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On Macdonald's formula for the volume of a compact Lie group

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Abstract. We give a simple proof of Macdonald's formula for the volume of a compact Lie group.

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It is well-known that a compact Lie group G of rank r is rational homotopy equivalent to a product of spheres

$$S = S^{2m_1+1} \times \cdots \times S^{2m_r+1}$$

where $(m_1, \ldots, m_r)$ is called the exponent of G. In [3] I. G. Macdonald calculated the volume of G with respect to the Haar measure μ induced from a Lebesgue measure λ on the Lie algebra $\mathfrak{g}$ of G. The result is as follows:

$$\mu(G) = \lambda(\mathfrak{g}/\mathfrak{g}_{\mathbb{Z}})\sigma(S)$$

where $\mathfrak{g}_{\mathbb{Z}}$ is the Chevalley lattice and σ is the product of the superficial measure of the unit spheres S^{2m_i+1} in $\mathbb{R}^{2m_i+2}$, that is

$$\sigma(S^{2m+1}) = \frac{2\pi^{m+1}}{m!}.$$

In this note we shall give a simpler proof of this formula.

Let T be a maximal subgroup of G, $\mathfrak{t}$ its Lie algebra and $\mathfrak{t}_{\mathbb{Z}}$ a lattice in $\mathfrak{t}$ such that the kernel of the exponential map from $\mathfrak{t}$ to T is $2\pi\mathfrak{t}_{\mathbb{Z}}$. We fix a positive definite inner product on $\mathfrak{g}$ which is invariant under the adjoint action of G such that the induced volume form coincides with λ. This inner product induces Lebesgue measures on $\mathfrak{t}$ and $\mathfrak{g}/\mathfrak{t}$ (also denoted by λ), which determine a Haar measure on T and a G-invariant measure on the flag manifold G/T (also denoted by μ). Then clearly

$$\mu(T) = (2\pi)^r\lambda(\mathfrak{t}/\mathfrak{t}_{\mathbb{Z}}), \quad \mu(G) = \mu(T)\mu(G/T).$$

The main problem is to compute the volume of the flag manifold. We compute at a point of the flag manifold the ratio of the volume form μ and another G-invariant top form, the top term of the Todd class, whose integral value is known to be 1 [1]. The adjoint action of G on $\mathfrak{g}$ induces trivialization of the direct sum of the tangent bundle of G/T and a trivial $\mathfrak{t}$ bundle. Hence the rational Pontrjagin classes of G/T all vanish and the Todd class equals $\exp \frac{1}{2}c_1(G/T)$. The top term is $\frac{1}{n!}(\frac{1}{2}c_1)^n$ where n is the complex dimension of G/T. Let $X_\alpha \in \mathfrak{g} \otimes \mathbb{C}$ be an eigenvector for a root α such that $\frac{1}{2}[X_\alpha, X_{-\alpha}] = \alpha^\vee$, the coroot for α, then

$$\frac{1}{2}c_1(X_\alpha, X_\beta) = \begin{cases} \frac{\langle\rho|\alpha^\vee\rangle}{\pi}, & \beta = -\alpha, \\ 0, & \beta \neq -\alpha. \end{cases}$$

where

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha,$$

on the other hand

$$\langle X_\alpha, X_{-\alpha}\rangle = |\alpha^\vee|^2.$$

Evaluating the volume form and $\frac{1}{n!}(\frac{1}{2}c_1)^n$ for $2n$ vectors $(X_\alpha, X_{-\alpha}; \alpha > 0)$, we obtain

$$\prod_{\alpha>0} |\alpha^\vee|^2, \quad \prod_{\alpha>0} \frac{\langle\rho|\alpha^\vee\rangle}{\pi}$$

respectively. Since $\alpha^\vee$ is the sum of $\langle\rho|\alpha^\vee\rangle$ simple coroots [2], it holds that

$$\prod_{\alpha>0} \langle\rho|\alpha^\vee\rangle = \prod_{i=1}^{r} (m_i!).$$

Hence

$$\mu(G/T) = \frac{\sigma(S)}{(2\pi)^r} \prod_{\alpha>0} |\alpha^\vee|^2.$$

The final result follows from

$$\lambda(\mathfrak{g}/\mathfrak{g}_\mathbb{Z}) = \lambda(\mathfrak{t}/\mathfrak{t}_\mathbb{Z}) \prod_{\alpha>0} |\alpha^\vee|^2.$$

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