Comment. Math. Helv. 72 (1997) 331–348
0010-2571/97/030331-18 $ 1.50+0.20/0

Commentarii Mathematici Helvetici

The asymptotic behavior of the set of rays

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to my daughter Ariana

Abstract. We introduce new invariants to study the asymptotic behavior of the set of rays and prove a splitting theorem for the radius of the ideal boundary of an open manifold with $K \geq 0$ (Shioya's Conjecture).

Mathematics Subject Classification (1991). Primary 53C20, 53C42.

Keywords. Ideal boundary, set of rays.

0. Introduction

Let M^n denote an n-dimensional complete and noncompact connected riemannian manifold with secctional curvature $K \geq 0$. In Theorem 2.2 in [CG] it was proved that M is diffeomorphic to the normal bundle of a totally geodesic compact submanifold S_0, which is called the soul of M. After rigidity theorems of Strake ([St]) and Walschap ([W]), Yim proved Theorem 0.1 below which extends these results ([Ym-2]). A subset $S \subset M$ is called a pseudosoul if S and S_0 are isometric and homologous. The space of souls $\mathbf{W(M)}$ is the union of the pseudosouls of M. Any soul is a pseudosoul and $W(M)$ does not depend on the choice of S_0 (see [Ym-1]).

0.1. Theorem. *The set $W(M)$ is a totally geodesic embedded submanifold which is isometric to a product manifold $S_0 \times V$, where V is a complete manifold of nonnegative curvature diffeomorphic to $\mathbb{R}^k$, and k is the dimension of the space of all parallel normal vector fields along the soul S_0. Furthermore any pseudosoul in M is of the form $S_0 \times \{p\}$ for some $p \in V$.*

Kasue obtained in [K] a compactification of M, in which the boundary $\mathbf{M(\infty)}$ is the set of equivalence classes of rays (see the second section in this paper).

* The author is indebted to Professor Manfredo do Carmo for his general assistance during the preparation of this paper.

Given a metric space (X,d) the radius of X, to be denoted by $\mathbf{r(X)}$, is defined by $\mathrm{r}(X)=\inf_{x\in X}\sup_{y\in X} d(x,y)$. By the triangle inequality we have $\frac{\mathrm{diam}(X)}{2}\le \mathrm{r}(X)\le \mathrm{diam}(X)$, where $\mathrm{diam}(X)$ is the diameter of X. By using Theorem 0.1 and estimating the dimension of the space of parallel normal vectors fields along S_0, Shioya proved the following result ([Sy-3], p. 224]).

0.2. Theorem. *There exists $\epsilon(n)>0$ so that if $\mathrm{r}\big(M(\infty)\big)>\pi-\epsilon(n)$, then M is isometric to $S^k\times V^{n-k}$, where S is a soul of M and V is diffeomorphic to $\mathbb{R}^{n-k}$.*

Again using Theorem 0.1 and proving that any point of M is contained in a soul, we proved the following result, that was conjectured by Shioya ([Sy-3], p. 224]). Perelman obtained, independently, another proof of it ([P]).

Theorem A. *If $\mathrm{r}\big(M(\infty)\big)>\pi/2$, then M is isometric to $S^k\times V^{n-k}$, where S is a soul of M and V is diffeomorphic to $\mathbb{R}^{n-k}$. Furthermore, every point of V is a soul of V.*

If $\mathrm{r}\big(M(\infty)\big)=\pi/2$, the conclusion of Theorem A does not hold. In fact by taking a product of a flat open Möbius band with $\mathbb{R}$ we have a counterexample ([Sy-3], p. 224]). The manifold V is not necessarily isometric to $\mathbb{R}^{n-k}$ because of Example 3.11 in this paper, that shows the existence of a surface M with $K\ge 0$, not isometric to $\mathbb{R}^2$, and so that all its points are souls.

All geodesics, unless otherwise stated, are supposed to be normalized. A geodesic $\gamma\colon[0,+\infty)\to M$ is called a **ray** starting at p, if $\gamma(0)=p$ and if the distance $d\big(p,\gamma(t)\big)=t$, for all $t>0$. Let $\mathbf{\Gamma_p}$ be the set of rays which start at p, and $\mathbf{\Gamma}$ be the set of all rays in M. Take $p\in M$ and $\gamma\in\Gamma_p$. Set $\mathbf{H}_\gamma=\{x\in M \mid \text{for each } t\ge 0,\ d\big(x,\gamma(t)\big)\ge t\}$, $\gamma_t(s)=\gamma(t+s)$, $s>0$, and $\mathbf{C_t(p)}=\bigcap_{\gamma\in\Gamma_p} H_{\gamma_t}$. With the same proof as in Theorem A we can prove the following result.

Theorem B. *Assume that there exists $R>0$ and a compact set $D\subset M$ so that $\mathrm{diam}\big(C_0(p)\big)\le R$, for all p in the complement $M\backslash D$. Then M is isometric to $S\times V$, where S and V are as in Theorem A.*

Let $R,S\subset M$ and $\mathbf{\Gamma(R,S)}$ be the set of geodesics which are minimal connections between R and S. For $p\in M$ and $\eta\in T_pM$ set $\boldsymbol{\gamma_\eta}\mathbf{(t)}=\exp_p(t\eta)$, $t\ge 0$, where exp is the exponencial map and T_pM is the tangent space at p. Set $\mathbf{A_p}=\{v\in T_pM \mid \|v\|=1 \text{ and } \gamma_v\in\Gamma_p\}$. The **mass of rays** at p, to be denoted by $\mathbf{m(A_p)}$, is the Lebesgue measure of the compact set $A_p\subset S^{n-1}\subset T_pM$. The mass of rays has been extensively studied in dimension two by several authors ([M], [Sg], [Sm-2], [Sy-1], [Sy-2], [SST] etc) who related it with the total curvature of complete noncompact surfaces and with the lenght of the ideal boundary $M(\infty)$. Shioya studied the mass of rays for dimensions higher than two ([Sy-4]).

Here we introduce two different invariants, the **critical function** and the **radial function**, which are more fruitful for dimensions higher than two. We study their asymptotic behavior and its topological and geometrical consequences.

For a closed subset L, a point $p \in M\backslash L$ is said to be a **critical point relative to the distance function** from L (see for example [G], p.205), if for every $v \in T_pM$ there exists $\sigma \in \Gamma(p, L)$ with the angle $\measuredangle\big(\sigma'(0), v\big) \leq \pi/2$. This definition will be used only in the proof of Proposition 3.6. Similarly we say that $p \in M$ is a **critical point of the infinity**, and we denote it by $\mathbf{p \prec \infty}$, if for all $v \in T_pM$ there exists $\gamma \in \Gamma_p$ such that $\measuredangle\big(v, \gamma'(0)\big) \leq \pi/2$. For each $p \in M$ and each $v \in T_pM$ set $\mathbf{C_p(v, \boldsymbol{\alpha})} = \{w \in T_pM \mid \measuredangle(v, w) \leq \alpha\}$, that is, $C_p(v, \alpha)$ is the cone with axis v and angle α. The **critical function** at p, to be denoted by $\boldsymbol{\theta}\mathbf{(p)}$, is given by $\theta(p) = \min \alpha$, where $A_p \subset C_p(v, \alpha)$ and $v \in T_pM$. By Proposition 3.8 $p \prec \infty$ if and only if $\theta(p) \geq \frac{\pi}{2}$. Thus $\theta(p)$ intents to measure how p is close to be a critical point of the infinity. By Proposition 3.9, if $\theta(p) > \frac{\pi}{2}$ then $\{p\}$ is a soul of M.

0.3. Proposition. *If $\theta \geq \frac{\pi}{2}$ in $M\backslash L$ for a certain compact set L, then $\theta \geq \frac{\pi}{2}$ in M, that is, $x \prec \infty$ for all x. Furthermore, if $\theta > \frac{\pi}{2}$ in $M\backslash L$, then $\theta > \frac{\pi}{2}$ in M. Thus $\{x\}$ is a soul for all x.*

0.4. Proposition. *If there exists a sequence $p_k \to \infty$, with $p_k \prec \infty$, then $\theta \geq \pi/4$ in M.*

Shiga proved (Theorem 2 in [Sg]) that if $K > 0$ in M^2 and $m(A_x)$ attains the infimum at $p \in M$, then $m(A_p) = 0$. We obtained a similar result for $\theta(p)$ without any restriction on the dimension.

0.5. Proposition. *Assume that $K > 0$ in $M\backslash L$ for a fixed compact set L. If $\theta(p) = \inf_{x\in M} \theta(x)$, then $\theta(p) = m(A_p) = 0$. In other words A_p has a unique element.*

Proposition 0.3 suggests that the infimum of the critical function is attained at infinity. This is the following result.

Theorem C. *It holds that $\inf_{p\in M} \theta(p) = \liminf_{p\to\infty} \theta(p)$.*

The **radial function** at p is the radius $\mathbf{r(A_p)}$, where we take $\measuredangle$ as a distance in A_p. It holds that $\mathrm{r}(A_p) = \inf \alpha$, where $A_p \subset C_p(v, \alpha)$ and $v \in A_p$. Clearly we have $\theta(p) \leq \mathrm{r}(A_p) \leq \pi$. Moreover $r(A_p) = \pi$ if and only if A_p is symmetric relative to $0 \in T_pM$. In Theorems D and E below M has only one end. There is no loss of generality in that, since $M(\infty)$ becomes trivial if M has more than one end (see the second section, after Lemma 2.4).

Theorem D. *Assume that M has only one end. Then we have* $\limsup_{q\to\infty} r(A_q) \leq \mathrm{diam}\big(M(\infty)\big)$.

0.6. Corollary. *Assume that M has only one end. Then we have* $\frac{1}{2}\mathrm{diam}\big(M(\infty)\big) \leq \liminf_{q\to\infty} \theta(q) \leq \limsup_{q\to\infty} \theta(q) \leq \mathrm{diam}\big(M(\infty)\big)$.

In dimension 2 we have (see the fifth section) $\frac{1}{2}m(A_q), \theta(q), \mathrm{r}(A_q) \to \frac{1}{2}\big(2\pi\mathcal{X}(M) - c(M)\big) = \mathrm{diam}\big(M(\infty)\big)$ as $q \to \infty$, where $\mathcal{X}(M)$ is the Euler characteristic of M and $c(M)$ is the total curvature of M to be defined in the fifth section. Example 4.3 will show that if $n \geq 3$ these limits do not necessarily exist, and that the inequalities in Theorem D and in Corollary 0.6 are sharp.

0.7. Corollary. *If* $\limsup_{q\to\infty} \mathrm{r}(A_q) = \pi$ *then M is isometric to $V \times \mathbb{R}$. In particular the same conclusion holds if A_{p_k} is symmetric for some sequence $p_k \to \infty$ or if* $\limsup_{q\to\infty} \theta(q) = \pi$.

Theorem E. *Assume that M has only one end. Then it holds that* $\inf_{q\in M} \mathrm{r}(A_q) = \liminf_{q\to\infty} \mathrm{r}(A_q) = \mathrm{r}\big(M(\infty)\big)$.

We describe next the contents of the various sections of this paper. Basic facts and notations are recalled in the first section. In the second one we recall the Kasue's compactification of M. In the third section we study the asymptotic behavior of $\theta(p)$ and prove Theorem C. Theorems A, B, D and E are proved in the fourth section. In the fifth one we restrict ourselves to surfaces M^2 which admit total curvature. We obtain for $\theta(p)$ and $\mathrm{r}(A_p)$ results similar to those that are known for $m(A_p)$. So the new invariants do not lose information in comparison with the mass of rays.

1. Basic facts and notations about nonnegatively curved manifolds

Consider a closed totally convex set C (that is, C is closed and any geodesic joining $p, q \in C$ is contained in C). By Theorem 1.6 in [CG], C is a k-dimensional submanifold with smooth interior and a boundary of C^0 class. Let $\mathbf{int(C)}$ be the interior of C and $\boldsymbol{\partial}\mathbf{C}$ be the boundary of C. Set $\mathbf{dim(C)} = k$. If C is compact, set $\mathbf{C^a} = \{x \in C; d(x, \partial C) \geq a\}$. By Theorem 1.10 in [CG], C^a is totally convex. Set $\mathbf{C^{max}} = \bigcap C^a$, for all $C^a \neq \emptyset$. The set $C^{\max}$ is nonempty, compact and totally convex. It holds that $\dim(C^{\max}) < \dim(M)$. From now on $\mathbf{C}$ denotes a closed totally convex subset of M with $\partial C \neq \emptyset$. The results in this section are simple and well-known.

1.1. Lemma. *Take $p \in M$ and $\gamma \in \Gamma_p$. Set* $H_t = \{x \,\big|\, d\big(x, \gamma(s)\big) \geq s - t \text{ for } s > t\}$.

If $t \geq 0$ *then* $H_t = H_{\gamma_t}$. *For* $t_1 \leq t_2$ *we have* $H_{t_1} = (H_{t_2})^{t_2 - t_1}$. *It holds that* $\bigcup_{t \geq 0} H_t = M$ *and* $p \in \partial H_0$.

Here ∂X is the topological boundary of X in M.

1.2. Lemma. *Take* $\gamma \in \Gamma_p,\ p \in M$. *Then* $\gamma \in \Gamma\big(p, \partial C_t(p)\big)$ *and* $\gamma \in \Gamma(p, \partial H_{\gamma_t})$, $t > 0$. *The set* H_{γ_t} *is closed and totally convex, and the set* $C_t(p)$ *is compact and totally convex.*

1.3. Lemma. *Take* $0 \leq a < b$ *and* $q \in \partial C^b$. *Consider a geodesic* $\gamma \colon [0, b] \to C,\ \gamma \in \Gamma(q, \partial C)$, $p = \gamma(c) = \gamma \cap \partial C^a$. *Then* $d(q, \partial C^a) = c = b - a$, $\gamma \in \Gamma(q, \partial C^a) \cap \Gamma(p, \partial C)$ *and* $\partial C^b = \partial\left((C^a)^{b-a}\right)$.

1.4. Lemma. *Take* $q \in \partial C^{a_0},\ a_0 > 0$ *and a geodesic* $\gamma \colon [0, \rho] \to M$ *with* $\gamma(0) = q,\ \gamma(\rho) \in \partial C$. *Suppose that* $\gamma \in \Gamma(q, \partial C^b)$, *for a certain* $b \in [0, a_0)$. *Then* $\gamma \in \Gamma(q, \partial C^s)$, *for all* $s \in [0, a_0)$.

The following lemma follows from Lemma 1.4.

1.5. Lemma. *Take* $p \in M$ *and set* $C_{t'} = C_{t'}(p)$. *Let* $t > 0$ *be so that* $q \in int(C_t)$. *Consider a geodesic* $\gamma \colon [0, +\infty) \to M,\ \gamma \in \Gamma(q, \partial C_t)$. *Then* $\gamma \in \Gamma(q, \partial C_s)$ *for all* s *so that* $q \in int(C_s)$. *In particular* γ *is a ray.*

2. The points at infinity on nonnegatively curved manifolds

Kasue obtained ([K]) a compactification of an asymptotically nonnegatively curved manifold. In the particular situation in that $K \geq 0$ the proofs become easier but we outline the construction in this case for completeness.

Take $\gamma, \sigma \in \Gamma$. We say that γ is **asymptotic** to σ, and we denote it by $\boldsymbol{\gamma \prec \sigma}$, if there exist sequences $p_k \to \gamma(0),\ t_k \to +\infty,\ \tau_k \in \Gamma\big(p_k, \sigma(t_k)\big)$, so that $\tau_k'(0) \to \gamma'(0)$. Consider $x \in M$ and set $h_\sigma(x) = \lim_{t \to +\infty} \big(t - d\big(x, \sigma(t)\big)\big)$. It is well-known that h_σ is well defined and that $h_\sigma(x) - h_\sigma(y) \leq d(x, y)$. The function h_σ is called the Buseman function associated with σ. The following proposition about Buseman functions and asymptotic rays is well-known.

2.1. Proposition. *Take* $\gamma, \sigma \in \Gamma,\ \gamma \prec \sigma$. *Set* $h = h_\sigma,\ \gamma_t(s) = \gamma(t + s), s \geq 0$, $H_t = \{x \mid d\big(x, \sigma(s)\big) \geq s - t,\ \textit{for } s > t\}$. *Then we have:*

(a) *for* $t \geq 0$ *it holds that* $h\big(\gamma(t)\big) = t + h\big(\gamma(0)\big)$;

(b) γ_t *is the unique ray starting at* $\gamma(t)$ *which is asymptotic to* σ;

(c) $q = \gamma(0) \in \partial H_a$, *for some* $a \in \mathbb{R}$ *and* $\gamma \in \Gamma(q, \partial H_t)$ *for all* $t > a$.

Take $\gamma, \sigma \in \Gamma_p$. Set $\ell_t = d\big(\gamma(t), \sigma(t)\big)$. Let α_t be the angle opposite to ℓ_t

in the triangle (t, t, ℓ_t) in the plane. Set $\measuredangle_{\boldsymbol{\infty}}\boldsymbol{(\gamma,\ \sigma)} = \lim_{t \to +\infty} \alpha_t$. Such a limit must exist, since Toponogov-Alexandrov Theorem ([Sm-3], Theorem 2.1) assures that the function $t \longmapsto \alpha_t$ is nonincreasing. By Toponogov Theorem ([CE], Theorem 2.2) we have $\measuredangle\big(\gamma'(0), \sigma'(0)\big) \geq \alpha_t \geq \measuredangle_\infty(\gamma,\ \sigma)$. Let $\gamma, \sigma \in \Gamma$. Set $\mathbf{L}_\infty\boldsymbol{(\gamma, \sigma)} = \lim_{t \to +\infty} \frac{d\big(\gamma(t), \sigma(t)\big)}{t}$, if there exists such a limit. By using Toponogov Theorem, the triangle inequality and Proposition 2.1 (b) it is not difficult to prove the following lemma.

2.2. Lemma. *Let $\gamma \prec \sigma$. Then $L_\infty(\gamma, \sigma) = 0$.*

The following lemma may be easily proved from Lemma 2.2 and the triangle inequality.

2.3. Lemma. *Take $\gamma, \sigma \in \Gamma$ and $\gamma_1, \sigma_1 \in \Gamma_p$ with $\gamma_1 \prec \gamma$ and $\sigma_1 \prec \sigma$. Then $L_\infty(\gamma, \sigma) = L_\infty(\gamma_1, \sigma_1) = 2 \sin \frac{\measuredangle_\infty(\gamma_1, \sigma_1)}{2}$.*

If $\gamma, \sigma \in \Gamma$ we define $\measuredangle_\infty(\gamma, \sigma) = \measuredangle_\infty(\gamma_1, \sigma_1)$, where $\gamma_1, \sigma_1 \in \Gamma_p$ for some p and $\gamma_1 \prec \gamma,\ \sigma_1 \prec \sigma$. By Lemma 2.3 $\measuredangle_\infty(\gamma, \sigma)$ does not depend on the choice of γ_1 and σ_1, and $L_\infty(\gamma, \sigma) = 2 \sin \frac{\measuredangle_\infty(\gamma, \sigma)}{2}$.

2.4. Lemma. *Let $\gamma, \sigma, \tau \in \Gamma_p$. Then $\measuredangle_\infty(\sigma, \tau) \leq \measuredangle_\infty(\sigma, \gamma) + \measuredangle_\infty(\gamma, \tau)$.*

Proof. Fix $s > 0$. Take sequences $t_j \to +\infty$ with $t_j > s$ and $\lambda_j \in \Gamma\big(\gamma(s), \sigma(t_j)\big)$. Set $\eta_j = \measuredangle\big(\gamma'(s), {\lambda_j}'(0)\big)$, $\beta_j = \pi - \eta_j$ and $d_j = d\big(\gamma(s), \sigma(t_j)\big)$. Consider the triangle (s, t_j, d_j) in the plane with corresponding angles $(\tilde{\theta}_j, \tilde{\beta}_j, \tilde{\alpha}_j)$. By Toponogov Theorem ([CE], Theorem 2.2) and Toponogov-Alexandrov Theorem ([Sm-3], Theorem 2.1) we have $\eta_j = \pi - \beta_j \leq \pi - \tilde{\beta}_j = \tilde{\alpha}_j + \tilde{\theta}_j \leq \alpha^s + \tilde{\theta}_j$, where α^s is the angle opposite to $d^s = d(\gamma(s), \sigma(s)\big)$ in the triangle (s, s, d^s) in the plane. By passing to a subsequence, we have $\eta_j \to \eta^s = \measuredangle\big(\gamma'(s), \lambda'(0)\big)$, where $\lambda \in \Gamma_{\gamma(s)}$ and $\lambda \prec \sigma$. It is easy to see that $\tilde{\theta}_j \to 0$. Then we have $\eta^s \leq \alpha^s$. By making $s \to \infty$ we have $\limsup \eta^s \leq \measuredangle_\infty(\gamma, \sigma)$.

For any s take a ray $\mu \in \Gamma_{\gamma(s)},\ \mu \prec \tau$. Set $\rho^s = \measuredangle\big(\mu'(0), \gamma'(s)\big)$. Then we have $\measuredangle_\infty(\sigma, \tau) = \measuredangle_\infty(\mu, \lambda) \leq \measuredangle\big(\mu'(0), \lambda'(0)\big) \leq \eta^s + \rho^s$. Then $\measuredangle_\infty(\sigma, \tau) \leq \limsup(\eta^s + \rho^s) \leq \measuredangle_\infty(\sigma, \gamma) + \measuredangle_\infty(\gamma, \tau)$. □

We say that the γ and σ are equivalent if $\measuredangle_\infty(\gamma, \sigma) = 0$. Let $\boldsymbol{\gamma(\infty)}$ be the equivalence class of γ and $\mathbf{M}\boldsymbol{(\infty)}$ be the set of all such classes. If for sufficiently large t the points $\gamma(t)$ and $\sigma(t)$ belong to the same end of M, we set $d_\infty\big(\gamma(\infty), \sigma(\infty)\big) = \measuredangle_\infty(\gamma, \sigma)$. Otherwise we set $d_\infty\big(\gamma(\infty), \sigma(\infty)\big) = +\infty$. By the Splitting Toponogov Theorem ([CE], Theorem 5.1), it can be shown that if M has more than one end, then M is isometric to $S \times \mathbb{R}$, where S is compact. Thus the study of $M(\infty)$ becomes trivial in this case. By Lemma 2.4, it is easy to see that $\big(M(\infty), d_\infty\big)$ is a metric space.

Take a divergent sequence $(p_k) \subset M$ and $\gamma \in \Gamma$. We say that $p_k \to \gamma(\infty)$ if $d_k/t_k \to 0$, where $t_k = d\big(p_k, \gamma(0)\big)$, and $d_k = d\big(p_k, \gamma(t_k)\big)$. By the triangle inequality it is not difficult to see that this definition does not depend on the choice of $\gamma \in \gamma(\infty)$. It is easy to prove the following lemmas.

2.5. Lemma. *Take a sequence $p_k \to \infty$, $p \in M$ and $\tau_k \in \Gamma(p, p_k)$. Suppose that $\tau_k'(0) \to \gamma'(0)$, $\gamma \in \Gamma_p$. Then $p_k \to \gamma(\infty)$.*

2.6. Lemma. *If $\gamma_k, \gamma \in \Gamma_p$ and $\gamma_k'(0) \to \gamma'(0)$ then $\gamma_k(\infty) \to \gamma(\infty)$.*

The reader can prove that with the topology introduced here the set $\tilde{M} = M \cup M(\infty)$ is compact. Our notion of convergence to a point at $M(\infty)$ is not standard. It agrees with the notion introduced by Kasue because of Lemma 1.5 in [Sy-3].

3. On the asymptotic behavior of θ(p); proof of Propositions 0.3 to 0.5 and of Theorem C

We initially prove results that will be used in the proof of Theorem C. Let $\sigma\colon [a,b] \to M$ be a geodesic, P_s be the parallel transport along σ and $L \subset [a,b] \times \mathbb{R}$. Set $\mathbf{f}[\boldsymbol{\eta}, \boldsymbol{\sigma}, \mathbf{L}](\mathbf{s}, \mathbf{t}) = \exp_{\sigma(s)} tP_s\eta$, $(s,t) \in L$, $\eta \in T_{\sigma(a)}M$. For $p \in M$ let $\triangle_{\boldsymbol{\eta\mu}}$ be the triangle determined by $\eta, \mu \in T_pM$.

3.1. Lemma. *Let $\sigma\colon [a,b] \to int(C)$ be a geodesic and $\gamma \in \Gamma\big(\sigma(a), \ \partial C\big)$. Set $\alpha = \measuredangle\big(\gamma'(0), \sigma'(a)\big)$. If $d\big(\sigma(a), \partial C\big) = d\big(\sigma(b), \partial C\big)$ then $\alpha \geq \pi/2$. Furthermore if $\alpha = \pi/2$ then $\varphi(s) = d\big(\sigma(s), \partial C\big)$ is constant.*

Proof. By Theorem 1.10 in [CG] φ is concave. Since $\varphi(a) = \varphi(b)$ we have $\varphi \geq \varphi(a)$. By the proof of Theorem 1.10 in [CG], for small $(s-a)$, it holds that $\varphi(s) \leq \varphi(a) - (s-a)\cos\alpha$. Since φ is concave, this inequality holds for all $s \in [a,b]$. It follows that $\alpha \geq \pi/2$. If $\alpha = \pi/2$, then φ is constant. Thus, Lemma 3.1 is proved. $\square$

3.2. Lemma. *Let $(\gamma_1, \gamma_2, \gamma_3)$ be a minimizing geodesic triangle in M, with angles $(\alpha_1, \alpha_2, \alpha_3)$ which is contained in a strongly convex ball B centered at $p = \gamma_1(0) = \gamma_3(\ell_3)$, where ℓ_i is the length $L(\gamma_i)$. Suppose that $\ell_3 = \ell_1 \cos\alpha_2$ and α_1, $\alpha_2 \leq \pi/2$. Set $L = \{(s,t) \mid 0 \leq s \leq \ell_1, 0 \leq t \leq (\ell_1 - s)\cos\alpha_2\}$. Then: $\alpha_1 = \frac{\pi}{2}$; $S = \exp_p \triangle_{vw}$ is flat and totally geodesic, where $v = \ell_1\gamma_1'(0)$ and $w = -\ell_3\gamma_3'(\ell_3)$; $\gamma_2 \subset \partial S$; by setting $f = f[w, \gamma_1, L]$ the field $\partial f/\partial t$ along γ_2 is parallel.*

Proof. Take in the plane the triangle $\triangle = (\ell_1, \ell_2, \ell_3)$ with angles $(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3)$. By Toponogov Theorem ([CE], Theorem 2.2) we have $\alpha_i \geq \tilde{\alpha}_i,\ i = 1, 2, 3$. Then $\tilde{\alpha}_1, \tilde{\alpha}_2 \leq \pi/2$. Therefore $\ell_1 \cos\tilde{\alpha}_2 \leq \ell_3 = \ell_1 \cos\alpha_2$. Then $\tilde{\alpha}_2 \geq \alpha_2$, hence $\tilde{\alpha}_2 = \alpha_2$. Thus, we obtain $\ell_3 = \ell_1 \cos\tilde{\alpha}_2$. Then $\tilde{\alpha}_1 = \pi/2$, hence $\alpha_1 = \pi/2$. Since $\alpha_2 = \tilde{\alpha}_2$, Toponogov Theorem ([CE], Corollary 2.3) implies that S is flat and totally geodesic. $\triangle_{vw}$ is isometric to $\triangle$. Let ν be the line segment in T_pM that joins v and w. By 3.1 Rauch I in [G] we have $\ell_2 = d\big(\gamma_3(0), \gamma_1(\ell_1)\big) \leq L(\exp_p \nu) \leq L(\nu) = \ell_2$. Then $L(\gamma_2) = L(\exp_p \nu)$. Since $\gamma_2 \subset B$ we have $\gamma_2 = \exp_p(\nu)$, hence $\gamma_2 \subset \partial S$. Since S is totally geodesic, the parallel transport of w along γ_1 is tangent to S. Set $f_s(t) = f(s,t)$. Fix $s \in (0, \ell_1)$. Since S is flat, the Gauss-Bonnet Theorem may be applied to the geodesic quadrilateral determined by $\gamma_1, \gamma_2, \gamma_3$ and f_s, thereby concluding that $\partial f/\partial t$ is ortogonal to γ_2. Thus the field $\partial f/\partial t$ along γ_2 is parallel. □

The following lemma improves Theorem 1.10 in [CG].

3.3. Lemma. *Let* $\sigma\colon [a_1, a_2] \to \mathrm{int}(C)$ *be a geodesic. Set* $\alpha(s) = \measuredangle\big(\gamma_s'(0), \sigma'(s)\big)$, *where* $\gamma_s \in \Gamma\big(\sigma(s), \partial C\big)$. *Set* $\eta = {\gamma_{a_1}}'(0)$ *and* $L = \{(s,t) \mid s \in [a_1, a_2], 0 \leq t \leq \mathrm{r}(s)\}$, *where* r *is the linear function that satisfies* $\mathrm{r}(a_i) = d\big(\sigma(a_i), \partial C\big)$. *Set* $f = f[\eta, \sigma, L]$ *and* $f_s(t) = f(s,t)$. *Then* $\alpha(a_1) \geq \alpha(a_2)$. *Moreover, if* $0 < \alpha(a_1) = \alpha(a_2) < \pi$ *then* $f(L)$ *is flat and totally geodesic, and* $f_s \in \Gamma\big(\sigma(s), \partial C\big)$ *for all* $s \in [a_1, a_2]$.

Proof. By the concavity of the function $\varphi(s) = d\big(\sigma(s), \partial C\big)$ it is not difficult to see that $\alpha(a_1) \geq \alpha(a_2)$. Assume then that $0 < \alpha(a_1) = \alpha(a_2) < \pi$. Then $\alpha(s)$ is constant. For simplicity, set $\alpha = \alpha(a_1)$.

Claim 1. $\varphi(s) = \varphi(a_1) - (s - a_1)\cos\alpha = \mathrm{r}(s)$ *for* $s \in [a_1, a_2]$.

Proof of Claim 1. By the proof of Theorem 1.10 in [CG], for small $s - a_1$ we have $\varphi(s) \leq \varphi(a_1) - (s - a_1)\cos\alpha$. Since φ is concave this inequality holds for $s \in [a_1, a_2]$. Similarly we have $\varphi(a_1) \leq \varphi(s) - (a_1 - s)\cos\alpha$, and Claim 1 follows.

Claim 2. *The statement of Lemma 3.3 holds in the case* $\alpha = \pi/2$.

Proof of Claim 2. By Claim 1 we have $\varphi(s) = \varphi(a_1)$, for all $s \in [a_1, a_2]$. Then Lemma 3.3 follows from Theorem 1.10 in [CG].

Claim 3. *The statement of Lemma 3.3 holds in the case* $\alpha < \pi/2$.

Proof of Claim 3. Without loss of generality, we may assume that σ is contained in a small strongly convex ball around $\sigma(a_1)$. Fix $s \in [a_1, a_2]$. Set $\gamma = \gamma_{a_1}$, $u = (s - a_1)\cos\alpha$, $\gamma^s = \gamma_{|\,[0,u]}$, $p = \gamma(u)$, $q = \sigma(s)$, $v = (s - a_1)\sigma'(a_1)$, $w =$

$u\eta$, $S(s) = \exp_{\gamma(0)} \triangle_{vw}$ and $L_s = L \cap ([0,s] \times \mathbb{R})$. Take $\mu \in \Gamma(p,q)$ and set $\beta = \measuredangle\big(-\gamma'(u), \mu'(0)\big)$. By Lemma 1.3 we have $\gamma \in \Gamma(p, \partial C)$. Then $d(p, \partial C) = \varphi(a_1) - u = d(q, \partial C)$. By Lemma 3.1 we obtain $\beta \leq \frac{\pi}{2}$. Thus Lemma 3.2 applies and we have: $\beta = \frac{\pi}{2}$, $S(s)$ is flat and totally geodesic, $\mu \subset \partial S(s)$, the parallel transport of η along σ is tangent to $S(s)$; the field $\partial f/\partial t$ along μ is parallel. Since $\beta = \frac{\pi}{2}$, Lemma 3.1 implies that the function $\psi(s') = d\big(\mu(s'), \partial C\big)$ is constant. Then Theorem 1.10 in [CG] implies that $f_s \in \Gamma(q, \partial C)$. Since s is arbitrary, we have $f(L) \subset C$. By Theorem 1.10 in [CG], we obtain that $f(L) \backslash S(a_2)$ is flat and totally geodesic. Since $S(a_2)$ is also flat and totally geodesic, we conclude that $f(L)$ is flat and totally geodesic. Thus Claim 3 is proved.

Claim 4. *The statement of Lemma 3.3 holds in the case $\alpha > \pi/2$.*

Proof of Claim 4. Set $x = \sigma(a_1)$ and $\gamma = \gamma_{a_1}$. Take $v \in T_x M$ contained in the plane generated by η and $\sigma'(a_1)$, which satisfies $|v| = (a_2 - a_1)\sin\alpha$, $\measuredangle\big(\eta, v\big) = \pi/2$, and $\measuredangle\big(\sigma'(a_1),\ v\big) < \pi/2$. Set $R = [0, \varphi(a_1)] \times [0,1]$, $g = f[v, \gamma, R]$, and $g_t(s) = g^s(t) = g(t,s)$. Without loss of generality, we may assume that σ is contained in a small strongly convex ball centered at x, and that g_t is free of focal points to $\gamma(t)$ for all t. By 3.2 Rauch II in [G] we have $L\big(g^1\big) \leq L(\gamma) = \varphi(a_1)$. From Lemma 1.7 in [CG], we obtain $g_{\varphi(a_1)} \subset \big(M \backslash \mathrm{int}(C)\big)$. Then $d\big(g_0(1), \partial C\big) \leq L\big(g^1\big) \leq \varphi(a_1)$. Consider in the plane the hinge $\big((a_2 - a_1)\sin\alpha, (a_2 - a_1), (\alpha - \pi/2)\big)$, which is a right triangle, whose third side is equal to $-(a_2 - a_1)\cos\alpha$. By Toponogov Theorem ([CE], Theorem 2.2) it holds that $d\big(g_0(1), \sigma(a_2)\big) \leq -(a_2 - a_1)\cos\alpha$. Then $\varphi(a_2) = d\big(\sigma(a_2), \partial C\big) \leq d\big(g_0(1), \partial C\big) + d\big(g_0(1), \sigma(a_2)\big) \leq \varphi(a_1) - (a_2 - a_1)\cos\alpha = \mathrm{r}(a_2)$. By Claim 1 all inequalities above become equalities. Set $w = (a_2 - a_1)\sigma'(a_1)$. Let $S = \exp_x \triangle_{vw}$. By Toponogov Theorem ([CE], Corollary 2.3) S is flat and totally geodesic and $f_{a_2} \in \Gamma\big(\sigma(a_2), g_0(1)\big)$. Set $\ell = -(a_2 - a_1)\cos\alpha$. Then $\varphi(a_2) = \varphi(a_1) - (a_2 - a_1)\cos\alpha = L\left(g^1\right) + L\big((f_{a_2})_{|[0,\ell]}\big)$. Then ${f_{a_2}}'(\ell) = (g^1)'(0)$, and $f_{a_2} \in \Gamma\big(\sigma(a_2), \partial C\big)$. Now we apply Claim 3, replacing α by $\pi - \alpha$ and changing the orientation of σ. We conclude that $f(L)$ is flat and totally geodesic, and that $f_s \in \Gamma\big(\sigma(s), \partial C\big)$ for all $s \in [a_1, a_2]$. Thus Claim 4 and Lemma 3.3 are proved. $\square$

3.4. Lemma. *Let $p, q \in M$ and $\sigma\colon [0,a] \to M$ be a geodesic with $\sigma(0) = p$ and $\sigma(a) = q$. Take $\gamma \in \Gamma_p$ and $\tau \in \Gamma_q$, $\tau \prec \gamma$. Set $\alpha = \measuredangle\big(\gamma'(0), \sigma'(0)\big)$, $\eta = \measuredangle\big(\tau'(0), \sigma'(a)\big)$, $L = [0,a] \times [0, +\infty)$, $f = f[\gamma'(0), \sigma, L]$ and $f_s(t) = f(s,t)$. Then $\alpha \geq \eta$. If $0 < \alpha = \eta < \pi$ then $f(L)$ is flat and totally geodesic, and $f_s \in \Gamma$ for all $s \in [0,a]$.*

Proof. We may assume that $0 < \alpha < \pi$. Set $H_t = H_{\gamma_t}$. By Lemma 1.1 for large $t > 0$ we have $q \in \mathrm{int}(H_t)$. By Proposition 2.1 (c), $\tau \in \Gamma(q, \partial H_t)$. Then we use Lemma 3.3 and conclude the proof. $\square$

3.5. Lemma. *Take $p, q \in M$ with $q \notin C_0(p)$. Let $\sigma\colon [0,a] \to M$ be a geodesic with $\sigma(0) = p$ and $\sigma(a) = q$. Then there exists $\gamma \in \Gamma_q$ such that $\measuredangle\big(\gamma'(0), \sigma'(a)\big) < \pi/2$.*

Proof. Set $C_t = C_t(p)$. By Proposition 1.3 in [CG] there exists $t > 0$ so that $q \in \operatorname{int}(C_t)$. Let $\gamma \in \Gamma(q, \partial C_t)$. By Lemma 1.5, we have $\gamma \in \Gamma_q$. Set $\alpha = \measuredangle\big(\gamma'(0), \sigma'(a)\big)$ and $\varphi(s) = d\big(\sigma(s), \partial C_t\big)$. By the proof of Theorem 1.10 in [CG] for small $|s - a|$ it holds that $\varphi(s) \leq \varphi(a) - (s-a)\cos\alpha$. Since φ is concave this inequality holds for $s \in [0,a]$. Then $\varphi(0) - \varphi(a) \leq a\cos\alpha$. Since $q \notin C_0$ Proposition 1.3 in [CG] implies that $\varphi(0) - \varphi(a) > 0$, hence $\alpha < \pi/2$. $\square$

Now we present some comments about critical points of the infinity.

3.6. Proposition. *Let $p \in M$ and $t > 0$. For each $q \in \big(C_t(p)\big)^{\max}$ we have $q \prec \infty$. In particular $q \prec \infty$ for any q in a soul.*

Proof. Fix $t > 0$. Set $C_t = C_t(p)$. At every $q \in {C_t}^{\max}$ the distance from ∂C_t as a function assumes a maximum. Thus q is a critical point of the distance function from ∂C_t. Take $w \in T_qM$. Since q is a critical point of the distance function from ∂C_t, there exists $\gamma \in \Gamma(q, \partial C_t)$ such that $\measuredangle\big(\gamma'(0), w\big) \leq \pi/2$. By Lemma 1.5 we have $\gamma \in \Gamma_q$, hence $q \prec \infty$. $\square$

Proposition 3.6 says that if some family $\{C_t(p)\}$ has a reduction of dimension at $q \in M$, (that is, if $q \in \big(C_t(p)\big)^{\max}$), then $q \prec \infty$. Conversely, we will see that if $p \prec \infty$, then the family $\{C_t(p)\}$ has a reduction of dimension at p, that is, $C_0(p) = \big(C_t(p)\big)^{\max}$. Thus, $p \prec \infty$ if and only if some family $\{C_t(q)\}_{t>0}$ has a reduction of dimension at p. More precisely, we have the following result.

3.7. Proposition. *We have $p \prec \infty$ if and only if $C_t(p)^{\max} = C_0(p)$, $t > 0$.*

Proof. Set $C_t = C_t(p)$. By Proposition 3.6 it suffices to show that if $p \prec \infty$, then $C_0 = {C_t}^{\max}$, that is, C_0 has no interior points in the sense of the topology of M. Let q be an interior point of $C = C_0$ and take $\sigma\colon [0,a] \to C, \sigma \in \Gamma(p,q)$. Set $\varphi(s) = d\big(\sigma(s), \partial C_0\big)$. We have $\varphi \geq 0, \varphi(0) = 0$ and $\varphi(a) > 0$. Since φ is concave we obtain $\varphi(s) > 0$, for $s \in (0,a]$. Then $\sigma(s) \in \operatorname{int}(C)$, for $s \in (0,a]$. Let $\gamma \in \Gamma_p$. By Lemma 1.2 we have $\gamma \in \Gamma(p, \partial C_t)$ and by Proposition 1.7 in [Ym-1] we obtain $\measuredangle\big(\gamma'(0), \sigma'(0)\big) > \pi/2$. Thus $p \not\prec \infty$ and the proof is complete. $\square$

3.8. Proposition. *We have $p \prec \infty$ if and only if $\theta(p) \geq \frac{\pi}{2}$.*

Proof. Assume that $\theta(p) < \frac{\pi}{2}$. Take $v \in T_pM$ such that $A_p \subset C_p\big(v, \theta(p)\big)$. Then $\measuredangle(-v, w) > \pi/2$, for all $w \in A_p$, hence $p \not\prec \infty$. Assume now that $p \not\prec \infty$. Then there exists $v \in T_pM$ such that $\measuredangle(v, w) > \pi/2$ for all $w \in A_p$. Therefore there exists $\alpha < \pi/2$ such that $A_p \subset C_p(-v, \alpha)$, hence $\theta(p) \leq \alpha < \pi/2$. Thus the proof is complete. $\square$

3.9. Proposition. *Let $p \in M$. If $\theta(p) > \frac{\pi}{2}$, then $C_0(p) = \{p\}$, hence $\{p\}$ is a soul.*

Proof. Suppose that there exists $q \in C_0(p)$ with $q \neq p$. Let $\sigma \in \Gamma(p,q)$. Since $\theta(p) > \frac{\pi}{2}$, we have $A_p \not\subset C_p\big(-\sigma'(0), \pi/2\big)$. Then there exists $v \in A_p$ such that $\measuredangle\big(v, \sigma'(0)\big) < \pi/2$. By the formula for the first variation, for small $s > 0$ we have $\sigma(s) \notin C_0(p)$, but this is false, since $C_0(p)$ is totally convex. Thus the proof is complete. $\square$

3.10. Lemma. *Let $p \in M$ and $v \in T_pM$ satisfy $A_p \subset C_p\big(v, \theta(p)\big)$. Then $A_{\gamma_v(t)} \subset C_{\gamma_v(t)}\big({\gamma_v}'(t), \theta(p)\big)$, $t \geq 0$, hence $\theta(p) \ \geq \ \theta\big(\gamma_v(t)\big)$, $t \geq 0$.*

Proof. Assume that there exists $w \in A_{\gamma_v(t)}$ with $w \notin C_{\gamma_v(t)}\big({\gamma_v}'(t), \theta(p)\big)$. Then $\measuredangle\big(w, {\gamma_v}'(t)\big) > \theta(p)$. By Lemma 3.4 there exists $\mu \in A_p$ such that $\measuredangle(\mu, v) > \theta(p)$, but this contradicts the hypotheses. $\square$

We now start proving Propositions 0.3 to 0.5 and Theorem C.

Proof of Proposition 0.3. Assume first that $\theta \geq \frac{\pi}{2}$ in $M \backslash L$ and suppose that there exists $p \in L$ such that $\theta(p) < \frac{\pi}{2}$. Take v as in Lemma 3.10 and set $\sigma = \gamma_v$. Let $\gamma \in \Gamma_p$. Then $\measuredangle\big(\gamma'(0), v\big) < \pi/2$. By Theorem 5.1 in [CG] σ goes to infinity, and for a large $t > 0$ we have $\sigma(t) \notin L$, hence $\theta\big(\sigma(t)\big) \geq \frac{\pi}{2}$. By Lemma 3.10 we have $\theta(p) \geq \theta\big(\sigma(t)\big) \geq \frac{\pi}{2}$, and this is a contradiction.

Assume now that $\theta > \frac{\pi}{2}$ in $M \backslash L$ and suppose that there exists $p \in L$ such that $\theta(p) \leq \frac{\pi}{2}$. Because of the first part we have $\theta(p) = \frac{\pi}{2}$. Let v, σ be as above. If σ goes to infinity, the proof is completed as in the first part. Thus we assume that $\sigma\big([0, +\infty)\big)$ stays in the compact set L. Take $\gamma \in \Gamma_p$. By Theorem 5.1 in [CG] we have $\measuredangle\big(\gamma'(0), v\big) \geq \pi/2$. Since $\theta(p) = \frac{\pi}{2}$ we have $\measuredangle\big(\gamma'(0), v\big) \leq \pi/2$, hence $\measuredangle\big(\gamma'(0), v\big) = \pi/2$. Set $H_t = H_{\gamma_t}$. By Theorem 8.22 in [CG] we have $\sigma\big([0, +\infty)\big) \subset \partial H_\sigma = \partial H_0$. From this and Lemma 1.1 we obtain $d\big(\sigma(t), \partial H_s\big) = s$, $s > 0$, $t \geq 0$. By Lemma 1.2 we have $\gamma \in \Gamma(p, \partial H_s)$, $s > 0$. Let P_t be the parallel transport along σ and set $\tau_s(t) = \exp_{\sigma(t)} sP_t\big(\gamma'(0)\big)$. Theorem 1.10 in [CG] implies that τ_s is a geodesic and $d\big(\sigma(t), \tau_s(t)\big) = s$ for all $t \geq 0$. Thus τ_s stays in a compact set. For large s we have $\theta\big(\gamma(s)\big) > \frac{\pi}{2}$ and $\{\gamma(s)\}$ is a soul. By Theorem 5.1 in [CG] τ_s goes to infinity in both directions, and we have a contradiction. $\square$

The following example has been mentioned in the Introduction.

3.11. Example. There exists a surface M with $K \geq 0$, not isometric to $\mathbb{R}^2$, such that all its points are souls of M. In fact, take $O, v \in \mathbb{R}^3$. Set $\mathcal{C} = \partial C_O(v, \beta)$, where $\beta > \pi/6$. By cutting $\mathcal{C}$ along a ray starting at O we obtain a sector with angle $2\pi \sin\beta$. Thus, it is easy to see that $\theta(p) = \pi \sin\beta > \pi/2$, for $p \neq 0$. Modify $\mathcal{C}$ in a neighborhood of O to obtain a C^∞ nonnegatively curved surface M. For large $d(O,p)$ we have still $\theta(p) = \pi \sin\beta > \pi/2$. By Proposition 0.3 all points of M are souls and M is not isometric to $\mathbb{R}^2$.

Proof of Proposition 0.4. Since $p_k \prec \infty$ Proposition 3.7 implies that $C_0(p_k) = C_t(p_k)^{\max}$, $t > 0$. Fix $p \in M$. If $p \in C_0(p_k)$ Proposition 3.6 implies that $p \prec \infty$, hence $\theta(p) \geq \frac{\pi}{2}$ and Proposition 0.4 follows. Thus we assume that $p \notin C_0(p_k)$, for all k. Let $\sigma_k \in \Gamma(p, p_k)$. Since $p \notin C_0(p_k)$ Lemma 3.5 implies that there exists $\gamma_k \in \Gamma_{p_k}$ such that $\measuredangle\big({\gamma_k}'(0), -{\sigma_k}'(0)\big) < \pi/2$. By passing to a subsequence, we may admit that ${\sigma_k}'(0) \to v \in A_p$ and that ${\gamma_k}'(0) \to w \in A_p$. Since $\measuredangle(v,w) \geq \pi/2$, we have $\theta(p) \geq \pi/4$, thus concluding the proof. □

Proof of Proposition 0.5. Assume that $\theta(p) = \inf_{x\in M} \theta(x)$ and suppose that $\theta(p) > 0$. If $\theta(p) = \pi$ then all geodesics of M are lines, hence M is isometric to $\mathbb{R}^n$, and this contradicts the hypotheses. Thus we may assume that $0 < \theta(p) < \pi$. Take v as in Lemma 3.10. Set $\theta_s = \theta\big(\gamma_v(s)\big)$, $s > 0$. Lemma 3.10 implies that $\theta(p) \geq \theta_s$. By the minimality of $\theta(p)$ we have $\theta(p) = \theta_s$. By Lemma 3.10 we obtain $A_{\gamma_v(s)} \subset C_{\gamma_v(s)}\big({\gamma_v}'(s), \theta(p)\big) = C_{\gamma_v(s)}\big({\gamma_v}'(s), \theta_s\big)$. By the definition of θ_s there exists $\gamma \in \Gamma_{\gamma_v(s)}$ so that $\measuredangle\big(\gamma'(0), {\gamma_v}'(s)\big) = \theta_s = \theta(p)$. Take $\tau \in \Gamma_p$, $\tau \prec \gamma$. Lemma 3.4 implies that $\beta = \measuredangle\big(\tau'(0), v\big) \geq \theta(p)$. By the definition of $\theta(p)$ we have $\beta = \theta(p)$. Since $0 < \theta(p) < \pi$ Lemma 3.4 assures that there exists $\tilde{\tau} \in \Gamma_p$ such that γ, $\tilde{\tau}$ and γ_v bound a flat totally geodesic surface, and this contradicts the hypotheses of the Proposition, thus concluding the proof. □

Proof of Theorem C. Let $(q_k) \subset M$ be so that $\theta(q_k) \to \eta = \inf_{x\in M} \theta(x)$. We must prove that there exists $p_k \to \infty$ with $\theta(p_k) \to \eta$.

Case 1. *There exists (q_{k_j}) such that $\theta(q_{k_j}) = \frac{\pi}{2}$, for all j.*

In this case we have $\eta = \frac{\pi}{2}$. Suppose that there exists a compact set L such that $\theta > \eta = \frac{\pi}{2}$ in $M \backslash L$. By Proposition 0.3 we have $\theta > \frac{\pi}{2}$ in M, but this contradicts the hypothesis of Case 1. Thus, such a set L does not exist, hence there exists $(p_k) \subset M$ with $p_k \to \infty$ such that $\theta(p_k) = \frac{\pi}{2} = \eta$, thus concluding the proof in this case.

Case 2. *There exists $k_0 \in \mathbb{N}$ such that $\theta(q_k) \neq \frac{\pi}{2}$, for each $k > k_0$.*

Take $k > k_0$ and $v_k \in T_{q_k}M$ so that $A_{q_k} \subset C_{q_k}\big(v_k, \theta(q_k)\big)$. If $\theta(q_k) > \frac{\pi}{2}$ Proposition 3.9 implies that $\{q_k\}$ is a soul, and by Theorem 5.1 in [CG] γ_{v_k} goes to infinity. If $\theta(q_k) < \frac{\pi}{2}$ then there exists $\gamma \in \Gamma_p$ such that $\measuredangle\big(\gamma'(0), v_k\big) < \pi/2$, and by Theorem 5.1 in [CG] γ_{v_k} goes to infinity. Fix $p \in M$. Choose $p_k = \gamma_{v_k}(t_k)$, where t_k is sufficiently large so that $d(p_k, p) > k$. By Lemma 3.10 we have $\theta(q_k) \geq \theta(p_k)$, hence $\theta(p_k) \to \eta$. Since $p_k \to \infty$, the proof is complete. □

4. On the radius and diameter of M(∞); proofs of Theorems A, B, D and E

Kasue proved (Remark of Theorem 4.3 in [K]) the following result.

4.1. Lemma. *Let N be an asymptotically nonnegatively curved manifold with only one end and $\gamma, \sigma \in \Gamma$. For each t take $\sigma_t \in \Gamma_{\gamma(t)}$, $\sigma_t \prec \sigma$. Then $\measuredangle\big(\gamma'(t), \sigma_t'(0)\big) \to \measuredangle_\infty\big(\gamma(\infty), \sigma(\infty)\big)$ as $t \to +\infty$.*

We need the following modification of Lemma 4.1

4.2. Lemma. *Let M have only one end, $p \in M$ and $(p_k) \subset M$ be so that $p_k \to x \in M(\infty)$. For all k choose $\sigma_k \in \Gamma_{p_k}$ in such a way that $\sigma_k(\infty) \to y \in M(\infty)$, and take $\tau_k\colon [0, s_k] \to M$ with $\tau_k \in \Gamma(p, p_k)$. Then $\eta_k = \measuredangle\big({\tau_k}'(s_k), {\sigma_k}'(0)\big) \to \measuredangle_\infty(x, y)$.*

Proof. Since (η_k) is bounded, it suffices to show that any convergent subsequence of (η_k) converges to $\measuredangle_\infty(x, y)$. Thus we assume that $\eta_k \to \eta$ and prove that $\boldsymbol{\eta} = \measuredangle_{\boldsymbol\infty}(\mathbf{x}, \mathbf{y})$. By passing to a subsequence we have ${\tau_k}'(0) \to \gamma'(0)$, $\gamma \in \Gamma_p$. Lemma 2.5 implies that $\gamma(\infty) = x$. Set $\epsilon_k = \measuredangle\big(\gamma'(0), {\tau_k}'(0)\big)$. Let $\tilde\gamma_k \in \Gamma_{p_k}, \tilde\gamma_k \prec \gamma_k$. By Lemma 2.2 we obtain $\tilde\gamma_k(\infty) = \gamma(\infty) = x$. Set $\tilde\epsilon_k = \measuredangle\big(\tilde\gamma_k'(0), {\tau_k}'(s_k)\big)$. By Lemma 3.4 we have $\tilde\epsilon_k \le \epsilon_k \to 0$. We obtain $\eta_k \ge \measuredangle\big({\sigma_k}'(0), \tilde\gamma_k'(0)\big) - \tilde\epsilon_k$. Then $\eta_k \ge \measuredangle_\infty\big(\sigma_k(\infty), x\big) - \tilde\epsilon_k$. By taking limits we obtain $\boldsymbol{\eta} \ \ge\ \measuredangle_{\boldsymbol\infty}(\mathbf{y},\ \mathbf{x})$.

Fix k. Take $t_j \to +\infty$. Set $q_j = \sigma_k(t_j)$. Let $\mu_j\colon [0, u_j] \to\ M$, $\ \mu_j \in \Gamma(p, q_j)$. Set $\beta_k = \pi - \eta_k$ and $d_{kj} = d\big(p_k, \mu_j(s_k)\big)$. Let (t_j, u_j, s_k) be a triangle in the plane with angles $(\alpha_{kj}, \beta_{kj}, \theta_{kj})$. By Toponogov Theorem we have $\eta_k = \pi - \beta_k \le \pi - \beta_{kj} = \alpha_{kj} + \theta_{kj}$. For large j we have $u_j > s_k$. By Theorem 2.1 in [Sm-3] we have $\eta_k \le \alpha_{kj}' + \theta_{kj}$, where α_{kj}' is the angle opposite to d_{kj} in the triangle $(s_k,\ s_k,\ d_{kj})$ in the plane. By passing to a subsequence we may admit that ${\mu_j}'(0) \to \tilde\sigma_k'(0)$, where $\tilde\sigma_k \in \Gamma_p$. Set $d_k = d\big(p_k, \tilde\sigma_k(s_k)\big)$. Take the triangle (s_k, s_k, d_k) in the plane. Let α_k be the angle opposite to d_k. Then $2\sin(\alpha_{kj}'/2) = d_{kj}/s_k \to d_k/s_k = 2\sin(\alpha_k/2)$ as $j \to +\infty$, hence $\alpha_{kj}' \to \alpha_k$. Since $\theta_{kj} \to 0$ we have $\eta_k \ \le\ \alpha_k$.

It remains to prove that $\boldsymbol{\alpha}_{\mathbf{k}} \to \measuredangle_{\boldsymbol\infty}(\mathbf{x}, \mathbf{y})$. By passing to a subsequence assume that $\tilde\sigma_k'(0) \to \sigma'(0)$, where $\sigma \in \Gamma_p$. Since $\tilde\sigma_k \prec \sigma_k$, it follows from Lemma 2.2 that $\tilde\sigma_k(\infty) = \sigma_k(\infty)$. Then $\tilde\sigma_k(\infty) \to y$. Since $\tilde\sigma_k'(0) \to \sigma'(0)$ Lemma 2.6 implies that $\tilde\sigma_k(\infty) \to \sigma(\infty)$. Thus we obtain $\sigma(\infty) = y$. Set $e_k = d\big(p_k, \gamma(s_k)\big)$, $f_k = d\big(\sigma(s_k), \tilde\sigma_k(s_k)\big)$, and $\tilde d_k = d\big(\gamma(s_k), \sigma(s_k)\big)$. By the triangle inequality we have $\tilde d_k/s_k - e_k/s_k - f_k/s_k \le d_k/s_k \le \tilde d_k/s_k + e_k/s_k + f_k/s_k$. Since $p_k \to x$ we have $p_k \to \gamma(\infty)$. Then $e_k/s_k \to 0$. Since $\tilde\sigma_k'(0) \to \sigma'(0)$ Lemma 2.5 implies that $\tilde\sigma_k(s_k) \to \sigma(\infty)$. Therefore $f_k/s_k \to 0$. Since $\gamma(\infty) = x$ and $\sigma(\infty) = y$ Lemma 2.3 implies that $\tilde d_k/s_k \to 2\sin\big(\measuredangle_\infty(x, y)/2\big)$. Then $d_k/s_k \to 2\sin\big(\measuredangle_\infty(x, y)/2\big)$, hence $2\sin(\alpha_k/2) = d_k/s_k \to 2\sin\big(\measuredangle_\infty(x, y)/2\big)$. Then $\boldsymbol{\alpha}_{\mathbf{k}} \to \measuredangle_{\boldsymbol\infty}(\mathbf{x}, \mathbf{y})$. Thus we have $\boldsymbol{\eta} \le \measuredangle_{\boldsymbol\infty}(\mathbf{x}, \mathbf{y})$, hence $\boldsymbol{\eta} = \measuredangle_{\boldsymbol\infty}(\mathbf{x}, \mathbf{y})$, thereby concluding the proof. $\square$

Proof of Theorems A and B

Claim 1. *Fix $p \in M$. There exists a compact set $\tilde{D} \subset M$ such that $p \notin C_0(z)$, for all $z \in M \backslash \tilde{D}$.*

Proof of Claim 1. The hypotheses of Theorem B clearly imply Claim 1. Assume then the hypotheses of Theorem A. Suppose that the statement of Claim 1 is false. Then there exists a sequence $p_k \to \infty$ such that $p \in C_0(p_k)$ for all k. Take $\tau_k\colon [0, t_k] \to M$, with $\tau_k \in \Gamma(p, p_k)$. By passing to a subsequence we assume that $p_k \to x \in M(\infty)$. Take $y \in M(\infty)$ so that $\measuredangle_\infty(x, y) = \max_{w \in M(\infty)} \measuredangle_\infty(x, w) \geq \mathrm{r}\big(M(\infty)\big)$. Choose $\sigma_k \in \Gamma_{p_k}$ so that $\sigma_k(\infty) = y$. If $\measuredangle\big(\tau_k'(t_k), \sigma_k'(0)\big) > \pi/2$ the first variation formula implies that $p \notin C_0(p_k)$ and this is false. Thus Lemma 4.2 implies that $\pi/2 \geq \measuredangle\big(\tau_k'(t_k), \sigma_k'(0)\big) \to \measuredangle_\infty(x, y)$, hence $\mathrm{r}\big(M(\infty)\big) \leq \pi/2$, and this contradicts the hypotheses of Theorem A. Thus Claim 1 is proved.

Claim 2. *For all $p \in M$, we have $C_0(p) = S(p)$, where $S(p)$ is the soul of the set $C_t(p), t > 0$.*

Proof of Claim 2. Assume that Claim 2 is false. If $p \in S(p)$, take $q \in C_0(p) \backslash S(p)$. If $p \notin S(p)$, take $q \in S(p)$. Let $\sigma\colon [-a, +\infty) \to M$ be a geodesic with $\sigma(-a) = q, a > 0$ and $\sigma(0) = p$. Theorem 5.1 in [CG] implies that σ goes to infinity. By Claim 1, for a large t we have $p \notin C_0\big(\sigma(t)\big)$, and by Lemma 3.5 there exists $\gamma \in \Gamma_p$ such that $\measuredangle\big(\gamma'(0), -\sigma'(0)\big) < \pi/2$. By the formula for the first variation, for small $s > 0$, we have $\sigma(-s) \notin H_\gamma$, hence $\sigma(-s) \notin C_0(p)$. This contradiction concludes the proof of Claim 2.

Claim 3. *M is isometric to $S \times V$, where S^k is a soul of M and V is diffeomorphic to $\mathbb{R}^{n-k}$.*

Proof of Claim 3. Let $p \in M$. By Proposition 1.3 in [CG] and by Claim 2 above we have $p \in C_0(p) = S(p)$. Thus $W(M) = M$. Claim 3 follows from Theorem 0.1.

Claim 4. *For all $p_2 \in V$ it holds that $\{p_2\}$ is a soul of V.*

Proof of Claim 4. It suffices to show that $C_0(p_2) = \{p_2\}$. Let $\tilde{d}(x, y)$ be the distance between x and y in V. Take $q_2 \in C_0(p_2)$. Fix $p_1 \in S$. Set $p = (p_1, p_2)$ and $q = (p_1, q_2)$. Let $\gamma \subset M$, $\gamma \in \Gamma_p$. Since S is compact and any minimizing geodesic in M is a product of minimizing geodesics, we have $\gamma(t) = \big(p_1, \gamma_2(t)\big)$, with $\gamma_2 \subset V$, $\gamma_2 \in \Gamma_{p_2}$. Since q and $\gamma(t)$ have the same first coordinate, it follows that $d\big(q, \gamma(t)\big) = \tilde{d}\big(q_2, \gamma_2(t)\big) \geq t$, for all $t \geq 0$, since $q_2 \in C_0(p_2)$. Then $q \in C_0(p) = S(p)$. By Theorem 0.1 $S(p)$ is of the form $S \times \{r_2\}$ for a certain $r_2 \in V$. Thus p and q have the same second coordinate, hence $p_2 = q_2$. $\square$

Proof of Theorem D. Consider a divergent sequence (p_k) which satisfies $\mathrm{r}(A_{p_k}) \to \eta = \limsup_{q\to\infty} \mathrm{r}(A_q)$. Fix $p \in M$. Take $\tau_k\colon [0, s_k]$

$\to M$, $\tau_k \in \Gamma(p, p_k)$. Let $\sigma_k \in \Gamma_{p_k}$ be so that $\eta_k = \measuredangle\big({\tau_k}'(s_k), {\sigma_k}'(0)\big)$ is maximal. By passing to a subsequence we assume that ${\tau_k}'(0) \to \gamma'(0)$, $\gamma \in \Gamma_p$ and $\sigma_k(\infty) \to y \in M(\infty)$. Take $\gamma_k \in \Gamma_{p_k}$, $\gamma_k \prec \gamma$. Then $p_k \to x = \gamma(\infty) = \gamma_k(\infty)$. Set $\alpha_k = \measuredangle\big({\tau_k}'(0), \gamma'(0)\big)$. By Lemma 3.4 we have $\delta_k = \measuredangle\big({\gamma_k}'(0), {\tau_k}'(s_k)\big) \le \alpha_k \to 0$. Take $v \in A_{p_k}$. Then $\measuredangle\big(v, {\gamma_k}'(0)\big) \le \measuredangle\big(v, {\tau_k}'(s_k)\big) + \delta_k \le \eta_k + \delta_k$, hence $\mathrm{r}(A_{p_k}) \le \eta_k + \delta_k$. Lemma 4.2 applies and we conclude that $\eta_k \to \measuredangle_\infty(x, y)$. By taking limits we obtain $\eta \le \measuredangle_\infty(x, y) \le \mathrm{diam}\big(M(\infty)\big)$. □

Proof of Corollary 0.6. Since $\theta(q) \le r(A_q)$ it remains only to show that $\liminf_{q\in M} \theta(q) \ge \frac{1}{2}\mathrm{diam}\big(M(\infty)\big)$. Let $q \in M$ and $v \in T_qM$ be so that $A_q \subset C_q\big(v, \theta(q)\big)$. For $\gamma, \sigma \in \Gamma_q$, we have $2\theta(q) \ge \measuredangle\big(\gamma'(0), v\big) + \measuredangle\big(v, \sigma'(0)\big) \ge \measuredangle\big(\gamma'(0), \sigma'(0)\big) \ge \measuredangle_\infty\big(\gamma(\infty), \sigma(\infty)\big)$.

Then $2\theta(q) \ge \mathrm{diam}\big(M(\infty)\big)$. Thus Corollary 0.6 follows. □

4.3. Example. Let $M = \mathcal{P} \times \mathbb{R}^m$, where $\mathcal{P}$ is a paraboloid. Let $p \in \mathcal{P}$ be the pole, $(q_k) \subset \mathbb{R}^m$, $q_k \to \infty$, and $(r_k) \subset \mathcal{P}$, $r_k \to \infty$. Since M contains a line we have $\mathrm{diam}\big(M(\infty)\big) = \pi$. A curve $\gamma = (\gamma_1, \gamma_2)$ is a ray in M if and only if either γ_i is a (not necessarily normalized) ray for $i = 1, 2$, or if γ_i is constant and γ_j is a ray, for $i \ne j$. Then we have $A_{(p,q_k)} = S^{m+1}$, hence $\theta\big((p, q_k)\big) = \mathrm{r}(A_{(p,q_k)}) = \pi = \mathrm{diam}\big(M(\infty)\big)$, $m(A_{(p,q_k)}) = \mathrm{vol}(S^{m+1})$, and we have that $A_{(r_k,q_k)}$ is an $(m+1)$-hemisphere, hence $\theta\big((r_k, q_k)\big) = \mathrm{r}(A_{(r_k,q_k)}) = \frac{\pi}{2} = \frac{1}{2}\mathrm{diam}\big(M(\infty)\big)$, $m(A_{(r_k,q_k)}) = \mathrm{vol}(S^{m+1})/2$. Thus the limits of $m(A_x), \mathrm{r}(A_x)$ and of $\theta(x)$ do not exist, and the inequalities in Theorem D and in Corollary 0.6 are sharp.

Proof of Corollary 0.7. By Theorem D we have $\mathrm{diam}\big(M(\infty)\big) \ge \pi$. Then there exists a line in M and Corollary 0.7 follows. □

Proof of Theorem E. With a proof similar and easier to that of Theorem C we have $\eta = \inf_{q\in M} \mathrm{r}(A_q) = \liminf_{q\to\infty} \mathrm{r}(A_q)$. Let $p \in M$ and $v, w \in A_p$ be so that $\measuredangle(v, w) = r(A_p)$. Take $u \in A_p$ so that $\measuredangle_\infty\big(\gamma_v(\infty), \gamma_u(\infty)\big)$ is maximal. We have $\mathrm{r}(A_p) = \measuredangle(v, w) \ge \measuredangle(v, u) \ge \measuredangle_\infty\big(\gamma_v(\infty), \gamma_u(\infty)\big) \ge \mathrm{r}\big(M(\infty)\big)$, hence $\eta \ge \mathrm{r}\big(M(\infty)\big)$. Take $x \in M(\infty)$ so that $\max_{y\in M(\infty)} \measuredangle_\infty(x, y) = \mathrm{r}\big(M(\infty)\big)$ and $\gamma \in \Gamma$ so that $\gamma(\infty) = x$. Let $p_k = \gamma(s_k)$, $s_k \to +\infty$. Take $\sigma_k \in \Gamma_{p_k}$ so that $\eta_k = \measuredangle\big(\gamma'(s_k), \sigma_k'(0)\big)$ is maximal. By passing to a subsequence we assume that $\sigma_k(\infty) \to y \in M(\infty)$. Clearly $p_k \to \gamma(\infty) = x$. By Lemma 4.2 we obtain $\eta_k \to \measuredangle_\infty(x, y)$. Since η_k is maximal we have $A_{p_k} \subset C_{p_k}\big(\gamma'(s_k), \eta_k\big)$, hence $\eta \le \mathrm{r}(A_{p_k}) \le \eta_k$. By taking limits we obtain $\eta \le \measuredangle_\infty(x, y) \le \mathrm{r}\big(M(\infty)\big)$, and this completes the proof. □

4.4. Corollary. *Let M have only one end. Take compact sets $K_j \subset K_{j+1} \subset M$ so* *that* $\bigcup_j K_j = M$. *Then* $r\big(M(\infty)\big) = \liminf_{j\to+\infty} \dfrac{\int_{K_j} \mathrm{r}(A_x)\, dM(x)}{\int_{K_j} dM(x)} \le \limsup_{j\to+\infty}$ $\dfrac{\int_{K_j} \mathrm{r}(A_x) dM(x)}{\int_{K_j} dM(x)} \le diam\big(M(\infty)\big)$, *where* dM *denotes the volume element of* M.

Proof. By [Ya] M has infinite volume. Thus we can do a proof entirely similar to that of Theorem 1 in [Sm-2].

5. The radial function in dimension 2

Along this section N will denote a complete and noncompact connected surface with finite topological type admitting total curvature. Let L be a compact subset of N. Set $\mathbf{c(L)} = \int_L K dN$. We say that N **admits total curvature** if there exists $\mathbf{c(N)} \in [-\infty, +\infty]$ such that, for each increasing sequence of compact sets $L_j \subset M$ with $\bigcup_j L_j = M$, we have $\lim_{j\to+\infty} c(L_j) = c(N)$. By Theorem 2.4 in [Sy-2] we have $\mathrm{diam}\big(M(\infty)\big) = \frac{1}{2}\big(2\pi\mathcal{X}(N) - c(N)\big)$. For any measurable subset $X \subset N$ we may define $\mathbf{c(X)}$ in a similar manner.

In dimension two, several results which are true for the mass of rays remain valid for $r(A_p)$ and $\theta(p)$. We will prove here one of these results. The other ones can be proved similarly.

Theorem F. (similar to [Sm-2], p.196, [SST], p.352 and [Sy-1], Theorem A) *If* N *has only one end, then we have* $\lim_{p\to\infty} 2r(A_p) = \lim_{p\to\infty} 2\theta(p) = \min\{2\pi\mathcal{X}(N) - c(N), 2\pi\}$.

5.1. Lemma. *If there is no line in* N *then* $2r(A_p) - m(A_p) \to 0$*, as* $p \to \infty$.

Proof. Initially we prove that $\mathbf{2\boldsymbol{\theta}(p) - m(A_p) \to 0}$ as $p \to \infty$.

Fix $\epsilon > 0$. Take a compact set L, so that $c(X) < \epsilon$ for each measurable set $X \subset N \backslash L$, and so that $N\backslash L$ is homeomorphic to a halfcylinder. Since there is no line in N, there exists a compact set Q with $L \subset Q$ satisfying that $N\backslash Q$ is also a halfcylinder and that, for each $p \in N\backslash Q$, if $v \in A_p$ then $\gamma_v \cap L = \emptyset$.

Fix $p \in N\backslash Q$. Take $u_1, u_2 \in A_p$ so that $\measuredangle(u_1, u_2) = 2\theta(p)$, that is, u_1 and u_2 make a maximal angle in A_p. Let E be the region bounded by γ_{u_1} and γ_{u_2} which is homeomorphic to a halfplane and $S = \{u \in T_pN \mid |u| = 1 \text{ and } \gamma_u(t) \in E \text{ for small } t > 0\}$. Then S is a closed arc and $S\backslash A_p = \bigcup_{j\in\mathbb{N}} V_j$, where V_j is an open arc. Set $\partial V_j = \{v_j, w_j\}$, $v_j, w_j \in A_p$. Let D_j be the closed region bounded by γ_{v_j} and γ_{w_j} which satisfies $\gamma_u(t) \in D_j$ for small $t > 0$ and $u \in V_j$. Lemma 1.2 in [Sm-2] applies (see also Theorem A, (4) in [Sm-1]) to each region D_j. Since D_j is homeomorphic to a half plane we have $m(V_j) = \measuredangle(v_j, w_j) = c(V_j)$, hence $0 \le 2\theta(p) - m(A_p) = m(S) - m(A_p) = \sum_j m(V_j) = \sum_j c(D_j) = c\big(\bigcup_j D_j\big) < \epsilon$, where the last inequality is due to the fact that $\bigcup_j D_j \subset N \backslash L$. Thus $\mathbf{2\theta(p) - m(A_p) \to 0}$ as $p \to \infty$.

Consider a sequence $p_k \to \infty$. Let $v_k \in T_{p_k}N$ be so that $A_{p_k} \subset C_{p_k}\big(v_k, \theta(p_k)\big)$. Take $w_k \in A_{p_k}$ so that $\delta_k = \measuredangle(v_k, w_k)$ is minimal. Then $\delta_k \to 0$, otherwise $2\theta(p_k) - m(A_{p_k}) \not\to 0$. We have $\mathrm{r}(A_{p_k}) \leq \theta(p_k) + \delta_k$ and Lemma 5.1 follows. $\square$

Proof of Theorem F. Consider initially the case in which $c(N) \leq 2\pi\big(\mathcal{X}(N) - 1\big)$. Then $2\pi\mathcal{X}(N) - c(N) \geq 2\pi$. By Theorem A in [Sy-1] we have $\lim_{p\to\infty} m(A_p) = \min\{2\pi\mathcal{X}(N) - c(N), 2\pi\} = 2\pi$. Since $2\pi \geq 2\mathrm{r}(A_p) \geq 2\theta(p) \geq m(A_p)$ we obtain $\lim_{p\to\infty} 2\mathrm{r}(A_p) = \lim_{p\to\infty} 2\theta(p) = 2\pi = \min\{2\pi\mathcal{X}(N) - c(N), 2\pi\}$, and the theorem is proved in this case.

Suppose that $c(N) > 2\pi\big(\mathcal{X}(N) - 1\big)$. By Theorem A in [Sg] (see also [Sm-2], p. 204) there is no line in N. Then Lemma 5.1 applies and we obtain $\lim_{p\to\infty} 2\mathrm{r}(A_p) - m(A_p) = 0$. By Theorem A in [Sy-1] we have $\lim_{p\to\infty} m(A_p) = \min\{2\pi\mathcal{X}(N) - c(N), 2\pi\}$, hence $\lim_{p\to\infty} 2\mathrm{r}(A_p) = \lim_{p\to\infty} 2\theta(p) = \min\{2\pi\mathcal{X}(N) - c(N), 2\pi\}$. $\square$

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(Received: October 26, 1995; revised: February 5, 1997)