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The zero-norm subspace of bounded cohomology

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Abstract. Let Σ be a closed, orientable surface of genus > 1. In this paper, non-trivial elements α of the third bounded cohomology $H_b^3(\Sigma;\mathbf{R})$ with $\|\alpha\| = 0$ are given constructively by using both a hyperbolic metric and a singular euclidean metric on $\Sigma \times \mathbf{R}$. Furthermore, it is shown that the dimension of the subspace $N^3(\Sigma)$ of $H_b^3(\Sigma;\mathbf{R})$ consisting of zero-norm elements is the cardinality of the continuum.

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Introduction

Let X be a topological space and $C^k(X)$ the k-cochain group of real coefficient. The $\mathbf{R}$-subspace $C_b^k(X)$ of $C^k(X)$ consists of elements $c \in C^k(X)$ with

$$\|c\| = \sup\{|c(\sigma)|; \sigma\colon \Delta^k \longrightarrow X \text{ is a singular } k\text{-simplex}\} < \infty.$$

Consider the restriction $\delta_b^k = \delta^k\big|_{C_b^k(X)}\colon C_b^k(X) \longrightarrow C_b^{k+1}(X)$ of the coboundary operator $\delta^k\colon C^k(X) \longrightarrow C^{k+1}(X)$. Then, the cochain complex $(C_b^*(X), \delta_b^*)$ defines the *bounded cohomology*

$$H_b^*(X;\mathbf{R}) = Z_b^*(X)/B_b^*(X),$$

where $Z_b^k(X) = \mathrm{Ker}(\delta_b^k)$, $B_b^k(X) = \mathrm{Im}(\delta_b^{k-1})$. We refer to Gromov [7] for fundamental results on bounded cohomology. The *pseudonorm* $\|\alpha\|$ of $\alpha \in H_b^k(X;\mathbf{R})$ is defined by

$$\|\alpha\| = \inf\{\|c\|; c \in Z_b^k(X) \text{ with } [c] = \alpha\}.$$

We say that $N^k(X) = \{\alpha \in H_b^k(X;\mathbf{R}); \|\alpha\| = 0\}$ is the *zero-norm subspace* of $H_b^k(X;\mathbf{R})$. For any topological space X, Matsumoto–Morita [9] and Ivanov [8] proved independently that $N^k(X) = \{0\}$ whenever $k \leq 2$. At that moment, any examples of non-trivial $N^k(X)$ were not known for $k \geq 3$.

Here, we are mainly concerned with the case where the space X is a closed, connected, orientable surface Σ of genus > 1. Then, the structure of the second bounded cohomology $H_b^2(\Sigma;\mathbf{R})$ was studied by Brooks–Series [2], Mitsumatsu [10], Barge–Ghys [1], Epstein–Fujiwara [4] and that of the third $H_b^3(\Sigma,\mathbf{R})$ by Yoshida [18], Soma [11], [12] and so on. We refer to Grigorchuk [6] for other useful references on bounded cohomology. Furthermore, the author showed in [13] that $N^3(\Sigma)$ is non-trivial by invoking Matsumoto–Morita [9, Theorem 2.3]. However, since the proof of their theorem relies on the Hahn–Banach theorem, we could not construct any non-trivial elements of $N^3(\Sigma)$ practically.

In this paper, non-trivial elements of $N^3(\Sigma)$ are given constructively by using both a hyperbolic metric and a singular euclidean metric on $\Sigma\times\mathbf{R}$, where the latter metric is defined by using a measured foliation associated to a pseudo-Anosov automorphism of Σ. A combination of these two metrics presents a continuous family $\{[c_{r,\varepsilon}]; 0 < r \leq 1\}$ of elements of $N^3(\Sigma\times\mathbf{R})$ which are linearly independent in $H_b^3(\Sigma\times\mathbf{R};\mathbf{R}) \cong H_b^3(\Sigma;\mathbf{R})$, see Theorems 1 and 2 in §2 for details. In particular, it is shown that the dimension of the $\mathbf{R}$-vector subspace $N^3(\Sigma)$ of $H_b^3(\Sigma;\mathbf{R})$ is the cardinality of the continuum.

The key fact in our arguments is that the bounded 3-cocycle $c_{r,\varepsilon}$ given in §2 is the coboundary of a certain unbounded 2-cochain. For the proof, it is crucial that the 3-dimensional euclidean space $\mathbf{E}^3$ is the product metric space $\mathbf{E}^2\times\mathbf{E}^1$. This is the main reason why we use a euclidean metric as well as a hyperbolic metric.

§1. Euclidean and hyperbolic structures on manifolds

Let Σ be a closed, connected and oriented surface of genus > 1. A *measured foliation* $\mathcal{F}$ on Σ is a topological foliation with finitely many prong singular points of degree ≥ 3 and equipped with the transverse measure. The set of singular points of $\mathcal{F}$ is denoted by $S_{\mathcal{F}}$. An orientation-preserving homeomorphism $f\colon \Sigma \longrightarrow \Sigma$ is called a *pseudo-Anosov automorphism* if there exists $\lambda = \lambda(f) > 1$ and a pair of mutually transverse, measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ with $S_{\mathcal{F}^s} = S_{\mathcal{F}^u} (= S(f))$ and $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$, $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$. We refer to [3], [5] and [16] for the existence and fundamental properties of such automorphisms and for typical pictures of $\mathcal{F}^{s(u)}$ near $p \in S(f)$.

Note that the pair of these measured foliations $\mathcal{F}^u$, $\mathcal{F}^s$ determines an incomplete, euclidean structure, a smooth structure on $\Sigma^\circ = \Sigma - S(f)$. We will define a smooth structure on Σ extending that on Σ°. For any $n \in \mathbf{N}$ with $n \geq 3$, the euclidean 2-space $\mathbf{R}^2 = \mathbf{C}$; $(x, y) = x + \sqrt{-1}\,y$ is divided into the n sectors $V_1, \dots, V_n$ such that

$$V_k = \left\{ r\exp\bigl(\sqrt{-1}\theta\bigr) \in \mathbf{C}; r \geq 0, \frac{2(k-1)\pi}{n} \leq \theta \leq \frac{2k\pi}{n} \right\}$$

for $k = 1, \dots, n$. The upper half plane $H = \{z \in \mathbf{C}; \operatorname{Im}(z) \geq 0\}$ admits the euclidean structure induced from that on $\mathbf{C} = \mathbf{R}^2$. Let $\chi_k\colon V_k \longrightarrow H$ be the homeomorphism defined by

$$\chi_k\big(r\exp(\sqrt{-1}\theta)\big) = r\exp\left(\sqrt{-1}\left(\frac{n\theta}{2} - (k-1)\pi\right)\right).$$

Note that the Jacobian of χ_k with respect to the standard euclidean coordinates on V_k and H is the constant $n/2$. Let $\mathcal{F}_H^s$, $\mathcal{F}_H^u$ be the measured foliation on H such that the set of leaves in $\mathcal{F}_H^s$ (resp. $\mathcal{F}_H^u$) consists of straight lines parallel to (resp. straight rays orthogonal to) the x-axis ∂H and such that the transverse measures are induced from the euclidean metric on H. Then, the pair $\{\mathcal{F}_n^s, \mathcal{F}_n^u\}$ of measured foliations on $\mathbf{R}^2$ with the prong singular point $(0,0)$ of degree n is defined by

$$\mathcal{F}_n^s = \bigcup_{k=1}^{n} \chi_k^{-1}(\mathcal{F}_H^s), \qquad \mathcal{F}_n^u = \bigcup_{k=1}^{n} \chi_k^{-1}(\mathcal{F}_H^u).$$

For a sufficiently small $\varepsilon > 0$, there exist mutually disjoint neighborhoods U_p of $p \in S(f)$ in Σ and homeomorphisms $\varphi_p\colon U_p \longrightarrow D(\varepsilon) = \{z \in \mathcal{C}; |z| < \varepsilon\}$ such that $\varphi_p(\mathcal{F}^s\big|_{U_p}) = \mathcal{F}_n^s\big|_{D(\varepsilon)}, \varphi_p(\mathcal{F}^u\big|_{U_p}) = \mathcal{F}_n^u\big|_{D(\varepsilon)}$. For $V_k(\varepsilon) = \varphi_p^{-1}(D(\varepsilon) \cap V_k)$, the composition $\chi_k \circ \varphi_p\big|_{V_k(\varepsilon)-\{p\}}\colon V_k(\varepsilon) - \{p\} \longrightarrow H - \{0\}$ is a locally isometric embedding if $V_k(\varepsilon) - \{p\}$ has the euclidean metric induced from that on Σ°. Regarding $\{(U_p, \varphi_p); p \in S(f)\}$ as a family of coordinate systems for Σ in $\cup_p U_p$, one can define the smooth structure on Σ extending that on Σ°. Then, $\Sigma \times I$ admits the product smooth structure, where I is the closed interval $[0,1]$. From now on, we identify $\cup_p U_p \times I$ with $\cup_p D_p(\varepsilon) \times I$ via $\varphi_p \times \mathrm{id}_I$'s, where $D_p(\varepsilon)$ are copies of $D(\varepsilon)$. Note that the homeomorphism $f\colon \Sigma \times \{0\} \longrightarrow \Sigma \times \{1\}$ is not a diffeomorphism with respect to this smooth structure. So, we need another smooth structure on $\Sigma \times I$. For any t with $0 \le t \le 1$, consider the elliptic half-disk

$$E_t = \Big\{(x,y) \in \mathbf{R}^2; \lambda^{2+2t}x^2 + \lambda^{2-2t}y^2 = \varepsilon^2, y \ge 0\Big\}$$

in H. Set $W_{p,t} = \bigcup_{k=1}^n \chi_k^{-1}(E_t) \subset D_p(\varepsilon)$, and

$$X_p = \{(q,t); t \in I, q \in W_{p,t}\} \subset D_p(\varepsilon) \times I \subset \Sigma \times I.$$

For simplicity, we denote the product homeomorphism $\chi_k \times \mathrm{id}_I\colon V_k \times I \longrightarrow H \times I$ by $\widehat{\chi}_k$. The homeomorphism $\psi_p\colon X_p \longrightarrow D_p(\varepsilon/\lambda) \times I$ is defined by

$$\psi_p(q,t) = \widehat{\chi}_k^{-1}(\lambda^t x, \lambda^{-t} y, t)$$

if $q \in \chi_k^{-1}(E_t)$ and $\chi_k(q) = (x,y)$. By taking $\{(X_p, \psi_p); p \in S(f)\}$ as a coordinte system for $\Sigma \times I$ instead of $\{(U_p \times I, \varphi_p \times \mathrm{id}_I); p \in S(f)\}$, we have a new smooth structure on $\Sigma \times I$, and denote this smooth manifold by $\Sigma \times I^{\mathrm{new}}$. Then, $f\colon \Sigma \times \{0\}^{\mathrm{new}} \longrightarrow \Sigma \times \{1\}^{\mathrm{new}}$ is a diffeomorphism. In particular, the mapping torus $M = \Sigma \times I^{\mathrm{new}}/\{(x,0) \sim (f(x),1)\}$ admits the induced smooth structure.

Let $\mathrm{Vol}_{(1)}(B)$ (resp. $\mathrm{Vol}_{(2)}(B)$) denote the volume of a compact 3-dimensional submanifold B in $X_p^\circ = X_p - \{p\} \times I$ (resp. in $D_p(\varepsilon/\lambda) \times I$) with respect to the incomplete euclidean metric on $X_p^\circ \subset \Sigma^\circ \times I$ (resp. the standard euclidean metric on $D_p(\varepsilon/\lambda) \times I$). Similarly, the areas of subsurfaces F in X_p° and $D_p(\varepsilon/\lambda) \times I$ are denoted by $\mathrm{Area}_{(1)}(F)$ and $\mathrm{Area}_{(2)}(F)$, respectively.

We denote the degree of $\mathcal{F}^s$ (or $\mathcal{F}^u$) at $p \in S(f)$ by $n(p)$. Then, the following lemma holds.

Lemma 1. (i) *For any compact* 3*-dimensional submanifold* B *of* X_p°,

$$\mathrm{Vol}_{(1)}(B) = \frac{n(p)}{2}\mathrm{Vol}_{(2)}(\psi_p(B)).$$

(ii) *For any compact subsurface* F *of* X_p°,

$$\mathrm{Area}_{(1)}(F) \leq \frac{n(p)\lambda}{2}\mathrm{Area}_{(2)}(\psi_p(F)).$$

Proof. Since $X_k^\circ \subset D_p(\varepsilon)^\circ \times I = \bigcup_{k=1}^n V_k(\varepsilon)^\circ \times I$, if necessary dividing B and F into smaller pieces, we may assume that B and F are contained in $V_k(\varepsilon)^\circ \times I$ for some $k \in \{1, \ldots, n(p)\}$, where $V_k(\varepsilon)^\circ = D_p(\varepsilon) \cap V_k - \{p\}$. Set $B' = \widehat{\chi}_k(B)$ and $F' = \widehat{\chi}_k(F)$. Recall that $\widehat{\chi}_k\big|_{V_k(\varepsilon)^\circ \times I} \colon V_k(\varepsilon)^\circ \times I \longrightarrow (H - \{0\}) \times I$ is a locally isometric embedding if $V_k(\varepsilon)^\circ$ has the incomplete euclidean metric induced from that on Σ°. For the diffeomorphism $\Psi \colon H \times I \longrightarrow H \times I$ with $\Psi(x, y, t) = (\lambda^t x, \lambda^{-t} y, t)$, we have

$$\begin{aligned} \mathrm{Vol}_{(1)}(B) = \mathrm{Vol}_{H \times I}(B') &= \mathrm{Vol}_{H \times I}(\Psi(B')) \quad \text{and} \\ \mathrm{Area}_{(1)}(F) = \mathrm{Area}_{H \times I}(F') &\leq \lambda\, \mathrm{Area}_{H \times I}(\Psi(F')). \end{aligned}$$

Since $\hat{\chi}_k(\psi_p(B)) = \Psi(B')$ and $\hat{\chi}_k(\psi_p(F)) = \Psi(F')$ and since the Jacobian of $\chi_k \colon V_k \longrightarrow H$ is $n(p)/2$, we have

$$\frac{n(p)}{2}\mathrm{Vol}_{(2)}(\psi_p(B)) = \mathrm{Vol}_{H \times I}(\Psi(B')) \text{ and } \frac{n(p)}{2}\mathrm{Area}_{(2)}(\psi_p(F)) \geq \mathrm{Area}_{H \times I}(\psi(F')).$$

This completes the proof. □

Let $\rho \colon \widetilde{M} = \Sigma \times \mathbf{R} \longrightarrow M$ be the infinite cyclic covering associated to $\pi_1(\Sigma) \subset \pi_1(M)$, and set $\widetilde{L} = S(f) \times \mathbf{R}$. Note that $\widetilde{M}^\circ = \Sigma^\circ \times \mathbf{R}$ has the product, incomplete euclidean metric induced from the euclidean metrics on Σ° and $\mathbf{R}$. For the euclidean area form η_{Σ° on Σ°, $\widetilde{\eta} = \zeta^*(\eta_{\Sigma^\circ})$ is a 2-form on $\widetilde{M}^\circ$, where $\zeta \colon \Sigma^\circ \times \mathbf{R} \longrightarrow \Sigma^\circ$ is the orthogonal projection. The volume form $\widetilde{\Omega}_E$ on $\widetilde{M}^\circ$ is given by $\widetilde{\Omega}_E = \widetilde{\eta} \wedge dt$. The diffeomorphism $\widetilde{f} \colon \widetilde{M} \longrightarrow \widetilde{M}$ with $\widetilde{f}(x,t) = (f(x), t+1)$ is the generator of the covering transformation group. Since $f\big|_{\Sigma^\circ} \colon \Sigma^\circ \longrightarrow \Sigma^\circ$ is a euclidean-area-preserving diffeomorphism, $\widetilde{f}\big|_{\widetilde{M}^\circ}$ is a volume-preserving diffeomorphism, that is, $\widetilde{f}^*(\widetilde{\Omega}_E) = \widetilde{f}^*(\widetilde{\eta}) \wedge \widetilde{f}^*(dt) = \widetilde{\eta} \wedge dt = \widetilde{\Omega}_E$. Thus, there exists the 3-form Ω_E in $M^\circ = M - L$ with $\rho^*(\Omega_E) = \widetilde{\Omega}_E$, where $L = \rho(\widetilde{L})$ is a link in M. Similarly, there exists a 2-form η_{M° on M° with $\rho^*(\eta_{M^\circ}) = \widetilde{\eta}$. According to Thurston [17] (see also Sullivan [14]), the smooth manifold M admits a hyperbolic structure. For the hyperbolic volume form Ω_H on M, there exists a positive, smooth function $h \colon M^\circ \longrightarrow \mathbf{R}$ with $\Omega_E = h\Omega_H$. We suppose that $\widetilde{M}$ admits the hyperbolic metric induced from that on M via ρ.

For the derivative $d\xi$ of the smooth embedding

$$\xi = \bigcup_{p \in S(f)} \psi_p^{-1} \colon \bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I \longrightarrow \Sigma \times I^{\mathrm{new}} \subset \widetilde{M},$$

we set

$$\iota(\xi) = \inf\Big\{ \|d\xi_x(v)\|_{\widetilde{M}}; x \in \bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I, v \in TU_x\Big(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I\Big)\Big\} > 0,$$

where $TU(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I)$ is the unit tangent bundle over the euclidean manifold $\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I$. We note that the image $Y = \rho(\bigcup_{p \in S(f)} D_p(\varepsilon/\lambda) \times I)$ is a union of solid tori in M, and the complement $M - \operatorname{int} Y$ of $\operatorname{int} Y$ is a compact manifold.

Lemma 2. $K_1 = \sup\{h(s); s \in M^\circ\} < \infty$.

Proof. For any compact 3-dimensional submanifold B of $Y - L$, we have $\iota(\xi)^3 \mathrm{Vol}_{(2)}(\widetilde{B}) \le \mathrm{Vol}_M(B)$, where $\widetilde{B} = \rho^{-1}(B) \cap (\Sigma \times I^{\mathrm{new}})$. Then, by Lemma 1 (i), we have

$$\sup\{h(s); s \in M^\circ\} \le \max\Big\{\max\{h(s); s \in M - \operatorname{int} Y\}, \frac{n(f)}{2\iota(\xi)^3}\Big\} < \infty,$$

where $n(f) = \max\{n(p); p \in S(f)\}$. This completes the proof. □

Note that, in general, for a sequence $\{s_m\}$ in M° converging to a point in L, the limit $\lim_{m\to\infty} h(s_m)$ does not exist. Then, we can not extend h to a continuous map on M.

Let Q be a 2-dimensional subspace of $T_s(M^\circ)$ for $s \in M^\circ$. There exists a small, hyperbolic disk D centered at $x_0 \in \mathbf{H}^2$ and an embedding $i_Q \colon D \longrightarrow M$ with $i_Q(x_0) = s$, $i_{Q*}(T_{x_0}(D)) = Q$ and such that i_Q is an isometry onto the image $i_Q(D)$ which is totally geodesic with respect to the hyperbolic metric on M. Let

$\varphi_Q\colon D^\circ \longrightarrow \mathbf{R}$ be the smooth function with $i_Q^*(\eta_{M^\circ}) = \varphi_Q \cdot \eta_H$ on D°, where $D^\circ = D - i_Q^{-1}(L)$ and η_H is the hyperbolic area form on D. Then, we have

$$\int_{D^\circ} |i_Q^*(\eta_{M^\circ})| \leq \sup\{|\varphi_Q(x)|; x \in D^\circ\}\mathrm{Area}_M(D^\circ), \tag{1.1}$$

where $\mathrm{Area}_M(D^\circ)$ $(= \mathrm{Area}_M(D))$ denotes the hyperbolic area of D°. Intuitively, $\varphi_Q(x_0)$ represents the ratio, in the cross section Q, of η_{M° to the hyperbolic metric at $s \in M$. It is easily seen that there exists the maximum

$$g(s) = \max\{|\varphi_Q(x_0)|; Q \text{ is a 2-dimensional subspace of } T_s(M^\circ)\},$$

and $g\colon M^\circ \longrightarrow \mathbf{R}$ is a continuous, non-negative function. The following lemma is proved by the argument similar to that in Lemma 2.

Lemma 3. $K_2 = \sup\{g(s); s \in M^\circ\} < \infty$.

Proof. As in the proof of Lemma 2, for any compact subsurface F of $Y - L$, the inequality $\iota(\xi)^2\mathrm{Area}_{(2)}(\widetilde{F}) \leq \mathrm{Area}_M(F)$ holds, where $\widetilde{F} = \rho^{-1}(F) \cap (\Sigma \times I^{\mathrm{new}})$. If necessary dividing $\widetilde{F}$ into smaller pieces, we may assume that, for the inclusion $i\colon \widetilde{F} \longrightarrow \Sigma^\circ \times I$, the composition $\zeta \circ i$ is injective. Then, by the definition of $\widetilde{\eta}$,

$$\int_{\widetilde{F}} |i^*(\widetilde{\eta})| = \mathrm{Area}_{\Sigma^\circ}(\zeta(\widetilde{F})) \leq \mathrm{Area}_{(1)}(\widetilde{F}).$$

By this inequality together with Lemma 1 (ii),

$$\int_F |i_F^*(\eta_{M^\circ})| \leq \mathrm{Area}_{(1)}(\widetilde{F}) \leq \frac{n(f)\lambda}{2\iota(\xi)^2}\mathrm{Area}_M(F),$$

where $i_F\colon F \longrightarrow Y - L \subset M^\circ$ is the inclusion. This shows that

$$\sup\{g(s); s \in M^\circ\} \leq \max\Big\{\max\{g(s); s \in M - \mathrm{int}\, Y\}, \frac{n(f)\lambda}{2\iota(\xi)^2}\Big\} < \infty.$$

This completes the proof. □

By Lemma 2, for any hyperbolically straight 3-simplex $\sigma\colon \Delta^3 \longrightarrow \widetilde{M}$,

$$\int_{\Delta_\sigma^{3\circ}} |\sigma^*(\widetilde{\Omega}_E)| = \int_{\Delta_\sigma^{3\circ}} |(\rho\circ\sigma)^*(\Omega_E)| \leq K_1 \int_{\Delta_\sigma^{3\circ}} |(\rho\circ\sigma)^*(\Omega_H)| = K_1\mathrm{Vol}(\Delta_\sigma^{3\circ}),$$

where Δ_σ^3 denotes the 3-simplex Δ^3 with the hyperbolic metric induced from that on $\widetilde{M}$ via σ and $\Delta_\sigma^{3\circ} = \Delta_\sigma^3 - \sigma^{-1}(\widetilde{L})$. Since the hyperbolic volume $\mathrm{Vol}(\Delta_\sigma^{3\circ}) = \mathrm{Vol}(\Delta_\sigma^3)$ is less than the volume $\mathbf{v}_3$ of a regular ideal simplex in $\mathbf{H}^3$,

$$\int_{\Delta_\sigma^{3\circ}} |\sigma^*(\widetilde{\Omega}_E)| < K_1\mathbf{v}_3. \tag{1.2}$$

Similarly, by Lemma 3 together with the equation (1.1), for any straight 2-simplex $\tau\colon \Delta^2 \longrightarrow \widetilde{M}$,

$$\int_{\Delta^{2\circ}_\tau} |\tau^*(\widetilde{\eta})| = \int_{\Delta^{2\circ}_\tau} |(\rho\circ\tau)^*(\eta_{M^\circ})| \le K_2 \mathrm{Area}(\Delta^{2\circ}_\tau),$$

where Δ^2_τ denotes the 2-simplex Δ^2 with the induced hyperbolic metric and $\Delta^{2\circ}_\tau = \Delta^2_\tau - \tau^{-1}(\widetilde{L})$. Since $\mathrm{Area}(\Delta^{2\circ}_\tau) = \mathrm{Area}(\Delta^2_\tau) < \pi$,

$$\int_{\Delta^{2\circ}_\tau} |\tau^*(\widetilde{\eta})| < \pi K_2. \tag{1.3}$$

The inequalities (1.2) and (1.3) will be used in the next section.

§2. Zero-norm elements of bounded cohomology

For a topological space X, the *Gromov norm* of a singular k-chain $z = \sum_{i=1}^n a_i\sigma_i^k \in C_k(X)$ with real coefficients $a_i \in \mathbf{R}$ is defined by

$$\|z\| = \sum_{i=1}^n |a_i|.$$

Then, for any bounded k-cochain $c \in C_b^k(X)$, we have $|c(z)| \le \|c\|\,\|z\|$.

For any $r \ge 0$, $\varepsilon > 0$, consider the continuous functions $\alpha_{r,\varepsilon}\colon \mathbf{R} \longrightarrow \mathbf{R}$ and $A_{r,\varepsilon}\colon \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$\alpha_{r,\varepsilon}(t) = \min\{\varepsilon, |t|^{-r}\}, \qquad A_{r,\varepsilon}(t) = \int_0^t \alpha_{r,\varepsilon}(u)du.$$

Note that $\lim_{t\to\infty}\alpha_{r,\varepsilon}(t) = 0$ if $r > 0$ and $\lim_{t\to\infty} A_{r,\varepsilon}(t) = \infty$ if $r \le 1$. The compositions of the projection $\widetilde{M} = \Sigma\times\mathbf{R} \longrightarrow \mathbf{R}$ with $\alpha_{r,\varepsilon}$, $A_{r,\varepsilon}$ are also denoted by $\alpha_{r,\varepsilon}\colon \widetilde{M} \longrightarrow \mathbf{R}$ and $A_{r,\varepsilon}\colon \widetilde{M} \longrightarrow \mathbf{R}$, that is, $\alpha_{r,\varepsilon}(p,t) = \alpha_{r,\varepsilon}(t)$ and $A_{r,\varepsilon}(p,t) = A_{r,\varepsilon}(t)$. For a singular n-simplex $\tau\colon \Delta^n \longrightarrow \widetilde{M}$, straight$(\tau)\colon \Delta^n \longrightarrow \widetilde{M}$ denotes the straight n-simplex obtained by straightening τ, see [15, Chapter 6] for details. Let $c_{r,\varepsilon} \in Z^3(\widetilde{M})$ be the 3-cycle defined by

$$c_{r,\varepsilon}(\sigma) = \int_{\Delta^{3\circ}_{\mathrm{straight}(\sigma)}} \mathrm{straight}(\sigma)^*(\alpha_{r,\varepsilon}\widetilde{\Omega}_E)$$

for any singular 3-simplex $\sigma\colon \Delta^3 \longrightarrow \widetilde{M}$. Intuitively, $c_{r,\varepsilon}(\sigma)$ represents the "euclidean" volume with weight $\alpha_{r,\varepsilon}$ of the "hyperbolically" straightened simplex. Since $\max\{|\alpha_{r,\varepsilon}(t)|; t\in\mathbf{R}\} = \varepsilon$, by (1.2),

$$|c_{r,\varepsilon}(\sigma)| \le \varepsilon\int_{\Delta^{3\circ}_{\mathrm{straight}(\sigma)}} |\mathrm{straight}(\sigma)^*(\widetilde{\Omega}_E)| < \varepsilon K_1 \mathbf{v}_3.$$

This shows that $c_{r,\varepsilon} \in Z_b^3(\widetilde{M})$ and $\|c_{r,\varepsilon}\| \le \varepsilon K_1 \mathbf{v}_3$.

In Theorem 1, we will show that the class $[c_{r,\varepsilon}] \in H_b^3(\widetilde{M};\mathbf{R})$ is independent of ε if $r > 0$. However, Theorem 2 implies that $[c_{r,\varepsilon}]$ strictly depends on r if $0 \le r \le 1$.

Theorem 1. *If $0 \le r \le 1$, then $[c_{r,\varepsilon}] \ne 0$ in $H_b^3(\widetilde{M};\mathbf{R})$. If $r > 0$, then for any $\varepsilon, \varepsilon' > 0$, $[c_{r,\varepsilon}] = [c_{r,\varepsilon'}]$ in $H_b^3(\widetilde{M};\mathbf{R})$. In particular, if $0 < r \le 1$, then $[c_{r,\varepsilon}]$ is a non-trivial element of $H_b^3(\widetilde{M};\mathbf{R})$ with $\big\|[c_{r,\varepsilon}]\big\| = 0$.*

Proof. We set $\Sigma_n = \Sigma \times \{n\} \subset \widetilde{M}$ for $n \in \mathbf{Z}$. For a sufficiently small $\delta > 0$, let $\widehat{\Sigma}_0$ be an oriented surface piecewise smoothly embedded in $\Sigma \times [-\delta, \delta]$ each piece of which is a totally geodesic triangle with respect to the hyperbolic metric on $\widetilde{M}$ and such that $\widehat{\Sigma}_0$ is isotopic to Σ_0 in $\Sigma \times [-\delta, \delta]$. Furthermore, we may take $\widehat{\Sigma}_0$ so that it satisfies (2.1).

$$\text{For any } p \in \Sigma,\ \widehat{\Sigma}_0 \text{ meets the line } \zeta^{-1}(p) \text{ in a single point.} \tag{2.1}$$

Let $z_0 \in Z_2(\widetilde{M})$ be a 2-cycle representing this hyperbolic triangulation of $\widehat{\Sigma}_0$. We set $\widehat{\Sigma}_n = \widetilde{f}^n(\widehat{\Sigma}_0)$ and $z_n = \widetilde{f}^n_*(z_0)$. Since $z_n - z_0$ is homologous to zero in $\widetilde{M}$, there exists a 3-chain $w_n \in C_3(\widetilde{M})$ consisting of straight 3-simplices and with $\partial w_n = z_n - z_0$. Note that $\widetilde{\eta} = \zeta^*(\eta_{\Sigma^\circ})$ and $\eta_{\Sigma^\circ} > 0$. Thus, we have

$$\int_{\widehat{\Sigma}_n^\circ} \widetilde{\eta} = \int_{\Sigma^\circ} \eta_{\Sigma^\circ} = \int_{\widehat{\Sigma}_n^\circ} |\widetilde{\eta}|,$$

where $\widehat{\Sigma}_n^\circ = \widehat{\Sigma}_n - \widehat{\Sigma}_n \cap \zeta^{-1}(S(f))$ and the second equality is derived from the property (2.1). We denote the value of these integrals by $K_3 > 0$.

If $[c_{r,\varepsilon}] = 0$ in $H_b^3(\widetilde{M};\mathbf{R})$ for some $0 \le r \le 1$, then there would exist a bounded 2-cochain $a \in C_b^2(\widetilde{M})$ with $\delta_b^2(a) = c_{r,\varepsilon}$. This implies that, for any $n \in \mathbf{N}$,

$$|c_{r,\varepsilon}(w_n)| = |a(z_n - z_0)| \le \|a\|\big(\|z_n\| + \|z_0\|\big) = 2\|a\|\,\|z_0\|.$$

Since $A_{r,\varepsilon}$ is an increasing function, $A_{r,\varepsilon}(n-\delta) \le A_{r,\varepsilon} \le A_{r,\varepsilon}(n+\delta)$ in $\Sigma_n \times [-\delta, \delta]$ and $A_{r,\varepsilon} \le \varepsilon\delta$ in $\Sigma_0 \times [-\delta, \delta]$. Consider the 2-form $\theta_{r,\varepsilon} = A_{r,\varepsilon}\widetilde{\eta}$ on $\widetilde{M}^\circ$. Since $d\theta_{r,\varepsilon} = \alpha_{r,\varepsilon} dt \wedge \widetilde{\eta} = \alpha_{r,\varepsilon}\widetilde{\Omega}_E$ and since $\mathrm{straight}(w_n) = w_n$, the Stokes Theorem shows that

$$\begin{aligned}
|c_{r,\varepsilon}(w_n)| &= \left| \int_{\widehat{\Sigma}_n^\circ} A_{r,\varepsilon}\widetilde{\eta} - \int_{\widehat{\Sigma}_0^\circ} A_{r,\varepsilon}\widetilde{\eta} \right| \\
&\ge A_{r,\varepsilon}(n-\delta) \int_{\widehat{\Sigma}_n^\circ} |\widetilde{\eta}| - \varepsilon\delta \int_{\widehat{\Sigma}_0^\circ} |\widetilde{\eta}| \\
&= (A_{r,\varepsilon}(n-\delta) - \varepsilon\delta) K_3.
\end{aligned}$$

The condition $0 \leq r \leq 1$ implies that $\lim_{n\to\infty} A_{r,\varepsilon}(n-\delta) = \infty$ and hence $\lim_{n\to\infty} |c_{r,\varepsilon}(w_n)| = \infty$, a contradiction. It follows that $[c_{r,\varepsilon}]$ is a non-trivial element of $H_b^3(\widetilde{M};\mathbf{R})$ for any $0 \leq r \leq 1$.

For any $\varepsilon, \varepsilon'$ with $\varepsilon > \varepsilon' > 0$, the 2-cochain $b \in C^2(\widetilde{M})$ is given by

$$b(\tau) = \int_{\Delta^{2\circ}_{\mathrm{straight}(\tau)}} \mathrm{straight}(\tau)^*(\theta_{r,\varepsilon} - \theta_{r,\varepsilon'})$$

for any singular 2-simplex $\tau\colon \Delta^2 \longrightarrow \widetilde{M}$. Then, the coboundary of b is $\delta^2(b) = c_{r,\varepsilon} - c_{r,\varepsilon'}$. If $r > 0$, then

$$K_4 = \max\{|A_{r,\varepsilon}(t) - A_{r,\varepsilon'}(t)|;\, t \in \mathbf{R}\} = \int_0^{(\varepsilon')^{-1/r}} (\alpha_{r,\varepsilon}(u) - \alpha_{r,\varepsilon'}(u))du < \infty.$$

By (1.3), we have

$$\begin{aligned} |b(\tau)| &= \left| \int_{\Delta^{2\circ}_{\mathrm{straight}(\tau)}} \mathrm{straight}(\tau)^*((A_{r,\varepsilon} - A_{r,\varepsilon'})\widetilde{\eta}) \right| \\ &\leq K_4 \int_{\Delta^{2\circ}_{\mathrm{straight}(\tau)}} |\mathrm{straight}(\tau)^*(\widetilde{\eta})| \\ &\leq \pi K_2 K_4. \end{aligned}$$

This shows that $b \in C_b^2(\widetilde{M})$ and hence $c_{r,\varepsilon} - c_{r,\varepsilon'} \in B_b^3(\widetilde{M})$ for $r > 0$. By the definition of the pseudonorm, for any $\varepsilon' > 0$, $\big\|[c_{r,\varepsilon}]\big\| = \big\|[c_{r,\varepsilon'}]\big\| \leq \varepsilon' K_1 \mathbf{v}_3$. Thus, we have $\big\|[c_{r,\varepsilon}]\big\| = 0$ whenever $r > 0$. □

For two sequences $\{a_n\}$, $\{b_n\}$ with $a_n, b_n > 0$ $(n \in \mathbf{N})$, $a_n \sim b_n$ means that

$$0 < \liminf_{n\to\infty} \frac{a_n}{b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n} < \infty.$$

The notation in the proof of Theorem 1 still works to prove Theorem 2.

Theorem 2. *For a fixed $\varepsilon > 0$, the elements $[c_{r,\varepsilon}]$ $(0 \leq r \leq 1)$ are linearly independent in $H_b^3(\widetilde{M};\mathbf{R})$.*

Proof. We suppose that

$$\gamma_1[c_{r_1,\varepsilon}] + \gamma_2[c_{r_2,\varepsilon}] + \cdots + \gamma_m[c_{r_m,\varepsilon}] = 0$$

for $0 \leq r_1 < r_2 < \cdots < r_m \leq 1$. Then, there exists a bounded 2-cochain $a \in C_b^2(\widetilde{M})$ with

$$\gamma_1 c_{r_1,\varepsilon} + \gamma_2 c_{r_2,\varepsilon} + \cdots + \gamma_m c_{r_1,\varepsilon} = \delta_b^2(a).$$

For the straight 3-chain $w_n \in C_3(\widetilde{M})$ given as above, we have

$$|\gamma_1 c_{r_1,\varepsilon}(w_n)| \leq \sum_{j=2}^{m} |\gamma_j c_{r_j,\varepsilon}(w_n)| + |\delta_b^2(a)(w_n)|.$$

The argument similar to that in the proof of Theorem 1 shows that

$$|\gamma_1| K_3(A_{r_1,\varepsilon}(n-\delta) - \varepsilon\delta) \leq \sum_{j=2}^{m} |\gamma_j| K_3(A_{r_j,\varepsilon}(n+\delta) + \varepsilon\delta) + 2\|a\|\,\|z_0\|,$$

and hence

$$|\gamma_1| \leq \frac{\sum_{j=2}^{m} |\gamma_j|(A_{r_j,\varepsilon}(n+\delta) + \varepsilon\delta) + 2K_3^{-1}\|a\|\,\|z_0\|}{A_{r_1,\varepsilon}(n-\delta) - \varepsilon\delta}. \tag{2.2}$$

Since $A_{r_1,\varepsilon}(n-\delta) \sim n^{1-r_1}$, $A_{r_j,\varepsilon}(n+\delta) \sim n^{1-r_j}$ if $r_j < 1$, and $A_{r_m,\varepsilon}(n+\delta) \sim \log n$ if $r_m = 1$, the right hand side of (2.2) converges to zero as $n \to \infty$. This shows that $\gamma_1 = 0$. Similarly, we have $\gamma_2 = \cdots = \gamma_m = 0$. Thus, $[c_{r,\varepsilon}]$ $(0 \leq r \leq 1)$ are linearly independent. □

By Theorems 1 and 2, the continuous family $\{[c_{r,\varepsilon}]; 0 < r \leq 1\}$ consists of linearly independent elements in $N^3(\widetilde{M})$. Since the inclusion $i \colon \Sigma = \Sigma_0 \longrightarrow \widetilde{M}$ is a homotopy equivalence, the induced homomorphism $i^* \colon (H_b^3(\widetilde{M};\mathbf{R}), \|\cdot\|) \longrightarrow (H_b^3(\Sigma;\mathbf{R}), \|\cdot\|)$ is isometrically isomorphic. Thus, we have the following corollary.

Corollary. *For any closed, connected, orientable surface Σ of genus > 1, the dimension of the zero-norm subspace $N^3(\Sigma)$ of $H_b^3(\Sigma;\mathbf{R})$ is the cardinality of the continuum.* □

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