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A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$

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Abstract. In this paper, we consider the following conformally invariant equations of fourth order

$$\begin{cases} \Delta^2 u = 6e^{4u} \qquad \text{in } \mathbf{R}^4, \\ e^{4u} \in L^1(\mathbf{R}^4), \end{cases} \tag{1}$$

and

$$\begin{cases} \Delta^2 u = u^{\frac{n+4}{n-4}}, \\ u > 0 \;\text{ in }\; \mathbf{R}^n \text{ for } n \geq 5, \end{cases} \tag{2}$$

where Δ^2 denotes the biharmonic operator in $\mathbf{R}^n$. By employing the method of moving planes, we are able to prove that all positive solutions of (2) are arised from the smooth conformal metrics on S^n by the stereograph projection. For equation (1), we prove a necessary and sufficient condition for solutions obtained from the smooth conformal metrics on S^4.

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1. Introduction

Recently, there have been much analytic work on the conformal geometry. A well known example is the Yamabe problem or, more generally, the problem of prescribling scalar curvature. Given a smooth function K defined in a compact Riemannian manifold (M, g_0) of dimension $n \geq 2$, we ask whether there exists a metric g conformal to g_0 such that K is the scalar curvature of the new metric g. Let $g = e^{2u} g_0$ for $n = 2$ or $g = u^{\frac{4}{n-2}} g_0$ for $n \geq 3$, then the problem is reduced to find solutions of the following nonlinear elliptic equations:

$$\Delta u + K e^{2u} = K_0 \tag{1.1}$$

for $n = 2$, or,

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta u + K u^{\frac{n+2}{n-2}} = K_0 u, \\ u > 0 \quad \text{in } M \end{cases} \tag{1.2}$$

for $n \geq 3$, where Δ denotes the Beltrami-Laplacian operator of (M, g_0) and K_0 is the scalar curvature of g_0. In studying equations (1.1) and (1.2), it is very important to understand the solution set of

$$\begin{cases} \Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n, \\ u > 0 \quad \text{in } \mathbf{R}^n \end{cases} \tag{1.3}$$

for $n \geq 3$, or,

$$\begin{cases} \Delta u + e^{2u} = 0 \quad \text{in } \mathbf{R}^2, \\ e^{2u} \in L^1(\mathbf{R}^2). \end{cases} \tag{1.4}$$

By employing the method of moving planes, Caffarelli-Gidas-Spruck [CGS] was able to classify all the solutions of (1.3) for $n \geq 3$, and, Chen-Li [CL] did the same thing for the equation (1.4).

There are another interesting examples arising from the conformal geometry. For a compact Riemannian manifold of dimension 4, Chang and Yang [CY] considered the existence of extremal functions of the variational problem:

$$II[w] = < Pw, w > + \int Q_0 w dV_{g_0} - (\int Q_0 dV_{g_0}) \log \int e^{4w} dV_{g_0}, \tag{1.5}$$

where P is the Paneitz operator on M, discovered by Paneitz:

$$P\varphi = \Delta^2 \varphi + \delta(\frac{2}{3}K_0 I - 2Ric)d\varphi,$$
$$Q_0 = \frac{1}{12}(K_0^2 - \Delta K_0 - 3|Ric|^2),$$

where Ric is the Ricci curvature of (M, g_0). The variational form (1.5) arises form the difference of log-determinants of conformally covariant operator with respect to metrics in a conformal class. For background material and other related problems, we refer [BCY], [CY] and the references therein. The extremal function u of $II(w)$ satisfies a conformal invariant elliptic equation of fourth order:

$$Pu + 2Q_0 = 2Qe^{4u}, \tag{1.6}$$

where Q is a constant. When (M, g_0) is the standard S^4, by using the coordinate of the stereographic projection in $\mathbf{R}^4$, the equation (1.6) can be reduced to

$$\begin{cases} \Delta^2 u = 6e^{4u} \qquad \text{in } \mathbf{R}^4, \\ e^{4u} \in L^1(\mathbf{R}^4), \end{cases} \tag{1.7}$$

where Δ^2 denotes the biharmonic operator. It is expected that in order to understand the equation (1.6), we should classify all the solutions of (1.7) completely.

The equation (1.7) looks very similar to the equation (1.4). In fact, there are many common properties shared by both equations. For example, the biharmonic operator Δ^2 in $\mathbf{R}^4$ has const. $\log \frac{1}{|x-y|}$ as its fundamental solution. And the equation (1.7) is invariant under the change of the conformal transformation. In particular, the new function $w(x) = u(\frac{x}{|x|^2}) - 2\log|x|$ satisfies the same equation as u does. However, the appearance of the biharmonic operator in (1.7) expects to make the equation (1.7) very different from (1.4). In fact, a study of radial solutions of (1.7) shows that there are solutions of (1.7) which do not come from the smooth functions on S^4 through the stereographic projection. This is not quite the same as the equation (1.4). But, under certain constraint on the behavior of u at ∞, we have

Theorem 1.1. *Suppose that u is a solution of (1.7) with $|u(x)| = o(|x|^2)$ at ∞. Then there exists some point $x_0 \in \mathbf{R}^4$ such that u is radially symmetric about x_0 and*

$$u(x) = \log \frac{2\lambda}{(1+\lambda^2|x-x_0|^2)}. \tag{1.8}$$

Let α be defined by

$$\alpha = \frac{3}{4\pi^2}\int_{\mathbf{R}^4} e^{4u(y)}dy. \tag{1.9}$$

Theorem 1.2. *Let u be a solution of (1.7). Then the following statements hold. (i) After an orthorgonal transformation, $u(x)$ can be represented by*

$$\begin{aligned} u(x) &= \frac{3}{4\pi^2}\int_{\mathbf{R}^4} \log(\frac{|y|}{|x-y|})e^{4u(y)}dy - \sum_{j=1}^{4} a_j(x_j - x_j^0)^2 + c_0 \\ &= -\sum_{j=1}^{4} a_j(x_j - x_j^0)^2 - \alpha\log|x| + c_0 + O(|x|^{-\tau}) \end{aligned} \tag{1.10}$$

for some $\tau > 0$ and for large $|x|$. The function Δu satisfies

$$\Delta u(x) = -\frac{3}{2\pi^2}\int_{\mathbf{R}^4} \frac{e^{4u(y)}}{|x-y|^2}dy - 2\sum_{j=1}^{4} a_j \tag{1.11}$$

where $a_j \geq 0$, c_0 are constants and $x^0 = (x_1^0, \ldots, x_4^0) \in \mathbf{R}^4$. Moreover, if $a_i \neq 0$ for all i, then u is symmetric with respect to the hyperplane $\{x \mid x_i = x_i^0\}$. If $a_1 = a_2 = a_3 = a_4 \neq 0$, then u is radially symmetric with respect to x^0.
(ii) The total integration $\alpha \leq 2$. If $\alpha = 2$, then $u(x)$ has the form of (1.8).

In this paper, we also consider the following equation analogue to the equation (1.3):

$$\begin{cases} \Delta^2 u = u^{\frac{n+4}{n-4}}, \\ u > 0 \quad \text{in} \quad \mathbf{R}^n \end{cases} \tag{1.12}$$

for $n \geq 5$. The equation (1.12) can be derived from the Sobolev embedding of H^2 into $L^{\frac{2n}{n-4}}$:

$$\sup_{u \in H^2(\mathbf{R}^4)} \frac{\int |\Delta u|^2}{\left(\int u^{\frac{2n}{n-4}}\right)^{\frac{n-4}{n}}}. \tag{1.13}$$

The existence of extremal functions of (1.13) was shown in [L] by P.L. Lions. In the same paper, Lions also proved the radial symmetry of any extremal function of (1.13). In general, the radial symmetry of solutions of (1.12) holds also.

Theorem 1.3. *Suppose that u is a smooth solution of (1.12). Then u is radially symmetric about some point $x_0 \in \mathbf{R}^n$ and u has the following form:*

$$u(x) = c_n \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2}\right)^{\frac{n-4}{2}} \tag{1.14}$$

for some constant $\lambda > 0$, where $c_n = [n(n-4)(n-2)(n+2)]^{-\frac{n-4}{8}}$.

Similarly, we also have

Theorem 1.4. *Suppose that u is a nonnegative solution of*

$$\Delta^2 u = u^p \quad \text{in} \quad \mathbf{R}^n \tag{1.15}$$

for $1 < p < \dfrac{n+4}{n-2}$. Then $u \equiv 0$ in $\mathbf{R}^n$.

As in equations (1.3) and (1.4), we will use the method of moving planes to prove the radial symmetry. In our situation, however, the maximum principle can not directly be applied to u without any information of Δu. Hence we have to get some informations about Δu from equations (1.7) and (1.12). First, we are going to prove that for any solution of (1.7), $\Delta u(x)$ can be reprensented by

$$\Delta u(x) = \frac{-3}{2\pi^2} \int_{\mathbf{R}^4} \frac{e^{4u(y)}}{|x-y|^2} dy - c_1 \tag{1.16}$$

for some nonnegative constant $c_1 \geq 0$. Thus, u satisfies $\Delta u < 0$ in $\mathbf{R}^4$. The representation (1.16) is an indication that we should apply the method of moving planes to $-\Delta u$, not u itself. The method of moving planes was first invented by A.D. Alexandrov, and was shown to be a powerful tool in studying equations (1.3) and (1.4) by Gidas-Ni-Nirenberg [GNN], Caffarelli-Gidas-Spruck [CGS], Chen-Li [CL] and many others. As usual, in order to start the process of moving planes at ∞, we have to understand the asymptotic behavior of both u and Δu at infinity. The analysis of asymptotic behaviors will be carried out in Section 2. In Section 3, we will establish the radial symmetry and prove Theorem 1.1 and Theorem 1.2. In Section 4, both Theorem 1.3 and Theorem 1.4 are proved.

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2. Asymptotic behavior

In this section, we want to study the asymptotic behavior for a solution u of (1.7). First, we note that the fundamental solution of the biharmonic operator Δ^2 in R^4 is

$$P(x,y) = \frac{1}{8\pi^2} \log \frac{1}{|x-y|}.$$

Let u be a solution of (1.7). Set

$$\alpha = \frac{3}{4\pi^2} \int_{\mathbf{R}^4} e^{4u(y)} dy, \tag{2.1}$$

and

$$v(x) = \frac{3}{4\pi^2} \int_{\mathbf{R}^4} \log(\frac{|x-y|}{|y|}) e^{4u(y)} dy. \tag{2.2}$$

Obviously, $v(x)$ satisfies

$$\Delta^2 v(x) = -e^{4u(x)} \quad \text{in} \quad R^4. \tag{2.3}$$

Lemma 2.1. *Suppose u is a solution of (1.7). Let α be given as in (2.1). Then there exists a constant C such that*

$$v(x) \leq \alpha \log |x| + C$$

Proof. For $|x| \geq 4$, we decompose $\mathbf{R}^4 = A_1 \cup A_2$, where $A_1 = \{y \mid |y-x| \leq \frac{|x|}{2}\}$ and $A_2 = \{y \mid |y-x| \geq \frac{|x|}{2}\}$. For $y \in A_1$, we have $|y| \geq |x| - |x-y| \geq \frac{|x|}{2} \geq |x-y|$, which implies

$$\log \frac{|x-y|}{|y|} \leq 0. \tag{2.4}$$

Since $|x-y| \le |x|+|y| \le |x||y|$ for $|x|,|y| \ge 2$ and $\log|x-y| \le \log|x| + C$ for $|x| \ge 4$ and $|y| \le 2$, we have

$$\begin{aligned} v(x) &\le \frac{3}{4\pi^2}\int_{A_2} \log\frac{|x-y|}{|y|} e^{4u(y)}dy \\ &\le \frac{3}{4\pi^2}(\log|x|\int_{A_2} e^{4u(y)}dy + \int_{|y|\le 2}\log\frac{|x-y|}{|y|}e^{4u(y)}dy \\ &\le \frac{3}{4\pi^2}(\int_{\mathbf{R}^4} e^{4u(y)}dy)\log|x| + C \\ &= \alpha\log|x| + C. \end{aligned}$$

□

Lemma 2.2. *Suppose u is a solution of (1.7). Then $\Delta u(x)$ can be represented by*

$$\Delta u(x) = \frac{-3}{2\pi^2}\int_{\mathbf{R}^4}\frac{e^{4u(y)}}{|x-y|^2}dy - C_1 \tag{2.5}$$

where $C_1 \ge 0$ is a constant.

Proof. Let $w(x) = u(x)+v(x)$. By (2.3), we have $\Delta^2 w(x) = 0$ in $\mathbf{R}^4$. Since $\Delta w(x)$ is a harmonic fumction in $\mathbf{R}^4$, we have for any $x_0 \in \mathbf{R}^4$ and $r > 0$,

$$\begin{aligned} \Delta w(x_0) &= \frac{2}{\pi^2 r^4}\int_{|y-x_0|\le r}\Delta w(y)dy \\ &= \frac{2}{\pi^2 r^4}\int_{|y-x_0|=r}\frac{\partial w}{\partial r}(y)d\sigma. \end{aligned} \tag{2.6}$$

where $\pi^2/2$ is the volume of the unit ball and $d\sigma$ denotes the area element of the sphere $|y-x_0| = r$.

Integrating (2.6) along r, we have

$$\frac{r^2}{8}\Delta w(x_0) = \fint_{|x-x_0|=r} w d\sigma - w(x_0),$$

where $\fint_{|x-x_0|=r} w d\sigma = \frac{1}{2\pi^2 r^3}\int_{|x-x_0|=r} w d\sigma$ is the integral average of w over the sphere $|x-x_0| = r$. Hence, by the Jensen inequality,

$$\begin{aligned} \exp(\frac{r^2}{2}\Delta w(x_0)) &\le e^{-4w(x_0)}\exp(4\fint_{|x-x_0|=r} w d\sigma) \\ &\le e^{-4w(x_0)}\fint_{|x-x_0|=r} e^{4w}d\sigma. \end{aligned}$$

Since $w(x) = u(x) + v(x) \le u(x) + \alpha \log |x| + c$, we have $r^{3-4\alpha} \exp(\frac{\Delta w(x_0)}{2} r^2) \in L^1[1, \infty)$. Thus $\Delta w(x_0) \le 0$ for all $x_0 \in \mathbf{R}^4$. By Liouville's Theorem, $\Delta w(x) \equiv -C_1$ in $\mathbf{R}^4$ for some constant $C_1 \ge 0$. Hence (2.5) follows immediately. □

Let $h(x)$ be the solution of

$$\begin{cases} \Delta^2 h(x) = f(x) \ \text{ in } \Omega, \\ \Delta h(x) = h(x) = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^4$. Following the argument of [BM], we have

Lemma 2.3. *Suppose $f \in L^1(\bar{\Omega})$. For any $\delta \in (0, 32\pi^2)$, there exists a constant $C_\delta > 0$ such that the inequality,*

$$\int_\Omega \exp(\frac{\delta |h|}{\|f\|_{L^1}}) dx \le C_\delta (\text{diam } \Omega)^4, \tag{2.7}$$

where diam Ω denotes the diameter of Ω.

Proof. Without loss of generality, we may assume $0 \in \Omega$. Let $R = \text{diam } \Omega$. Set

$$v(x) = \frac{1}{8\pi^2} \int_{B_R(0)} \log(\frac{2R}{|x-y|}) |\tilde{f}(y)| dy$$

where $\tilde{f}(y) = f(y)$ for $y \in \Omega$ and $\tilde{f}(y) = 0$ for $y \notin \Omega$. By a direct computation, we have

$$\Delta v(x) = \frac{-1}{4\pi^2} \int_{B_R(0)} |x-y|^{-2} |\tilde{f}(y)| dy \le 0 \tag{2.8}$$

for $x \in \Omega$. Since both v and $-\Delta v$ are positive on $\partial\Omega$, we have by the maximum principle,

$$|h(x)| \le v(x) \ \text{ for } \ x \in \Omega.$$

Applying the Jensen inequality, we have

$$\begin{aligned} &\int_\Omega \exp(\frac{\delta |h(x)|}{\|f\|_{L^1}}) dx \\ &\quad \le \int_\Omega \exp(\frac{\delta}{8\pi^2} \int_{B_R(0)} \log(\frac{2R}{|x-y|}) d\mu(y)) \\ &\quad \le \int_\Omega \int_{B_R(0)} (\frac{2R}{|x-y|})^{\frac{\delta}{8\pi^2}} d\mu(y) dx \le C_\delta R^4, \end{aligned}$$

where $d\mu(y) = \frac{1}{\|f\|_{L^1}} |\tilde{f}(y)| dy$. Hence Lemma 2.3 is proved. □

Lemma 2.4. *Let u be a solution of (1.7) and v is defined by (2.2). Then, given any $\varepsilon > 0$, there exists a $R = R(\varepsilon)$ such that for $|x| \geq R$, $v(x)$ satisfies*

$$v(x) \geq (\alpha - \varepsilon)\log|x|, \quad \textit{and}, \tag{2.9}$$

$$\lim_{|x|\longrightarrow+\infty} \Delta v(x) = 0. \tag{2.10}$$

Proof. To prove (2.9), we first claim that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that

$$v(x) \geq (\alpha - \frac{\varepsilon}{2})\log|x| + \frac{3}{4\pi^2}\int_{B(x,1)} \log|x-y|e^{4u(y)}dy. \tag{2.11}$$

To prove (2.11), we decompose $\mathbf{R}^4 = A_1 \cup A_2 \cup A_3$, where $A_1 = \{y \mid |y| < R_0\}$, $A_2 = \{y \mid |x-y| \leq \frac{|x|}{2}, |y| \geq R_0\}$, and $A_3 = \{y \mid |x-y| \geq \dfrac{|x|}{2}, |y| \geq R_0\}$. Let $R_0 = R_0(\varepsilon)$ be sufficiently large such that

$$\frac{3}{4\pi^2}\int_{A_1} \log\frac{|x-y|}{|y|}e^{4u(y)}dy \geq (\alpha - \frac{\varepsilon}{4})\log|x| \tag{2.12}$$

for large $|x|$.

For $|x|$ large, we have

$$\begin{aligned}
&\int_{A_2} \log(\frac{|x-y|}{|y|})e^{4u(y)}dy \\
&\quad = \int_{A_2} \log|x-y|e^{4u(y)}dy - \int_{A_2} \log|y|e^{4u(y)}dy \\
&\quad \geq \int_{B(x,1)} \log|x-y|e^{4u(y)}dy - \log(2|x|)\int_{A_2} e^{4u(y)}dy.
\end{aligned}$$

For $y \in A_3$ and $|y| \leq 2|x|$ we have $|x-y| \geq \dfrac{|x|}{2} \geq \dfrac{|y|}{4}$. For $x \in A_3$ and $|y| \geq 2|x|$ we have $|x-y| \geq |y| - |x| \geq \dfrac{|y|}{2}$. In any case, we have for $y \in A_3$,

$$\frac{|x-y|}{|y|} \geq \frac{1}{4}.$$

Hence

$$\int_{A_3} \log\frac{|x-y|}{|y|}e^{4u(y)}dy \geq \log(\frac{1}{4})\int_{A_3} e^{4u(y)}dy. \tag{2.14}$$

By (2.12), (2.13) and (2.14), we have (2.11) for large $|x|$.

Let $0 < \varepsilon_0 < \pi^2$ and $R_0 = R_0(\varepsilon_0) > 0$ be sufficiently large such that

$$6 \int_{B(x,4)} e^{4u} dy \le \varepsilon_0 \tag{2.15}$$

for $|x| \ge R_0$. Let h be the solution of

$$\begin{cases} \Delta^2 h = 6e^{4u(y)} & \text{in} \quad B(x,4), \\ h = \Delta h = 0 & \text{on } \partial B(x,4). \end{cases}$$

By Lemma 2.3, we have for small ε_0,

$$\int_{B(x,4)} e^{12|h|} dy \le c_1, \tag{2.16}$$

for some constant c_1 independent of x.

Set $q(y) = u(y) - h(y)$ for $y \in B(x,4)$. Then q satisfies

$$\begin{cases} \Delta^2 q(y) = 0 & \text{on} \quad B(x,4), \\ \Delta q = \Delta u \quad \text{and} \quad q = u & \text{on} \quad \partial B(x,4). \end{cases} \tag{2.17}$$

Let $\tilde{q}(y) = -\Delta q(y)$. By Lemma 2.2, $\tilde{q}(y)$ is harmonic with positive boundary value on $\partial B(x,2)$. Applying the maximum principle, we have $\tilde{q}(y) > 0$ in $B(x,4)$. Thus, by the Harnack inequality, we have

$$\tilde{q}(y) \le c_2 \tilde{q}(x) = -c_2 \fint_{\partial B(x,4)} \Delta u d\sigma \tag{2.18}$$

for $y \in \bar{B}(x,2)$ where c_2 is a constant depending on n only.

Integrating the equation (1.1), we have for any $r > 0$,

$$\int_{\partial B(x,r)} \frac{\partial}{\partial r}(\Delta u) d\sigma = 6 \int_{B(x,r)} e^{4u} dy.$$

Integrating the identity above along r, we have

$$\fint_{\partial B(x,r)} \Delta u - \Delta u(x) = \frac{3}{2\pi^2} \int_{B(x,r)} \left(\frac{1}{|x-y|^2} - \frac{1}{r^2}\right) e^{4u(y)} dy \tag{2.19}$$

Applying Lemma 2.2 and (2.19), we have

$$-\fint_{\partial B(x,r)} \Delta u = \frac{3}{2\pi^2} \int_{|x-y| \ge r} \frac{e^{4u(y)}}{|x-y|^2} dy + \frac{3}{2\pi^2 r^2} \int_{B(x,r)} e^{4u(y)} dy + C_1.$$

In particular, we have $r = 4$,

$$-\!\!\!\!\!\!\int_{\partial B(x,4)} \Delta u \le c_3. \tag{2.20}$$

Hence, by (2.18), we have

$$\tilde{q}(y) \le c_4 \quad \text{for} \quad y \in \overline{B(x,2)}. \tag{2.21}$$

Since q satisfies

$$\begin{cases} \Delta q(y) = -\tilde{q}(y) & \text{in } B(x,4), \\ q = u & \text{on } \partial B(x,4), \end{cases}$$

by estimates for linear elliptic equations (e.g. see Theorem 8.17 in [GT]), we have for any $p > 1$ and $\sigma > 2$

$$\sup_{B(x,1)} q \le c\ (\|q^+\|_{L^p(B(x,2))} + \|\tilde{q}\|_{L^\sigma(B(x,2))}), \tag{2.22}$$

where $q^+ = \max(q, 0)$ and $c = c(p, \sigma)$. Recall $q = u - h$. Thus, $q^+(y) \le u^+(y) + |h(y)|$ for $y \in B(x,4)$. By (2.15), we have

$$\int_{B(x,2)} q^{+^p} \le c_5 \int_{B(x,2)} e^{2q^+} \le c_5 (\int_{B(x,2)} e^{4u^+})^{\frac{1}{2}} (\int_{B(x,2)} e^{4|h|})^{\frac{1}{2}}.$$

Since $e^{4u^+} \le 1 + e^{4u}$, we have together with (2.21),

$$\sup_{B(x,1)} q \le c_6. \tag{2.23}$$

Since $u = h + q$, we have

$$u(y) \le h(y) + q(y) \le c_6 + |h(y)|$$

for $y \in B(x,1)$. Therefore,

$$\int_{B(x,1)} e^{12u} \le c_7 \int_{B(x,1)} e^{12|h|} dy \le c_8, \tag{2.24}$$

and then,

$$\begin{aligned} |\int_{B(x,1)} \log|x-y| e^{4u(y)} dy| \le & (\int_{B(x,1)} (\log \frac{1}{|x-y|})^2 dy)^{\frac{1}{2}} \\ & (\int_{B(x,1)} (e^{8u(y)} dy)^{\frac{1}{2}} \le c_9, \end{aligned}$$

where c_9 is a constant in dependent of x. By (2.11), (2.9) follows immediately. By (2.24), it is an elementary exercise to prove $\lim_{|x|\longrightarrow\infty} \Delta v(x) = 0$. □

Lemma 2.5. *Suppose* $|u(x)| = o(|x|^2)$ *at* ∞. *Then*

$$u(x) = \frac{3}{2\pi^2}\int_{\mathbf{R}^4} \log(\frac{|y|}{|x-y|})e^{4u(y)}dy + C_0 \tag{2.25}$$

where C_0 *is a constant. Furthermore, for any* $\varepsilon > 0$, *there exists* $R = R(\varepsilon) > 0$ *such that* $u(x)$ *satisfies*

$$-\alpha \log|x| \le u(x) \le (-\alpha + \varepsilon)\log|x| \tag{2.26}$$

for $|x| \ge R(\varepsilon)$.

Proof. By Lemma 2.2, we have

$$\Delta u(x) = \frac{-3}{2\pi^2}\int_{2\pi^2}\int_{\mathbf{R}^4} |x-y|^{-2}e^{4u(y)}dy - C_1.$$

Suppose $|u(x)| = o(|x|^2)$. First, we claim $C_1 = 0$. Otherwise, we have $\Delta u(x) \le -C_1 < 0$ for $|x| \ge R_0$ where R_0 is sufficiently large.

Let

$$h(y) = u(y) + \varepsilon|y|^2 + A(|y|^{-2} - R_0^{2-n}) \tag{2.27}$$

where ε is small such that

$$\Delta h(y) = \Delta u + 8\varepsilon < -\frac{C_1}{2} < 0 \tag{2.28}$$

for $|y| \ge R_0$, and A is sufficiently large so that $\inf_{|y|\ge R_0} h(y)$ is achieved by some $y_0 \in R^4$ with $|y_0| > R_0$. This can be done because $\lim_{|y|\longrightarrow+\infty} h(y) = +\infty$ for any $A > 0$. Applying the maximum principle to (2.28) at y_0, we have a contradiction. Hence the claim is proved.

By the claim, we have $\Delta(u+v) = 0$ in $\mathbf{R}^4$. By the assumption and Lemma 2.1, we have $|u+v(x)| = o(|x|^2)$ at ∞. Since $u+v$ is a harmonic function, by the gradient estimates of harmonic functions, we have $u(x)+v(x) = \sum_{j=1}^{4} a_j x_j + a_0$ for some constants $a_j \in \mathbf{R}$, $0 \le j \le 4$. Thus,

$$e^{4u(x)} = e^{a_0}e^{-4v(x)}e^{\sum_{j=1}^4 a_j x_j} \ge \text{const. } |x|^{-4\alpha}e^{\sum_{j=1}^4 a_j x_j}.$$

Since $e^{4u(x)} \in L^1(\mathbf{R}^4)$, we have $a_j = 0$ for $1 \le j \le 4$. Hence, we have proved (2.25). Obviously, (2.26) immediately follows from (2.25), (2.9) and Lemma 2.1. The proof of Lemma 2.5 is finished. □

Now suppose u is a smooth solution of

$$\Delta^2 u = Q(x)e^{4u} \quad \text{in} \quad \mathbf{R}^4 \tag{2.29}$$

where $Q(x) \in C^1(\mathbf{R}^4)$. Then we have the following Pohozaev identity.

Lemma 2.6. *Suppose u is an entire smooth function of (2.29). Then for any $R > 0$, we have*

$$\begin{aligned}
&\int_{B_R} Q(x)e^{4u}dx + \frac{1}{4}\int_{B_R}(x\cdot\nabla Q)e^{4u}dx \\
&\quad = \frac{1}{4}\int_{\partial B_R} Q(x)|x|e^{4u}d\sigma - \int_{\partial B_R}|x|[\frac{(\Delta u)^2}{2} + |x|\frac{\partial u}{\partial r}\frac{\partial}{\partial r}\Delta u]d\sigma \\
&\quad + \int_{\partial B_R}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})\Delta u d\sigma.
\end{aligned} \tag{2.30}$$

Proof. The proof of Lemma 2.6 goes exactly the same as in the case of the semilinear elliptic equations of second order. For the sake of completeness, we give a proof here.

Multiplying $x \cdot \nabla u$ on the both sides of the equation (2.29), we have

$$\begin{aligned}
&\frac{1}{4}\int_{\partial B_R} Q|x|e^{4u}d\sigma - \frac{1}{4}[\int_{B_R}[(x\cdot\nabla Q) + 4Q]e^{4u}dx \\
&\quad = \int_{B_R}(x\cdot\nabla u)Qe^{4u}dx = \int_{B_R}(x\cdot\nabla u)\Delta^2 u dx \\
&\quad = \int_{B_R}\Delta(x\cdot\nabla u)\Delta u dx + \int_{\partial B_R}[r\frac{\partial u}{\partial r}\frac{\partial}{\partial r}(\Delta u) - \Delta u\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})]d\sigma \\
&\quad = \int_{\partial B_R}|x|[\frac{|\Delta u|^2}{2} + \frac{\partial u}{\partial r}\frac{\partial}{\partial r}(\Delta u)]d\sigma - \int_{\partial B_R}\Delta u\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})d\sigma,
\end{aligned}$$

where we have ultilized $\frac{1}{2}$ div $(x(\Delta u)^2) = \Delta(x\cdot\nabla u)\Delta u$. Obviously, (2.30) follows immediately. □

Lemma 2.7. *Let u be a solution of (1.7) and $u(x) = o(|x|^2)$ at ∞. Then $\alpha = 2$.*

Proof. By Lemma 2.5, we have

$$u(x) = \frac{3}{4\pi^2}\int_{\mathbf{R}^4}\log(\frac{|y|}{|x-y|})e^{4u(y)}dy + C_0.$$

By elementary calculations, we have

$$|x|\frac{\partial u}{\partial r}(x) = -\frac{3}{4\pi^2}\int_{\mathbf{R}^4} \frac{x\cdot(x-y)}{|x-y|^2}e^{4u(y)}dy, \tag{2.31}$$

$$\begin{aligned}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})(x) = &-\frac{3}{4\pi^2}\int_{\mathbf{R}^4} \frac{2r^2 - x\cdot y}{r|x-y|^2}e^{4u(y)}dy \\ &+ \frac{3}{2\pi^2}\int_{\mathbf{R}^4} \frac{(x\cdot(x-y))^2}{r|x-y|^4}e^{4u(y)}dy,\end{aligned} \tag{2.32}$$

and,

$$\Delta u(x) = -\frac{3}{2\pi^2}\int_{\mathbf{R}^4} \frac{e^{4u(y)}}{|x-y|^2}dy \tag{2.33}$$

Since $e^{4u(y)} \geq |y|^{-4\alpha}$ for large $|y|$ by lemma 2.5, we have $\alpha > 1$. Therefore, it is easy to calculate from (2.31) $\sim$ (2.33) that

$$\lim_{|x|\longrightarrow+\infty} |x|\frac{\partial u}{\partial r}(x) = -\alpha, \tag{2.34}$$

$$\lim_{|x|\longrightarrow+\infty} \frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})(x)|x| = 0, \tag{2.35}$$

$$\lim_{|x|\longrightarrow+\infty} \Delta u(x)|x|^2 = -2\alpha, \tag{2.36}$$

and

$$\lim_{|x|\longrightarrow+\infty} \frac{\partial}{\partial r}(\Delta u(x))|x|^3 = 4\alpha. \tag{2.37}$$

Applying the Pohozaev identity and (2.34) $\sim$ (2.37), the right hand side of (2.30) (Here, $Q(x) = 6$) tends to $4\pi^2\alpha^2$. Hence, we have

$$8\pi^2\alpha = 4\pi^2\alpha^2,$$

which implies $\alpha = 2$. □

Lemma 2.8. *Let u satisfy the assumption of Lemma 2.5. Then $u(x)$ satisfies*

$$u(x) = -2\log|x| + c + O(|x|^{-1}), \tag{2.38}$$

and

$$\begin{cases} -\Delta u(x) = 4|x|^{-2} + \sum_{j=1}^4 a_j x_j |x|^{-4} + O(x^{-4}), \\ -\dfrac{\partial}{\partial x_i}\Delta u(x) = -8x_i|x|^{-4} + O(|x|^{-4}), \\ -\dfrac{\partial^2}{\partial x_i \partial x_j}\Delta u(x) = O(|x|^{-4}) \end{cases} \tag{2.39}$$

for large $|x|$, *where* $c, a_j, 1 \le j \le 4$ *are constant.*

Proof. Let $w(x) = u(\frac{x}{|x|^2}) - 2\log|x|$. By a straightforward computation, $w(x)$ satisfies

$$\begin{cases} \Delta^2 w(x) = 6e^{4w(x)} \ \text{ in } \ \mathbf{R}^4\backslash\{0\}, \\ |w(x)| = o(\log\frac{1}{|x|}) \ \text{ and } \ |\Delta w(x)| = o(\frac{1}{|x|^2}) \ \text{ as } \ |x| \longrightarrow 0. \end{cases} \tag{2.40}$$

Set $h(x)$ be the solution of

$$\begin{cases} \Delta^2 h(x) = 6e^{4w(x)} \ \text{ in } B_1 \\ h(x) = w(x) \ \text{ on } \ \partial B_1, \ \ \Delta h(x) = \Delta w(x), \ \text{on } \partial B_1. \end{cases} \tag{2.41}$$

Since Lemma 2.5 implies $e^{4w(x)} \in L^p(\bar{B}_1)$ for any $p > 1$, by the regularity theorems of linear elliptic equations, $h(x) \in C^{3,\tau}(\bar{B}_1)$ for any $0 < \tau < 1$. Let $q(x) = w(x) - h(x)$. Then $q(x)$ satisfies

$$\begin{cases} \Delta^2 q = 0 \ \text{ in } B_1\backslash\{0\}, \\ q = \Delta q = 0 \ \text{ on } \partial B_1, \\ |q(x)| = o(\log\frac{1}{|x|}), |\Delta q(x)| = o(\frac{1}{|x|^2}) \ \text{ as } |x| \longrightarrow 0. \end{cases} \tag{2.42}$$

By the maximum principle, we have, for any $\varepsilon > 0$

$$|\Delta q(x)| \le \varepsilon/|x|^2$$

for $x \in \bar{B}_1$. Applying the maximum principle again, we have

$$|q(x)| \le \varepsilon \log\frac{1}{|x|}.$$

Thus, $q(x) \equiv 0$. Namely, $w(x) = h(x) \in C^{3,\tau}(\bar{B}_1)$. By the regularity of the linear elliptic equation again, we have $w(x) \in C^\infty(\bar{B}_1)$. It is not difficult to see that (2.39) follows immediately. □

3. Radial symmetry

Now we are in the position to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that u is a smooth entire solution of (1.7) such that $u(x) = o(|x|^2)$ at ∞. Let $v(x) = -\Delta u(x)$. By Lemma 2.8, $v(x)$ has a harmonic asymptotic expansion at ∞:

$$\begin{cases} v(x) &= \frac{1}{|x|^2}(4 + \sum_{j=1}^4 \frac{a_j x_j}{|x|^2}) + O(\frac{1}{|x|^4}), \\ v_{x_i} &= \frac{-8x_i}{|x|^4} + O(\frac{1}{|x|^4}), \\ v_{x_i x_j} &= O(\frac{1}{|x|^4}). \end{cases} \tag{3.1}$$

We want to apply the method of moving planes to prove that u is symmetric about some point in $\mathbf{R}^4$. Following conventional notations, we let for any λ, $T_\lambda = \{x = (x_1, \dots, x_4) \mid x_1 = \lambda\}, \Sigma_\lambda = \{x \mid x_1 > \lambda\}$ and $x^\lambda = (2\lambda - x_1, x_2, \dots, x_4)$ be the reflection point of x with respect to T_λ. To start the process of moving planes along the x_1-direction, we need two lemmas below.

Lemma 3.1. *Let v be a positive function defined in a neighborhood at infinity satisfying the asymptotic expansion (3.1). Then there exists $\bar{\lambda}_0 < 0$ and $R > 0$ such that the inequality*

$$v(x) > v(x^\lambda)$$

holds for $\lambda \le \bar{\lambda}_0, |x| \ge R$ and $x \in \Sigma_\lambda$.

Lemma 3.2. *Suppose v satisfies the assumption of Lemma 3.1, and $v(x) > v(x^{\lambda_0})$ for $x \in \Sigma_{\lambda_0}$. Assume $v(x) - v(x^{\lambda_0})$ is superharmonic in Σ_{λ_0}. Then there exist $\varepsilon > 0$, $S > 0$ such that the followings hold.*

(i) $v_{x_1} > 0$ in $|x_1 - \lambda_0| < \varepsilon$ and $|x| > S$.
(ii) $v(x) > v(x^\lambda)$ in $x_1 \ge \lambda_0 + \dfrac{\varepsilon}{2} > \lambda$ and $|x| > S$

for all $x \in \Sigma_\lambda, \lambda \le \lambda_1$ with $|\lambda_1 - \lambda_0| < c_0\varepsilon$, where c_0 is a small positive number depending on λ_0 and v only.

The proofs of both lemmas are contained in [CGS]. Please see Lemma 2.3 and Lemma 2.4 in [CGS] for their proofs.

For any λ, we consider $w_\lambda(x) = u(x) - u(x^\lambda)$ in Σ_λ. Then $w_\lambda(x)$ satisfies

$$\begin{cases} \Delta^2 w_\lambda(x) = b_\lambda(x) w_\lambda & \text{in } \Sigma_\lambda, \\ w_\lambda(x) = \Delta w_\lambda(x) = 0 & \text{on } T_\lambda, \end{cases}$$

where $b_\lambda(x) = 6\dfrac{e^{4u(x)} - e^{4u(x^\lambda)}}{u(x) - u(x^\lambda)} > 0$ in $\bar{\Sigma}_\lambda$. By Lemma 3.1, $\Delta w_\lambda(x) = v(x^\lambda) - v(x) < 0$ for $x \in \Sigma_\lambda$, $\lambda \le \bar{\lambda}_0$ and $|x| \ge R$. Since $v(x) > 0$ in $\mathbf{R}^4$, there exists $\bar{\lambda}_1 < \bar{\lambda}_0$ such that

$$v(x^\lambda) < v(x)$$

for $|x| \le R$ and $\lambda \le \bar{\lambda}_1$. Therefore, we have

$$\Delta w_\lambda(x) < 0$$

in Σ_λ for $\lambda \le \bar{\lambda}_1$. By Lemma 2.8, $\lim\limits_{|x| \longrightarrow +\infty} w_\lambda(x) = 0$. Applying the maximum principle, we have $w_\lambda(x) > 0$ in Σ_λ for all $\lambda \le \bar{\lambda}_1$.

Let $\lambda_0 = \sup\{\lambda \mid v(x^\mu) \leq v(x) \text{ for } x \in \Sigma_\mu \text{ and } \mu \leq \lambda\}$. Since $v(x)$ tends to zero at ∞, it is not difficult to see that $\lambda_0 < +\infty$. We claim that

$$u(x) \equiv u(x^{\lambda_0})$$

for all $x \in \Sigma_{\lambda_0}$.

The claim will be proved by contradiction. Suppose $w_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}. By continuity, $\Delta w_{\lambda_0}(x) \leq 0$ in Σ_{λ_0}. Since $w_{\lambda_0}(x)$ tends to 0 as $|x| \longrightarrow +\infty$ by (2.38), the strong maximum principle implies $w_{\lambda_0}(x) > 0$ in Σ_{λ_0}. By applying equaiton (1.7), we have $\Delta^2 w_{\lambda_0}(x) = 6(e^{4u(x)} - e^{4u(x^{\lambda_0})}) > 0$, which implies Δw_{λ_0} is a subharmonic function. Applying the strong maximum principle again, we have $\Delta w_{\lambda_0}(x) < 0$ in Σ_{λ_0}.

By the definition of λ_0, there exists a sequence $\lambda_n \uparrow \lambda_0$ such that $\sup\limits_{\Sigma_{\lambda_n}} \Delta w_{\lambda_n}(x) > 0$. Since $\lim\limits_{|x|\longrightarrow+\infty} \Delta w_{\lambda_n}(x) = 0$, there exists $x_n \in \Sigma_{\lambda_n}$ such that $\Delta w_{\lambda_n}(x_n) = \sup\limits_{\Sigma_{\lambda_n}} \Delta w_{\lambda_n}(x) > 0$. By Lemma 3.2. x_n is bounded. Without loss of generality, we may assume $x_0 = \lim\limits_{n\longrightarrow+\infty} x_n$. If $x_0 \in \Sigma_{\lambda_0}$, then by the continuity, we have $\Delta w_{\lambda_0}(x_0) = 0$, which yields a contradition to $\Delta w_{\lambda_0}(x) < 0$ in Σ_{λ_0}. If $x_0 \in T_{\lambda_0}$, then $\bigtriangledown(\Delta w_{\lambda_0}(x_0)) = 0$, which yields a contradiction to the Hopf boundary Lemma because Δw_{λ_0} is a negative subharmonic function in Σ_{λ_0}. Therefore, the claim is proved. Obviously, the radial symmetry of u follows from the claim.

By a straightforward computation, $u_\lambda(|x|) \equiv \log(\dfrac{2\lambda}{1+\lambda^2|x|^2})$ is a family of solutions of (1.7) for $\lambda > 0$. Now let $\omega(r)$ be a radial solution of (1.7). From the uniqueness of ODE, $\omega(r)$ is completely determined by $\omega(0)$ and $\Delta\omega(0) = 4\omega''(0)$ (ω always satisfies $\omega'(0) = \omega'''(0) = 0$). Without loss of generality, we may assume $\omega(0) = u_{\lambda_0}(0)$ for some $\lambda_0 > 0$. If $\omega''(0) < u''_{\lambda_0}(0)$, then $\omega(r) < u_{\lambda_0}(r)$ for small $r > 0$. We first claim $u_{\lambda_0}(r) > \omega(r)$ for all $r > 0$.

Suppose there exists $r_0 > 0$ such that $u_{\lambda_0}(r_0) = \omega(r_0)$ and $u_{\lambda_0}(r) > \omega(r)$ for $0 \leq r < r_0$. Then, by (1.7),

$$\frac{\partial}{\partial r}\Delta(u_{\lambda_0}(r) - \omega(r)) > 0$$

for $0 < r \leq r_0$. In particular, $\Delta(u_{\lambda_0}(r) - \omega(r)) > 0$ for $0 \leq r \leq r_0$. Since $u_{\lambda_0}(r) - \omega(r) = 0$ on $r = r_0$, the maximum principle implies $u_{\lambda_0}(r) - \omega(r) < 0$ for all $0 \leq r \leq r_0$, which yields a contradiction to $u_{\lambda_0}(0) = \omega(0)$. Thus, the claim is proved.

From the proof above, we also have $\Delta u_{\lambda_0}(r) - \Delta\omega(r)$ is increasing in r. Thus, $\omega(r) \sim -cr^2$ as $r \longrightarrow +\infty$ for some constant $c > 0$.

If $\omega''(0) > u''_{\lambda_0}(0)$, then we have $\omega(r) > u_{\lambda_0}(r)$ for all $r > 0$. By the equation (1.7), $\Delta\omega(r) - \Delta u_{\lambda_0}(r)$ is increasing in r. Thus, if $\omega(r)$ exists for all $r > 0$ then

$\omega(r) \geq cr^2$ for r large and for some $c > 0$. Hence, $\int_{\mathbf{R}^4} e^{4\omega(|x|)} dx = +\infty$, and the proof of Theorem 1.1 is completely finished. □

Lemma 3.3. *Suppose that u is a harmonic function in $\mathbf{R}^n$ such that* $\exp(u - c|x|^2) \in L^1(\mathbf{R}^n)$ *for some $c > 0$. Then u is a polynomial of order at most 2.*

Proof. For any unit vector $\xi \in \mathbf{R}^4$, we want to prove $u_{\xi\xi}(x) \equiv a$ constant. By Liouville's Theorem, it suffices to prove $u_{\xi\xi}(x)$ is bounded from above by a constant independent of x. Without loss of generality, we may take $x = 0$ and $\xi = e_1$.

Since $u_{x_1x_1}$ is harmonic, we have for any $r > 0$,

$$\begin{aligned} u_{x_1x_1}(0) &= \frac{1}{\sigma_n r^n} \int_{B_r(0)} u_{x_1x_1}(x) dx \\ &= \frac{1}{\sigma_n r^n} \int_{\partial B_r(0)} u_{x_1} \frac{x_1}{|x|} d\sigma \end{aligned}$$

where σ_n is the volume of the unit ball in $\mathbf{R}^n$. Integrating the identity along r, we have

$$\begin{aligned} &\frac{\sigma_n}{n+1} r^{n+1} u_{x_1x_1}(0) \qquad (3.2) \\ &\quad = \int_{B_r} u_{x_1} \frac{x_1}{|x|} dx \\ &\quad = -\int_{B_r} u \frac{\partial}{\partial x_1}(\frac{x_1}{|x|}) dx + \int_{\partial B_r(0)} u \frac{x_1^2}{|x|^2} d\sigma \\ &\quad = -\int_{B_r} u (\frac{1}{|x|} - \frac{x_1^2}{|x|^3}) dx + \int_{\partial B_r(0)} u \frac{x_1^2}{|x|^2} d\sigma \\ &\quad = -\int_{B_r} \frac{u}{|x|} dx + \int_{B_r} u \frac{x_1^2}{|x|^3} dx + \int_{\partial B_r(0)} u \frac{x_1^2}{|x|^2} d\sigma \end{aligned}$$

The first integration can be written as

$$\begin{aligned} \int_{B_r} \frac{u}{|x|} dx &= \int_0^s (⨍_{\partial B_s(0)} u d\sigma)(n\sigma_n) s^{n-2} ds \\ &= n\sigma_n u(0) \int_0^r s^{n-2} ds \\ &= n\sigma_n u(0) \frac{r^{n-1}}{n-1} \\ &= (\frac{n\sigma_n}{n-1}) r^{n-1} u(0). \qquad (3.3) \end{aligned}$$

By a direct computation, we have

$$\int_{B_r} \frac{x_1^2}{|x|^3} dx = \frac{\sigma_n}{n-1} r^{n-1}, \tag{3.4}$$

and

$$\int_{\partial B_r} \frac{x_1^2}{|x|^2} d\sigma = \sigma_n r^{n-1}. \tag{3.5}$$

By (3.2), we have

$$\frac{r^2}{n+1} u_{x_1x_1}(0) = -\frac{n}{n-1} u(0) + \frac{1}{n-1} \fint_{B_r(0)} u d\mu_1 + \fint_{\partial B_r(0)} u d\mu_2,$$

where $d\mu_1 = \dfrac{x_1^2}{|x|^3} dx$ and $d\mu_2 = \nu_1^2 d\sigma$ on $= \partial B_r(0)$. By Jensen's inequality, we have

$$\begin{aligned} &\exp\left(\frac{r^2}{2(n+1)} u_{x_1x_1}(0)\right) \\ &\quad \le \exp\left(-\frac{n}{2(n-1)} u(0)\right)\left(\fint_{B_r(0)} e^{\overline{\frac{u}{2(n-1)}}} d\mu_1\right) \cdot \left(\fint_{\partial B_r(0)} e^{\frac{u}{2}} d\mu_2\right) \end{aligned}$$

For any positive $c_1 > 0$, we have

$$\begin{aligned} &\int_1^\infty \exp\left[\left(\frac{1}{2(n+1)} u_{x_1x_1}(0) - c_1\right) r^2\right] dr \\ &\quad \le \exp\left(-\frac{n}{2n-1} u(0)\right)\left(\int_1^\infty \left(\fint_{B_r(0)} u e^{\overline{\frac{u}{n-1}}} d\mu_1\right) e^{-c_1 r^2} dr\right)^{\frac{1}{2}} \\ &\quad \left(\int_1^\infty \left(\fint_{\partial B_r(0)} u e^u d\mu_1 e^{-c_1 r^2} dr\right)^{\frac{1}{2}}. \end{aligned} \tag{3.6}$$

By the assumption, we can choose a large c_1 such that the right hand side of (3.6) is finite. Thus, we have

$$u_{x_1x_1}(0) \le 2(n+1)c_1.$$

By Liouville's Theorem, we have $u_{x_1x_1}(x) \equiv$ constant. Obviously, the conclusion of Lemma 3.3 follows immediately. □

Proof of Theorem 1.2. Suppose that u is a solution of (1.7) with $e^{4u} \in L^1(\mathbf{R}^4)$. Let

$$v(x) = \frac{3}{4\pi^2} \int_{\mathbf{R}^4} \log\left(\frac{|x-y|}{|y|}\right) e^{4u(y)} dy,$$

and $w(x) \equiv u(x) + v(x)$. By Lemma 2.2, we have $\Delta w(x) \equiv -C_1$ in $\mathbf{R}^4$. Applying Lemma 3.3, we have $w(x) \equiv \Sigma(a_{ij}x_ix_j + b_kx_k) + c_0$, where $a_{ij} = a_{ji}$. After a change of coordinate by an orthorgonal transformation, we may assume

$$u(x) = \frac{3}{4\pi^2}\int_{\mathbf{R}^4} \log\frac{|y|}{|x-y|} e^{4u(y)}dy - \sum_{i=1}^{4}(a_ix_i^2 + b_ix_i) + c_0,$$

where $a_i \geq 0, b_i$ and c_0 are constants. Since $e^{4u} \in L^1(\mathbf{R}^4)$, we have $b_i = 0$ whenever $a_i = 0$. Thus $u(x)$ can be written as

$$u(x) = \frac{3}{4\pi^2}\int_{\mathbf{R}^4} \log\frac{|y|}{|x-y|} e^{4u(y)}dy - \sum_{i=1}^{n} a_i(x_i - x_i^0)^2 + c_0 \tag{3.7}$$

After a translation, we may assume $x^0 = 0$. Let $\tilde{u}(x) \equiv u(x) + \sum_{i=1}^{n} a_ix_i^2$. Then $\tilde{u}(x)$ satisfies

$$\Delta^2\tilde{u}(x) = Q(x)e^{4\tilde{u}(x)} \quad \text{in} \quad \mathbf{R}^4 \tag{3.8}$$

where $Q(x) = 6e^{-4\sum_{i=1}^{n} a_ix_i^2}$.

If $a_i = 0$ for all i, then it is the case of Theorem 1.1. Thus, we assume $a_i \neq 0$ for $1 \leq i \leq k$, $a_i = 0$ for $i > k$ where $1 \leq k \leq 4$. Lemma 2.1 implies $\alpha > 1 - \frac{k}{4}$. As in Lemma 2.8, we let $\tilde{w}(x) = u(\frac{x}{|x|^2}) - \alpha\log|x| = o(\log\frac{1}{|x|})$. Then $\tilde{w}$ satisfies

$$\Delta\tilde{w} + \tilde{Q}(x)e^{4\tilde{w}} = 0 \quad \text{in} \quad \mathbf{R}^4\backslash\{0\} \tag{3.9}$$

where $\tilde{Q}(x) = 6e^{-\sum_j a_j(\frac{x_j}{|x|^2})^2}|x|^{4(\alpha-2)}$.

Since $\alpha > 1 - \frac{k}{4}$, we have $\tilde{Q}(x)e^{4\tilde{w}} \in L^p(B_1)$ for some $p > 1$. By the same proof of Lemma 2.8, we have $\tilde{w} \in C^{0,\tau}(\bar{B}_1)$ for some $1 > \tau > 0$. In particular, we have

$$\tilde{u}(x) = -\alpha\log|x| + c_0 + o(|x|^{-r})$$

at ∞, which together with (3.7), yields (1.10).

If $a_i < 0$ for all i, then it is easily to see $\tilde{Q}(x)e^{4\tilde{w}} \in L^p(\bar{B}_1)$ for any $p > 1$. Thus $\tilde{w} \in C^\infty(\bar{B}_1)$. Therefore, $\tilde{u}$ satisfies both (2.38) and (2.39) for large $|x|$, i.e., we have for large $|x|$,

$$\tilde{u}(x) = -\alpha\log|x| + c_0 + O(|x|^{-1}), \tag{3.10}$$

$$\begin{cases} -\Delta\tilde{u}(x) = \dfrac{2\alpha}{|x|^2} + \displaystyle\sum_{j=1}^{n}\frac{c_jx_j}{|x|^4} + O(|x|^{-4}), \\ -(\Delta\tilde{u})_{x_i}(x) = -\dfrac{4\alpha x_i}{|x|^4} + O(|x|^{-4}), \\ (\Delta\tilde{u})_{x_ix_j} = O(|x|^{-4}). \end{cases} \tag{3.11}$$

Employing (3.10) and (3.11), we can use, as in the proof of Theorem 1.1, the method of moving planes to show that $\tilde{u}(x)$ is symmetric with respect to hyperplane $\{x \mid x_i = 0\}$ for $1 \leq i \leq 4$. In particular, if $a_1 = \ldots = a_4 \neq 0$, then u is radially symmetric with respect to 0. Hence, we have proved (i) of Theorem 1.2.

If $\alpha \geq 2$, then $\tilde{Q}(x)e^{4\tilde{w}} \in L^p(\bar{B}_1)$ for any $p > 1$ also. Therefore $\tilde{w} \in C^\infty(\bar{B}_i)$, and, $e^{4\tilde{u}} = O(|x|^{-8})$ at ∞. By (2.31) $\sim$ (2.33), we can prove without difficulty:

$$\lim_{|x|\longrightarrow +\infty} |x|\frac{\partial \tilde{u}}{\partial r}(x) = -\alpha, \tag{3.12}$$

$$\lim_{|x|\longrightarrow +\infty} r\frac{\partial}{\partial r}(r\frac{\partial \tilde{u}}{\partial r})(x) = 0, \tag{3.13}$$

$$\lim_{r\longrightarrow +\infty} \Delta\tilde{u}(x)|x|^2 = -2\alpha, \tag{3.14}$$

and,

$$\lim_{r\longrightarrow +\infty} \frac{\partial}{\partial r}(\Delta\tilde{u})r^3 = 4\alpha. \tag{3.15}$$

Applying the Pohozaev identity, we have

$$8\pi^2\alpha + \frac{1}{32\pi^2}\int_{\mathbf{R}^4}(x, \bigtriangledown Q)e^{4\tilde{u}}dx = 4\pi^2\alpha^2.$$

Since $\alpha \geq 2$, we have $8\pi^2\alpha \leq 4\pi^2\alpha^2$. Thus,

$$\int_{\mathbf{R}^4}(x, \bigtriangledown Q)e^{4u}dx \geq 0.$$

Since

$$x \cdot \bigtriangledown Q = -\sum a_j x_j^2 e^{-\Sigma a_j x_j^2} \leq 0,$$

we have $a_j = 0$ for all j. Then, by Theorem 1.1, we have $\alpha = 2$ and $u(x)$ has a form of (1.8). Hence, (ii) of Theorem 1.2 is proved. □

4. Proof of Theorem 1.3

Let u be a smooth positive solution of

$$\Delta^2 u = u^p \quad \text{in } \mathbf{R}^n,$$

for $1 < p \leq \dfrac{n+4}{n-4}$ and $n \geq 5$. As in the case of the equation (1.3), we let

$$u^*(x) = |x|^{4-n}u(\frac{x}{|x|^2}) \tag{4.1}$$

By a direct computation, u^* satisfies

$$\Delta^2 u^* = |x|^{-\tau} u^{*p} \quad \text{in} \quad \mathbf{R}^n \backslash \{0\}, \tag{4.2}$$

where $\tau = n + 4 - p(n-4) \geq 0$. Let $v(x) = -\Delta u^*(x)$. By (4.1), we have

$$\begin{cases} v(x) = c_0 |x|^{2-n} + \sum_{j=1}^n \frac{a_j x_j}{|x|^n} + O(\frac{1}{|x|^n}) \\ v_{x_i} = -(n-2) c_0 |x|^{-n} x_i + O(\frac{1}{|x|^n}) \\ v_{x_i x_j} = O(\frac{1}{|x|^n}) \end{cases} \tag{4.3}$$

at ∞, where $c_0 > 0$ and $a_j \in \mathbf{R}$. In particular, we have for large $|x|$,

$$\Delta u^*(x) < 0. \tag{4.4}$$

As in Theorem 1.1, we need to prove $\Delta u^*(x) < 0$ in $\mathbf{R}^n \backslash \{0\}$.

Lemma 4.1. *Let u be a smooth positive solution of*

$$\Delta^2 u = |x|^{-\tau} u^p \quad \text{in } B_1 \backslash \{0\} \tag{4.5}$$

where $1 < p \leq \dfrac{n+4}{n-4}$, $\tau = (n+4) - p(n-4)$ and $n \geq 5$. Then Δu is a subharmonic function in B_1 in the distributional sense.

Proof. First, we want to prove $|x|^{-\tau} u^p \in L^1(\bar{B}_{\frac{1}{2}})$. Suppose $|x|^{-\tau} u^p \notin L^1(\bar{B}_{\frac{1}{2}})$. Then we have

$$\int_{\partial B_r} \frac{\partial}{\partial r} (\Delta u) d\sigma - \int_{\partial B_s} \frac{\partial}{\partial r} (\Delta u) d\sigma = \int_{B_r \backslash B_s} |x|^{-\tau} u^p > 0 \tag{4.6}$$

for all $0 < s \leq r \leq \dfrac{1}{2}$. Since the right hand side of (4.6) tends to $+\infty$ as $s \longrightarrow 0$, there exists $r_1 > 0$ such that

$$⨍_{\partial B_r} \frac{\partial}{\partial r} (\Delta u) d\sigma \leq -c_1 r^{1-n} \int_{B_{\frac{1}{2}} \backslash B_r} |x|^{-\tau} u^p, \tag{4.7}$$

which implies

$$\int_{\partial B_r} \Delta u d\sigma - \int_{\partial B_s} \Delta u d\sigma \leq -c_1 \int_s^r \tau^{1-n} \int_{B_{\frac{1}{2}} \backslash B_r} |x|^{-\tau} u^p dx d\tau, \tag{4.8}$$

and

$$\int_{\partial B_r} \Delta u d\sigma \geq c_2 r^{-n+2} \tag{4.9}$$

for $0 < r \le r_2$ and for some $0 < r_2 < r_1$. Let $\bar{u} = \fint_{\partial B_r} u d\sigma$. By (4.9),

$$(r^{n-1}\bar{u}'(r))' \ge c_2 r. \tag{4.10}$$

If $\lim_{r \longrightarrow 0} r^{n-1}\bar{u}'(r) \ge 0$, then we have for any $r > 0$,

$$r^{n-1}\bar{u}'(r) \ge \frac{1}{2} c_2 r^2, \tag{4.11}$$

which yields,

$$\bar{u}(r) \ge \int_0^r \bar{u}'(t)dt \ge \frac{1}{2} c_2 \int_0^r t^{3-n} dt = +\infty,$$

a contradiction. Therefore we may assume there exists $0 < r_3 < r_2$ such that for all $r \le r_3$, we have

$$r^{n-1}\bar{u}'(r) \le -c_3, \tag{4.12}$$

where c_3 is a positive constant. Therefore,

$$\bar{u}(r) \ge c_4 r^{2-n}. \tag{4.13}$$

Suppose $\bar{u}(r) \ge c_4 r^{-s}$ for some $\sigma \ge n-2$. Then, by (4.7) and (4.8), we have for small $r > 0$,

$$(\Delta \bar{u}(r))' \le -c_1 r^{-\tilde{\sigma}-n+2}, \tag{4.14}$$

$$\Delta \bar{u}(r) \ge c_2 r^{-\tilde{\sigma}-n+3}, \tag{4.15}$$

$$r^{n-1}\bar{u}'(r) \le -c_3 r^{-\tilde{\sigma}+3}, \tag{4.16}$$

and,

$$\bar{u}(r) \ge c_4 r^{-\tilde{\sigma}+5-n}, \tag{4.17}$$

where $\tilde{\sigma} = \tau + p\sigma$. We note that, in order to have (4.16) held, we need $\tilde{\sigma} > 3$. Since $\tau = n + 4 - p(n-4)$ and $\sigma \ge n-2$, we have $\tilde{\sigma} \ge \tau + p(n-2) \ge n + 6 > 5$. Since $\tilde{\sigma} + n - 5 - p\sigma = \tau + n - 5 \ge 0$ for $n \ge 5$, after a finite time of iterations, we have

$$\bar{u}(r) \ge r^{-(1+\beta)}, \tag{4.18}$$

$$\bar{u}'(r) \le -r^{-(2+\beta)}, \tag{4.19}$$

$$\bar{u}''(r) \ge r^{-(3+\beta)}, \tag{4.20}$$

and,

$$\bar{u}'''(r) \le -r^{-(4+\beta)} \tag{4.21}$$

for $0 \le r \le 2r_0$ where $\beta = \dfrac{4}{p-1}$ (Since $\bar{u}' \le 0$, we have $0 < \Delta\bar{u}(r) \le \bar{u}''$ and $\bar{u}^{(3)}(r) \le (\Delta\bar{u})' < 0$). By (4.19) $\sim$ (4.21), we have by Jensen inequality,

$$\bar{u}^{(4)}(r) \ge \Delta^2\bar{u}(r) \ge |x|^{-\tau}\bar{u}^p(r). \tag{4.22}$$

Let $v(r) = A(r-r_0)^{-\beta}$ for $r_0 \le r \le 2r_0$. By direct computations, we have

$$\begin{aligned} v^{(4)}(r) &= A\beta(\beta+1)(\beta+2)(\beta+3)(r-r_0)-(\beta+4) \\ &= A\beta(\beta+1)(\beta+2)(\beta+3)v^p(r) < r^{-\tau}v^p(r) \end{aligned}$$

for $r_0 \le r \le 2r_0$ if A is large. If r_0 is sufficiently small, then by (4.18) $\sim$ (4.21), we have $v(r) \le \bar{u}(r)$ for all $r_0 \le r \le 2r_0$. However, $\lim_{r \longrightarrow r_0} \bar{u} \ge \lim_{r \longrightarrow r_0} v(r) = +\infty$ yields a contradiction. Therefore, we have proved $|x|^{-\tau}u^p \in L^1(\bar{B}_{\frac{1}{2}})$.

Let $\varphi \in C_0^\infty(B_{\frac{1}{2}})$ be a nonnegative function. We want to prove

$$\int \Delta\varphi\Delta u dx \ge 0. \tag{4.23}$$

Let $\eta_\varepsilon \in C_0^\infty(B_{\frac{1}{2}})$ satisfy $\eta_\varepsilon(x) \equiv 1$ for $|x| \ge 2\varepsilon$, and $\eta_\varepsilon(x) \equiv 0$ for $|x| \le \varepsilon$. We also assume

$$|D^j\eta_\varepsilon(x)| \le \frac{c}{\varepsilon^j}$$

for $1 \le j \le 4$. Multiplying (4.5) by $\varphi(x)\eta_\varepsilon$, we have

$$\begin{aligned} 0 &< \int \varphi(x)\eta_\varepsilon(x)|x|^{-\tau}u^p(x)dx \\ &= \int \Delta(\varphi(x)\eta_\varepsilon(x))\Delta u(x)dx \\ &= \int \Delta u(x)\{\Delta\varphi(x)\eta_\varepsilon(x) + 2\bigtriangledown\varphi(x)\bigtriangledown\eta_\varepsilon + \varphi(x)\Delta\eta_s\}dx \end{aligned} \tag{4.24}$$

Let $\psi(x) = 2\bigtriangledown\varphi(x)\bigtriangledown\eta_\varepsilon + \varphi(x)\Delta\eta_\varepsilon(x)$. We have $\psi(x) \equiv 0$ for $|x| \le \varepsilon$ and for $|x| \ge 2\varepsilon$, and $|\Delta\psi(x)| \le c\,\varepsilon^{-4}$.

Since $\dfrac{n}{q} + \dfrac{\tau}{p} = \dfrac{4}{p} + 4 > 4$ where $\dfrac{1}{q} = 1 - \dfrac{1}{p}$, we have

$$\begin{aligned} |\int \Delta u(x)\psi(x)dx| &\le \int u(x)|\Delta\psi(x)|dx \\ &\le c\,\varepsilon^{-4}\Big(\int_{\varepsilon\le|x|\le2\varepsilon}|x|^{-\tau}u^p(x)dx\Big)^{1/p}\varepsilon^{\frac{n}{q}+\frac{\tau}{p}} \\ &\le c\,\varepsilon^{\frac{n}{q}+\frac{\tau}{p}-4} \longrightarrow 0, \end{aligned}$$

as $\varepsilon \longrightarrow 0$. Therefore, by (4.24), we have

$$\begin{aligned}\int \Delta u(x)\Delta\varphi(x)dx &= \lim_{\varepsilon\longrightarrow 0}\int \eta_\varepsilon(x)\Delta u(x)\Delta\varphi(x)dx \\ &= \int \varphi(x)|x|^{-\tau}u^p(x)dx > 0.\end{aligned}$$

Thus, $\Delta u(x)$ is a subharmonic in $B_{\frac{1}{2}}$. □

Proofs of Theorem 1.3. and Theorem 1.4. Let $v(x) = -\Delta u^*(x)$. By Lemma 4.1 and (4.3), we have $v(x) > 0$ in $\mathbf{R}^4\backslash\{0\}$ and $v(x)$ satisfies for any $r > 0$,

$$v(x) \geq \inf_{\partial B_r(0)} v(x) > 0, \quad \text{for} \quad x \in B_r(0). \tag{4.25}$$

Since $u^*(x)$ is a superharmonic function in $B_r(0)\backslash\{0\}$ and $u^*(x) > 0$, then we have

$$u^*(x) \geq \inf_{\partial B_r(0)} u^*(y) \quad \text{for} \quad x \in B_r(0) \tag{4.26}$$

(For a proof of (4.26), please see Lemma 2.1 in [CLn]).

Following notations in Section 3, we let $w_\lambda(x) = u^*(x) - u^*(x^\lambda)$ in Σ_λ. Since $v(x) = -\Delta u^*$ has a harmonic expansion (4.3) at infinity, by Lemma 3.1 and (4.25), there exists a $\bar{\lambda}_0 < 0$ such that

$$\Delta w_\lambda(x) < 0 \quad \text{in} \quad \Sigma_\lambda$$

for all $\lambda \leq \bar{\lambda}_0$. By the maximum principle, we have

$$w_\lambda(x) > 0 \quad \text{in} \quad \Sigma_\lambda$$

for all $\lambda \leq \bar{\lambda}_0$.

We consider the case $p < \dfrac{n+4}{n-4}$ first. Let

$$\lambda_0 = \sup\{\lambda < 0 \mid \Delta w_\mu(x) < 0 \quad \text{in } \Sigma_\mu \text{ for } \mu \leq \lambda\}.$$

Suppose $\lambda_0 < 0$. Although u^* may has a singularity at 0, by (4.25) and (4.26), we still can apply the same arguments as in Theorem 1.1 to prove $w_{\lambda_0}(x) \equiv 0$ in Σ_{λ_0}. Since $\tau < 0$, it yields a contradiction. Thus we must have $\lambda_0 = 0$ and

$$u(-x_1, x_2, \ldots, x_n) \leq u(x_1, x_2, \ldots, x_n) \quad \text{for} \quad x_1 \geq 0.$$

By applying the method of moving planes along any direction in $\mathbf{R}^n$, $u^*(x)$ is radially symmetric with respect to 0. Since we can take any point in $\mathbf{R}^n$ as the origin, we conclude that if u is a positive smooth solution in $\mathbf{R}^n$, then $u \equiv$ constant in $\mathbf{R}^n$ which implies $u \equiv 0$ in $\mathbf{R}^n$, a contradiction. Thus, Theorem 1.4 is proved.

For the case $p = \dfrac{n+4}{n-4}$, we also let

$$\lambda_0 = \sup\{\lambda < 0 \mid \Delta w_\mu(x) < 0 \ \text{ in } \Sigma_\mu \text{ for } \mu \le \lambda\}.$$

If $\lambda_0 < 0$, by applying the same arguments again, we can show $w_{\lambda_0}(x) \equiv 0$. Thus, $u^*(x)$ has a removable singularity at 0 and u itself satisfies (4.3) at infinity. Therefore, we can directly apply the method of moving plane to u itself to yield the radial symmetry of u about some point x_0 in $\mathbf{R}^n$. If $\lambda_0 = 0$, then we can do the same procedure by moving the hyperplane T_λ from positive direction of x_1. Thus, we can prove either u^* has a removable singularity at 0 or $u^*(x)$ is symmetric with respect to the hyperplane $\{x \mid x_1 = 0\}$. In any case, the radial symmetry of u follows immediately.

Suppose that u is radially symmetric with respect to 0. We can take another point $x_0 \neq 0$ as the origin of the "Kelvin" transformation, and do the same procedure as the above. Since u is not radially symmetric about x_1, we have $\lambda_0 \neq 0$, namely, $u(x)$ satisfies (4.3) at infinity. In particular, we have $\Delta u(x) \longrightarrow 0$ as $|x| \longrightarrow +\infty$.

By a direct computation, we can see that $u_\lambda(x) = c_n(\dfrac{\lambda}{1+\lambda^2|x|^2})^{\frac{n-4}{2}}$ is a solution of (1.12) for any $\lambda > 0$. Suppose $\omega(r)$ is a radial solution of (1.12) and $\omega(0) = u_{\lambda_0}(0)$ for some $\lambda_0 > 0$. If $\Delta\omega(0) > \Delta u_{\lambda_0}(0)$, then we can prove $\omega(r)$ blows up in finite r. Because, if $\omega(r)$ exists for all $r > 0$, as in the proof of Theorem 1.1, then we can show $\omega(r) > u_{\lambda_0}(r)$ for all $r > 0$ and $(\Delta\omega - \Delta u)'(r) > 0$ for all $r > 0$. Therefore

$$\Delta\omega(\infty) = \lim_{r \longrightarrow +\infty} (\Delta\omega - \Delta u_{\lambda_0})(r) > (\Delta\omega - \Delta u_{\lambda_0})(0) > 0,$$

which yields a contradition to $\lim\limits_{|x| \longrightarrow +\infty} \Delta\omega(x) = 0$ which was already proved for any solution of (1.12). If $\Delta\omega(0) < \Delta u_{\lambda_0}(0)$, then, by the same proof, $\omega(r)$ must become zero at a finite r. Thus the proof of Theorem 1.3 is considered completely finished. □

In fact, the same proof can imply

Theorem 4.2. *Suppose u is a positive smooth solution of*

$$\Delta^2 u = u^p \ \text{ in } \ R^n \backslash \{0\},$$

where $1 < p \le \dfrac{n+2}{n-4}$. Assume 0 is a nonremovable singularity, then u is radially symmetric with respect to the origin.

Corollary 4.3. *Let u be a solution of*

$$\begin{cases} \Delta^2 u = u^p & \text{in } B_1\backslash\{0\}, \\ u > 0, \end{cases}$$

where $1 < p < \frac{n+4}{n-4}$. Then $u(x) \le c|x|^{-\frac{4}{p-1}}$ for $|x| \le \frac{1}{2}$, where c is a constant, depending on n a nd p only.

Corollary 4.3 is an immediate consequence of Theorem 1.3 and a blow-up argument due to R. Schoen for the equation (1.3) (for example, please see [P].) We omit the details of the proof.

References

[BCY] T. Branson, S-Y. A. Chang and P. C. Yang, Estimates and extremal problems for the log-determinant on 4-manifolds, *Communication Math. Physics* **149** (2) (1992), 241–262.

[BM] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations* **16** (1991), 1223–1253.

[BO] T. Branson and B. Orsted, Explicit functional determinants in four dimensions, Proc. *Amer. Math. Soc.* **113** (1991), 669–682.

[CGS] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic Symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, *Comm. Pure Appl. Math.* **42** (1989), 271–297.

[CL] Chen, E. and Li, C., Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (3) (1991), 615–622.

[CLn] C.C. Chen and C.S. Lin, Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent, *Duke Math. J.* **78** (1995), 315–334.

[CY] S-Y. A. Chang and P. C. Yang, Extremal metrics of zeta functional determinants on 4-manifolds, *Annals of Math.* **142** (1995), 1171–212.

[GNN] B. Gidas, W.M. Ni, and L. Nirenberg, *Symmetry of positive solutions of nonlinear equations in $\mathbf{R}^n$. Analysis and Applications* part A, pp. 369–402, Advances in Math. Supp. Stud. 79, Academic Press, New York–London 1981.

[GT] D. Gilbarg, N. S, Trudinger, *Elliptic partial differential equations*, 2nd edition, Grundlehren der mathematischen Wissenschaften 224, Springer-Verlag, 1983.

[L] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, parts 1 and 2. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **1** (1984), no. 2, 109–145, no. 4, 223–283.

[P] D. Pollack, Compactness results for complete metrics of constant positive scalar curvature on subdomains of S^n, *Indiana Univ. Math. J.* **42** (1993), 1441–1456.

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