism between local rings. Therefore

$$L_{\mathcal{F}}(A)/M \subseteq L_{\mathcal{F}}(A/p)/(M/p).$$

It is easy to find examples where $L_{\mathcal{F}}(A/p)/(M/p) \subset V/N$. Thus, $L_{\mathcal{F}}(A/p)/(M/p)$ is a better upper bound for $L_{\mathcal{F}}(A)/M$ than V/N.

Theorem 5.7. *Suppose that A is a real closed domain in which there is a largest prime ideal p that is properly contained in M. Then $L_{\mathcal{F}}(A)/p$ is the largest convex subring of $qf(A/p)$ which dominates A/p. In particular, $L_{\mathcal{F}}(A)/M$ is isomorphic to $L_{\overline{\mathcal{F}}}(A/p)/(M/p)$.*

Proof. Since A/p is a pseudo valuation domain the ring $L_{\overline{\mathcal{F}}}(A/p)$ has been determined in Proposition 5.6. It is only necessary to show that $L_{\mathcal{F}}(A) \to L_{\overline{\mathcal{F}}}(A/p)$ is surjective. Since M/p is the maximal ideal of $L_{\overline{\mathcal{F}}}(A/p)$ it suffices to prove that $L_{\overline{\mathcal{F}}}(A/p)^*$ belongs to the image. Let $\pi : A_p \to qf(A/p)$ be the canonical homomorphism. In $qf(A/p)$ the largest convex subring dominating A/p is $L_{\overline{\mathcal{F}}}(A/p)$. Let $W = \pi^{-1}(L_{\overline{\mathcal{F}}}(A/p)) \subseteq A_p$. Then $\pi(W) = L_{\overline{\mathcal{F}}}(A/p)$, and it suffices to show that $W = L_{\mathcal{F}}(A)$. It is clear that $L_{\mathcal{F}}(A) \subseteq W$. For the reverse inclusion, pick $a \in W$. If $x \in M$ then $\pi(a)\pi(x) \in A/p$ (since $\pi(a) \in L_{\overline{\mathcal{F}}}(A/p)$ and $\pi(x) \in M/p$), say $\pi(ax) = \pi(b)$ with $b \in A$. But this implies $ax - b \in p \subseteq A$, hence $ax \in A$. □

The theorem applies to all real closed domains of finite dimension, in particular to the factor domains of any ring of semi–algebraic functions on an affine semi–algebraic set.

6. Localizations of real closed rings are real closed

The localizations that were considered in the preceding two sections are real closed rings. This is a special case of a far more general phenomenon which is studied in this section. It will be shown, for example, that the ring $A_I = l_{\mathcal{F}}(A)$ is real closed whenever A is real closed and $\mathcal{F} = \{J \subseteq A; \exists n : I^n \subseteq J\}$ (see section 2).

Lemma 6.1. *Suppose that A is a reduced f–ring with bounded inversion and that $B \subseteq A$ is a convex subring. Then there is a multiplicative subset $S \subseteq B$ such that $A = B_S$.*

Proof. Define
$$S = \{s \in B; 0 \leq s \in A^*\}.$$
The inclusion $i : B \to A$ induces $i_S : B_S \to A$. Evidently, i_S is injective. For surjectivity, pick $a \in A$ and write
$$a = \sup(a, 1)\sup(-a, 1)\sup(\inf(a, 1), -1).$$
Since A has bounded inversion, $\sup(a, 1), \sup(-a, 1) \in A^*$. But then $\sup(a, 1)^{-1}$, $\sup(-a, 1)^{-1} \in S$. Since $\sup(\inf(a, 1), -1) \in B$ the claim follows immediately. □

Corollary 6.2. *With A and B as in Lemma 6.1, A is real closed if and only if B is real closed.*

Proof. If A is real closed then so is the convex subring B ([39], Theorem I 7.8). If B is real closed then so is any classical ring quotients of B, in particular A ([40], Theorem I 3.29, Theorem I 4.8). □

Now consider following situation: $A \subseteq C$ is an extension of real closed rings, $I \subseteq A$ is an ideal and let $\mathcal{F} = \{J \subseteq A; \exists n : I^n \subseteq J\}$. Define $B = \{b \in C; \exists n : bI^n \subseteq A\}$. Obviously, this is a subring of C. The main result of the present section is that B is a real closed ring. The proof will eventually be done by using Corollary 6.2. This requires that first a couple of properties of B are established. Because of the results of section 1 it may and will be assumed that I is an l–ideal of A.

Lemma 6.3. *B is a sub–f–ring of C.*

Proof. (With help by Warren McGovern) Given $b \in B$ with $b = b^+ - b^-$ in C it will be shown that $b^+ \in B$. It suffices to prove that $bI^n \subseteq A$ implies $b^+I^n \subseteq A$. By assumption I is an l–ideal, hence so is I^n. If $x = x^+ - x^- \in I^n$ then $b^+x = b^+x^+ - b^+x^-$ and b^+x^+, $b^+x^- \in A$ imply $b^+x \in A$. Therefore one may assume that $0 \leq x$. But then $bx = b^+x - b^-x$ with $inf(b^+x, b^-x) = 0$, i.e., $(bx)^+ = b^+x$. Because $A \subseteq C$ is a sub–f–ring and $bx \in A$ one concludes $b^+x = (bx)^+ \in A$. □

Lemma 6.4. *The f–ring B has bounded inversion.*

Proof. Suppose that $1 \leq b \in B$ and that $bI^n \subseteq A$. Being real closed, the ring C has bounded inversion, hence b^{-1} exists in C. It is only necessary to show that $b^{-1}J \subseteq A$ for some ideal $J \in \mathcal{F}$. It is certainly true that $b^{-1}(bI^n) \subseteq A$. So it suffices to show that $bI^n \in \mathcal{F}$. It will be shown that $I^{2n} \subseteq bI^n$. It is enough to deal with nonnegative elements of I^{2n}. So, pick $0 \leq x \in I^{2n}$ and write $x = \sum y_i z_i$ with

$y_i, z_i \in I^n$. Then $t = \sum |y_i| + |z_i| \in I^n$ (since I is an l–ideal) and $0 \leq x \leq t^2 \leq (bt)^2$. The 2^{nd} convexity property implies that $x \in (bt)$ in A, hence $x \in bI^n$. □

Lemma 6.5. *In B the squares are exactly the nonnegative elements.*

Proof. The squares are nonnegative since B is an f–ring. Now suppose that $0 \leq b \in B$ and that $bI^n \subseteq A$. In C there exists some $0 \leq c$ with $b = c^2$. Because I^n is an l–ideal it suffices to show that $cx \in A$ for every $0 \leq x \in I^n$. Because of $0 \leq bx^2 \in A \subseteq C$ there is a unique nonnegative square root of this element both in A and in C, namely cx. This implies $cx \in A$. □

Corollary 6.6. *Every prime ideal $q \subseteq B$ is an l–ideal and the residue domain B/q is totally ordered. The positive cone of B/q is formed by the squares. In particular, the support map $supp : Sper(B) \to Spec(B)$ is a homeomorphism.* □

The preparations for the proof of the main result of this section are finished now:

Theorem 6.7. *B is a real closed ring.*

Proof. The results proved so far show that Corollary 6.2 is applicable to the ring B. Thus, it suffices to show that the convex hull $H \subseteq B$ of A is real closed. Let $\rho_H : H \to \rho(H)$ be the real closure of H ([40], Definition I 4.1). It will be shown that ρ_H is an isomorphism. By [40], p. 10/11, there is a unique homomorphism $\tau : \rho(H) \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\subseteq} & B \\ {\scriptstyle \rho_H}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ \rho(H) & \xrightarrow[\tau]{} & C \end{array}$$

Being a convex subring of B, H shares all the properties with B that were established above. In particular, $supp : Sper(H) \to Spec(H)$ is a homeomorphism, hence ρ_H is an essential epimorphic extension in the category of reduced partially ordered rings ([45], Corollary 2.14). Therefore τ is a monomorphism and all the rings in the diagram can be considered as subrings of C. Note that the convex hull of H in $\rho(H)$ is all of $\rho(H)$. This implies that $H = B \cap \rho(H)$. Since ρ_H is a monomorphism one only needs to show that ρ_H is surjective. So, pick $h \in \rho(H)$. It will be shown that $h \in B$, i.e., that $hI^n \subseteq A$ for some $n \in \mathbb{N}$.

The inclusion $\varphi : A \to H$ and the multiplicative filter $\mathcal{F}$ satisfy the hypotheses of Theorem 2.6. Let Y be the open subscheme $D(\mathcal{F}) = D(I) \subseteq Spec(A)$,

Z the open subscheme $D(\varphi_* \mathcal{F}) \subseteq Spec(H)$. The functorial map $Spec(\varphi)$ restricts to an isomorphism $Z \to Y$. If $x \in I$ then $Spec(\varphi)$ restricts further to an isomorphism $D_H(x) \to D_A(x)$. Since $D_A(x)$ is a real closed scheme, so is $D_H(x)$. But then the restriction $D_{\rho(H)}(x) \to D_H(x)$ of the functorial morphism $Spec(\rho_H) : Spec(\rho(H)) \to Spec(H)$ is an isomorphism as well. In particular, $hx|_{D_{\rho(H)}(x)}$ may be considered as an element of $\Gamma(D_A(x))$. A constructible section $\alpha(x)$ on $Spec(A)$ over A is defined by setting $\alpha(x)(p) = hx(p)$ for $p \in D_A(x)$, $\alpha(x)(p) = 0$ otherwise.

It is claimed that $\alpha(x)$ is also a compatible secton. So, pick $p, q \in Spec(A)$, $p \subseteq q$. If $p, q \in Y$ then there exists some $y \in I$ such that $p, q \in D_A(y) \subseteq D(I)$. Under the isomorphism $D_{\rho(H)}(y) \to D_A(y)$ the sections $\alpha(x)|_{D_A(y)}$ and $hx|_{D_{\rho(H)}(y)}$ correspond to each other. Since $hx|_{D_{\rho(H)}(y)}$ is compatible, so is $\alpha(x)|_{D_A(y)}$. Now suppose that $p, q \notin Y$. In this case $\alpha(x)(p) = 0$, $\alpha(x)(q) = 0$, and there is nothing to prove. Finally, suppose that $p \in Y$ and $q \notin Y$. Let r be the closed point in $\overline{\{p\}} \cap D_A(x)$, s the generic point in $\overline{\{p\}} \setminus D_A(x)$. Then compatibility holds for $\alpha(x)$ between p and r and also between s and q. If $\alpha(x)$ is also compatible with respect to r and s then transitivity of the compatibility condition (cf. [36], Lemma 5.5) implies that $\alpha(x)$ is compatible with respect to p and q. So, one may assume that p is a closed point both of $D_A(x)$ and Y and that q is a generic point of $Spec(A) \setminus Y$. Let $p' \subseteq H$ be the unique prime ideal with $p' \cap A = p$. Since $D_H(x) \to D_A(x)$ is an isomorphism the canonical homomorphism $\rho(p) \to \rho(p')$ of the residue fields is an isomorphism. These fields are identified. As $\rho(H)$ is the convex hull of H in $\rho(H)$ and by the definition of H there exists some $a \in A$ such that $0 \leq |h| \leq a$. In particular, evaluating at p and p' one gets $0 \leq |h(p')| \leq a(p)$. Therefore $h(p')$ belongs to the convex hull of A/p in $\rho(p)$. In A/p one has $ax(p) \in I + p/p$. The ideal $I + p \subseteq A$ is convex and $I + p \subseteq q$. Thus, $I + p/p \subseteq q/p$ in A/p and $I + p/p$ is convex. By [38], Lemma 8, q/p is not only convex in A/p, but also in $\rho(p)$. Because of $0 \leq |hx(p')| \leq |ax(p)|$ one concludes that $hx(p') \in q/p$. Now $\alpha(x)(p) = hx(p')$ belongs to the maximal ideal of the largest convex valuation ring $C_{qp} \subseteq \rho(p)$ with center q/p in A/p. Hence the residue map $\lambda_{qp} : C_{qp} \to C_{qp}/M_{qp}$ maps $\alpha(x)(p)$ to $\alpha(x)(q) = 0$. This finishes the proof that $\alpha(x)$ is compatible.

Being a constructible and compatible section on $Spec(A)$ over A, $\alpha(x) \in A$. Considering $\alpha(x)$ as an element of $\rho(H)$, it is clear that $\alpha(x) = hx$. This proves that $hI \subseteq A$, hence that $h \in B \cap \rho(H)$, and the proof is finished. $\square$

In the rest of this section, Theorem 6.7 will be applied to show that a number of important ring theoretic constructions applied to real closed rings always yield real closed rings. For the notion of an *ideal transform* see e.g. [2]; [9]; [25], p. 30.

Corollary 6.8. *Let A be a real closed ring, $I \subseteq A$ an ideal. Then the ideal transform of A with respect to I is real closed.*

Proof. The total quotient ring $Tot(A)$ of A is a real closed ring ([40], Theorem I

3.29, Theorem I 4.8). Since

$$B = \{a \in Tot(A); \exists n : aI^n \subseteq A\}$$

is the ideal transform, the assertion follows immediately from Theorem 6.7. □

Suppose that A is real closed and $I \subseteq A$ is an ideal. It will be shown now that the localization A_I is real closed. Let $\mathcal{F} = \{J \subseteq A; \exists n : I^n \subseteq J\}$ and set $Y = D(\mathcal{F}) = D(I)$. In the proof of Lemma 3.5 the homomorphisms $\sigma_J : Hom_A(J, A) \to \Gamma(Y)$ where defined for every $J \in \mathcal{F}$. Let $\sigma : A_I = \lim\limits_{\to} Hom_A(J, A) \to \Gamma(Y)$ be the limit of the σ_J. It is clear that σ is injective. The ring $\Gamma(Y)$ is the global ring of sections of the real closed scheme Y, hence $\Gamma(Y)$ is real closed ([40], Theorem I 4.12). As before, let $\rho : A \to \Gamma(Y)$ be the canonical restriction homomorphism, let $\overline{A} = \rho(A)$. This is a real closed ring as well ([43], Lemma 3.7). One checks easily that ρ maps I bijectively onto $\rho(I)$. Thus, I will be identified with $\rho(I)$.

Theorem 6.9. *For the extension $\overline{A} \subseteq \Gamma(Y)$ of real closed rings one has*

$$A_I = \{\gamma \in \Gamma(Y); \exists n : \gamma I^n \subseteq \overline{A}\}.$$

In particular, A_I is a real closed ring.

Proof. Suppose that $\gamma \in A_I$. Then γ has a representative $\alpha : I^n \to A$. If $x \in I^n$ then $\gamma x = \rho(\alpha(x)) \in \overline{A}$ (cf. [46], Lemma 7.4). This proves one inclusion; for the other one pick $\gamma \in \Gamma(Y)$ with $\gamma I^n \subseteq \overline{A}$. But then $\gamma I^{n+1} \subseteq I$, and $\alpha : I^{n+1} \to I \subseteq A : x \to \gamma x$ is an A–linear map representing an element $a \in A_I$. From the definition of σ it is clear that $\sigma(a) = \gamma$. □

Corollary 6.10. *If A is a real closed ring and $\mathcal{F}$ is multiplicative filter then $l_{\mathcal{F}}(A)$ is real closed. If $\mathcal{F}$ is a Gabriel filter then also $L_{\mathcal{F}}(A)$ is real closed.* □

In a reduced ring the set of all dense ideals forms a Gabriel filter $\mathcal{F}$ ([7], Chapitre 2, p.164, Exercise 24). The localization $l_{\mathcal{F}}(A) = L_{\mathcal{F}}(A)$ is the *complete ring of quotients* ([29], section 2.3). Therefore:

Corollary 6.11. *Let A be real a closed ring, $Q(A)$ its complete ring of quotients. Then $Q(A)$ is real closed.* □

There are other ways to prove this result. For example, coming from another direction, this is a special case of the following general result: Let **C** be a monoreflective subcategory of the category **PO/N** of reduced partially ordered rings. Then for any object A of **C** the complete ring of quotients also belongs to **C**. The category of real closed rings is a monoreflective subcategory of **PO/N**, hence complete rings of quotients of real closed rings are real closed. (Monoreflectors of **PO/N** are investigated in [34]). Or, one may note that $Q(A)$ is the direct limit of all rings of global sections of dense open subschemes of $Spec(A)$ if A is reduced (cf. [4], section 3; or, Theorem 3.4). If A is real closed then every such ring of sections is real closed, and $Q(A)$, being the direct limit of real closed rings, is real closed ([40], Theorem I 3.29, Theorem I 4.8).

7. Multiplicative filters and the Keimel spectrum

Filters of the form $\mathcal{F}(Y)$, $Y \subseteq Spec(A)$ (section 2), have always played a particularly important rôle in the investigation of Gabriel filters. Let $\mathcal{G}(A)$ and $\mathcal{M}(A)$ be the sets of Gabriel filters and multiplicative filters of A, let $\mathcal{S}(A)$ be the set of subsets of $Spec(A)$ closed under specialization. Each one of these sets is a complete lattice. Define

$$V : \mathcal{M}(A) \to \mathcal{S}(A) : \mathcal{F} \to \mathcal{F} \cap Spec(A),$$
$$\mathcal{E} : \mathcal{S}(A) \to \mathcal{G}(A) : Z \to \mathcal{F}(Spec(A)\backslash Z).$$

It is clear that V is a homomorphism of complete lattices and that $\mathcal{E}(\cap Z_i) = \cap\mathcal{E}(Z_i)$ for all families $(Z_i)_i$ in $\mathcal{S}(A)$. But $\mathcal{E}$ is not homomorphic with respect to join, in general. Also, one has $V\mathcal{E} = id$, but $\mathcal{E}V \neq id$.

For real closed rings this technique for studying the sets $\mathcal{M}(A)$ and $\mathcal{G}(A)$ can be refined by using the Keimel spectrum. Let $\mathcal{S}_K(A)$ be the set of subsets $Z \subseteq SpeK(A)$ which are closed with respect to specialization and for which $I \in Z$ implies $I^n \in Z$. It is clear that $V_K(\mathcal{F}) = \mathcal{F} \cap SpeK(A) \in \mathcal{S}_K(A)$ for every $\mathcal{F} \in \mathcal{M}(A)$. Thus, there is a map

$$V_K : \mathcal{M}(A) \longrightarrow \mathcal{S}_K(A).$$

One checks immediately that $\mathcal{S}_K(A)$ is a complete lattice and that V_K is a homomorphism of complete lattices. If A is even a real closed domain then it is evident from section 1 that V_K is a bijection. This is not true in general, but recall that $\mathcal{F}$ always has a basis consisting of l–ideals and that each l–ideal is an intersection of irreducible l–ideals. Thus, $\mathcal{F}$ has a basis which consists of intersections of elements of $\mathcal{S}_K(A)$.

There are two natural maps in the other direction. To define them, pick a set $Z \in \mathcal{S}_K(A)$ and let $p \subset A$ be a prime ideal. As usual, $SpeK(A/p)$ is considered as as subset of $SpeK(A)$. Then $Z_p = Z \cap SpeK(A/p) \in \mathcal{S}_K(A/p)$ is the basis of a Gabriel filter $\mathcal{Z}_p$ on A/p. Let $\varphi_p : A \to A/p$ be the canonical map. Then both

$$\varphi_p^*\mathcal{Z}_p = \{I \subseteq A; \exists J \in Z : J \subseteq I + p\}$$

(see Lemma 2.1) and

$$\varphi_p^{-1}\mathcal{Z}_p = \{I \subseteq A; \exists J \in Z : p \subseteq J \subseteq I\}$$

(see Lemma 2.9) are Gabriel filters of A. Let $\mathcal{E}_0(Z)$ be the smallest multiplicative filter containing $\bigcup_p \varphi_p^{-1}\mathcal{Z}_p$, let $\mathcal{E}_1(Z) = \bigcap_p \varphi_p^{*}\mathcal{Z}_p$. Both $\mathcal{E}_0(Z)$ and $\mathcal{E}_1(Z)$ are multiplicative filters, $\mathcal{E}_1(Z)$ is even a Gabriel filter. Thus, two maps

$$\mathcal{E}_0 : \mathcal{S}_K(A) \longrightarrow \mathcal{M}(A),$$
$$\mathcal{E}_1 : \mathcal{S}_K(A) \longrightarrow \mathcal{G}(A) \subseteq \mathcal{M}(A)$$

have been defined. The principal properties of $\mathcal{E}_0$ and $\mathcal{E}_1$ are contained in

Proposition 7.1.

(a) If $(Z_i)_i$ is any family in $\mathcal{S}_K(A)$, then $\mathcal{E}_0(\cap Z_i) = \cap\mathcal{E}_0(Z_i)$, $\mathcal{E}_0(\cup Z_i)$ is the smallest multiplicative filter containing every $\mathcal{E}(Z_i)$ and $\mathcal{E}_1(\cap Z_i) = \cap\mathcal{E}_1(Z_i)$.

(b) $V_K\mathcal{E}_0 = id$, $V_K\mathcal{E}_1 = id$.

(c) $\mathcal{E}_0(Z)$ is the smallest multiplicative filter $\mathcal{F}$ with $V_K(\mathcal{F}) \supseteq Z$; $\mathcal{E}_1(Z)$ is the largest multiplicative filter $\mathcal{F}$ with $V_K(\mathcal{F}) \subseteq Z$.

(d) $\mathcal{E}_0(Z) = \{I \subseteq A; \exists I_1, \ldots, I_r \in Z : I_1 \cap \ldots \cap I_r \subseteq I\}$; $\mathcal{E}_1(Z) = \{I \subseteq A; \forall J \in SpeK(A)\backslash Z : I \not\subseteq J\}$.

Proof. Suppose that $p, q \in Spec(A)$ and that $I \in \varphi_p^{-1}\mathcal{Z}_p$. Pick $J \in Z$ with $p \subseteq J \subseteq I$. Then also $J \subseteq I + q$, hence $I \in \varphi_q^{*}\mathcal{Z}_q$. This shows that $\mathcal{E}_0(Z) \subseteq \mathcal{E}_1(Z)$. Next, if $I \in Z$ then there is a prime ideal $p \subseteq I$ ([5], Théorème 9.3.2), hence $I \in \varphi_p^{-1}\mathcal{Z}_p \subseteq \mathcal{E}_0(Z)$. On the other hand, suppose that $I \in \mathcal{E}_1(Z) \cap SpeK(A)$. Once again, pick a prime ideal $p \subseteq I$ ([5], Théorème 9.3.2) and note that $I \in \varphi_p^{*}\mathcal{Z}_p$, hence $I = I + p \in Z$. This shows that

$$Z \subseteq V_K(\mathcal{E}_0(Z)) \subseteq V_K(\mathcal{E}_1(Z)) \subseteq Z,$$

i.e., (b) has been proved.

If $\mathcal{F}$ is some multiplicative filter containing Z then $Z_p \subseteq \mathcal{F}$ for every p. Since Z_p is a basis for the filter $\varphi_p^{-1}\mathcal{Z}_p$ one concludes that $\mathcal{E}_0(Z) \subseteq \mathcal{F}$, i.e., $\mathcal{E}_0(Z)$ is the smallest multiplicative filter containing Z. Let $\mathcal{F}$ be some multiplicative filter with $Z' = V_K(\mathcal{F}) \subseteq Z$. Let $p \in Spec(A)$ and $I \in \mathcal{F}$. Since $\mathcal{F}$ has a basis consisting of l–ideals one may assume that $I \in LId(A)$. Then $I + p \in V_K(\mathcal{F})$, hence $I + p \in Z$. This shows that $I \in \varphi_p^{*}\mathcal{Z}_p$. Therefore $\mathcal{F} \subseteq \varphi_p^{*}\mathcal{Z}_p$ for every p, i.e., $\mathcal{F} \subseteq \mathcal{E}_1(Z)$, and the proof of (c) is complete.

For the description of $\mathcal{E}_0(Z)$ in (d), first note that the set is clearly contained in $\mathcal{E}_0(Z)$. Also, it contains Z and is a filter. So it remains prove that it is a multiplicative filter. For this it suffices to show that a product II' with $I = I_1 \cap \ldots \cap I_r$ and $I' = I'_1 \cap \ldots \cap I'_s$, $I_\rho, I'_\sigma \in Z$, belongs to the set. Since $I \cap I'$ belongs to the set and $(I \cap I')^2 \subseteq II'$ the problem is further reduced to proving that I^2 is in the set. If p is any prime ideal then $I + p \in Z$. For, $I_1 + p, \ldots, I_r + p \in Z$, hence also $I_p = \bigcap_\rho (I_\rho + p) \in Z$. Suppose that $a_1, \ldots, a_r \in I_p$ and write $a_\rho = b_\rho + c_\rho$ with $b_\rho \in I_\rho$, $c_\rho \in p$. Then

$$a_1 \cdot \ldots \cdot a_r = b_1 \cdot \ldots \cdot b_r + c \in I_1 \cdot \ldots \cdot I_r + p \subseteq I + p.$$

This implies that $I_p^r \subseteq I+p$, hence that $I+p \in Z$. Now, for each ρ choose some prime ideal $p_\rho \subseteq I_\rho$ and observe that $\bigcap_\rho (I+p_\rho) = I$. Since it has just been shown that $I+p_\rho \in Z$ one may assume that $I_\rho = I+p_\rho$.

For every p one has $(I+p)^2 = I^2+p$, hence $I^2+p \in Z$. For, it is clear that $(I+p)^2 \subseteq I^2+p$. The other inclusion is proved as follows: If $a \in I^2+p$ then write $a = \sum a_i b_i + x$ with $a_i, b_i \in I$, $x \in p$. In the real closed ring A, $x^+ = y^2$ and $x^- = z^2$ with $y, z \in p$. Now $a = \sum a_i b_i + y^2 - z^2 \in (I+p)^2$.

It is claimed now that $I^2 = \bigcap_\rho (I^2+p_\rho)$. One inclusion is obvious. For the other one note that both ideals are l–ideals. Therefore, given $a \in \cap(I^2+p_\rho)$ it suffices to find some $b \in I$ such that $0 \le |a| \le b^2$. First write $a = \sum_i u_{\rho i} v_{\rho i} + w_\rho$ for every ρ, where $u_{\rho i}, v_{\rho i} \in I$, $w_\rho \in p_\rho$. Define $u = \sum_{\rho,i} |u_{\rho i}| + |v_{\rho i}| \in I$ and $x_\rho = |w_\rho|^{\frac{1}{2}} \in p_\rho$. It is then clear that $0 \le |a| \le u^2 + x_\rho^2$ for each ρ. With $w = \inf\{x_\rho; \rho\}$ one gets $0 \le |a| \le u^2+w^2 \le (u+w)^2$. Since each p_ρ is an l–ideal one knows that $w \in \cap p_\rho$. From $I \subseteq I + \cap p_\rho \subseteq \cap(I+p_\rho) = I$ it follows that $0 \le |a| \le b^2$ with $b = u+w \in I$. This finishes the description of $\mathcal{E}_0(Z)$.

As for $\mathcal{E}_1(Z)$, first suppose that $I \in \mathcal{E}_1(Z)$ and assume that $J \in SpeK(A)\backslash Z$, $I \subseteq J$. Then there is a minimal prime ideal $p \subseteq J$ ([5], Théorème 9.3.2) and there is some $K \in Z$ such that $K \subseteq I+p \subseteq J$. This implies $J \in Z$, a contradicton. Now pick $I \notin \mathcal{E}_1(Z)$. Then there is some p such that $I+p$ does not contain any element of Z. In particular, $I+p \notin \mathcal{E}_1(Z)$. Therefore one may assume that $I = I+p$. If $L(I) = I$ then $I \subseteq I \in SpeK(A)\backslash Z$ and the proof is finished. Therefore, suppose that $L(I) \subset I$. Let $\overline{I}$ be the l–ideal generated by I. Then $\overline{I} \in SpeK(A)$ and $\overline{I}^2$ is a convex ideal contained in I (because of the 2^{nd} convexity property), hence $\overline{I}^2 \subseteq L(I)$. Therefore $\overline{I} \notin Z$. Altogether, $I \subseteq \overline{I}$ and $\overline{I} \in SpeK(A)\backslash Z$. This finishes the proof of (d).

For the proof of (a), note that $\cap Z_i \subseteq Z_j$ for every j and $Z_j \subseteq \mathcal{E}_0(Z_j)$, $Z_j \subseteq \mathcal{E}_1(Z_j)$. Now (c) implies that $\mathcal{E}_0(\cap Z_i) \subseteq \cap \mathcal{E}_0(Z_i)$ and $\mathcal{E}_1(\cap Z_i) \subseteq \cap \mathcal{E}_1(Z_i)$. It follows from

$$V_K(\cap \mathcal{E}_1(Z_i)) = \cap V_K(\mathcal{E}_1(Z_i)) = \cap Z_i$$

and from (c) that $\cap \mathcal{E}_1(Z_i) \subseteq \mathcal{E}_1(\cap Z_i)$. Finally, pick some l–ideal $I \in \cap \mathcal{E}_0(Z_i)$. By (d), for any j there are $I_1, \ldots, I_r \in Z_j$ with $I_1 \cap \ldots \cap I_r \subseteq I$. Then

$$I = I + I_1 \cap \ldots \cap I_r \subseteq \cap(I+I_\rho).$$

It will be shown that these l–ideals are equal. So, choose some $a \in \cap(I+I_\rho)$. It suffices to find some $b \in I$ such that $0 \le |a| \le b$. There are $u_\rho \in I$, $v_\rho \in I_\rho$ such that $a = u_\rho + v_\rho$. With $u = \sum |u_\rho|$ one has $0 \le |a| \le u + |v_\rho|$ for each ρ. Setting $v = \inf\{|v_\rho|; \rho\}$ one gets $0 \le |a| \le u+v$. Since $v \in I_1 \cap \ldots \cap I_r \subseteq I$ it follows that $u+v \in I$, hence $a \in I$. For all indices i, it now follows from $I \in \mathcal{E}_0(Z_i)$ and $I \subseteq I+I_\rho$ that $I+I_\rho \in V_K(\mathcal{E}_0(Z_i)) = Z_i$, hence $I+I_\rho \in \cap Z_i$. Now (d) implies that $I = \cap(I+I_\rho) \in \mathcal{E}(\cap Z_i)$.

To finish the proof of (a), let $\mathcal{F}$ be the smallest multiplicative filter containing each $\mathcal{E}(Z_i)$. Since $V_K(\mathcal{E}_0(\cup Z_i)) = \cup Z_i \supseteq Z_j$ it follows from (c) that $\mathcal{E}_0(\cup Z_i) \supseteq \mathcal{E}_0(Z_j)$, hence $\mathcal{E}_0(\cup Z_i) \supseteq \mathcal{F}$. The other inlcusion also follows immediately from (c) since $V_K(\mathcal{F}) \supseteq \cup V_K(\mathcal{E}_0(Z_i)) = \cup Z_i$. □

According to Proposition 7.1 (a), $\mathcal{E}_0$ is a homomorphism of complete lattices. Let $\mathcal{M}_0(A) \subseteq \mathcal{M}(A)$ be the image of $\mathcal{E}_0$. Then $\mathcal{E}_0 V_K$ is a retraction of the complete lattice $\mathcal{M}(A)$ onto its complete sublattice $\mathcal{M}_0(A)$.

The set of multiplicative filters of the ring A is most accessible in cases where $\mathcal{E}_0 = \mathcal{E}_1$. For example, it is known from section 5 that this is the case if A is a real closed domain. More generally one has

Corollary 7.2. *If A has only finitely many minimal prime ideals then $\mathcal{E}_0 = \mathcal{E}_1$. In particular, every multiplicative filter is a Gabriel filter.*

Proof. Suppose that $Z \in \mathcal{S}_K(A)$ and let $I \in \mathcal{E}_1(Z) \cap LId(A)$. If $p_1, \ldots, p_r$ are the minimal prime ideals then $I = \cap(I + p_\rho)$. It follows that $I + p_1, \ldots, I + p_r \in V_K(\mathcal{E}_1(Z)) = Z$. By Proposition 7.1 (d) one concludes that $I \in \mathcal{E}_0(Z)$. □

Continuing with the situation of Corollary 7.2, the set of Gabriel filters of A can be described completely by using the maps φ_{p^*} where $p \in Spec(A)$ and $\varphi_p : A \to A/p$ is canonical. The description is reminiscent of a description given for *h–local domains* in [8], section 2. There is a map

$$\varphi_* : \mathcal{G}(A) \longrightarrow \prod_{\rho=1}^{r} \mathcal{G}(A/p_\rho) : \mathcal{F} \longrightarrow (\varphi_{p_\rho *}\mathcal{F})_\rho.$$

Because of Corollary 7.2 this map is injective. To determine its image, call a tuple $(\mathcal{F}_1, \ldots, \mathcal{F}_r) \in \prod_\rho \mathcal{G}(A/p_\rho)$ *compatible* if the following holds: Whenever $p_i, p_j \subseteq p \in Spec(A)$ then $\varphi_{pp_i *}\mathcal{F}_i = \varphi_{pp_j *}\mathcal{F}_j$ (with $\varphi_{pp_\rho} : A/p_\rho \to A/p$ the canonical map). Since $\mathcal{F} = \mathcal{E}_1(V_K(\mathcal{F}))$ for every Gabriel filter it is easy to see that $im(\varphi_*)$ is exactly the set of compatible tuples.

If $\mathcal{F} \in \mathcal{G}(A)$ then it is also possible to determine $L_\mathcal{F}(A)$ to the same extent as the $\varphi_{p_\rho *}\mathcal{F}$ can be determined (cf. section 5). Let $Y = Spec(A) \backslash V_K(\mathcal{F})$ and let $Y_1, \ldots, Y_s$ be the connected components of Y. Each Y_σ is a generically closed subset of $Spec(A)$. Since $|Y_\sigma \cap Min(A)| \leq |Min(A)| < \infty$ the closure $\overline{Y}_\sigma$ consists of the specializations of the elements of Y_σ. Define $I_\sigma = \cap Y_\sigma$ and let $\varphi_\sigma : A \to A/I_\sigma$ be canonical. Then $L_\mathcal{F}(A) \cong \prod_\sigma L_{\varphi_{\sigma *}\mathcal{F}}(A/I_\sigma)$. So it suffices to handle the case that Y is connected. Now $p_1, \ldots, p_r$ have a common specialization $p \in Y$. Let $\psi : A_p \to \rho(p)$ be the canonical residue map. Then $L_{\varphi_{p^*}\mathcal{F}}(A) \subseteq \rho(p)$ and $L_\mathcal{F}(A) = \psi^{-1}(L_{\varphi_{p^*}\mathcal{F}}(A))$.

The class of rings covered by the hypotheses of Corollary 7.2 includes, of course, all real closed rings with $|Spec(A)| < \infty$. A less trivial class of examples is provided by the localizations of rings $C(X)$ of continuous functions from an SV–space X into the real numbers ([22], Theorem 4.1).

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Niels Schwartz
Fakultät für Mathematik und Informatik
Universität Passau
Postfach 2540
D-94030 Passau
Germany

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