Comment. Math. Helv. 73 (1998) 237–258
0010-2571/98/020237-22 \$ 1.50+0.20/0

Commentarii Mathematici Helvetici

The double Coxeter arrangement

Louis Solomon and Hiroaki Terao*

Abstract. Let V be Euclidean space. Let $W \subset \mathbf{GL}(V)$ be a finite irreducible reflection group. Let $\mathcal{A}$ be the corresponding Coxeter arrangement. Let S be the algebra of polynomial functions on V. For $H \in \mathcal{A}$ choose $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. The arrangement $\mathcal{A}$ is known to be free: the derivation module $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H\}$ is a free S-module with generators of degrees equal to the exponents of W. In this paper we prove an analogous theorem for the submodule $E(\mathcal{A})$ of $D(\mathcal{A})$ defined by $E(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^2\}$. The degrees of the basis elements are all equal to the Coxeter number. The module $E(\mathcal{A})$ may be considered a deformation of the derivation module for the Shi arrangement, which is conjectured to be free. The proof is by explicit construction using a derivation introduced by K. Saito in his theory of flat generators.

Mathematics Subject Classification (1991). Primary 52B30; secondary 05E15.

Keywords. Hyperplane arrangement, free arrangement, Shi arrangement, reflection group, basic invariants, Coxeter number, Jacobian matrix.

§ 1. Introduction

Let V be a Euclidean space of dimension l over $\mathbf{R}$. Let $(\ ,\ )$ denote the positive definite symmetric bilinear form on V. Let $W \subseteq \mathbf{GL}(V)$ be a finite group generated by orthogonal reflections [Bou, V.2.3]. Let $\mathcal{A}$ be the corresponding Coxeter arrangement, the set of hyperplanes $H \subset V$ such that W contains the orthogonal reflection which fixes H. Let S be the algebra of polynomial functions on V. The algebra S is naturally graded by $S = \bigoplus_{q \geq 0} S_q$ where S_q is the space of homogeneous polynomials of degree q. Thus $S_1 = V^*$ is the dual space of V. Let Der_S be the S-module of $\mathbf{R}$-derivations of S. We say that $\theta \in \mathrm{Der}_S$ is homogeneous of degree q if $\theta(S_1) \subseteq S_q$. Choose for each hyperplane $H \in \mathcal{A}$ a linear form $\alpha_H \in V^*$ such that $H = \ker(\alpha_H)$. Define $Q \in S$ by

$$Q = \prod_{H \in \mathcal{A}} \alpha_H \,. \tag{1.1}$$

*Second author partially supported by NSF Grant DMS9504457.

The polynomial Q is uniquely determined, up to a constant multiple, by the group W. Let

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H\} . \tag{1.2}$$

K. Saito [Sai1, Theorem], [Ter,Theorem 2] proved that $D(\mathcal{A})$ is a free S-module of rank l and that a set of basis elements for $D(\mathcal{A})$ as S-module may be described as follows. Let $R = S^W$ be the algebra of W-invariant polynomials on V. By a theorem of Shephard, Todd, and Chevalley [Bou, V.5.3, Theorem 3] there exist algebraically independent homogeneous polynomials $f_1, \dots , f_l \in R$ such that $R = \mathbf{R}[f_1, \dots , f_l]$. Let $x_1, \dots , x_l$ be an orthonormal basis for V^*. Let ∂_i be partial differentiation with respect to x_i. Define $\theta_j \in \mathrm{Der}_S$ by $\theta_j = \sum_{i=1}^{l} (\partial_i f_j)\, \partial_i$ for $1 \le j \le l$. Then $\{\theta_1, \dots , \theta_l\}$ is an S-basis for $D(\mathcal{A})$. Note that θ_j is homogeneous of degree $\deg(f_j)-1$. The integers $m_j = \deg(f_j)-1$ for $1 \le j \le l$ are the exponents of W [Bou, V.6.2, Proposition 3].

In this paper we will prove an analogous theorem for the submodule $E(\mathcal{A})$ of $D(\mathcal{A})$ defined by

$$E(\mathcal{A}) = \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in S\alpha_H^2\} . \tag{1.3}$$

Note that we have replaced $S\alpha_H$ in (1.2) by $S\alpha_H^2$ in (1.3), which explains the phrase "double Coxeter arrangement" in the title of this paper. If $\theta \in D(\mathcal{A})$ and $\alpha = \alpha_H$ then $\theta(Q) = \theta(\alpha \cdot Q/\alpha) = (Q/\alpha)\theta(\alpha) + \alpha\theta(Q/\alpha) \in S\alpha$ so that $\theta(Q) \in SQ$. On the other hand, it may happen that $\theta \in E(\mathcal{A})$, but $\theta(Q) \notin SQ^2$.

To state our theorem we need some preliminary definitions. Assume that W is an irreducible subgroup of $\mathbf{GL}(V)$. The form $(\ ,\ )$ on V induces a positive definite symmetric bilinear form on V^*, sometimes called the inverse form, which we also write as $(\ ,\ )$. Let $e_1, e_2, \dots , e_l$ be a basis for V. We do not assume that $e_1, e_2, \dots , e_l$ is an orthonormal basis unless orthonormality is explicitly stated. Let $x_1, x_2, \dots , x_l$ be the dual basis for V^*. Let Γ be the matrix of the inverse form with respect to the chosen basis $x_1, \dots , x_l$. Thus $\Gamma_{ij} = (x_i, x_j)$. Number the invariant polynomials f_j so that $\deg(f_1) \le \cdots \le \deg(f_l)$. Since W is irreducible, the Coxeter number h of W is defined [Bou, V.6.1] and $h = \deg(f_l)$ [Bou, V.6.2]. Let K be the quotient field of S. K. Saito [Sai2, 2.2], [SYS, (1.6)] studied an $\mathbf{R}$-derivation $D \in \mathrm{Der}_K$ such that $Df_j = 0$ for $1 \le j \le l-1$ and $Df_l \in \mathbf{R}^*$. This derivation is uniquely determined, up to a constant multiple, by the group W and does not depend on choice of basic invariants $f_1, \dots , f_l$. Define rational fuctions $h_j \in K$ for $1 \le j \le l$ by

$$h_j = Dx_j .$$

Let $J(h_1, \dots , h_l)$ be the Jacobian matrix, labeled so that $\partial_i h_j$ is its (i, j) entry. We will prove in Corollary 3.32 that $J(h_1, \dots , h_l)$ is invertible over K. This is perhaps the most difficult point in the paper. Furthermore $J(h_1, \dots , h_l)^{-1}$ has entries in S. The structure of the S-module $E(\mathcal{A})$ is given by the following theorem.

Theorem 1.4. *Let $W \subseteq \mathbf{GL}(V)$ be a finite irreducible group generated by reflec-*

tions. Define an $l \times l$ matrix P by

$$P = \Gamma J(h_1, \dots , h_l)^{-1} . \tag{1.5}$$

Define $\xi_1, \dots , \xi_l \in \mathrm{Der}_S$ by $\xi_j = \sum_{i=1}^{l} p_{ij}\, \partial_i$ for $1 \le j \le l$, where p_{ij} is the (i, j) entry of P. Then $\xi_i \in E(\mathcal{A})$, and $E(\mathcal{A})$ is a free S-module with basis $\xi_1, \dots , \xi_l$.

Note that if $x_1, \dots , x_l$ is an orthonormal basis for V^* then Γ is the identity matrix and (1.5) becomes $P = J(h_1, \dots , h_l)^{-1}$. We will see that all entries of P are homogeneous of degree equal to the Coxeter number h. Thus all derivations $\xi_1, \dots , \xi_l$ are homogeneous of degree h. We will prove in Proposition 4.7 that the homogeneous component $E(\mathcal{A})_h$ of degree h is isomorphic to V^* as W-module. The rational functions $h_1, \dots , h_l$ may be computed as follows. Let $J(f_1, \dots , f_l)$ be the Jacobian matrix of $f_1, \dots , f_l$. Since $f_1, \dots , f_l$ are algebraically independent, $J(f_1, \dots , f_l)$ is invertible over K. Then $[h_1, \dots , h_l]$ is, up to constant multiple, the l-th row of $J(f_1, \dots , f_l)^{-1}$.

Remark 1.6. Define polynomials $u_1, \dots , u_l \in S$ by $u_i = Qh_i$ for $1 \le i \le l$. Invertibility of the matrix $J(h_1, \dots , h_l)$ is equivalent to invertibility of the matrix $J(u_1, \dots , u_l)$, which was conjectured in [Sol2].

Remark 1.7. The definition (1.3) of $E(\mathcal{A})$ is due to Ziegler [Zie, Definition 4] who developed the theory of multiarrangements. A double Coxeter arrangement is a multiarrangement with multiplicity two for each hyperplane belonging to the Coxeter arrangement.

Remark 1.8. We were led to study the double Coxeter arrangements by an attempt to understand the Shi arrangements [Shi1], [Sshi2]. Suppose that W is a Weyl group. Choose a crystallographic root system in V^* and choose the linear forms α_H so that $\pm\alpha_H$ is a root for each $H \in \mathcal{A}$. Let $\alpha_1, \dots , \alpha_n \in V^*$ be a system of positive roots. The Shi arrangement $\tilde{\mathcal{A}}$ of type W is an affine arrangement with $2n$ hyperplanes whose defining polynomial is $\tilde{Q} = \prod_{i=1}^{n}(\alpha_i - 1) \prod_{i=1}^{n} \alpha_i$. Shi arrangements have been studied by Stanley [Sta1], [Sta2] and others. A special case of a conjecture due to Edelman and Reiner [EdR, Conjecture 3.3] states that the cone [OrT, p.14] $\mathbf{c}\tilde{\mathcal{A}}$ of each Shi arrangement is a free arrangement with exponents $\{1, h, \dots , h\}$ [OrT, Definition 4.15, Definition 4.25]; the module $D(\mathbf{c}\tilde{\mathcal{A}})$ is a free module over $\mathbf{R}[x_0, \dots , x_l]$. Athanasiadis [Ath] verified this conjecture for type A_l. Note that the restriction (as a multiarrangement) of $\mathbf{c}\tilde{\mathcal{A}}$ to the infinite hyperplane $x_0 = 0$ is the double Coxeter arrangement. Therefore, if the conjecture is true, then, by Ziegler's theorem [Zie, Theorem 11], we may conclude that the double Coxeter arrangement is a free arrangement with exponents $\{h, h, \dots , h\}$, which is true by our main result, Theorem 1.4. So Theorem 1.4 may be regarded as a piece of evidence supporting the conjecture.

Here is an outline of the paper. In Section 2 we introduce more notation and state some elementary facts. In Section 3 we prove the invertibility of $J(h_1, \dots, h_l)$. In Section 4 we complete the proof of Theorem 1.4. In Section 5 we compute the matrix P in case $l = 2$ and in case W has type B_l. In Section 6 we use the invertibility of the matrix $J(u_1, \dots, u_l)$ to describe the differential 1-forms which are anti-invariant under W.

§2. Notation and preliminary definitions

In this Section we fix more notation, state some elementary facts about derivations and differential forms, and introduce some of the main constructs in the argument. We often use the notation of Section 1 without comment. When convenient we choose a basis $e_1, \dots, e_l$ for V and let $x_1, \dots, x_l$ denote the dual basis for V^*. Let $\langle\ ,\ \rangle : V^* \times V \to \mathbf{R}$ denote the natural pairing. Thus $\langle x_i, e_j\rangle = \delta_{ij}$. Let Der_S be the S-module of $\mathbf{R}$-derivations of S. For each $v \in V$ let $\partial_v \in \mathrm{Der}_S$ be the unique derivation such that $\partial_v x = \langle x, v\rangle$ for $x \in V^*$. Define $\partial_i \in \mathrm{Der}_S$ by $\partial_i = \partial_{e_i}$. Then $\partial_i x_j = \delta_{ij}$ and Der_S is a free S-module with basis $\partial_1, \dots, \partial_l$. There is a natural isomorphism $S \otimes V \to \mathrm{Der}_S$ of S-modules given by

$$f \otimes v \mapsto f\partial_v \tag{2.1}$$

for $f \in S$ and $v \in V$. Let $\Omega^1_S = \mathrm{Hom}_S(\mathrm{Der}_S, S)$ be the S-module dual to Der_S. Define $d : S \to \Omega^1_S$ by $df(\theta) = \theta(f)$ for $f \in S$ and $\theta \in \mathrm{Der}_S$. Then $d(ff') = (df)f' + f(df')$ for $f, f' \in S$. Furthermore, Ω^1_S is a free S-module with basis $dx_1, \dots, dx_l$ and $df = \sum_{i=1}^{l} (\partial_i f)dx_i$. There is a natural isomorphism $S \otimes V^* \to \Omega^1_S$ of S-modules given by

$$f \otimes x \mapsto f dx \tag{2.2}$$

for $f \in S$ and $x \in V^*$. The modules Der_S and Ω^1_S inherit gradings from S which are defined by $\deg(f\partial_v) = \deg(f)$ and $\deg(f dx) = \deg(f)$ if $f \in S$ is homogeneous.

We define several W-module structures which stem from the given W-module structure on V. If $f \in S$ define $wf \in S$ by $(wf)(v) = f(w^{-1}v)$ for $v \in V$. This makes S a W-module and W acts as a group of $\mathbf{R}$-algebra automorphisms of S. In particular $V^* = S_1$ has a W-module structure, and $\langle wx, wv\rangle = \langle x, v\rangle$ for $w \in W$, $x \in V^*$ and $v \in V$. The spaces $S \otimes V$ and $S \otimes V^*$ have W-module structures given by $w(f \otimes v) = wf \otimes wv$ and $w(f \otimes x) = wf \otimes wx$. We give Der_S a W-module structure by defining $(w\theta)(f) = w(\theta(w^{-1}f))$ for $w \in W$, $\theta \in \mathrm{Der}_S$ and $f \in S$. Then $w\partial_v = \partial_{wv}$ for $w \in W$ and $v \in V$. To check this it suffices to check that both derivations $w\partial_v$ and ∂_{wv} have the same effect on V^*. This is so since $(w\partial_v)(x) = \partial_v(w^{-1}x) = \langle w^{-1}x, v\rangle = \langle x, wv\rangle = \partial_{wv}(x)$. We give Ω^1_S a W-module structure by defining $w(f dx) = (wf)d(wx)$ for $w \in W$, $f \in S$ and $x \in V^*$. In particular, $w(dx) = d(wx)$. The isomorphisms in (2.1) and (2.2) are W-module isomorphisms.

Define an S-bilinear form $(\ ,\ ) : \Omega^1_S \times \Omega^1_S \to S$ by

$$(f dx, f' dx') = f f'(x, x') \tag{2.3}$$

for $f, f' \in S$ and $x, x' \in V^*$ where (x, x') denotes the form on V^* inverse to the given form on V. In particular, $(dx, dx') = (x, x')$ for $x, x' \in V^*$. If $w \in W$ then, since $(wx, wx') = (x, x')$ for $w \in W$, it follows from (2.3) that $w(\omega, \omega') = (w\omega, w\omega')$ for $\omega, \omega' \in \Omega^1_S$.

Let K be the quotient field of S. We make various conventions about matrices over K which will be used throughout the paper. Let $\mathbf{M}_l(K)$ denote the set of $l \times l$ matrices over K. We use similar notation for matrices over other rings. If A is any rectangular matrix over K we let A_{ij} denote the (i, j) entry of A and let $A^\top$ denote the transpose of A. It is sometimes convenient to define a matrix as $A = [a_{ij}]$. When we do this, it is understood that i is the row index and j is the column index, so that $A_{ij} = a_{ij}$. If $w \in W$ we define the matrix $w[A]$ by

$$w[A]_{ij} = w(A_{ij})\,.$$

Then $w[AB] = w[A]w[B]$ when the matrix products are defined, and $w[A]^\top = w[A^\top]$. Row vectors $\mathbf{y} \in K^l$ are viewed as matrices $\mathbf{y} = [y_1, \dots, y_l]$. Column vectors are viewed as matrices $\mathbf{y}^\top = [y_1, \dots, y_l]^\top$. If A is a rectangular matrix over K and $\partial \in \mathrm{Der}_K$ we define the matrix $\partial[A]$ by

$$\partial[A]_{ij} = \partial(A_{ij})\,.$$

Then $\partial[AB] = \partial[A]B + A\partial[B]$ when the matrix products are defined. If $\mathbf{y} = [y_1, \dots, y_l] \in K^l$ we let $J(\mathbf{y})$ denote the Jacobian matrix defined by

$$J(\mathbf{y})_{ij} = \partial_i y_j.$$

Let $R = \{f \in S \mid wf = f \text{ for all } w \in W\}$ be the algebra of W-invariant polynomial functions on V. As in Section 1, choose algebraically independent homogeneous polynomials $f_1, \dots, f_l \in R$ such that $R = \mathbf{R}[f_1, \dots, f_l]$. Let $\mathbf{f} = [f_1, \dots, f_l] \in S^l$. For $1 \le j \le l$ define $\theta_j \in \mathrm{Der}_S$ by

$$\theta_j(g) = (dg, df_j)$$

for $g \in S$, where $(\ ,\ )$ is the bilinear form on Ω^1_S defined by (2.3). It is known [Sai1], [Ter, Theorem 2] that $D(\mathcal{A})$ is a free S-module with basis $\theta_1, \dots, \theta_l$.

Let Der_R denote the R-module of $\mathbf{R}$-derivations of R and define Der_K in similar manner. Then $\mathrm{Der}_K = K\partial_1 \oplus \cdots \oplus K\partial_l$. Define $D^{(1)}, \dots, D^{(l)} \in \mathrm{Der}_R$ by $D^{(i)} f_j = \delta_{ij}$. Let Der_K denote the K-vector space of $\mathbf{R}$-derivations of K. We may extend $D^{(i)} : R \to R$ uniquely to an element of Der_K which we also call $D^{(i)}$. Since $D^{(i)} = \sum_{j=1}^l (D^{(i)} x_j)\partial_j$ we have $\delta_{ik} = D^{(i)} f_k = \sum_{j=1}^l (D^{(i)} x_j)(\partial_j f_k)$. Thus

$$[D^{(i)} x_j] = J(\mathbf{f})^{-1}\,. \tag{2.4}$$

Recall, from the Introduction, that we number the exponents $m_j = \deg(f_j) - 1$ so that $m_1 \leq \cdots \leq m_l$. Since W is irreducible we have $m_{l-1} < m_l$ [Bou, V.6.2, Corollary 2], and $m_l + 1 = h$ is the Coxeter number of W [Bou, V.6.2, Theorem 1]. It follows from the inequality $m_{l-1} < m_l$ that the one-dimensional space $\mathbf{R}D^{(l)}$ is uniquely determined by W and is independent of the choice of $f_1, \dots, f_l$ [Sai2, (2.2)], [SYS, (1.6)]. This remark of Saito is fundamental for the proof of our theorem. We make the following

Definition 2.5. *A Saito derivation is a nonzero element of* $\mathbf{R}D^{(l)}$.

Thus Saito derivations are characterized by the property

$$Df_1 = Df_2 = \cdots = Df_{l-1} = 0,\ Df_l \in \mathbf{R}^* . \tag{2.6}$$

We choose a Saito derivation D and fix it throughout the paper. Define $h_j \in K$ for $1 \leq j \leq l$ and $\mathbf{h} \in K^l$ by

$$h_j = Dx_j \text{ and } \mathbf{h} = [h_1, \dots, h_l] . \tag{2.7}$$

It follows from (2.4) that $\mathbf{h}$ is, up to constant multiple, the last row of $J(\mathbf{f})^{-1}$:

$$\mathbf{h} \doteq [J(\mathbf{f})^{-1}_{l1}, \dots, J(\mathbf{f})^{-1}_{ll}] . \tag{2.8}$$

Here and elsewhere $\doteq$ means equality of vectors (or matrices or polynomials) up to a nonzero constant multiple. By [Bou, , Proposition 6 (ii)] we have

$$\det J(\mathbf{f}) \doteq Q . \tag{2.9}$$

It follows from (2.9) that $D^{(i)}(S) \subseteq Q^{-1}S$ for $1 \leq i \leq l$. Thus

$$h_j \in Q^{-1}S \tag{2.10}$$

for $1 \leq j \leq l$. If $g \in S$ is homogeneous, we define the degree of $Q^{-1}g \in K$ by $\deg(Q^{-1}g) = \deg(g) - \deg(Q) = \deg(g) - \sum_{i=1}^{l} m_i$; the second equality follows from (2.9). From (2.8) we have

$$\deg(h_j) = -m_l . \tag{2.11}$$

Define $L_1, \dots, L_l \in \mathrm{Der}_K$ by

$$L_i = [\partial_i, D] = \partial_i D - D\partial_i . \tag{2.12}$$

Then $L_i x_j = \partial_i D x_j - D\partial_i x_j = \partial_i D x_j - D\delta_{ij} = \partial_i D x_j = \partial_i h_j$ so that

$$J(\mathbf{h}) = [L_i x_j] . \tag{2.13}$$

Define a matrix $N \in \mathbf{M}_l(S)$ by

$$N_{ij} = (dx_i, df_j) = \theta_j(x_i)\,. \tag{2.14}$$

Let

$$\Gamma = [(x_i, x_j)] \tag{2.15}$$

be the matrix of the form $(\ ,\ )$ on V^* with respect to the basis $x_1, \ldots, x_l$. Then

$$N = \Gamma J(\mathbf{f})\,. \tag{2.16}$$

Thus, if $x_1, \ldots, x_l$ is an orthonormal basis then $N = J(\mathbf{f})$. Define a matrix $B \in \mathbf{M}_l(K)$ by

$$B = -N^\top J(\mathbf{h}) J(\mathbf{f}) = -J(\mathbf{f})^\top \Gamma J(\mathbf{h}) J(\mathbf{f})\,. \tag{2.17}$$

The matrices $J(\mathbf{h})$ and B are the key constructs in our argument. Note that $\mathbf{h}$ depends only on the chosen derivation D and not on the chosen basic invariants $f_1, \ldots, f_l$. On the other hand B does depend on $f_1, \ldots, f_l$. We will prove in Corollary 3.33 that if W is not of type D_l with l even, and we replace D by $-D$ if necessary, then it is possible to choose a basis $x_1, \ldots, x_l$ for V^* and basic invariants $f_1, \ldots, f_l$ so that B has the form

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & m_l \\ 0 & 0 & \cdots & m_{l-1} & * \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & m_2 & \cdots & * & * \\ m_1 & * & \cdots & * & * \end{bmatrix}. \tag{2.18}$$

where the entries $*$ lie in R. The reason for the possible sign change in D will become clear in the proof of Corollary 3.33. In Section 5 we give examples of a matrix B of the form (2.18) in case $l = 2$ and in case W has type B_l.

Remark 2.19. K. Saito introduced the concept of flat generators for the ring of polynomial invariants of an irreducible real reflection group W [Sai2]. A system of basic invariants $f_1, \ldots, f_l$ is called a system of flat generators if the matrix $D[(df_i, df_j)]$ is a constant matrix. It is known [Sai2] that the space $\mathbf{R}f_1 + \cdots + \mathbf{R}f_l$ is uniquely determined by W. In [SYS], Saito, Yano and Sekiguchi explicitly determined a system of flat generators for each irreducible Coxeter group except E_7 and E_8. We will see in (3.28) that $D[(df_i, df_j)] = B + B^\top$. So the matrix B may be regarded as a refinement of $D[(df_i, df_j)]$ in the sense that B determines $D[(df_i, df_j)]$. The study of B therefore seems intriguing. For example we do not know if B is a constant matrix for a system of flat generators. It is known [Sai2, (5.1)], [SYS, (1.12)] that $D[(df_i, df_j)]$ is an invertible matrix. The invertibility is important because it gives a linear structure on the quotient variety $V/W = \operatorname{Spec} \mathbf{R}[f_1, \ldots, f_l]$ [Sai2]. In Lemma 3.9 we will show that B is also invertible.

§ 3. Invertibility of B and $J(\mathbf{h})$

In this Section we will prove that the entries of B are W-invariant polynomials and that $\det B \in R^*$ is a unit. It follows then from (2.9), (2.16) and (2.17) that $\det J(\mathbf{h}) \doteq Q^{-2}$.

Lemma 3.1. $B \in \mathbf{M}_l(R)$.

Proof. Let $\rho : W \to \mathbf{GL}_l(\mathbf{R})$ be the matrix representation of W afforded by the W-module V relative to the basis $e_1, \dots, e_l$. Thus

$$we_j = \sum_{i=1}^{l} \rho(w)_{ij} e_i \text{ and } wx_j = \sum_{i=1}^{l} \rho(w^{-1})_{ji} x_i \ . \tag{3.2}$$

Since $w\partial_v = \partial_{wv}$ for $w \in W$ and $v \in V$ we also have

$$w\partial_j = \sum_{i=1}^{l} \rho(w)_{ij} \partial_i \, . \tag{3.3}$$

To prove that the entries of B are W-invariant we need transformation rules for the action of $w \in W$ on certain matrices defined by basic invariants $f_1, \dots, f_l$. These rules are

$$w[N] = \rho(w^{-1}) N \, , \tag{3.4}$$

$$w[J(\mathbf{f})] = \rho(w)^\top J(\mathbf{f}) \, , \tag{3.5}$$

$$w[\mathbf{h}] = \mathbf{h} \rho(w^{-1})^\top \, , \tag{3.6}$$

$$w[J(\mathbf{h})] = \rho(w)^\top J(\mathbf{h}) \rho(w^{-1})^\top \, . \tag{3.7}$$

We sketch the proofs of these formulas. To prove (3.4) note that $w[N]_{ij} = w(N_{ij}) = w((dx_i, df_j)) = (w(dx_i), w(df_j)) = (d(wx_i), d(wf_j)) = (d(wx_i), df_j) = \sum_{k=1}^{l} \rho(w^{-1})_{ik} (dx_k, df_j) = (\rho(w^{-1})N)_{ij}$. To prove (3.5), note that $w[\mathbf{f}] = \mathbf{f}$ and use (3.3). To prove (3.6), note that $w[J(\mathbf{f})^{-1}] = J(\mathbf{f})^{-1} \rho(w^{-1})^\top$ and use (2.8). The last transformation rule (3.7) follows from (3.6) and (3.3). It follows from (3.4)–(3.7) that $w[B] = w[N^\top] w[J(\mathbf{h})] w[J(\mathbf{f})] = N^\top J(\mathbf{h}) J(\mathbf{f}) = B$. Thus the entries of B are W-invariant. To complete the proof we must show that the entries of B are polynomials. It follows from (2.13) that $L_i = \sum_{k=1}^{l} (L_i x_k) \partial_k = \sum_{k=1}^{l} (\partial_i h_k) \partial_k$. Then $(J(\mathbf{h}) J(\mathbf{f}))_{ij} = \sum_k (\partial_i h_k)(\partial_k f_j) = L_i f_j = [\partial_i, D] f_j = \partial_i D f_j - D \partial_i f_j = -D \partial_i f_j = -D[J(\mathbf{f})_{ij}]$. This shows that

$$J(\mathbf{h}) J(\mathbf{f}) = -D[J(\mathbf{f})] \ . \tag{3.8}$$

Since $J(\mathbf{f}) \in \mathbf{M}_l(S)$, it follows from (2.10) that $D[J(\mathbf{f})] \in \mathbf{M}_l(Q^{-1}S)$, so $QB \in \mathbf{M}_l(S)$. Since $B_{ij} \in R$ and Q is an anti-invariant polynomial, it follows that QB_{ij}

is an anti-invariant polynomial and hence [Bou, V.5.5, Proposition 6(iv)] lies in QR. Thus $B_{ij} \in R$. $\square$

The next lemma asserts, in particular, that $\det B$ is a non-zero real number.

Lemma 3.9. *1) If W is not of type D_l with l even then*

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & B_{1l} \\ 0 & 0 & \cdots & B_{2,l-1} & B_{2l} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & B_{l-1,2} & & B_{l-1,l-1} & B_{l-1,l} \\ B_{l1} & B_{l2} & \cdots & B_{l,l-1} & B_{ll} \end{bmatrix} \tag{3.10}$$

where

$$\begin{aligned} B_{ij} &= 0 \text{ if } i+j < l+1 \\ B_{ij} &\in \mathbf{R}^* \text{ if } i+j = l+1 \\ m_i B_{ij} &= m_j B_{ji} \text{ if } i+j = l+1\,. \end{aligned} \tag{3.11}$$

2) If W is of type D_l with $l = 2k$ then the 2×2 block in rows and columns $k, k+1$ of the matrix (3.10) – the center of the matrix – is to be replaced by a 2×2 block

$$B_0 = \begin{bmatrix} B_{k,k} & B_{k,k+1} \\ B_{k+1,k} & B_{k+1,k+1} \end{bmatrix} \tag{3.12}$$

with constant entries, where $B_{k,k+1} = B_{k+1,k}$ and $\det B_0 \in \mathbf{R}^*$. The statement (3.11) still holds true outside the 2×2 block B_0.

Proof. We agree in this argument that summation indices range over $1, \dots, l$. From (2.17) and (3.8) we have

$$B = J(\mathbf{f})^\top \Gamma D[J(\mathbf{f})]. \tag{3.13}$$

If $y \in K$ let $\operatorname{grad}(y) = [\partial_1 y, \dots, \partial_l y] \in K^l$ denote the gradient vector and let $\operatorname{Hess}(y) \in \mathbf{M}_l(K)$ denote the Hessian matrix, defined by $\operatorname{Hess}(y)_{ij} = \partial_i \partial_j y$. Then

$$\begin{aligned} B_{ij} &= \sum_{p,q} (\partial_p f_i)\,(dx_p, dx_q)\, D(\partial_q f_j) \\ &= \sum_{p,q} (\partial_p f_i)\,(dx_p, dx_q) \sum_r h_r (\partial_r \partial_q f_j) \\ &= \sum_{p,q,r} (\partial_p f_i)\,(dx_p, dx_q)\,(\partial_q \partial_r f_j) h_r \\ &= \operatorname{grad}(f_i)\, \Gamma \operatorname{Hess}(f_j)\, \mathbf{h}^\top\,. \end{aligned} \tag{3.14}$$

It follows from (3.14) and (2.11) that

$$\deg(B_{ij}) = m_i + m_j - 1 - m_l = m_i + m_j - h \tag{3.15}$$

when $B_{ij} \neq 0$. Thus $B_{ij} = 0$ whenever $m_i + m_j < h$. Also $B_{ij} \in \mathbf{R}$ by (3.15) whenever $m_i + m_j = h$.

We remark, parenthetically, that if $i + j \leq l$ then $j < l - i + 1$ so $m_i + m_j \leq m_i + m_{l-i+1} = h$ by duality in the exponents [Bou, V.6.2]. If equality holds in the last formula then, since $j \neq l - i + 1$, it follows from list of exponents in [Bou, VI.4] that W is of type D_l with l even. Thus in Case 1) the matrix B has the form (3.10). We do *not* know, at this stage, that the entries $B_{i,l-i+1}$ on the second diagonal are nonzero. Now return to the main line of argument. Define the row vector

$$\mathbf{g}^{(i,j)} = \mathrm{grad}(f_i)\Gamma\mathrm{Hess}(f_j) \in S^l\,. \tag{3.16}$$

Then $\deg(\mathbf{g}^{(i,j)}) = m_i + m_j - 1$ when $\mathbf{g}^{(i,j)} \neq 0$. By arguments like those used in the proofs of (3.4)–(3.7) we have the following transformation rules:

$$\Gamma = w[\Gamma] = \rho(w^{-1})\Gamma\rho(w^{-1})^\top\,, \tag{3.17}$$

$$w[\mathrm{grad}(f_i)] = \mathrm{grad}(f_i)\rho(w)\,, \tag{3.18}$$

$$w[\mathrm{Hess}(f_j)] = \rho(w)^\top\mathrm{Hess}(f_j)\rho(w)\,. \tag{3.19}$$

From these transformation rules, we have

$$w[\mathbf{g}^{(i,j)}] = \mathbf{g}^{(i,j)}\rho(w)\,. \tag{3.20}$$

If $\mathbf{g} = [g_1, \dots, g_l] \in S^l$ and $w[\mathbf{g}] = \mathbf{g}\rho(w)$ for all $w \in W$ then $g_1dx_1 + \cdots + g_ldx_l = [\mathbf{g}][dx_1, \dots, dx_l]^\top$ is W-invariant. It is shown in [Sol1, Theorem] that every W-invariant 1-form $g_1dx_1 + \cdots + g_ldx_l$ with $g_i \in S$ lies in $\sum_k R\,df_k$. Thus, by (3.20), we may write

$$\mathbf{g}^{(i,j)} = \sum_k r_k^{(i,j)}\mathrm{grad}(f_k) \tag{3.21}$$

with homogeneous $r_k^{(i,j)} \in R$. It follows from (3.14) and (3.15) that

$$B_{ij} = \mathbf{g}^{(i,j)}\mathbf{h}^\top = \sum_k r_k^{(i,j)}\mathrm{grad}(f_k)\mathbf{h}^\top = \sum_k r_k^{(i,j)}Df_k = r_l^{(i,j)}Df_l \tag{3.22}$$

and

$$\deg(r_k^{(i,j)}) = m_i + m_j - m_k - 1 \tag{3.23}$$

when $r_k^{(i,j)} \neq 0$. Let $\mathbf{x} = [x_1, \dots, x_l]$. Since $\deg(\partial_j f_i) = m_i$ for $1 \leq j \leq l$, it follows from (3.16) and the Euler formula that

$$\begin{aligned} m_i\,\mathbf{g}^{(i,j)}\mathbf{x}^\top &= m_i\,\mathrm{grad}(f_i)\,\Gamma\,m_j\,\mathrm{grad}(f_j)^\top \\ &= m_j\,\mathrm{grad}(f_j)\,\Gamma\,m_i\,\mathrm{grad}(f_i)^\top = m_j\,\mathbf{g}^{(j,i)}\mathbf{x}^\top \end{aligned} \tag{3.24}$$

because Γ is a symmetric matrix. On the other hand, by (3.21) and the Euler formula, we have

$$m_i \mathbf{g}^{(i,j)} \mathbf{x}^\top = m_i \sum_k r_k^{(i,j)} \mathrm{grad}(f_k) \mathbf{x}^\top = m_i \sum_k r_k^{(i,j)} (m_k + 1) f_k. \tag{3.25}$$

Combine (3.25) with (3.24). This gives

$$m_i \sum_k r_k^{(i,j)} (m_k + 1) f_k = m_j \sum_k r_k^{(j,i)} (m_k + 1) f_k \tag{3.26}$$

for all $1 \leq i, j \leq l$. It follows from (3.23) that both sides of (3.26) are homogeneous polynomials of degree $m_i + m_j$. Suppose now that i, j satisfy $m_i + m_j = h = \deg(f_l)$. Then $\deg(r_k^{(i,j)}) < h$ and thus $r_k^{(i,j)} \in \mathbf{R}[f_1, \ldots, f_{l-1}]$. Since the invariant polynomials $f_1, \ldots, f_l$ are algebraically independent, we can equate the coefficients of f_l on both sides of (3.26) and conclude that $m_i\, r_l^{(i,j)}(m_l + 1) = m_j\, r_l^{(j,i)}(m_l + 1)$. This proves

$$m_i B_{ij} = m_j B_{ji} \text{ whenever } m_i + m_j = h \tag{3.27}$$

because of (3.22). Note that $B_{ij} \in \mathbf{R}$ by (3.15) whenever $m_i + m_j = h$. On the other hand

$$B + B^\top = J(\mathbf{f})^\top \Gamma D[J(\mathbf{f})] + D[J(\mathbf{f})^\top] \Gamma J(\mathbf{f}) = D[J(\mathbf{f})^\top \Gamma J(\mathbf{f})] = D[(df_i, df_j)] \tag{3.28}$$

where the last equality follows from (2.3). The matrix on the right is non-singular, as shown in [Sai2, (5.1)], [SYS, (1.13)]. Thus

$$\det(B + B^\top) \neq 0. \tag{3.29}$$

Case 1) Assume that W is not of type D_l with l even. If $i + j < l + 1$, then $m_i + m_j < h$ and thus, as we have already remarked, $B_{ij} = 0$. Note that

$$\det(B + B^\top) \doteq \prod_{i+j=l+1} (B_{ij} + B_{ji}). \tag{3.30}$$

By (3.29) and (3.30) we have $B_{ij} + B_{ji} \neq 0$ whenever $i + j = l + 1$. It follows from (3.27) that $B_{ij} \neq 0$ whenever $i + j = l + 1$. This proves the desired result in Case 1).

Case 2) Assume that W is of type D_l with $l = 2k$ even. If $i + j < l + 1$ with $(i, j) \neq (k, k)$, then $m_i + m_j < h$ and thus $B_{ij} = 0$. Let B_0 be as in (3.12). Then

$$\det(B + B^\top) \doteq \det(B_0 + B_0^\top) \prod (B_{ij} + B_{ji}), \tag{3.31}$$

where the product is over the set $\{(i, j) \mid i + j = l + 1 \text{ and } |i - j| > 1\}$. By (3.29) we have $B_{ij} + B_{ji} \neq 0$ whenever $i + j = l + 1$ and $|i - j| > 1$. It follows from (3.27) that $B_{ij} \neq 0$ whenever $i + j = l + 1$ and $|i - j| > 1$. Since $m_k = m_{k+1}$, we have $B_{k,k+1} = B_{k+1,k}$ by (3.27). Thus B_0 is a symmetric matrix. By (3.29) and (3.31), we have $4 \det B_0 = \det(2B_0) = \det(B_0 + B_0^\top) \neq 0$. Thus $\det B_0 \neq 0$. □

Corollary 3.32. *The matrix $J(\mathbf{h})$ has determinant $\det J(\mathbf{h}) \doteq Q^{-2}$. Thus $J(\mathbf{h})$ is invertible and $h_1, \ldots , h_l$ are algebraically independent.*

Proof. We have $\det J(\mathbf{h}) = (\det \Gamma)^{-1}(\det J(\mathbf{f}))^{-2}(\det B) \doteq Q^{-2}$ by (2.17), (2.9) and Lemma 3.9. □

Corollary 3.33. *If W is not of type D_l with l even, then it is possible to choose a Saito derivation D and basic invariants $f_1, \ldots , f_l$ so that B has the form (2.18).*

Proof. Choose any basis $x_1, \ldots , x_l$, and basic invariants $f_1, \ldots , f_l$. Then B has the form (3.10) where

$$m_i B_{i,l+1-i} = m_{l+1-i} B_{l+1-i,i} \text{ for } 1 \leq i \leq l\,. \tag{3.34}$$

Suppose first that $l = 2k$ is even. Define $c_i = m_i/B_{l+1-i,i}$ for $1 \leq i \leq k$ and $c_i = 1$ for $k + 1 \leq i \leq l$. Then $c_i c_{l+1-i} B_{l+1-i,i} = m_i$ for $1 \leq i \leq l$ by (3.34). Define $f'_i = c_i f_i$ for $1 \leq i \leq l$ and let $\mathbf{f}' = [f'_1, \ldots , f'_l]$. Let $B' = -J(\mathbf{f}')^\top \Gamma J(\mathbf{h})J(\mathbf{f}')$ and let $C = \operatorname{diag}(c_1, \ldots , c_l)$. Since $J(\mathbf{f}') = J(\mathbf{f})C$ we have $B' = CBC$ so $B'_{l+1-i,i} = c_i c_{l+1-i} B_{l+1-i,i} = m_i$. Thus replacement of $\mathbf{f}$ by $\mathbf{f}'$ gives us (2.18). If $l = 2k + 1$ is odd we must modify the argument slightly. Note that the condition (3.34) is vacuous for $i = k + 1$. If $B_{k+1,k+1} < 0$ we replace D by $-D$. Thus, by (3.13), we may assume that $B_{k+1,k+1} > 0$. Define $c_i = m_i/B_{l+1-i,i}$ for $1 \leq i \leq k$ and $c_i = 1$ for $k + 2 \leq i \leq l$ by analogy with the case $l = 2k$. Choose c_{k+1} so that $c_{k+1}^2 B_{k+1,k+1} = m_{k+1}$. Let $f'_i = c_i f_i$ for $1 \leq i \leq l$. Then D and the basic invariants $f'_1, \ldots , f'_l$ have the desired property. □

We use the fact that B is invertible to give the following alternative expression for the matrix P in Theorem 1.4.

Proposition 3.35. *Define $A = B^{-1} \in M_l(R)$. Then*

$$P = -NAN^\top.$$

Proof. By (2.16) and (2.17) we have $N = \Gamma J(\mathbf{f})$ and $B = -N^\top J(\mathbf{h})J(\mathbf{f})$. Thus $NAN^\top = NB^{-1}N^\top = -\Gamma J(\mathbf{f})J(\mathbf{f})^{-1}J(\mathbf{h})^{-1}(N^\top)^{-1}N^\top = -\Gamma J(\mathbf{h})^{-1} = -P$. □

§ 4. Proof of Theorem 1.4

In this Section we will prove Theorem 1.4. We will also determine the graded W-module structure of $E(\mathcal{A})$. It turns out that its homogeneous component of degree h is W-isomorphic to V^*.

Recall that the matrix $P = \Gamma J(h_1, \dots, h_l)^{-1}$ of Theorem 1.4 is defined using a basis $e_1, \dots, e_l$ for V and the dual basis $x_1, \dots, x_l$ for V^* together with Saito's derivation D. We will study how P is transformed if $e_1, \dots, e_l$ is replaced by another basis for V. Suppose a basis $e'_1, \dots, e'_l$ for V is connected with $e_1, \dots, e_l$ through an invertible matrix $M \in \mathbf{GL}_l(\mathbf{R})$:

$$e'_j = \sum_{i=1}^{l} M_{ij} e_i. \tag{4.1}$$

The new objects, which are defined using the new basis $e'_1, \dots, e'_l$, will be denoted by $x'_j, \partial'_j, \Gamma'$ etc.. As in (3.2)–(3.7) and (3.17)–(3.19), we have

$$x'_j = \sum_{i=1}^{l} (M^{-1})_{ji} x_i\,,\ \partial'_j = \sum_{i=1}^{l} M_{ij} \partial_i\,, \tag{4.2}$$

$$\Gamma' = M^{-1}\Gamma(M^{\top})^{-1}\,,\ \mathbf{h}' = \mathbf{h}(M^{\top})^{-1}\,,\ J'(\mathbf{h}') = M^{\top} J(\mathbf{h})(M^{\top})^{-1}\,. \tag{4.3}$$

Thus

$$P' = M^{-1} P (M^{\top})^{-1}. \tag{4.4}$$

Recall that the derivations $\xi_1, \dots, \xi_l \in \mathrm{Der}_S$ of Theorem 1.4 are defined by $\xi_j = \sum_{i=1}^{l} p_{ij}\, \partial_i$ where p_{ij} is the (i,j) entry of P. By (4.2) and (4.4), we have

$$\xi'_j = \sum_{i=1}^{l} (M^{-1})_{ji} \xi_i. \tag{4.5}$$

In other words, $\xi_1, \dots, \xi_l$ satisfy the same base change rule as $x_1, \dots, x_l$.

Lemma 4.6. *If $H \in \mathcal{A}$ then $\xi_i(\alpha_H) \in S\alpha_H^2$. Thus $\xi_i \in E(\mathcal{A})$ for $1 \le i \le l$.*

Proof. Because of (4.5), we may assume that $\alpha_H = x_1$ and that $x_1, \dots, x_l$ is an orthonormal basis. Then $P = J(\mathbf{h})^{-1}$. It is thus enough to show that each entry of the first row of P is divisible by x_1^2. Since $x_1, \dots, x_l$ is an orthonormal basis we have $\theta_j = \sum_{i=1}^{l} (\partial_i f_j) \partial_i \in D(\mathcal{A})$, as remarked in the Introduction. Thus $\partial_1 f_j = \theta_j(x_1) \in Sx_1$, so each entry of the first row of $J(\mathbf{f})$ is divisible by x_1. Thus, outside the first column, each entry of $\operatorname{adj} J(\mathbf{f})$, is divisible by x_1. Since $\det J(\mathbf{f}) \doteq Q$ is divisible by x_1 exactly once, each entry of $J(\mathbf{f})^{-1} \doteq Q^{-1} \mathrm{adj} J(\mathbf{f})$, outside the first column, has no pole along $x_1 = 0$. In particular, h_j $(2 \le j \le l)$ has no pole along $x_1 = 0$. It follows that each entry of $J(\mathbf{h})$ outside the first column has no pole along $x_1 = 0$. Therefore, each entry of the first row of $\operatorname{adj} J(\mathbf{h})$ has no pole along $x_1 = 0$. Recall that $J(\mathbf{h}) \doteq Q^{-2}$ from Corollary 3.32. This implies that each entry of the first row of $J(\mathbf{h})^{-1} \doteq Q^2 \operatorname{adj} J(\mathbf{h})$ is divisible by x_1^2. □

Now we may complete the proof of Theorem 1.4. By Corollary 3.32 we have

$$\det[\xi_j(x_i)] = \det P = \det(\Gamma J(\mathbf{h})^{-1}) \doteq Q^2.$$

By Ziegler's generalization [Zie, p.351] of Saito's criterion [Sai3, p.270], [OrT, Theorem 4.19] to multiarrangements, we can conclude that $\xi_1, \dots, \xi_l \in E(\mathcal{A})$ form a basis for the S-module $E(\mathcal{A})$. This completes the proof of Theorem 1.4.

The space $E(\mathcal{A})$ inherits a grading from Der_S. Let $E(\mathcal{A})_q \subset E(\mathcal{A})$ denote the space of homogeneous elements of degree q. Then $E(\mathcal{A}) = \bigoplus_{q \geq 0} E(\mathcal{A})_q$. It follows from Theorem 1.4 that

$$E(\mathcal{A}) = S \otimes_{\mathbf{R}} E(\mathcal{A})_h$$

and that

$$E(\mathcal{A})_h = \bigoplus\nolimits_k \mathbf{R}\, \xi_k \,.$$

Thus the W-module structure of $E(\mathcal{A})$ is determined by that of $E(\mathcal{A})_h$. The W-module structure of $E(\mathcal{A})_h$ is given by the following:

Proposition 4.7. *The $\mathbf{R}$-linear map $\Xi : V^* \longrightarrow E(\mathcal{A})_h$ defined by $\Xi(x_i) = \xi_i$ for $1 \leq i \leq l$, is a W-isomorphism.*

Proof. We have already remarked in (4.5) that $\xi_1, \dots, \xi_l$ satisfy the same base change rule as $x_1, \dots, x_l$. Thus the assertion follows from (4.5) with $M = \rho(w)$. □

Since W is assumed irreducible, it follows from Schur's lemma that an arbitrary W-isomorphism from V^* to $E(\mathcal{A})_h$ is a nonzero constant multiple of the map Ξ.

Proposition 4.8. *If $H \in \mathcal{A}$ then $\Xi(\alpha_H) \in \alpha_H \operatorname{Der}_S$.*

Proof. Write $\alpha_H = c_1 x_1 + \cdots + c_l x_l$ with $c_i \in \mathbf{R}$. Then $\Xi(\alpha_H) = \sum_k c_k\, \Xi(x_k) = \sum_k c_k \xi_k$. For $1 \leq i \leq l$, let $\mathbf{e}_i \in \mathbf{R}^l$ be the i-th elementary unit vector. Then, by Proposition 3.35, $\Xi(\alpha_H)(x_i) = \sum_k c_k \xi_k(x_i) = \mathbf{e}_i P \mathrm{grad}(\alpha_H)^\top = -\mathbf{e}_i N A N^\top \mathrm{grad}(\alpha_H)^\top = -\mathbf{e}_i N A (\mathrm{grad}(\alpha_H) N)^\top \in S\alpha_H$ because $\mathrm{grad}(\alpha_H) N = [\theta_1(\alpha_H), \dots, \theta_l(\alpha_H)] \in (S\alpha_H)^l$. □

§ 5. Examples

In this Section we will study two examples: the two-dimensional double Coxeter arrangements and the double Coxeter arrangements of type B_l.

1. The two-dimensional case: Let V be two dimensional Euclidean space. Let $W \subset \mathbf{GL}(V)$ be a finite irreducible reflection group. Thus W is a dihedral group of order $2n$ where $n > 2$. Let $\mathcal{A}$ be the corresponding Coxeter arrangement. Choose Q as in (1.1). Then $\deg(Q) = n$. Choose an orthonormal basis e_1, e_2 for V. Let x_1, x_2 be the dual basis for V^*. Then Γ is the identity matrix. The exponents of W are $m_1 = 1, m_2 = n-1$. To construct the matrix P we must find a Saito derivation. Define $\lambda \in \Omega^1_S$ by $\lambda = x_1 dx_1 + x_2 dx_2$. Define $\sigma \in \Omega^2_S$ by $\sigma = dx_1 \wedge dx_2$. Let $w \in W$. Then $w\lambda = \lambda$ and $w\sigma = \det(w)\sigma$. Define the star operator $* : V^* \to V^*$ by $x \wedge y = (*x, y)\sigma$ for $x, y \in V^*$. Extend $*$ to an S-module map $* : \Omega^1_S \to \Omega^1_S$ by S-linearity. Then $*(df) = -(\partial_2 f)dx_1 + (\partial_1 f)dx_2$ for $f \in S$. Since $w\sigma = \det(w)\sigma$ we have $w(*\theta) = \det(w) *(w\theta)$ for $\theta \in \Omega^1_S$. Suppose $f \in R$. Then $w(\lambda, *df) = (w\lambda, w(*df)) = \det(w)(\lambda, *df)$ so $-x_2(\partial_1 f) + x_1(\partial_2 f) = (\lambda, *df) \in QR$. Define $D \in \mathrm{Der}_K$ by

$$D = \frac{1}{nQ}(-x_2\partial_1 + x_1\partial_2). \tag{5.1}$$

Then D maps $R \to R$. Now let f_1, f_2 be basic invariants with $\deg(f_1) = 2$ and $\deg(f_2) = n$. Since $\deg(Q) = n > 2 = \deg(f_1)$ we have $Df_1 = 0$. Since $\deg(f_2) = n = \deg(Q)$ we have $Df_2 \in \mathbf{R}$. If $Df_2 = 0$ then $Dx_1 = 0 = Dx_2$, a contradiction because $Df_k = (Dx_1)(\partial_1 f_k) + (Dx_2)(\partial_2 f_k)$ for $k = 1, 2$ and $J(f_1, f_2) \neq 0$. Thus $Df_1 = 0$ and $Df_2 \in \mathbf{R}^*$ so D is a Saito derivation by (2.6). We use this D and follow the procedure in Sections 2 and 3 to construct the matrix P, the derivations ξ_1, ξ_2 and the matrix B. From (2.7) we have

$$\mathbf{h} = \frac{1}{nQ}\left[-x_2, x_1\right].$$

We compute

$$J(\mathbf{h}) = \frac{1}{nQ^2}\begin{bmatrix} x_2 Q_1 & Q - x_1 Q_1 \\ -Q + x_2 Q_2 & -x_1 Q_2 \end{bmatrix},$$

where $Q_i = \partial_i Q$ for $i = 1, 2$. Since $\det J(\mathbf{h}) = (1-n)/n^2 Q^2$ we have

$$P = J(\mathbf{h})^{-1} = \frac{n}{n-1}\begin{bmatrix} x_1 Q_2 & Q - x_1 Q_1, \\ -Q + x_2 Q_2 & -x_2 Q_1 \end{bmatrix}, \tag{5.2}$$

and

$$\begin{aligned} \xi_1 &= \frac{n}{n-1}\left\{x_1 Q_2 \partial_1 + (x_2 Q_2 - Q)\partial_2\right\} \\ \xi_2 &= \frac{n}{n-1}\left\{(Q - x_1 Q_1)\partial_1 - x_2 Q_1 \partial_2\right\}. \end{aligned} \tag{5.3}$$

By Theorem 1.4, the derivations ξ_1 and ξ_2 form a basis for $E(\mathcal{A})$. Note in (5.2) that the matrix P depends only upon choice of x_1, x_2 and Q. The reader may have noticed that the group W is peripheral to the computations in this section. In fact we can use the derivations defined by (5.3) to prove a proposition about *any* central arrangement in a real two dimensional vector space V.

Proposition 5.4. *Suppose V is a real vector space of dimension 2. Let $\mathcal{A}$ be an arbitrary central arrangement in V and let Q be its defining polynomial. Let $n = |\mathcal{A}| = \deg Q$. Define an S-module $E(\mathcal{A})$ as in (1.3). Then the derivations ξ_1 and ξ_2 given by (5.3) form a basis for $E(\mathcal{A})$.*

Proof. Note that $\det[\xi_j(x_i)] = \det P \doteq Q^2$. Thanks to Ziegler's generalization [Zie, p.351] of Saito's criterion [Sai3, p.270], [OrT, Theorem 4.19] to multiarrangements, it is enough to show that $\xi_i \in E(\mathcal{A})$ for $i = 1, 2$. Let $H \in \mathcal{A}$. Write $\alpha_H = ax_1 + bx_2$ with $a, b \in \mathbf{R}$. Then

$$\xi_1(\alpha_H) \doteq ax_1Q_2 + b(x_2Q_2 - Q) = \alpha_H Q_2 - bQ = \alpha_H^2 \partial_2(Q/\alpha_H) \in \alpha_H^2 S.$$

So $\xi_1 \in E(\mathcal{A})$. Similarly $\xi_2 \in E(\mathcal{A})$. □

Now we return to the case of Coxeter arrangements. Since $f_1 \doteq x_1^2 + x_2^2$ we may choose $f_1 = (x_1^2 + x_2^2)/2$. If n is even, then the invariant f_2 is not uniquely determined up to a constant multiple. We make a special choice of f_2. Define

$$f_2 = -Q(DQ)\,.$$

We will find the matrix $B = J(\mathbf{f})^\top \Gamma D[J(\mathbf{f})]$ in (3.13) and check that f_1, f_2 is a system of flat generators in the sense of K. Saito; see Remark 2.19. First note that f_2 is an invariant because $Q^2 \in R$ and $D : R \to R$ since D is a Saito derivation. Since the Laplacian $\Delta = \partial_1^2 + \partial_2^2$ commutes with the action of W, and Q is an anti-invariant, ΔQ is also an anti-invariant. Since Q is an anti-invariant of minimal degree, we have

$$0 = \Delta Q = Q_{11} + Q_{22}. \tag{5.5}$$

To compute $J(\mathbf{f})$ use (5.5). Calculate $n\partial_1 f_2 = -\partial_1(-x_2Q_1 + x_1Q_2) = x_2Q_{11} - Q_2 - x_1Q_{12} = -(x_2Q_{22} + Q_2 + x_1Q_{12}) = -((n-1)Q_2 + Q_2) = -nQ_2$. Thus $\partial_1 f_2 = -Q_2$. Similarly $\partial_2 f_2 = Q_1$. Thus

$$J(\mathbf{f}) = \begin{bmatrix} x_1 & -Q_2 \\ x_2 & Q_1 \end{bmatrix}.$$

To compute $D[J(\mathbf{f})]$ use (5.1) and (5.5). Calculate $nQ(DQ_1) = -x_2Q_{11} + x_1Q_{12} = x_2Q_{22} + x_1Q_{12} = (n-1)Q_2$. Thus $DQ_1 = (n-1)Q_2/nQ$. Similarly $DQ_2 = -(n-1)Q_1/nQ$. Thus

$$B = J(\mathbf{f})^\top D[J(\mathbf{f})] = \begin{bmatrix} x_1 & x_2 \\ -Q_2 & Q_1 \end{bmatrix} \frac{1}{nQ} \begin{bmatrix} -x_2 & (n-1)Q_1 \\ x_1 & (n-1)Q_2 \end{bmatrix} = \begin{bmatrix} 0 & n-1 \\ 1 & 0 \end{bmatrix},$$

and

$$A = B^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{n-1} & 0 \end{bmatrix}.$$

The alternative expression of P given in Proposition 3.35 is:

$$P = -\begin{bmatrix} x_1 & -Q_2 \\ x_2 & Q_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \frac{1}{n-1} & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ -Q_2 & Q_1 \end{bmatrix}.$$

This agrees with (5.2) via the Euler formula. By (3.28),

$$D[(df_i, df_j)] = B + B^\top = \begin{bmatrix} 0 & n \\ n & 0 \end{bmatrix}.$$

It follows that f_1, f_2 is a system of flat generators. Note, by (5.5), that $\Delta f_2 = \partial_1(-Q_2) + \partial_2 Q_1 = 0$. Thus f_2 is harmonic.

2. The case B_l: Let W be the Coxeter group of type B_l acting on an l-dimensional Euclidean space V by signed permutations of an orthonormal basis $e_1, \ldots, e_l$. Let $\mathcal{A}$ be the corresponding Coxeter arrangement. Let $x_1, \ldots, x_l$ be the dual basis for V^*. Then Γ is the identity matrix. Define

$$p_i = p_i(x_1, \ldots, x_l) = \frac{1}{i} \sum_{k=1}^{l} x_k^i$$

for $i \geq 1$. Define $p_0 = 1$. Let $f_i = p_{2i}$. We will use the basic invariants $f_1, \ldots, f_l$ to find the matrices B, A, and P. To simplify formulas we use the following notation [Mac, pps.26-27]: if $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{N}^l$, let $A_\alpha = [x_j^{\alpha_i}]$ and let $a_\alpha = \det A_\alpha$. Then

$$J(\mathbf{f}) = \begin{bmatrix} x_1 & x_1^3 & \cdots & x_1^{2l-1} \\ \vdots & \vdots & & \vdots \\ x_l & x_l^3 & \cdots & x_l^{2l-1} \end{bmatrix} = A_{(1,3,\ldots,2l-1)}.$$

Define a derivation $D \in \mathrm{Der}_K$ by

$$Dy = \frac{1}{a_{(1,3,\ldots,2l-1)}} \begin{vmatrix} x_1 & x_1^3 & \cdots & x_1^{2l-3} & \partial_1 y \\ \vdots & \vdots & & \vdots & \vdots \\ x_l & x_l^3 & \cdots & x_l^{2l-3} & \partial_l y \end{vmatrix}, \tag{5.6}$$

for $y \in K$. Since $Df_1 = Df_2 = \cdots = Df_{l-1} = 0$ and $Df_l = 1$, D is a Saito derivation. For $i \geq 0$ let

$$c_i(x_1, \ldots, x_l) = \sum_{i_1 + \cdots + i_l = i} x_1^{i_1} \cdots x_l^{i_l}$$

be the i-th complete symmetric polynomial; it is not possible to use the now standard notation h_i of [Mac] since h_i has already been used. Let

$$\tilde{c}_i = \tilde{c}_i(x_1, \dots, x_l) = c_i(x_1^2, \dots, x_l^2)\,.$$

Then $\tilde{c}_i$ is a W-invariant polynomial of degree $2i$. Define $\tilde{c}_i(x_1, \dots, x_l) = 0$ if $i < 0$.

Lemma 5.7. *The derivation D satisfies $D(p_{2l+2i}) = \tilde{c}_i(x_1, \dots, x_l)$ for $i \geq -l$.*

Proof. From (5.6) we have

$$D(p_{2l+2i}) = \frac{a_{(1,3,\dots,2l-3,2l+2i-1)}}{a_{(1,3,\dots,2l-1)}} = \frac{a_{(0,2,\dots,2l-4,2l+2i-2)}}{a_{(0,2,\dots,2l-2)}}\,. \tag{5.8}$$

Define $\delta = (l-1, l-2, \dots, 1, 0) \in \mathbf{N}^l$ and define $\lambda = (i, 0, \dots, 0, 0) \in \mathbf{N}^l$. Then $\lambda + \delta = (l+i-1, l-2, \dots, 1, 0)$. The right hand side of (5.8) is thus $a_{\lambda+\delta}/a_\delta$ with x_i replaced by x_i^2. By [Mac, (I.3.1), (I.3.9)] we have $a_{\lambda+\delta}/a_\delta = c_i(x_1, \dots, x_l)$. Thus the right hand side of (5.8) is $\tilde{c}_i(x_1, \dots, x_l)$. $\square$

Since $J(\mathbf{f})_{ij} = x_i^{2j-1}$, the entries of $B = J(\mathbf{f})^\top D[J(\mathbf{f})]$ in (3.13) are

$$\begin{aligned} B_{ij} &= \sum_k x_k^{2i-1} D(x_k^{2j-1}) = (2j-1) \sum_k x_k^{2i+2j-3} D(x_k) \\ &= (2j-1) D(p_{2i+2j-2}) = (2j-1)\tilde{c}_{i+j-l-1}(x_1, \dots, x_l) \end{aligned}$$

by Lemma 5.7. Thus

$$B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 2l-1 \\ 0 & 0 & 0 & \cdots & 2l-3 & (2l-1)\tilde{c}_1 \\ 0 & 0 & 0 & \cdots & (2l-3)\tilde{c}_1 & (2l-1)\tilde{c}_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 5 & \cdots & (2l-3)\tilde{c}_{l-4} & (2l-1)\tilde{c}_{l-3} \\ 0 & 3 & 5\tilde{c}_1 & \cdots & (2l-3)\tilde{c}_{l-3} & (2l-1)\tilde{c}_{l-2} \\ 1 & 3\tilde{c}_1 & 5\tilde{c}_2 & \cdots & (2l-3)\tilde{c}_{l-2} & (2l-1)\tilde{c}_{l-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{c}_1 & 1 & \cdots & 0 & 0 & 0 \\ \tilde{c}_2 & \tilde{c}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \tilde{c}_{l-3} & \tilde{c}_{l-4} & \cdots & 1 & 0 & 0 \\ \tilde{c}_{l-2} & \tilde{c}_{l-3} & \cdots & \tilde{c}_1 & 1 & 0 \\ \tilde{c}_{l-1} & \tilde{c}_{l-2} & \cdots & \tilde{c}_2 & \tilde{c}_1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 2l-1 \\ 0 & 0 & \cdots & 2l-3 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 3 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

On the other hand, it is known [Mac, p.21] that

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{c}_1 & 1 & \cdots & 0 & 0 & 0 \\ \tilde{c}_2 & \tilde{c}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \tilde{c}_{l-3} & \tilde{c}_{l-4} & \cdots & 1 & 0 & 0 \\ \tilde{c}_{l-2} & \tilde{c}_{l-3} & \cdots & \tilde{c}_1 & 1 & 0 \\ \tilde{c}_{l-1} & \tilde{c}_{l-2} & \cdots & \tilde{c}_2 & \tilde{c}_1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ -\tilde{e}_1 & 1 & \cdots & 0 & 0 & 0 \\ \tilde{e}_2 & -\tilde{e}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^{l-3}\tilde{e}_{l-3} & (-1)^{l-4}\tilde{e}_{l-4} & \cdots & 1 & 0 & 0 \\ (-1)^{l-2}\tilde{e}_{l-2} & (-1)^{l-3}\tilde{e}_{l-3} & \cdots & -\tilde{e}_1 & 1 & 0 \\ (-1)^{l-1}\tilde{e}_{l-1} & (-1)^{l-2}\tilde{e}_{l-2} & \cdots & \tilde{e}_2 & -\tilde{e}_1 & 1 \end{bmatrix},$$

where $\tilde{e}_i = \tilde{e}_i(x_1, \dots, x_l) = e_i(x_1^2, \dots, x_l^2)$ is the i-th elementary symmetric polynomial in $x_1^2, \dots, x_l^2$. Therefore $A = B^{-1}$ is equal to

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1/1 \\ 0 & 0 & \cdots & 1/3 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1/(2l-3) & \cdots & 0 & 0 \\ 1/(2l-1) & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ -\tilde{e}_1 & 1 & \cdots & 0 & 0 & 0 \\ \tilde{e}_2 & -\tilde{e}_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ (-1)^{l-3}\tilde{e}_{l-3} & (-1)^{l-4}\tilde{e}_{l-4} & \cdots & 1 & 0 & 0 \\ (-1)^{l-2}\tilde{e}_{l-2} & (-1)^{l-3}\tilde{e}_{l-3} & \cdots & -\tilde{e}_1 & 1 & 0 \\ (-1)^{l-1}\tilde{e}_{l-1} & (-1)^{l-2}\tilde{e}_{l-2} & \cdots & \tilde{e}_2 & -\tilde{e}_1 & 1 \end{bmatrix}.$$

Since Γ is the identity matrix we have $N = J(\mathbf{f})$. Thus, by Proposition 3.35, the matrix P in Theorem 1.4 is given by

$$P = -NAN^\top = -J(\mathbf{f})AJ(\mathbf{f})^\top$$

with A as above.

§ 6. Anti-invariant differential 1-forms

If M is an $\mathbf{R}[W]$-module let $M^W = \{x \in M \mid wx = x \text{ for all } w \in W\}$ denote the space of invariant elements in M. Let $M^{\det} = \{x \in M \mid wx = \det(w)x \text{ for all } w \in W\}$ denote the space of anti-invariant elements in M. In this section we use the fact that $\det J(\mathbf{h}) \neq 0$ to prove the following Proposition. Recall that D denotes a Saito derivation and that $h_i = Dx_i$.

Proposition 6.1. *Let $W \subset \mathbf{GL}(V)$ be a finite irreducible group generated by reflections. Let $u_i = Qh_i$. Define an $\mathbf{R}$-linear map $\hat{d} : S \to \Omega^1_S$ by*

$$\hat{d}f = \sum_{i=1}^{l} (\partial_i f) du_i \tag{6.2}$$

for $f \in S^l$. Let $f_1, \dots, f_l$ be basic invariants. Then

$$(\Omega^1_S)^{\det} = R\,\hat{d}f_1 \oplus \cdots \oplus R\,\hat{d}f_l\,.$$

Proof. Choose an orthonormal basis $x_1, \dots, x_l$ for V^*. By Corollary 3.32, $h_1, \dots, h_l$ are algebraically independent. By (2.10) we have $Qh_i \in S$. Let $\mathbf{u} = [u_1, \dots, u_l] \in S^l$. Since $\mathbf{u} = Q\mathbf{h}$ it follows that $u_1, \dots, u_l$ are algebraically independent. Thus $\det J(\mathbf{u}) \neq 0$. To show that $\hat{d}f_i \in (\Omega^1_S)^{\det}$ we must check

$$w(\hat{d}f) = \det(w)\hat{d}(wf) \tag{6.3}$$

for all $w \in W$ and $f \in S$. Let $\mathbf{x} = [x_1, \dots, x_l]$. Let $\rho : W \to \mathbf{GL}_l(\mathbf{R})$ be the matrix representation of W defined in (3.2). If $w \in W$ then $w[\mathbf{x}] = \mathbf{x}\rho(w^{-1})^\top$. Since Q is anti-invariant and $\mathbf{u} = Q\mathbf{h}$, it follows from (3.6) that $w[\mathbf{u}] = \det(w)\,\mathbf{u}\rho(w^{-1})^\top$. Thus $w(\hat{d}x_j) = \det(w)\hat{d}(wx_j)$ for $j = 1, \dots, l$. This proves (6.3) for $f = x_j$. Since the map $f \mapsto \hat{d}f$ is $\mathbf{R}$-linear and $\hat{d}(fg) = f\,\hat{d}g + g\,\hat{d}f$, for all $f, g \in S$, the set of all $f \in S$ which satisfy (6.3) is an $\mathbf{R}$-subalgebra of S which contains $x_1, \dots, x_l$ and is thus equal to S. This proves (6.3). Thus $(\Omega^1_S)^{\det} \supseteq R\,\hat{d}f_1 + \cdots + R\,\hat{d}f_l$.

Now argue as in [Sol2, Theorem 3] to show that $\hat{d}f_1, \dots, \hat{d}f_l$ are linearly independent over S. If not, then we have a relation $\sum_{i=1}^{l} g_i\,\hat{d}f_i = 0$ where $g_i \in S$ and g_1, say, is not zero. Multiply the relation by $\hat{d}f_1$. This gives $\hat{d}f_1 \wedge \cdots \wedge \hat{d}f_l = 0$. Let $\mathbf{f} = [f_1, \dots, f_l]$. It follows from (6.2) that

$$\hat{d}f_1 \wedge \cdots \wedge \hat{d}f_l = \det(J(\mathbf{f}))\det(J(\mathbf{u}))\,dx_1 \wedge \cdots \wedge dx_l\,,$$

which is not zero since $\det J(\mathbf{u}) \neq 0$. This contradiction proves the linear independence. Thus the sum $R\,\hat{d}f_1 + \cdots + R\,\hat{d}f_l$ is direct, so

$$(\Omega^1_S)^{\det} \supseteq R\,\hat{d}f_1 \oplus \cdots \oplus R\,\hat{d}f_l\,. \tag{6.4}$$

To prove equality in (6.4) we show that both graded vector spaces have the same Poincaré series. Let $n = \deg(Q)$. By (2.11) we have $\deg(u_i) = n - m_l$ for $1 \le i \le l$. Thus $\deg(du_i) = n - h$ where h is the Coxeter number. Since $\deg(df_i) = m_i$ we have

$$\mathrm{Poin}(\bigoplus_{i=1}^{l} R\,\hat{d}f_i, t) = t^{n-h}(\sum_{i=1}^{l} t^{m_i})\,\mathrm{Poin}(R,t)\,. \tag{6.5}$$

Let Ω_S^{l-1} be the space of differential $l-1$ forms on V with coefficients in S. Grade Ω_S^{l-1} in the natural way. Define the star-operator $* : \Omega_S^1 \to \Omega_S^{l-1}$ by $*(f dx_i) = (-1)^{i-1} f\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_l$ [Fla, p.15, p.82]. Then $*(w\theta) = \det(w)\, w(*\theta)$ for $w \in W$. Since $* : \Omega_S^1 \to \Omega_S^{l-1}$ is an isomorphism of graded S-modules, it follows that the restriction of $*$ to $(\Omega_S^1)^{\det}$ defines an isomorphism

$$(\Omega_S^1)^{\det} \simeq (\Omega_S^{l-1})^W \tag{6.6}$$

of graded vector spaces. It is shown in [Sol1, Theorem] that $(\Omega_S^{l-1})^W$ is a free R-module with basis $\psi_i = df_1 \wedge \cdots \wedge df_{i-1} \wedge df_{i+1} \wedge \cdots \wedge df_l$ for $1 \le i \le l$. Define $\varphi_i \in \Omega_S^1$ by $*\varphi_i = \psi_i$. It follows that $\varphi_1, \ldots, \varphi_l$ is an R-module basis for $(\Omega_S^1)^{\det}$. Since $\deg(\varphi_i) = n - m_i$ we have

$$\mathrm{Poin}((\Omega_S^1)^{\det}, t) = (\sum_{i=1}^{l} t^{n-m_i})\,\mathrm{Poin}(R,t)\,. \tag{6.7}$$

Compare (6.5) and (6.7). By duality in the exponents we have $n - h + m_i = n - m_{l-i+1}$ for $1 \le i \le l$. Thus $\mathrm{Poin}(\bigoplus_{i=1}^{l} R\,\hat{d}f_i, t) = \mathrm{Poin}((\Omega_S^1)^{\det}, t)$. This completes the proof. □

Bibliography

[Ath] C. Athanasiadis, On free deformations of the braid arrangement, *European J. of Combinatorics* **19** (1998), 7–18.

[Bou] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapitres 4,5 et 6, Hermann, Paris 1968.

[EdR] P. Edelman and V. Reiner, Free arrangements and rhombic tilings, *Discrete and Computational Geometry* **15** (1996), 307–340.

[Fla] H. Flanders, *Differential Forms, with Applications to the Physical Sciences,* Academic Press, 1963.

[Mac] I. G. Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Univ. Press, 1995.

[OrT] P. Orlik and H. Terao, *Arrangements of Hyperplanes*. Grundlehren der Math. Wiss. **300**, Springer Verlag, 1992.

[Sai1] K. Saito, On the uniformization of complements of discriminant loci. In: *Conference Notes, Amer. Math. Soc. Summer Institute*, Williamstown, 1975.

[Sai2] K. Saito, On a linear structure of a quotient variety by a finite reflexion group. *RIMS Kyoto preprint* **288**, 1979 = *Publ. Res. Inst. Math. Sci.* **29** (1993), 535–579.

[Sai3] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect.IA Math.* **27** (1980), 265–291.

[SYS] K. Saito, T. Yano and J. Sekiguchi, On a certain generator system of the ring of invariants of a finite reflection group, *Communications in Algebra* **8** (1980), 373–408.

[Shi1] J.-Y. Shi, The Kazhdan-Lusztig cells in certain affine Weyl groups, *Lecture Notes in Math.* **1179**, Springer Verlag, 1986.

[Shi2] J.-Y. Shi, Sign types corresponding to an affine Weyl group, *J. London Math. Soc.* **35** (1987) 56–74.

[Sol1] L. Solomon, Invariants of finite reflection groups, *Nagoya Math. J.* **22** (1963), 57–64.

[Sol2] L. Solomon, Invariants of Euclidean reflection groups, *Trans. Amer. Math. Soc.* **113** (1964), 274–286.

[Sta1] R. Stanley, Hyperplane arrangements, interval orders, and trees, *Proc. Nat. Acad. Sci.* **93** (1996), 2620–2625.

[Sta2] R. Stanley, Hyperplane arrangements, parking functions, and tree inversions, preprint, 1996.

[Ter] H. Terao, Free arrangements of hyperplanes and unitary reflection groups, *Proc. Japan Acad. Ser. A* **56** (1980), 389–392.

[Zie] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness, In: Singularities, *Contemporary Math.* **90**, Amer. Math. Soc. (1989) 345–359.

Louis Solomon
Mathematics Department
University of Wisconsin
Madison WI 53706, USA

Hiroaki Terao
Mathematics Department
University of Wisconsin
Madison WI 53706, USA
e-mail: hterao@facstaff.wisc.edu
Current address:
Mathematics Department
Hokkaido University
Sapporo 060 JAPAN
e-mail: terao@math.sci.hokudai.ac.jp

(Received: March 13, 1997)