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The K-theory of p-compact groups

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Abstract. In this paper, we show that the p-adic K-theory of a connected p-compact is the ring of invariants of the Weyl group action on the K-theory of a maximal torus. We apply this result to show that a connected finite loop space admits a maximal torus if and only if its complex K-theory is λ-isomorphic to the K-theory of some BG, where G is a compact connected Lie group.

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Introduction

Let G be a compact Lie group and BG its classifying space. In [5] Atiyah and Segal have shown that the complex K-theory ring $K^*(BG;\mathbb{Z})$ is isomorphic to the I-adic completion of the complex representation ring $R(G)$. Assume G is connected and fix a maximal torus $T \subset G$ with Weyl group W. The preceding result is equivalent to the isomorphism

$$K^*(BG;\mathbb{Z}) \cong K^*(BT;\mathbb{Z})^W, \qquad (*)$$

where the last term stands for the ring of invariants of the natural W-action on $K^*(BT;\mathbb{Z})$ (see [4]).

In [15] Dwyer and Wilkerson introduce the concept of p-compact group, where p stands for a prime. Their original results and subsequent works ([16], [23], [24]) show that these objects constitute a natural homotopy theoretic generalization of compact Lie groups. For instance, p-compact groups have maximal tori, Weyl groups, etc. We refer to Section 1 for the precise definitions. In this introduction, we would like to emphasize the fact that all the structure of a p-compact group is concentrated at the single prime p. Therefore it is natural to consider p-adic K-theory rather than ordinary K-theory. In this framework, our main result is the following generalization of the isomorphism $(*)$:

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Theorem. *Let X be a connected p-compact group, $i\colon T \longrightarrow X$ a maximal torus and W the corresponding Weyl group. The classifying map $Bi\colon BT \longrightarrow BX$ induces a ring isomorphism*

$$K^*(BX;\mathbb{Z}_{\hat{p}}) \cong K^*(BT;\mathbb{Z}_{\hat{p}})^W.$$

As a first application we study the ordinary complex K-theory of finite loop spaces. More precisely, if L is a connected finite loop space, we show that $K^1(BL;\mathbb{Z}) = 0$ and $K^0(BL;\mathbb{Z})$ is torsion free and without zero divisors. Our results are much more complete for finite loop spaces with maximal tori. In the case of a compact connected Lie group, we obtain a non-analytical proof of the isomorphism $(*)$. Moreover we have the following generalization of a result due to Notbohm and Smith ([27, Theorem 5.1]):

Theorem. *Let L be a connected finite loop space. Then L admits a maximal torus if and only if there exists a compact connected Lie group G such that $K^*(BL;\mathbb{Z})$ is λ-isomorphic to $K^*(BG;\mathbb{Z})$.*

In Section 1, we recall the basic definitions of the theory of p-compact groups. In the same section we use a theorem of Kane and Lin to deduce that the p-adic K-theory of a 1-connected p-compact group is an exterior algebra. As the reader will see, our arguments depend heavily on this result. Section 2 contains the crucial step of the proof of our main result, namely the map $Bi\colon BT \longrightarrow BX$ above induces a finite ring homomorphism in mod p K-theory. To achieve this goal, we appeal to Dwyer's transfer and we show that any p-compact toral group P "embeds" into some unitary group $U(N)$. The technical Section 3 deals with the reduction of the general situation to the 1-connected case. In Section 4 we combine Dwyer's transfer with Kane and Lin's theorem and the finiteness result of Section 2 to conclude. Some consequences of our main result are given in Section 5. The last section is devoted to the announced application to finite loop spaces.

Notations. Throughout the paper, p is a fixed prime number, $\mathbb{Z}_{\hat{p}}$ the ring of p-adic integers and $\mathbb{Q}_{\hat{p}} = \mathbb{Q}\otimes\mathbb{Z}_{\hat{p}}$ the fraction field of $\mathbb{Z}_{\hat{p}}$. For any space Y the symbol $H^*_{\mathbb{Q}_{\hat{p}}}(Y)$ stands for $\mathbb{Q}\otimes H^*(Y;\mathbb{Z}_{\hat{p}})$ and $Y_{\hat{p}}$ for the Bousfield–Kan p-completion of Y.

1. Backgrounds

The purpose of this section is to fix the notation and to recall the definitions and the results we are going to use. The interested reader is referred to the seminal paper of Dwyer and Wilkerson ([15]) for a much more complete presentation.

A *loop space* X is a triple $(X, BX; e)$, where X is a space, BX a connected pointed space and $e\colon X \longrightarrow \Omega BX$ a homotopy equivalence; BX is called the classifying space of X. Such a loop space will be called a *p-compact group* if the following additional conditions are satisfied:

1. X is $\mathbb{F}_p$-finite, i.e., $H^*(X;\mathbb{F}_p)$ is a finite dimensional $\mathbb{F}_p$-vector space;
2. $\pi_0(X)$ is a finite p-group and $\pi_n(X)$ is a finitely generated $\mathbb{Z}_{\hat{p}}$-module for any $n \geq 1$.

A *morphism* $f\colon X \longrightarrow Y$ between two p-compact groups is a pointed map $Bf\colon BX \longrightarrow BY$. The morphism f is a *monomorphism* (respectively, an *epimorphism*) if the homotopy fiber Y/X of Bf is $\mathbb{F}_p$-finite (respectively, the classifying space of a p-compact group). A *short exact sequence* $X \xrightarrow{f} Y \xrightarrow{g} Z$ of p-compact groups is a sequence such that $BX \xrightarrow{Bf} BY \xrightarrow{Bg} BZ$ is a fibration up to homotopy. Two morphisms $f, g\colon X \longrightarrow Y$ are conjugate if the maps Bf and Bg are freely homotopic.

The *centralizer* of a morphism $f\colon X \longrightarrow Y$ of p-compact groups is the loop space

$$C_Y(f(X)) := (\Omega \mathrm{Map}(BX, BY)_{Bf}, \mathrm{Map}(BX, BY)_{Bf}; id).$$

The morphism f will be called *central* if the basepoint evaluation map

$$\mathrm{ev}\colon \mathrm{Map}(BX, BY)_{Bf} \longrightarrow BY$$

is a homotopy equivalence.

A *p-compact torus* (of rank n) is a p-compact group T such that $BT \simeq K(\mathbb{Z}_{\hat{p}}^n, 2)$. A *p-compact toral group* P is a p-compact group fitting into a short exact sequence $T \longrightarrow P \longrightarrow \pi$, where T is a p-compact torus and π a finite p-group. A *maximal torus* for a p-compact group X is a monomorphism $i\colon T \longrightarrow X$ whose centralizer is a p-compact toral group. One of the fundamental results of [15] says that any p-compact group admits a maximal torus, unique up to conjugacy.

Let $i\colon T \longrightarrow X$ be a maximal torus for a p-compact group X. We replace the map $Bi\colon BT \longrightarrow BX$ by an equivalent fibration $BT' \longrightarrow BX$; the *Weyl space* $\mathcal{W}_T(x)$ is defined as the space of self-maps of BT' over BX. In Proposition 9.5 of [15], it is shown that the space $\mathcal{W}_T(X)$ is homotopically discrete and $W_T(X) := \pi_0(\mathcal{W}_T(X))$ is a finite group under composition; $W_T(X)$ (or simply W_X) is called the *Weyl group* of X (with respect to the maximal torus i). By construction the Weyl space $\mathcal{W}_T(X)$ acts on BT'; the Borel construction of this action gives a loop space called the *normalizer* of T and denoted $\mathcal{N}(T)$. Thus we have a homotopy fibration sequence $BT \longrightarrow B\mathcal{N}(T) \longrightarrow BW_T(X)$. In general, the loop space $\mathcal{N}(T)$ is not a p-compact group: $\pi_0(\mathcal{N}(T)) \cong W_T(X)$ is seldom a p-group. However, one obtains a p-compact group $\mathcal{N}_p(T)$ by performing the Borel construction for the action of the submonoid $\mathcal{W}_T(X)_p$ given by the union of components of $\mathcal{W}_T(X)$ which project onto a p-Sylow subgroup of $W_T(X)$; $\mathcal{N}_p(T)$ is called the p-normalizer of T.

For future use, some other important results of [15] are recorded in the following

Theorem 1.1. *Let $i\colon T \longrightarrow X$ be a maximal torus for a connected p-compact group X, with Weyl group W_X.*

1. *The homotopy action of W_X on BT induces a faithful representation of W_X as a reflection group in the $\mathbb{Q}_{\hat{p}}$-vector space $H^2_{\mathbb{Q}_{\hat{p}}}(BT)$.*
2. *The map Bi induces a ring isomorphism*

$$H^*_{\mathbb{Q}_{\hat{p}}}(BX) \cong H^*_{\mathbb{Q}_{\hat{p}}}(BT)^{W_X}.$$

Throughout the paper we will be dealing with $\mathbb{Z}/2$-graded K-theories. The ordinary periodic complex K-theory will be denoted $K^*(-;\mathbb{Z})$. For any abelian group A, $K^*(-;A)$ denotes the K-theory with coefficients in A, as defined by Adams in [1, p. 220]. In the sequel we will be mainly interested in the cases $A = \mathbb{Z}/p^r$ and $\mathbb{Z}_{\hat{p}}$. Apart their relevance to our problem, these theories enjoy the following property ([17]). Assume that $A = \mathbb{Z}/p^r$ or $\mathbb{Z}_{\hat{p}}$. If Y is a CW-complex and $\{Y_\alpha\}$ the family of its finite subcomplexes, then

$$K^*(Y;A) \cong \lim_{\leftarrow} K^*(Y_\alpha;A).$$

In other words, there are no phantom maps for these K-theories. From this result, it is straightforward to check that

$$K^*(-;\mathbb{Z}_{\hat{p}}) \cong \lim_{\leftarrow} K^*(-;\mathbb{Z}/p^r).$$

We also observe that $K^0(-;\mathbb{Z}_{\hat{p}})$ is represented by $\mathbb{Z}_{\hat{p}} \times BU_{\hat{p}}$, while $K^1(-;\mathbb{Z}_{\hat{p}})$ is represented by $U_{\hat{p}}$; as usual U stands for the infinite unitary group.

The K-theory of 1-connected mod p finite H-spaces has been computed by Kane and Lin (see [18, §44-1]). For the p-compact groups, their result implies

Theorem 1.2. *Let X be a 1-connected p-compact group.*

1. *The K-theory of X is an exterior algebra*

$$K^*(X;\mathbb{Z}_{\hat{p}}) \cong E_{\mathbb{Z}_{\hat{p}}}(\eta_1,\ldots,\eta_r),$$

where the generators $\eta_1,\ldots,\eta_r$ are in $K^1(X;\mathbb{Z}_{\hat{p}})$.

2. *The K-theory of BX is a power series ring*

$$K^*(BX;\mathbb{Z}_{\hat{p}}) \cong \mathbb{Z}_{\hat{p}}[[\xi_1,\ldots,\xi_r]],$$

where the generators $\xi_1,\ldots,\xi_r$ are in $K^0(BX;\mathbb{Z}_{\hat{p}})$.

Proof. By Theorem 1.1. above and Theorem 3.1 in [7], there is a connected CW-complex Y of finite type with $BX \simeq Y_{\hat{p}}$. Since BX is 2-connected, the construction in [7] also shows that Y is 1-connected. Given this fact, Proposition VI.6.5 of [12] implies that $(\Omega Y)_{\hat{p}} \simeq \Omega(Y_{\hat{p}}) \simeq X$. Hence standard arguments show that

$\oplus_{i\geq 0} H_i(\Omega Y;\mathbb{Z}_{(p)})$ is finitely generated over the ring $\mathbb{Z}_{(p)}$ of p-local integers. As ΩY is an associative H-space, we can use Corollary 10.4 of [10] to show that $K^*(X;\mathbb{Z}_{\hat{p}}) \cong K^*(\Omega Y;\mathbb{Z}_{\hat{p}})$ is an exterior algebra over $\mathbb{Z}_{\hat{p}}$. The second assertion follows from a Rothenberg–Steenrod spectral sequence argument and the fact that there are no phantoms in p-adic K-theory. □

We write $H^{**}_{\mathbb{Q}_{\hat{p}}}(-)$ for the direct product $\prod_{n\geq 0} H^n_{\mathbb{Q}_{\hat{p}}}(-)$; it is a $\mathbb{Z}/2$-graded $\mathbb{Q}_{\hat{p}}$-algebra, graded by:

$$H^{\text{even}}_{\mathbb{Q}_{\hat{p}}}(-) = \prod_{n\geq 0} H^{2n}_{\mathbb{Q}_{\hat{p}}}(-), \qquad H^{\text{odd}}_{\mathbb{Q}_{\hat{p}}}(-) = \prod_{n\geq 0} H^{2n+1}_{\mathbb{Q}_{\hat{p}}}(-).$$

The Chern character

$$\text{ch}\colon K^*(-;\mathbb{Z}_{\hat{p}}) \longrightarrow H^{**}_{\mathbb{Q}_{\hat{p}}}(-)$$

is a $\mathbb{Z}/2$-graded ring homomorphism whose definition is as in [19, p. 282].

Proposition 1.3. *Let X be a 1-connected p-compact group.*

1. *The rationalization of the Chern character*

$$\text{ch} \bigotimes \mathbb{Q}_{\hat{p}}\colon K^*(X;\mathbb{Z}_{\hat{p}}) \bigotimes \mathbb{Q}_{\hat{p}} \longrightarrow H^{**}_{\mathbb{Q}_{\hat{p}}}(X)$$

is an isomorphism.

2. *The Chern character*

$$\text{ch}\colon K^*(BX;\mathbb{Z}_{\hat{p}}) \longrightarrow H^{**}_{\mathbb{Q}_{\hat{p}}}(BX)$$

is a monomorphism.

Note. The rationalization of the Chern character for BX is not necessarily onto. This is due to the non-surjectivity of the inclusion $\mathbb{Z}_{\hat{p}}[[\xi]] \otimes \mathbb{Q} \hookrightarrow \mathbb{Q}_{\hat{p}}[[\xi]]$.

Proof. Let us first consider the case of the space X. Proceeding as above, there is a 1-connected CW-complex of finite type V whose p-completion is homotopy equivalent to X. Take any finite subcomplex $\widetilde{V}$ of V such that $\widetilde{V} \hookrightarrow V$ induces an isomorphism in cohomology with $\mathbb{Z}_{\hat{p}}$-coefficients and observe that the composition

$$\widetilde{V} \hookrightarrow V \to V_{\hat{p}} \simeq X$$

induces an isomorphism in p-adic K-theory. The problem is now reduced to the case of a finite complex where the assertion is true (see [4]).

To treat the case of BX, let us fix the exterior generators $\eta_1, \ldots, \eta_r$ of $K^*(X;\mathbb{Z}_{\hat{p}})$. By the first part of the proof, we have $H^*_{\mathbb{Q}_{\hat{p}}}(X) \cong E_{\mathbb{Q}_{\hat{p}}}(y_1, \ldots, y_r)$, where $y_i = \mathrm{ch}(\eta_i)$ for $i = 1, \dots, r$. The map $\alpha\colon \Sigma X \longrightarrow BX$, adjoint to the homotopy equivalence $e\colon X \longrightarrow \Omega BX$, induces a commutative diagram

$$\begin{array}{ccc} K^*(BX;\mathbb{Z}_{\hat{p}}) & \xrightarrow{\alpha^*} & K^*(\Sigma X;\mathbb{Z}_{\hat{p}}) \\ {\scriptstyle\mathrm{ch}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{ch}} \\ H^{**}_{\mathbb{Q}_{\hat{p}}}(BX) & \xrightarrow{\alpha^{**}} & H^{**}_{\mathbb{Q}_{\hat{p}}}(\Sigma X). \end{array}$$

Now observe that the generators $\xi_1, \dots, \xi_r$ of $K^*(BX;\mathbb{Z}_{\hat{p}})$ may be chosen so that $\alpha^*(\xi_i) = \sigma(\eta_i)$, with σ the suspension isomorphism. Similarly, it is possible to choose generators $x_1, \dots, x_r$ of $H^{**}_{\mathbb{Q}_{\hat{p}}}(BX)$ such that $\alpha^{**}(x_i) = \sigma(y_i)$. The commutative diagram above implies that $\mathrm{ch}(\xi_i)$ is congruent, modulo decomposables, to x_i. We now invoke Theorem 1.2 to conclude the proof. □

2. A finiteness result

Let R and S be graded commutative rings; a ring homomorphism $\phi\colon R \longrightarrow S$ is called *finite* if S is a finitely generated $\phi(R)$-module. The present section is devoted to the proof of

Theorem 2.1. *Let X be a p-compact group and $i\colon T \to X$ a maximal torus. Then the map Bi induces a finite ring homomorphism*

$$Bi^*\colon K^*(BX;\mathbb{F}_p) \longrightarrow K^*(BT;\mathbb{F}_p).$$

This result will follow from a sequence of five propositions. The principal ingredients are the "main theorem" of [15] and Dwyer's transfer ([14]).

Proposition 2.2. *If X is a p-compact group, then there are homogeneous classes $b_1, \dots, b_r \in H^*(BX;\mathbb{F}_p)$ which generate a polynomial subalgebra $R = \mathbb{F}_p[b_1, \dots, b_r]$ in $H^*(BX;\mathbb{F}_p)$ and such that the inclusion $R \subset H^*(BX;\mathbb{F}_p)$ is finite.*

Proof. By Theorem 2.4 of [15] (alias the "main theorem"), $H^*(BX;\mathbb{F}_p)$ is a finitely generated graded algebra over $\mathbb{F}_p$. It suffices then to invoke the graded version of Noether normalization theorem (as stated for example in [8, Theorem 2.2.7]). □

Before going farther, a well-known observation is in order. Suppose that there exists a monomorphism $h\colon X \to U(n)_{\hat{p}}$, where $U(n)_{\hat{p}}$ denotes the p-compact group obtained by p-completing the unitary group $U(n)$. Let $j\colon T(n) \longrightarrow U(n)_{\hat{p}}$ be a maximal torus. By [15, Prop. 8.11], there exists a monomorphism $g\colon T \longrightarrow T(n)$ making the following diagram commutative:

$$\begin{array}{ccccc} & & & & BT(n) \\ & & & {}^{Bg}\nearrow & \downarrow{\scriptstyle Bj} \\ BT & \xrightarrow{Bi} & BX & \xrightarrow{Bh} & BU(n)_{\hat{p}} \end{array}$$

The map Bg induces a surjective (hence finite!) homomorphism in mod p K-theory. Using the Atiyah–Hirzebruch spectral sequence, one shows that $Bj^*\colon K^*(BU(n)_{\hat{p}};\mathbb{F}_p) \longrightarrow K^*(BT(n);\mathbb{F}_p)$ is also finite. It follows that the morphism $Bi^*\colon K^*(BX;\mathbb{F}_p) \longrightarrow K^*(BT;\mathbb{F}_p)$ is finite.

Unfortunately we do not know yet if every p-compact group admit a monomorphism into some $U(n)_{\hat{p}}$. But for the present purpose, the following proposition will be sufficient.

Proposition 2.3. *Let P be a p-compact toral group. Then there exists a monomorphism $\phi\colon P \longrightarrow U(n)_{\hat{p}}$ for some integer n.*

Proof. If P is finite the answer is well-known; thus we may assume that P fits into a short exact sequence $T \longrightarrow P \longrightarrow \pi$, where T is a p-compact torus of rank $r \geq 1$ and π a finite p-group. The discrete approximation of this sequence gives rise to a short exact sequence of discrete groups (see Proposition 3.7 in [16])

$$\{1\} \longrightarrow \breve{T} \longrightarrow \breve{P} \longrightarrow \pi \longrightarrow \{1\},$$

with $\breve{T} = (\mathbb{Z}/p^\infty)^r$. We write l for the order of π and embed the latter into the symmetric group Σ_l, via the regular representation. By a theorem of Kaloujnine and Krasner (Theorem 7.37 in [29]), there exists an embedding of $\breve{P}$ into the wreath product $\breve{T} \wr \Sigma_l \cong (\mathbb{Z}/p^\infty \wr \Sigma_l)^r$. Our candidate for the monomorphism ϕ is the p-completion of the composite

$$\breve{P} \overset{\alpha}{\hookrightarrow} (\mathbb{Z}/p^\infty \wr \Sigma_l)^r \overset{\beta}{\hookrightarrow} (\mathbb{S}^1 \wr \Sigma_l)^r \overset{\gamma}{\hookrightarrow} U(l)^r \subset U(lr).$$

The morphism γ is just the inclusion of the normalizer of the standard maximal torus of $U(l)^r$. By a result of Quillen (see [28]), $B\gamma$ and the classifying map of the last inclusion induce finite homomorphisms on mod p cohomology. The morphism β is given by the natural inclusion $\mathbb{Z}/p^\infty \subset \mathbb{S}^1$; therefore $B\beta$ induces an isomorphism on mod p cohomology. By Proposition 9.11 of [15], we will be done if we can prove that the composite

$$B(\beta \circ \alpha)\colon B\breve{P} \longrightarrow B(\mathbb{S}^1 \wr \Sigma_l)^r$$

induces a finite morphism on mod p cohomology.

In the proof of Proposition 12.1 of [15], it is shown that there exists a finite subgroup inclusion $\nu\colon P_m \hookrightarrow \breve{P}$ with $B\nu^*\colon H^*(B\breve{P};\mathbb{F}_p) \longrightarrow H^*(BP_m;\mathbb{F}_p)$ injective and finite. Consider now the diagram

$$\begin{array}{ccc} P_m & & \\ {\scriptstyle \nu}\big\downarrow & \searrow^{\beta\circ\alpha\circ\nu} & \\ \breve{P} & \xhookrightarrow[\beta\circ\alpha]{} & (\mathbb{S}^1 \wr \Sigma_l)^r \end{array}$$

The ring homomorphism $B(\beta\circ\alpha\circ\nu)^*$ is finite, by Quillen's result ([28]). Since $H^*(B(\mathbb{S}^1 \wr \Sigma_l)^r;\mathbb{F}_p)$ is noetherian and $B\nu^*$ injective, the ring homomorphism $B(\beta\circ\alpha)^*$ is finite; it follows that $B\phi^*$ is also finite. □

Proposition 2.4. *Let X be a p-compact group and $i\colon T \longrightarrow X$ a maximal torus. If the Atiyah–Hirzebruch spectral sequence*

$$E_2^{*,*}(BX) = H^*(BX; K^*(pt;\mathbb{F}_p)) \Longrightarrow K^*(BX;\mathbb{F}_p)$$

degenerates (i.e., there exists an integer n_0 such that $E_{n_0}(BX) \cong E_\infty(BX)$), then the ring homomorphism $Bi^*\colon K^*(BX;\mathbb{F}_p) \longrightarrow K^*(BT;\mathbb{F}_p)$ is finite.

Proof. We write $Z_n(BX) \subset E_2(BX)$ for the pre-image of the submodule of cycles in $E_n(BX)$. We clearly have a chain of inclusions

$$E_2(BX) \supset Z_2(BX) \supset \cdots \supset Z_n(BX) \supset \cdots \supset Z_\infty(BX) = \bigcap_{n\geq 2} Z_n(BX).$$

For $n \geq 2$, set

$$R_n = \mathbb{F}_p[b_1^{p^{n-1}}, \ldots, b_r^{p^{n-1}}] \otimes K^*(pt;\mathbb{F}_p) \subset E_2(BX),$$

where $b_1, \ldots, b_r$ are as in Proposition 2.2. For all $n \geq 2$, the inclusion $R_n \subset E_2(BX)$ is finite and (by construction) R_n is contained in $Z_n(BX)$. By hypothesis there is an integer n_0 such that $Z_{n_0}(BX) = Z_\infty(BX)$, thus $R_{n_0} \subset Z_\infty(BX)$. Since $i\colon T \longrightarrow X$ is a monomorphism, Bi induces a finite ring homomorphism in mod p cohomology; it follows that the composite

$$R_{n_0} \hookrightarrow E_2(BX) \xrightarrow{Bi^*} E_2(BT)$$

is also finite. At the E_∞-level, we obtain a diagram

$$R_{n_0} \longrightarrow E_\infty(BX) \xrightarrow{E_\infty(Bi^*)} E_\infty(BT),$$

whose composite coincides with the preceding one (since $E_2(BT) = E_\infty(BT)$). In particular the morphism $R_{n_0} \longrightarrow E_\infty(BT)$ is finite; this imply the finiteness of the map $E_\infty(Bi^*)\colon E_\infty(BX) \longrightarrow E_\infty(BT)$. We now invoke Corollary 1 (p. 41) of [9] to conclude that $Bi^*\colon K^*(BX;\mathbb{F}_p) \longrightarrow K^*(BT;\mathbb{F}_p)$ is finite as claimed. □

Proposition 2.5. *Let X be a p-compact group and $i\colon T \longrightarrow X$ a maximal torus. Let N_p denote the p-normalizer of this maximal torus. If the Atiyah–Hirzebruch spectral sequence degenerates for BN_p, then it also degenerates for BX.*

Proof. Let $j\colon N_p \longrightarrow X$ be the canonical monomorphism and

$$\tau\colon (BX_+)_{\hat{p}} \longrightarrow ((BN_p)_+)_{\hat{p}}$$

be the transfer associated to the map Bj (see [14, Example 1.1]). Here the notation $(Y_+)_{\hat{p}}$ stands for the p-completion of the suspension spectrum of the space $Y_+ = Y \coprod pt$. We consider the diagram

$$\begin{array}{ccccc}
E_2(BX) & \supset Z_2(BX) & \supset \cdots \supset Z_n(BX) \supset \dots & & \supset Z_\infty(BX) \\
{\scriptstyle \tau^*}\big\uparrow \; \big\downarrow{\scriptstyle Bj^*} & & & & \\
E_2(BN_p) & \supset Z_2(BN_p) & \supset \cdots \supset Z_n(BN_p) \supset \dots & & \supset Z_\infty(BN_p)
\end{array}$$

Since τ^* is induced by a stable map, we have $\tau^*(Z_n(BN_p)) \subset Z_n(BX)$ for all $n \geq 2$. By hypothesis there exists an integer n_0 such that $Z_{n_0}(BN_p) = Z_\infty(BN_p)$. Let $x \in Z_{n_0}(BX)$, then $Bj^*(x) \in Z_{n_0}(BN_p) = Z_\infty(BN_p)$ and

$$x = \chi^{-1}(\tau^* \circ Bj^*)(x) \in Z_\infty(BX),$$

where χ is the Euler characteristic of the space X/N_p. Dwyer and Wilkerson have shown that χ is invertible mod p (see the proof of 2.4 (p. 431) in [15]). Thus we have proved that $Z_{n_0}(BX) = Z_\infty(BX)$. □

Proposition 2.6. *For any p-compact toral group P, the Atiyah–Hirzebruch spectral sequence for BP degenerates.*

Proof. By Proposition 2.3 there exists a monomorphism $\phi\colon P \longrightarrow U(n)_{\hat{p}}$. Hence the induced homomorphism $B\phi^*\colon H^*(BU(n);\mathbb{F}_p) \longrightarrow H^*(BP;\mathbb{F}_p)$ is finite. Since $H^*(BU(n);\mathbb{F}_p)$ is concentrated in even degrees, the naturality of the spectral sequence implies that $R = \mathrm{Im}(B\phi^*) \otimes K^*(pt;\mathbb{F}_p)$ consists of permanent cycles. We can now invoke Proposition 4.1 of [6] to conclude. □

To complete the proof of Theorem 2.1, it suffices to recall that the p-normalizer N_p is p-toral. We close the section by our main application of this theorem.

Theorem 2.7. *Let X be a 1-connected p-compact group and $i\colon T \longrightarrow X$ a maximal torus. The ring homomorphism*

$$Bi^*\colon K^*(BX;\mathbb{Z}_{\hat{p}}) \longrightarrow K^*(BT;\mathbb{Z}_{\hat{p}})$$

makes $K^*(BT;\mathbb{Z}_{\hat{p}})$ into a free and finitely generated $K^*(BX;\mathbb{Z}_{\hat{p}})$-module.

Proof. For simplicity, we set

$$S_X := K^*(BX;\mathbb{Z}_{\hat{p}}) \quad \text{and} \quad S_T := K^*(BT;\mathbb{Z}_{\hat{p}}).$$

It is well-known that S_T is a power series ring over $Z_{\hat{p}}$. By Theorem 1.2, the same is true for S_X. Consequently we have the universal coefficient formulas (see [1, p. 201])

$$K^*(BX;\mathbb{F}_p) = S_X \otimes_{\mathbb{Z}_{\hat{p}}} \mathbb{F}_p \quad \text{and} \quad K^*(BT;\mathbb{F}_p) = S_T \otimes_{\mathbb{Z}_{\hat{p}}} \mathbb{F}_p.$$

Our Theorem 2.1 and Theorem 8.4 (p. 58) of [21], imply that the homomorphism $Bi^*\colon S_X \longrightarrow S_T$ is finite. We now observe that both rings are noetherian, local, regular and of the same dimension. And so we can apply Proposition 22 (p. IV-37) of [30] to conclude that S_T is free over S_X. □

3. Reduction to the one-connected case

The following results of [24] and [23] will play an important role in this section.

Theorem 3.1. *Let X be a connected p-compact group and π the torsion subgroup of $\pi_1(X)$. Set $Y = X\langle 1\rangle \times S$, where $X\langle 1\rangle$ denotes the 1-connected cover of X and S a p-compact torus of rank $\dim_{\mathbb{Q}_{\hat{p}}}(\pi_1(X) \otimes \mathbb{Q}_{\hat{p}})$.*

1. *There is a short exact sequence of p-compact groups*

$$\{1\} \longrightarrow \pi \xrightarrow{f} Y \xrightarrow{g} X \longrightarrow \{1\},$$

where the monomorphism $f\colon \pi \to Y$ is central.

2. *Let $i_1\colon T_1 \longrightarrow X\langle 1\rangle$ and $i\colon T_X \longrightarrow X$ be maximal tori for the respective p-compact groups. Set also $T_Y = T_1 \times S$, then*

$$j = i_1 \times id\colon T_Y \longrightarrow Y$$

is a maximal torus and there exists a short exact sequence

$$\{1\} \longrightarrow \pi \xrightarrow{\varphi} T_Y \xrightarrow{\gamma} T_X \longrightarrow \{1\}$$

which makes the following diagram commutative (up to homotopy):

$$\begin{array}{ccc} B\pi & = & B\pi \\ {\scriptstyle B\varphi}\downarrow & & \downarrow{\scriptstyle Bf} \\ BT_Y & \xrightarrow{Bj} & BY \\ {\scriptstyle B\gamma}\downarrow & & \downarrow{\scriptstyle Bg} \\ BT_X & \xrightarrow{Bi} & BX. \end{array}$$

3. *Let W_Y (respectively W_X) denote the Weyl group of the maximal torus j (respectively i); note that, by construction, the group W_Y acts trivially on BS. There exists an isomorphism $\Phi\colon W_Y \to W_X$ such that, for any w in the Weyl group W_Y, the diagram*

$$\begin{array}{ccc} BT_Y & \xrightarrow{w} & BT_Y \\ {\scriptstyle B\gamma}\downarrow & & \downarrow{\scriptstyle B\gamma} \\ BT_X & \xrightarrow{\Phi(w)} & BT_X \end{array}$$

commutes (up to homotopy).

Proof. The first part is proved in [24, Theorem 5.4], while the last two are contained in [23, Theorem 2.5]. □

In order to exploit fully the theorem above, we must study some properties of central monomorphisms. The main tool of this study is an observation of Dwyer and Wilkerson ([16, Lemma 5.3]):

Lemma 3.2. *Let*

$$\begin{array}{ccc} & & Z_1 \\ & & \downarrow{\scriptstyle f_1} \\ Z_2 & \xrightarrow{f_2} & Z_0 \end{array}$$

be a diagram of p-compact groups and suppose that the morphism f_2 is central. Then there is up to conjugacy a unique morphism *of p-compact groups $\mu(f_1, f_2)\colon Z_1 \times Z_2 \to Z_2$ which is conjugate to f_1 on $Z_1 \times \{1\}$ and to f_2 on $\{1\} \times Z_2$.*

Until further notice we will keep the data and notation of Theorem 3.1. The classifying space $B\pi$ possesses a multiplication

$$B\mu\colon B\pi \times B\pi \to B\pi;$$

the unit for $B\mu$ will be chosen to be the basepoint of $B\pi$. Let R denote either a ring $\mathbb{Z}/p^r$ $(r \geq 1)$ or the p-adic ring $\mathbb{Z}_{\hat{p}}$. By a well-known result of Atiyah (see [3]), $K^*(B\pi; R)$ is free and finitely generated over R. This fact implies for any p-compact group Z, that $K^*(B\pi \times BZ; R)$ is naturally isomorphic to $K^*(B\pi; R) \otimes_R K^*(BZ; R)$ (see [3, Lemma 1.4]). Thus the multiplication of $B\pi$ induces a Hopf algebra structure on $K^*(B\pi; R)$.

By Lemma 3.2 there is up to conjugacy a unique morphism $k\colon \pi \times Y \to Y$ which is conjugate to f on $\pi \times \{1\}$ and to the identity on $\{1\} \times Y$. It is easily checked that the homomorphism

$$Bk^*\colon K^*(BY; R) \longrightarrow K^*(B\pi; R) \otimes_R K^*(BY; R)$$

defines a $K^*(B\pi; R)$-comodule structure on $K^*(BY; R)$.

Let us now consider the maximal torus $j\colon T_Y \to Y$, with Weyl group W_Y. By Lemma 6.5 of [16], φ is up to conjugacy the unique morphism making the following diagram commutative

$$\begin{array}{ccc} & & BT_Y \\ & \overset{B\varphi}{\nearrow} & \big\downarrow {\scriptstyle Bj} \\ B\pi & \xrightarrow{Bf} & BY \end{array}$$

This uniqueness implies that the morphis φ is W_Y-invariant, i.e., for any w in the Weyl group W_Y, the morphisms φ and $w \circ \varphi$ are conjugate.

We proceed as above to obtain a morphism $\kappa\colon \pi \times T_Y \to T_Y$ (unique up to conjugacy) such that

$$B\kappa^*\colon K^*(BT_Y; R) \longrightarrow K^*(B\pi; R) \otimes_R K^*(BT_Y; R)$$

defines a $K^*(B\pi; R)$-comodule structure on $K^*(BT_Y; R)$. The uniqueness of κ and the W_Y-invariance of φ imply that, for all $w \in W_Y$, the induced map $w^*\colon K^*(BT_Y; R) \longrightarrow K^*(BT_Y; R)$ is a $K^*(B\pi; R)$-comodule homomorphism. Once again Lemma 3.2 implies that $Bj^*\colon K^*(BY; R) \longrightarrow K^*(BT_Y; R)$ is a $K^*(B\pi; R)$-comodule homomorphism.

We write A for $K^*(B\pi; \mathbb{Z}_{\hat{p}})$ and A_r for $K^*(B\pi; \mathbb{Z}/p^r)$. The universal coefficient formula implies $K^*(B\pi; \mathbb{Z}/p^r) \cong K^*(B\pi; \mathbb{Z}_{\hat{p}}) \bigotimes \mathbb{Z}/p^r$, that is, $A_r \cong A \bigotimes \mathbb{Z}/p^r$.

Since BX is 1-connected, we may and we will assume that the fibration $B\pi \longrightarrow BY \longrightarrow BX$ is principal. This allows us to use the Rothenberg–Steenrod spectral sequence for this fibration. Its E_2-term is given by

$$E_2^{*,*} = \mathrm{Cotor}^*_{A_r}(K^*(BY; \mathbb{Z}/p^r); \mathbb{Z}/p^r).$$

Recall that $\mathrm{Cotor}^n_{A_r}(-;-)$ is the n-th derived functor of the cotensor product $-\square_{A_r}-$ (we refer to [3] for more details).

The next proposition is due to Anderson and Hodgkin ([3, Prop. 3.5]). It is the main ingredient for the study of this spectral sequence.

Proposition 3.3. *For all A_r-comodules B and all $n > 0$,*

$$\operatorname{Cotor}^n_{A_r}(B; \mathbb{Z}/p^r) = 0.$$

The problem with the above fibration is that $K^*(BY; \mathbb{Z}/p^r)$ is not a finite $\mathbb{Z}/p^r$-module, hence the spectral sequence may not converge. This difficulty can be solved by replacing BX by the p-completion $Z_{\hat{p}}$ of a CW-complex of finite type Z ([7]). For the m-th skeleton $Z^{(m)}$ of Z, consider the induced principal fibration

$$\begin{array}{ccc} B\pi & \hookrightarrow & p^{-1}(Z^{(m)}) \\ & & \downarrow p \\ & & Z^{(m)} \end{array}$$

By Proposition 3.3, the Rothenberg–Steenrod spectral sequence of this fibration collapses. It follows that

$$\begin{aligned} K^*(BX; \mathbb{Z}/p^r) &\cong \varprojlim K^*(Z^{(m)}; \mathbb{Z}/p^r) \\ &\cong \varprojlim K^*(p^{-1}(Z^{(m)}); \mathbb{Z}/p^r) \square_{A_r} \mathbb{Z}/p^r \\ &\cong K^*(BY; \mathbb{Z}/p^r) \square_{A_r} \mathbb{Z}/p^r. \end{aligned}$$

Note that the third isomorphism is due to the fact that A_r is a free $\mathbb{Z}/p^r$-module.

Theorem 3.4. *Let X be a connected p-compact group, π the torsion subgroup of $\pi_1(X)$ and $A = K^*(B\pi; \mathbb{Z}_{\hat{p}})$. Let $B\pi \xrightarrow{Bf} By \xrightarrow{Bg} BX$ be the fibration of Theorem 3.1. The map Bg induces an isomorphism*

$$Bg^*\colon K^*(BX; \mathbb{Z}_{\hat{p}}) \xrightarrow{\cong} K^*(BY; \mathbb{Z}_{\hat{p}}) \square_A \mathbb{Z}_{\hat{p}}. \tag{1}$$

Note. The isomorphism (1) is equivalent to the exactness of the sequence

$$0 \longrightarrow K^*(BX; \mathbb{Z}_{\hat{p}}) \xrightarrow{Bg^*} K^*(BY; \mathbb{Z}_{\hat{p}}) \xrightarrow{\Psi} K^*(BY; \mathbb{Z}_{\hat{p}}) \otimes A,$$

where Ψ is the definig morphism for the cotensor product.

Proof. We give only the main steps. To begin with, recall that $A \cong \varprojlim A_r$ and $K^*(BY; \mathbb{Z}_{\hat{p}}) \cong \varprojlim K^*(BY; \mathbb{Z}/p^r)$. Moreover the modules A and A_r are free over their respective ground rings. Apply the left exact functor $\varprojlim(-)$ to the inverse system of exact sequences

$$0 \longrightarrow K^*(BX;\mathbb{Z}/p^r) \xrightarrow{Bg^*} K^*(BY;\mathbb{Z}/p^r) \xrightarrow{\Psi} K^*(BY;\mathbb{Z}/p^r) \otimes A_r$$

to conclude. □

Our next step is to investigate the relationship between the K-theory of BX and the K-theory of its maximal torus BT_X. Of course Theorem 3.4 applies to the fibration $B\pi \xrightarrow{B\varphi} BT_Y \xrightarrow{B\gamma} BT_X$. By naturality we obtain, for any w in the Weyl group W_X, the commutative diagram (with exact nows):

$$\begin{array}{ccccc} 0 \to K^*(BT_X;\mathbb{Z}_{\hat{p}}) & \xrightarrow{B\gamma^*} & K^*(BT_Y;\mathbb{Z}_{\hat{p}}) & \xrightarrow{\psi} & K^*(BT_Y;\mathbb{Z}_{\hat{p}}) \otimes A \\ \downarrow{\scriptstyle w^*} & & \downarrow{\scriptstyle \Phi(w)^*} & & \downarrow{\scriptstyle \Phi(w)^*\otimes id} \\ 0 \to K^*(BT_X;\mathbb{Z}_{\hat{p}}) & \xrightarrow{B\gamma^*} & K^*(BT_Y;\mathbb{Z}_{\hat{p}}) & \xrightarrow{\psi} & K^*(BT_Y;\mathbb{Z}_{\hat{p}}) \otimes A. \end{array}$$

It follows that the projection $B\gamma\colon BT_Y \longrightarrow BT_X$ induces an isomorphism

$$K^*(BT_X;\mathbb{Z}_{\hat{p}})^{W_X} \cong K^*(BT_Y;\mathbb{Z}_{\hat{p}})^{W_Y} \Box_A \mathbb{Z}_{\hat{p}}. \tag{2}$$

The isomorphisms (1) and (2) fit together in the commutative diagram:

$$\begin{array}{ccc} K^*(BX;\mathbb{Z}_{\hat{p}}) & \cong & K^*(BY;\mathbb{Z}_{\hat{p}})\Box_A \mathbb{Z}_{\hat{p}} \\ \downarrow{\scriptstyle Bi^*} & & \downarrow{\scriptstyle Bj^*} \\ K^*(BT_X;\mathbb{Z}_{\hat{p}})^{W_X} & \cong & K^*(BT_Y;\mathbb{Z}_{\hat{p}})^{W_Y}\Box_A \mathbb{Z}_{\hat{p}} \end{array}$$

The theorem we want to prove asserts that the ring homomorphism $Bi^*\colon K^*(BX;\mathbb{Z}_{\hat{p}}) \longrightarrow K^*(BT_X;\mathbb{Z}_{\hat{p}})^{W_X}$ is an isomorphism. Obviously this will be achieved if we can prove that Bj^* is an isomorphism onto the ring of invariants $K^*(BT_Y;\mathbb{Z}_{\hat{p}})^{W_Y}$. We recall here that

$$BY = BX\langle 1\rangle \times BS, \quad BT_Y = BT_1 \times BS \quad \text{and} \quad Bj = Bi_1 \times id.$$

As the Weyl group W_Y acts trivially on S, it suffices to show that Bi_1 induces an isomorphism onto the ring of invariants. In other words it is enough to deal with the 1-connected case. This is settled in the next section.

4. The one-connected case

In this section X is a 1-connected p-compact group, $i\colon T \longrightarrow X$ a maximal torus and $j\colon N \longrightarrow X$ the normalizer of i. Thus we have a commutative diagram

$$\begin{array}{ccc} BT & & \\ {\scriptstyle Bh}\downarrow & \searrow{\scriptstyle Bi} & \\ BN & \xrightarrow[Bj]{} & BX \end{array}$$

Let $\tau\colon (BX_+)_{\hat{p}} \longrightarrow (BN_+)_{\hat{p}}$ be the transfer associated to the map Bj ([14, Example 1.1]). Since the Euler characteristic of X/N is equal to 1 (this is a consequence of Proposition 9.5 in [15]), the composite

$$(BX_+)_{\hat{p}} \xrightarrow{\tau} (BN_+)_{\hat{p}} \xrightarrow{(Bj)_{\hat{p}}} (BX_+)_{\hat{p}}$$

is a homotopy equivalence; its inverse will be denoted ψ. We introduce further notations:

i) $\tilde{\tau}$ is the composite

$$(BX_+)_{\hat{p}} \xrightarrow{\psi} (BX_+)_{\hat{p}} \xrightarrow{\tau} (BN_+)_{\hat{p}};$$

ii) F is defined by the cofibre sequence

$$(BX_+)_{\hat{p}} \xrightarrow{\tilde{\tau}} (BN_+)_{\hat{p}} \xrightarrow{\nu} F;$$

iii) θ denotes the composite

$$(BN_+)_{\hat{p}} \xrightarrow{\nabla} (BN_+)_{\hat{p}} \vee (BN_+)_{\hat{p}} \xrightarrow{\nu\vee(Bj)_{\hat{p}}} F \vee (BX_+)_{\hat{p}}.$$

Proposition 4.1.

1 . *For $E^*(-) = H^*(-;\mathbb{Z}_{\hat{p}})$ or $K^*(-;\mathbb{Z}_{\hat{p}})$, the map*

$$E^*(\theta)\colon E^*(F)\bigoplus E^*(BX_+) \longrightarrow E^*(BN_+), \qquad (y,z) \mapsto \nu^*(y) + (Bj)^*(z)$$

is an isomorphism.

2 . *For any $y \in K^*(F;\mathbb{Z}_{\hat{p}})$, $(Bh^* \circ \nu^*)(y) = 0$.*

Proof. Let us first notice that $E^*(BX_+) \cong E^*((BX_+)_{\hat{p}})$, and similarly for BN_+. By the definition of $\tilde{\tau}$, we have

$$(Bj)_{\hat{p}} \circ \tilde{\tau} \sim id_{(BX_+)_{\hat{p}}}. \qquad (3)$$

Granted this fact, the long exact sequence in $E^*(-)$ of the cofibration

$$(BX_+)_{\hat{p}} \xrightarrow{\tilde{\tau}} (BN_+)_{\hat{p}} \xrightarrow{\nu} F$$

yields an inverse for $E^*(\theta)$.

By the second part of Theorem 11, $Bj^*\colon H^*_{\mathbb{Q}_{\hat{p}}}(BX) \longrightarrow H^*_{\mathbb{Q}_{\hat{p}}}(BN)$ is an isomorphism; hence (by what we have just proved), $\widetilde{H}^*_{\mathbb{Q}_{\hat{p}}}(F) = 0$. The second claim follows from a diagram chase in

$$\begin{array}{ccccc} K^*(F;\mathbb{Z}_{\hat{p}}) \oplus K^*(BX;\mathbb{Z}_{\hat{p}}) & \xrightarrow{\cong} & K^*(BN;\mathbb{Z}_{\hat{p}}) & \xrightarrow{Bh^*} & K^*(BT;\mathbb{Z}_{\hat{p}}) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{**}_{\mathbb{Q}_{\hat{p}}}(BX) & \xrightarrow{\cong} & H^{**}_{\mathbb{Q}_{\hat{p}}}(BN) & \xrightarrow{Bh^{**}} & H^{**}_{\mathbb{Q}_{\hat{p}}}(BT) \end{array}$$

□

Proposition 4.2. *The maps $\tilde{\tau} \circ Bi$ and Bh induce the same homomorphism in p-adic K-theory, hence*

$$\mathrm{Im}(Bi^*) = \mathrm{Im}(Bh^*) \subset K^*(BT;\mathbb{Z}_{\hat{p}}).$$

Proof. Let $x \in \widetilde{K}^*(BN;\mathbb{Z}_{\hat{p}})$. By Proposition 4.1, there exist $y \in \widetilde{K}^*(F;\mathbb{Z}_{\hat{p}})$ and $z \in \widetilde{K}^*(BX;\mathbb{Z}_{\hat{p}})$ such that $x = \nu^*(y) + Bj^*(z)$. On the one hand, Proposition 4.1 implies

$$\begin{aligned} Bh^*(x) &= Bh^*(\nu^*(y) + Bj^*(z)) \\ &= (Bh^* \circ \nu^*)(y) + (Bj \circ Bh)^*(z) \\ &= 0 + Bi^*(z). \end{aligned}$$

On the other hand, (3) implies

$$\begin{aligned} (Bi^* \circ \tilde{\tau}^*)(x) &= (Bi^* \circ \tilde{\tau}^*)(\nu^*(y) + Bj^*(z)) \\ &= (Bi^* \circ \tilde{\tau}^* \circ \nu^*)(y) + Bi^*((Bj \circ \tilde{\tau})^*)(z) \\ &= 0 + Bi^*(z). \end{aligned}$$

And we have obtained $Bi^* \circ \tilde{\tau}^* = Bh^*$. □

The preceding proposition reduces the determination of the image of Bi^* to that of Bh^*. A partial description of $\mathrm{Im}(Bh^*)$, which is sufficient for our purpose, is provided by

Proposition 4.3. *Let $|W|$ denote the order of the Weyl group $W = \pi_0(N)$, then*

$$\mathrm{Im}(Bh^*) \otimes \mathbb{Z}_{\hat{p}}[1/|W|] = K^*(BT;\mathbb{Z}_{\hat{p}})^W \otimes \mathbb{Z}_{\hat{p}}[1/|W|].$$

Proof. As easily checked, there is (up to homotopy) a fibre square

$$\begin{array}{ccc} BT \times \mathcal{W} & \xrightarrow{p} & BT \\ {\scriptstyle pr_1}\big\downarrow & & \big\downarrow{\scriptstyle Bh} \\ BT & \xrightarrow{Bh} & BN \end{array}$$

where $p(x,\omega) = \omega(x)$ and $pr_1(x,\omega) = x$, for any $x \in BT$ and $\omega \in \mathcal{W} = \mathcal{W}_T(X)$. Let tr be the transfer associated to the map Bh. Using the naturality and the product formulae for the transfer (see Theorem 2.6 and Theorem 2.8 in [14]) and applying p-adic K-theory, we obtain, for any $\xi \in K^*(BT;\mathbb{Z}_{\hat{p}})$:

$$(Bh^* \circ tr^*)(\xi) = \sum_{w \in W} w^*(\xi). \tag{4}$$

The proposition is an immediate consequence of this equation. □

Note. As the reader may have noticed, the equality (4) is the well known double coset formula for finite coverings. If one could prove such a formula for the fibration $Bi\colon BT \longrightarrow BX$, then most of the arguments in this section would drastically simplify.

We are now in position to complete the proof of our main result:

Proposition 4.4. *Let X be a 1-connected p-compact group, $i\colon T \longrightarrow X$ a maximal torus and W the Weyl group. Then the classifying map Bi induces an isomorphism*

$$K^*(BX;\mathbb{Z}_{\hat{p}}) \cong K^*(BT;\mathbb{Z}_{\hat{p}})^W.$$

Proof. The injectivity of Bi^* follows from Proposition 1.3 and the commutativity of the diagram

$$\begin{array}{ccc} K^*(BX;\mathbb{Z}_{\hat{p}}) & \xrightarrow{Bi^*} & K^*(BT;\mathbb{Z}_{\hat{p}}) \\ {\scriptstyle \mathrm{ch}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ch}} \\ H^{**}_{\mathbb{Q}_{\hat{p}}}(BX) & \xhookrightarrow{Bi^{**}} & H^{**}_{\mathbb{Q}_{\hat{p}}}(BT) \end{array}$$

To show that the image of Bi^* is the ring of invariants, we consider the diagram

$$\begin{array}{ccc} K^*(BT;\mathbb{Z}_{\hat{p}})^W & \hookrightarrow & \mathrm{Frac}(K^*(BT;\mathbb{Z}_{\hat{p}})^W) \\ {\scriptstyle Bi^*}\big\uparrow & & \big\uparrow \\ K^*(BX;\mathbb{Z}_{\hat{p}}) & \hookrightarrow & \mathrm{Frac}(K^*(BX;\mathbb{Z}_{\hat{p}})), \end{array}$$

where Frac(−) denotes the field of fractions of the corresponding integral domain. By Propositions 4.2 and 4.3, the right vertical map is an isomorphism. In Theorem 2.7, we showed that $Bi^*\colon K^*(BX;\mathbb{Z}_{\hat{p}}) \hookrightarrow K^*(BT;\mathbb{Z}_{\hat{p}})$ is finite, in particular $K^*(BX;\mathbb{Z}_{\hat{p}}) \hookrightarrow K^*(BT;\mathbb{Z}_{\hat{p}})^W$ is integral. As $K^*(BX;\mathbb{Z}_{\hat{p}})$ is integrally closed (it is a power series ring over $\mathbb{Z}_{\hat{p}}$), we obtain that the left vertical map is also an isomorphism. □

5. Applications to p-compact groups

Having completed the proof of our main result, we can safely turn to some of its consequences. The first one generalizes Proposition 1.2 in [27]:

Theorem 5.1. *Let T be a p-compact torus and X a connected p-compact group. The natural map*

$$[BT, BX] \longrightarrow \mathrm{Hom}_\lambda(K^*(BX;\mathbb{Z}_{\hat{p}}), K^*(BT;\mathbb{Z}_{\hat{p}}))$$

is a bijection, where $\mathrm{Hom}_\lambda(-,-)$ *stands for the set of λ-ring homomorphisms.*

Proof. Let $i_X\colon T_X \longrightarrow X$ be a maximal torus for X and W_X the corresponding Weyl group. It is not difficult to check (or see Proposition 1.2 of [27]) the bijectivity of the natural map

$$[BT, BT_X] \xrightarrow{\cong} \mathrm{Hom}_\lambda(K^*(BT_X;\mathbb{Z}_{\hat{p}}), K^*(BT;\mathbb{Z}_{\hat{p}})).$$

The Weyl group W_X acts on the domain and codomain of this map, and the latter is equivariant with respect to these actions. As easily seen, left composition with Bi_X induces a commutative diagram

$$\begin{array}{ccc} [BT, BT_X]/W_X & \xrightarrow{\cong} & \mathrm{Hom}_\lambda(K^*(BT_X;\mathbb{Z}_{\hat{p}}), K^*(BT;\mathbb{Z}_{\hat{p}}))/W_X \\ \downarrow & & \downarrow \\ [BT, BX] & \longrightarrow & \mathrm{Hom}_\lambda(K^*(BX;\mathbb{Z}_{\hat{p}}), K^*(BT;\mathbb{Z}_{\hat{p}})). \end{array}$$

Proposition 8.11 of [15] and Proposition 4.1 of [25] imply that the left vertical arrow of the diagram is bijective. Next we combine our main result with Theorem 4.1 of [31] to obtain the surjectivity of the right vertical arrow. To show its injectivity, we first observe that a nonzero vector space over an infinite field cannot be a union of a finite number of its proper subspaces (see [20, p. 78]). It follows that we can adapt the proof of [26, Lemma 7.1] to obtain our injectivity result. This completes the proof of our theorem. □

Our second application extends Proposition 1.3 to all connected p-compact groups.

Theorem 5.2. *Let X be a connected p-compact group.*

1. *The Chern character*

$$\mathrm{ch}\colon K^*(BX;\mathbb{Z}_{\hat{p}}) \longrightarrow H^{**}_{\mathbb{Q}_{\hat{p}}}(BX)$$

is a monomorphism.

2. *For all $x \in H^*_{\mathbb{Q}_{\hat{p}}}(BX)$, there exists $\xi \in K^*(BX;\mathbb{Z}_{\hat{p}})$ such that*

$$\mathrm{ch}(\xi) = Mx + \text{higher terms},$$

where M denotes the order of the Weyl group of X.

Proof. Let $i_X\colon T_X \longrightarrow X$ be a maximal torus and W_X the corresponding Weyl group. The naturality of the Chern character and of the action of W_X implies the commutativity of the diagram

$$\begin{array}{ccc} K^*(BX;\mathbb{Z}_{\hat{p}}) & \xrightarrow{Bi_X^*} & K^*(BT;\mathbb{Z}_{\hat{p}})^{W_X} \\ \mathrm{ch}\downarrow & & \downarrow\mathrm{ch} \\ H^{**}_{\mathbb{Q}_{\hat{p}}}(BX) & \xrightarrow{Bi_X^{**}} & H^{**}_{\mathbb{Q}_{\hat{p}}}(BT)^{W_X}. \end{array}$$

To complete the proof, we invoke our main result and the fact that the claims are true for tori. □

The third application will be useful in the sequel:

Proposition 5.3. *Let X be a connected p-compact group and let*

$$E_2^{s,t} = H^s(BX;K^t(pt;\mathbb{Z}_{\hat{p}})) \Longrightarrow K^{s+t}(BX;\mathbb{Z}_{\hat{p}})$$

be the p-adic Atiyah–Hirzebruch spectral sequence for BX. Given s,t, there exists an integer $r = r(s,t)$ such that $E_r^{s,t} = E_\infty^{s,t}$.

Proof. Recall that there are no phantoms in p-adic K-theory. Thus one can adapt the "proof of necessity" of Theorem 3.3 in [13], but their reference to Theorem 3.2 of that paper has to be replaced by our Theorem 5.2. This change in their argument is needed because of the following example: the work of Anderson and Hodgkin ([3]) imply that $\widetilde{K}^*(K(\mathbb{Z},3);\mathbb{Z}_{\hat{p}}) = 0$, but $H^3_{\mathbb{Q}_{\hat{p}}}(K(\mathbb{Z},3)) = \mathbb{Q}_{\hat{p}}$. □

6. Applications to finite loop spaces

In this section we study the K-theory of finite loop spaces; let us first recall the basic definitions.

A loop space $(L, BL; e)$ is said *finite* if $H_*(L;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module. By a well-known result of Hopf, $H^*(L;\mathbb{Q})$ is an exterior algebra whose number of generators is called the *rank* of the finite loop space. A connected finite loop space L admits a *maximal torus* if there is a pointed map $Bi\colon BT \longrightarrow BL$, with $BT = K(\mathbb{Z}^n; 2)$, satisfying the two conditions:

1. The homotopy fiber L/T of Bi is $\mathbb{Z}$-finite, that is, $H_*(L/T;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module.
2. The ranks of T and L are equal.

The *Weyl group* of a maximal torus $i\colon T \longrightarrow L$ is defined exactly as in Section 1. The relationship between finite loop spaces and p-compact groups has been investigated by Møller and Notbohm:

Proposition 6.1 ([25]). *Let L be a connected finite loop space with maximal torus $i\colon T \longrightarrow L$. Then for any prime p:*

1. *The triple $(L_{\hat{p}}, BL_{\hat{p}}, e_{\hat{p}})$ is a p-compact group and the p-completed map $i_{\hat{p}}\colon T_{\hat{p}} \longrightarrow L_{\hat{p}}$ is a maximal torus.*
2. *Let W (respectively, $W_{\hat{p}}$) be the Weyl group of the maximal torus i (respectively, $i_{\hat{p}}$). The p-completion induces an isomorphism $W_{\hat{p}} \cong W$.*

Let us begin with a general result on the K-theory of finite loop spaces.

Theorem 6.2. *Let L be a connected finite loop space.*

1. *The Chern character*
$$\mathrm{ch}\colon K^0(BL;\mathbb{Z}) \longrightarrow H^{\mathrm{even}}(BL;\mathbb{Q})$$
is injective. Consequently the ring $K^0(BL;\mathbb{Z})$ is torsion free and has no zero divisors.
2. *$K^1(BL;\mathbb{Z}) = 0$.*

Proof. For the first point, consider the commutative diagram

$$\begin{array}{ccc} K^0(BL;\mathbb{Z}) & \xrightarrow{\widehat{e}} & \prod_p K^0(BL;\mathbb{Z}_{\hat{p}}) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle (\mathrm{ch}_p)} \\ H^{\mathrm{even}}(BL;\mathbb{Q}) & \longrightarrow & \prod_p H^{\mathrm{even}}_{\mathbb{Q}_{\hat{p}}}(BL). \end{array}$$

By Theorem 5.2, the right vertical map is injective. Since

$$H_*(BL;\pi_{*+1}(BU)\otimes\mathbb{Q}) = 0,$$

Lemma 2 in [22] implies that the map $\widehat{e}$ is also injective.

Let us deal with the second point. For any sufficiently large prime number p, $H^*(BL;\mathbb{Z}_{\hat{p}})$ is concentrated in even degrees; hence the p-adic Atiyah–Hirzebruch spectral sequence is trivial for such p. This observation, our Proposition 5.3, Lemma 4.1 in [2] and Theorem 3.3 in [13] imply that:

$$K^1(BL;\mathbb{Z}) \cong \varprojlim K^1(BL^{(n)};\mathbb{Z}),$$

where the inverse limit is taken over the skeleta of BL. Now Lemma 2 in [22] implies that $K^1(BL;\mathbb{Z})$ injects into $\prod K^1(BL;\mathbb{Z}_{\hat{p}})$ and this product is zero by our main result. □

For finite loop spaces with maximal tori, we have an integral version of our main result:

Theorem 6.3. *Let L be a connected finite loop space with maximal torus $i\colon T \longrightarrow L$ and W the corresponding Weyl group. Then the map Bi induces a ring isomorphism:*

$$K^*(BL;\mathbb{Z}) \cong K^*(BT;\mathbb{Z})^W.$$

Proof. The version of Sullivan's arithmetic square given by W. Meier (in [22]) applies in our case. It gives rise to the following commutative diagram, with exact rows:

$$\begin{array}{ccccc} 0 \to K^0(BL;\mathbb{Z}) & \longrightarrow & K^0(BL;\mathbb{Q}) \oplus K^0(BL;\widehat{\mathbb{Z}}) & \longrightarrow & K^0(BL;A_f) \\ {\scriptstyle Bi^*}\downarrow & & {\scriptstyle Bi^*}\downarrow & & {\scriptstyle Bi^*}\downarrow \\ 0 \to K^0(BT;\mathbb{Z}) & \longrightarrow & K^0(BT;\mathbb{Q}) \oplus K^0(BT;\widehat{\mathbb{Z}}) & \longrightarrow & K^0(BT;A_f) \end{array} \tag{5}$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ and $A_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adeles.

Recall that if E is a $\mathbb{Q}$-algebra (in particular, $E = \mathbb{Q}$ or $E = A_f$), then

$$K^0(-;E) \cong \prod_{n\geq 0} H^{2n}(-;E).$$

In our case, Theorem 1.2 of [25] implies that $K^0(BL;E) \cong K^0(BT;E)^W$. On the one hand the map $Bi\colon BT \longrightarrow BL$, p-completion and Proposition 6.3 provide a commutative square

$$\begin{array}{ccc} K^0(BL_{\hat{p}};\mathbb{Z}_{\hat{p}}) & \xrightarrow{\cong} & K^0(BL;\mathbb{Z}_{\hat{p}}) \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ K^0(BT_{\hat{p}};\mathbb{Z}_{\hat{p}})^{W_{\hat{p}}} & \xrightarrow{\cong} & K^0(BT;\mathbb{Z}_{\hat{p}})^W. \end{array}$$

On the other hand, $K^0(-;\widehat{\mathbb{Z}}) \cong \prod K^0(-;\mathbb{Z}_{\hat{p}})$, where the product is taken over all the prime numbers. With all these observations, an easy diagram chase in (5) yields $K^0(BL;\mathbb{Z}) \cong K^0(BT;\mathbb{Z})^W$. □

In [25, Theorem 1.2] Møller and Notbohm showed that the Weyl group of a connected finite loop space with maximal torus is crystallographic. This combines with the preceding theorem to imply that a connected finite loop space with a maximal torus has the K-theory of the classifying space of some compact connected Lie group. We will now show that this condition characterizes the finite loop spaces which have maximal tori.

Theorem 6.4. *Let L be a connected finite loop space. Then L admits a maximal torus if and only if there exists a compact connected Lie group G such that $K^*(BL;\mathbb{Z})$ is λ-isomorphic to $K^*(BG;\mathbb{Z})$.*

This result can be viewed as a generalization of a theorem of Notbohm–Smith ([27, Theorem 5.1]). In fact, our proof will follow the same pattern as theirs. We deal first with some preliminaries.

Let L be a connected finite loop space. Consider the two filtrations of $K^*(BL;\mathbb{Z})$ defined by setting, for $n = 0, 1, \ldots$

i) $\mathcal{S}_n(L) = \operatorname{Ker}(K^*(BL;\mathbb{Z}) \longrightarrow K^*(BL^{(n-1)};\mathbb{Z}))$, where $BL^{(n)}$ is the n-th skeleton of BL.

ii) $\mathcal{C}_n(L) = \{\xi \in K^*(BL;\mathbb{Z}) \text{ s.t. } \mathrm{ch}_r(\xi) = 0, \text{ for } r = (0, 1, \ldots, n-1\}$, where $\mathrm{ch}_r(-)$ denotes the r-th component of the Chern character.

Then one easily checks that:

1. For any n, $\mathcal{S}_n(L)$ and $\mathcal{C}_n(L)$ are λ-ideals of $K^*(BL;\mathbb{Z})$ and $\mathcal{S}_n(L) \subset \mathcal{C}_n(L)$. Moreover if L is a torus T, then $\mathcal{S}_n(T) = \mathcal{C}_n(T) = I^n$ where I is the augmentation ideal.
2. For any prime p,
$$K^*(BL;\mathbb{Z}_{\hat{p}}) \cong \lim_{\leftarrow}(K^*(BL;\mathbb{Z})/\mathcal{S}_n(L) \otimes \mathbb{Z}_{\hat{p}}).$$
3. For any prime p,
$$\begin{aligned} K^*(BL;\mathbb{Q}_{\hat{p}}) &\cong \lim_{\leftarrow}(K^*(BL;\mathbb{Z})/\mathcal{S}_n(L) \otimes \mathbb{Q}_{\hat{p}}) \\ &\cong \lim_{\leftarrow}(K^*(BL;\mathbb{Z})/\mathcal{C}_n(L) \otimes \mathbb{Q}_{\hat{p}}). \end{aligned}$$

Next we consider the integral version of Theorem 5.1.

Proposition 6.5. *Let L be a connected finite loop space and T a torus. Then the natural map*
$$\alpha\colon [BT, BL] \longrightarrow \operatorname{Hom}_\lambda(K^*(BL;\mathbb{Z}), K^*(BT;\mathbb{Z}))$$
is a bijection.

Proof. Let $\varphi\colon K^*(BL;\mathbb{Z}) \longrightarrow K^*(BT;\mathbb{Z})$ be a λ-homomorphism. Consider the composite

$$\bar{\varphi}_n\colon K^*(BL;\mathbb{Z})/\mathcal{S}_n(L) \longrightarrow K^*(BL;\mathbb{Z})/\mathcal{C}_n(L) \longrightarrow K^*(BT;\mathbb{Z})/I^n,$$

where the first arrow is the canonical surjection and the second arrow is induced by φ (the notation is as above). Then define

$$\varphi_{\hat{p}}\colon K^*(BL;\mathbb{Z}_{\hat{p}}) \longrightarrow K^*(BT;\mathbb{Z}_{\hat{p}})$$

as the inverse limit of the $\bar{\varphi}_n \otimes \mathbb{Z}_{\hat{p}}$'s. Since $\varphi_{\hat{p}}$ is a λ-homomorphism, it is induced by a map $f_{\hat{p}}\colon BT_{\hat{p}} \longrightarrow BL_{\hat{p}}$ (see Theorem 5.1). On the rational side, the λ-homomorphism φ induces (via the Chern character) a graded homomorphism

$$\varphi_{\mathbb{Q}}\colon H^*(BL;\mathbb{Q}) \longrightarrow H^*(BT;\mathbb{Q}).$$

As BL and BT are rationally products of Eilenberg–MacLane spaces, $\varphi_{\mathbb{Q}}$ is induced by a map $f_{\mathbb{Q}}\colon BT_{\mathbb{Q}} \longrightarrow BL_{\mathbb{Q}}$.

To obtain a map $f\colon BT \longrightarrow BL$ out of the $f_{\hat{p}}$'s and $f_{\mathbb{Q}}$, we will use Sullivan's arithmetic square (as presented in [22, Theorem 4]). Thus it suffices to check that, for any prime p, the two composites

$$\begin{aligned} BT &\longrightarrow BT_{\hat{p}} \xrightarrow{\ f_{\hat{p}}\ } BL_{\hat{p}} \longrightarrow (BL_{\hat{p}})_{\mathbb{Q}},\\ BT &\longrightarrow BT_{\mathbb{Q}} \xrightarrow{\ f_{\mathbb{Q}}\ } BL_{\mathbb{Q}} \longrightarrow (BL_{\hat{p}})_{\mathbb{Q}} \end{aligned}$$

are homotopic. By choosing a rational equivalence $BL \longrightarrow \prod_{i=1}^k K(\mathbb{Z},2n_i)$ and observing that the two composites represent elements of the product $\prod_{i=1}^k H^{2n_i}(BT;\mathbb{Q}_{\hat{p}})$, it is sufficient to show the commutativity of the following diagram

$$\begin{array}{ccccc} & H^*_{\mathbb{Q}_{\hat{p}}}(BL_{\hat{p}}) & \xrightarrow{f^*_{\hat{p}}} & H^*_{\mathbb{Q}_{\hat{p}}}(BT_{\hat{p}}) & \\ {\scriptstyle c_p(L)}\swarrow\cong & & & & \cong\searrow{\scriptstyle c_p(T)} \\ H^*(BL;\mathbb{Q}_{\hat{p}}) & & & & H^*(BT;\mathbb{Q}_{\hat{p}}) \\ \cong\nwarrow{\scriptstyle c_0(L)} & & & & {\scriptstyle c_0(T)}\nearrow\cong \\ & H^*(BL_{\mathbb{Q}};\mathbb{Q}_{\hat{p}}) & \xrightarrow{f^*_{\mathbb{Q}}} & H^*(BT_{\mathbb{Q}};Q_{\hat{p}}). & \end{array}$$

This is equivalent to showing that the two homomorphisms

$$\psi_1 = c_p(T)\circ f^*_{\hat{p}} \circ c_p(L)^{-1} \quad \text{and} \quad \psi_2 = c_0(T)\circ f^*_{\mathbb{Q}} \circ c_0(L)^{-1}$$

are equal. Since ψ_1 and ψ_2 are $\mathbb{Q}_{\hat{p}}$-linear, it is sufficient to check the equality on the subring $H^*(BL;\mathbb{Q}) \subset H^*(BL;\mathbb{Q}_{\hat{p}})$. Assume that we are given $x \in H^k(BL;\mathbb{Q})$. As $K^1(BL;\mathbb{Z}) = 0$, there exists $\eta \in K^0(BL;\mathbb{Z})$ and an integer M such that $\mathrm{ch}(\eta) = Mx +$ higher terms (see [13, Theorem 3.2]). By construction the following diagram is commutative

$$\begin{array}{ccccccccc}
H^{**}(BL;\mathbb{Q}_{\hat{p}}) & \overset{\mathrm{ch}}{\leftarrow} & K^*(BL;\mathbb{Z}_{\hat{p}}) & \overset{i_p(L)}{\longleftarrow} & K^*(BL;\mathbb{Z}) & \overset{\mathrm{ch}}{\rightarrow} & H^{**}(BL;\mathbb{Q}) & \overset{i_0(L)}{\longrightarrow} & H^{**}(BL;\mathbb{Q}_{\hat{p}}) \\
\psi_1 \downarrow & & \varphi_{\hat{p}} \downarrow & & \varphi \downarrow & & \varphi_{\mathbb{Q}} \downarrow & & \psi_1 \downarrow \\
H^{**}(BT;\mathbb{Q}_{\hat{p}}) & \overset{\mathrm{ch}}{\leftarrow} & K^*(BT;\mathbb{Z}_{\hat{p}}) & \overset{i_p(T)}{\longleftarrow} & K^*(BT;\mathbb{Z}) & \overset{\mathrm{ch}}{\rightarrow} & H^{**}(BT;\mathbb{Q}) & \overset{i_0(T)}{\longrightarrow} & H^{**}(BT;\mathbb{Q}_{\hat{p}}).
\end{array}$$

By Theorem 4 in [22], we have $\mathrm{ch} \circ i_p(L) = i_0(L) \circ \mathrm{ch}$ on the upper horizontal line and $\mathrm{ch} \circ i_p(T) = i_0(T) \circ \mathrm{ch}$ on the lower one. This implies the desired equality; and we have shown that α is surjective. Its injectivity is a consequence of Theorem 2 and Lemma 2 in [22] and of our Theorem 5.1. □

Proof of Theorem 6.4. One of the implications has been established in Theorem 6.3; thus we are left with the second one. By hypothesis, there is a torus T and a finite group W of λ-automorphisms of $K^*(BT;\mathbb{Z})$ such that

$$K^*(BL;\mathbb{Z}) \overset{\lambda}{\cong} K^*(BT;\mathbb{Z})^W.$$

Let φ denote the composite $K^*(BL;\mathbb{Z}) \cong K^*(BT;\mathbb{Z})^W \subset K^*(BT;\mathbb{Z})$; the preceding theorem provides us with a map $f\colon BT \longrightarrow BL$ inducing φ in (integral) K-theory. Clearly the ranks of T and L are equal. Thus we will be done if we can show that the homotopy fiber V of f is $\mathbb{Z}$-finite. We have the following properties:

1. V is homotopy equivalent to a CW-complex of finite type;
2. For any prime p, V is $\mathbb{F}_p$-good and $\pi_1(V_{\hat{p}}) \cong \pi_1(V) \otimes \mathbb{Z}_{\hat{p}}$ (see Section 7 in [11]).

Let us now fix a prime p and consider the p-completion $f_{\hat{p}}\colon BT_{\hat{p}} \longrightarrow BL_{\hat{p}}$. Let also $Bi_L\colon BT_L \longrightarrow BL_{\hat{p}}$ be a maximal p-torus. By Proposition 8.11 in [15], there exists a map $B\psi\colon BT_{\hat{p}} \longrightarrow BT_L$ such that $Bi_L \circ B\psi = f_{\hat{p}}$. In p-adic K-theory this yields the following commutative diagram

$$\begin{array}{ccc}
K^*(BT_L;\mathbb{Z}_{\hat{p}}) & & \\
\uparrow Bi_L^* & \searrow^{B\psi^*} & \\
K^*(BL_{\hat{p}};\mathbb{Z}_{\hat{p}}) & \underset{f_{\hat{p}}^*}{\longrightarrow} & K^*(BT_{\hat{p}});\mathbb{Z}_{\hat{p}})
\end{array}$$

Since $K^*(BL;\mathbb{Z}_{\hat{p}}) \cong K^*(BT;\mathbb{Z}_{\hat{p}})^W$ (via $f^*_{\hat{p}}$), Theorem 4.1 in [31] provides us with a λ-ring homomorphism $\Phi\colon K^*(BT_{\hat{p}};\mathbb{Z}_{\hat{p}}) \longrightarrow K^*(BT_L;\mathbb{Z}_{\hat{p}})$ such that $\Phi \circ f^*_{\hat{p}} = Bi^*_L$ Passing to fields of fractions and using some Galois theory, one checks that the composites $\Phi \circ B\psi^*$ and $B\psi^* \circ \Phi$ are λ-isomorphisms. In other words, $B\psi$ induces an isomorphism in p-adic K-theory. It follows that $B\psi$ is a homotopy equivalence, so that $f_{\hat{p}}\colon BT_{\hat{p}} \longrightarrow BL_{\hat{p}}$ is also a maximal torus. Since the homotopy fiber of $f_{\hat{p}}$ is $V_{\hat{p}}$ (see [11, Proposition 4.2]), we obtain that $H_*(V_{\hat{p}};\mathbb{F}_p) \cong H_*(V;\mathbb{F}_p)$ is a finite dimensional $\mathbb{F}_{\hat{p}}$-vector space. Property 1) above now combines with the universal coefficient theorem to imply that $H_*(V;\mathbb{Z})$ is a finitely generated abelian group.$\square$

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