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The orbit space of the p-subgroup complex is contractible

Peter Symonds

Abstract. We show that the quotient space of the p-subgroup complex of a finite group by the action of the group is contractible. This was conjectured by Webb.

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The p-subgroup complex (or Brown complex or Quillen complex) was introduced by K.S. Brown [B]. It is defined for a group G and a prime p and will be denoted by S_p. It is a simplicial complex in which the n-simplices are chains of non-trivial finite p-groups (with strict inclusions):

$$Q_0 < Q_1 < Q_2 < \cdots < Q_n,$$

with the face maps corresponding to inclusion of subchains. In other words, S_p is the geometric realisation of the poset of non-trivial p-subgroups of G.

This complex has played a prominent role in finite group theory since its introduction and the fundamental work of Quillen [Q]. For some more recent contributions see [ASe, ASm, KR, TW, W1, W2]. This paper consists of a proof of the following result.

Theorem. *Let G be a finite group and p a prime which divides $|G|$. Let S_p denote the p-subgroup complex for G (considered as a topological space). Then S_p/G is contractible.*

This was conjectured by Webb [W1, W2], who proved that S_p/G is mod-p acyclic. When G is a group of Lie type in characteristic p, then S_p is equivariantly homotopy equivalent to the Tits building of G, for which the orbit space consists of just one simplex, so the conjecture was known to be true. Various cases were also considered by Thévenaz [T], who showed that the conjecture held when G was p-solvable, or when the Sylow p-subgroup was either abelian, generalized quaternion or TI.

Instead of S_p we shall consider a subcomplex Δ, introduced by Robinson, in which the n-simplices are chains of p-groups (with strict inclusions), each one normal in the others:

$$Q_0 \lhd Q_1 \lhd Q_2 \cdots \lhd Q_n, \quad Q_i \lhd Q_n, \quad 0 \leq i < n,$$

which we denote by $(Q_0, ..., Q_n)$. This complex Δ does not arise from a partially ordered set, but it is equivariantly homotopy equivalent to S_p (and to various other subgroup complexes too) [TW], and we actually prove that Δ/G is contractible.

Now Δ is a simplicial complex, but Δ/G is naturally only a CW-complex. Each simplex of Δ is naturally oriented, because it is a chain. This orientation is preserved by G, and so induces an orientation on Δ/G.

Proof. We show that
a) $\pi_1(\Delta/G) = 1$
and
b) $\tilde{H}_*(\Delta/G; \mathbb{Z}) = 0$,
and invoke Whitehead's Theorem.

a) Let P be a Sylow p-subgroup of G. Any class $x \in \pi_1(\Delta/G, P)$ can be represented by a cellular loop s, i.e. a loop in the 1-skeleton which traverses each 1-cell at constant speed. This loop is determined by the sequence of directed 1-cells along which it travels.

Lift s to a cellular path $\tilde{s}$ in Δ starting at P and ending at some Sylow p-subgroup $P^{'}$. Since Δ is a simplicial complex, $\tilde{s}$ is determined by the sequence of its vertices:

$$P \to Q_1 \to Q_2 \to \cdots \to Q_n \to P^{'}.$$

There are two operations that we can perform on $\tilde{s}$ which do not change its image in $\pi_1(\Delta/G, P)$.

i) *Homotopy.* Change $\tilde{s}$ by a homotopy in Δ that fixes its endpoints.
ii) *Change of Lift.* If $g \in N_G(Q_j)$ then we can replace

$$P \to Q_1 \to Q_2 \to \cdots \to Q_j \to \cdots \to Q_n \to P^{'}$$

by

$$P \to Q_1 \to \cdots \to Q_{j-1} \to Q_j \to Q_{j+1}^g \to \cdots \to Q_n^g \to P^{'g}.$$

Define a height function $h: \ \Delta \to \mathbb{R}$ by starting on the vertices with $h(Q) = \log_p|Q|$ and then extending linearly on each simplex. Define the depth of a path $\tilde{s}$ in Δ to be $d(\tilde{s}) = \min\{h(Q) | Q \text{ a vertex of } \tilde{s}\}$.

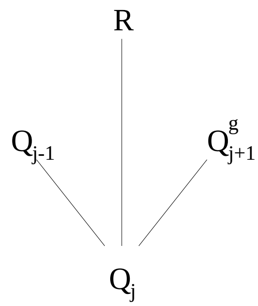
Now, for a given class $c \in \pi_1(\Delta/G, P)$, consider all the lifts starting at P of all the cellular paths representing c. Amongst these, restrict attention to those of maximal depth, and then choose one with the least possible number of vertices of minimal height. Call it $\tilde{s}$.

$$\tilde{s}: \ P \to Q_1 \to \cdots \to Q_n \to P^{'}.$$

Assume that $c \neq 1$ so that there are at least three vertices. Let Q_j be a vertex of minimal height and let R be a Sylow p-subgroup of $N_G(Q_j)$ containing Q_{j-1} (clearly $Q_j \lhd Q_{j-1}$ since Q_j is of minimal height). Then for some $g \in N_G(Q_j)$, ${}^gR$ contains Q_{j+1}, and we can change the lift to obtain

$$s^{'}: \ P \to Q_1 \to \cdots \to Q_{j-1} \to Q_j \to Q_{j+1}^g \to \cdots \to Q_n^g \to P^{'g}.$$

We now have 1-simplices:



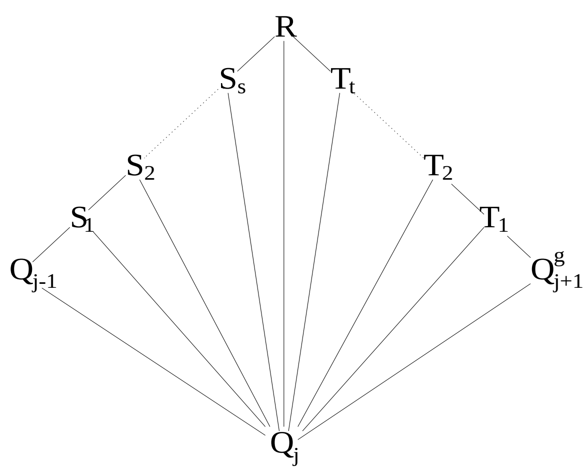
where $Q_{j-1},\ Q_{j+1}^g \leq R$ but they need not be normal. However there are sequences

$$Q_{j-1} \lhd S_1 \lhd \cdots \lhd S_s \lhd R$$

and

$$Q_{j+1}^g \lhd T_1 \lhd \cdots \lhd T_t \lhd R,$$

so we have 2-simplices:



We can now change the path $s^{'}$ by a homotopy to $s^{''}$:

$$P \to Q_1 \to \cdots \to Q_{j-1} \to S_1 \to \cdots$$
$$\to S_s \to R \to T_t \to \cdots \to T_1 \to Q^g_{j+1} \to \cdots \to Q^g_n \to P^{'g}.$$

But $s^{''}$ has fewer vertices of minimal height, a contradiction.

b) The case of homology is similar but a little more complicated. Clearly $\tilde{H}_0(\Delta/G;\mathbb{Z}) = 0$, i.e. Δ/G is connected, because for every p-subgroup Q there is a sequence $Q \lhd Q_1 \lhd \cdots \lhd Q_n$, where Q_n is a Sylow p-subgroup of G. This yields a path from Q to Q_n, and all Sylow p-subgroups are conjugate. From now on we assume that $n \geq 1$.

Each n-cycle in the CW-homology of Δ/G can be regarded as a linear combination s of oriented n-cells. This can be lifted to a linear combination $\tilde{s}$ of n-simplices of Δ, $\tilde{s} = \sum n_\sigma \sigma$. We do not assume that this lifting is necessarily done in such a way that only one σ appears from each G-orbit.

There are two operations that we can perform on $\tilde{s}$ which do not change its image in $H_n(\Delta/G,\mathbb{Z})$.

i) *Homology.* Add a boundary (i.e. something homologous to zero).
ii) *Change of Lift.* Any of the simplices can be replaced by another in the same G-orbit.

Define the height $h(\sigma)$ of a simplex to be the height of its barycentre (i.e. the average height of its vertices) and its depth to be the minimum height of its codimension 1 faces. The depth of a chain is defined by $d(\sum n_\sigma \sigma) = \min\{d(\sigma) | n_\sigma \neq 0\}$.

Given a class $c \in H_n(\Delta/G;\mathbb{Z})$ consider all the liftings $\tilde{s}$ to Δ of all cycles s representing c. Amongst these consider only those of maximal depth d, and write $\tilde{s} = \sum n_\sigma \sigma$. Now pick an $\tilde{s}$ that minimizes the multiplicity,

$$m(\tilde{s}) = \sum_{d(\sigma)=d} |n_\sigma|.$$

Assume that $c \neq 0$, so there must be a simplex ρ_1 with $n_{\rho_1} \neq 0$ and $d(\rho_1) = d$. Now ρ_1 has a face $\mu = (Q_0 \lhd \cdots \lhd Q_{n-1})$ with $h(\mu) = d$. Let R_1 be the vertex of ρ_1 not in μ. Then $h(R_1) > h(Q_i)$ for any i, otherwise ρ_1 would have a face of depth less than d, so $\rho_1 = (\mu, R_1)$.

Since the image of $\tilde{s}$ in Δ/G is a cycle, there must be another simplex $\rho^{'}$ with $n_{\rho^{'}} \neq 0$ such that some conjugate $\rho_2 = h\rho^{'}$ $(h \in G)$ also has a face μ, and n_{ρ_1} and $n_{\rho^{'}}$ have opposite signs (but not necessarily the same absolute value). Again, $\rho_2 = (\mu, R_2)$ by minimality and, by changing our attention to $-c$ if necessary, we can assume that $n_{\rho_1} > 0$ and $n_{\rho^{'}} < 0$. Note that minimality under change of lift implies that the coefficient function n_σ can not take both positive and negative values on the same orbit, so $\rho' \neq \rho_1 \neq \rho_2$ and also $n_{\rho_2} \leq 0$. A change of lift alters $\tilde{s}$ to

$$s^{'} = \tilde{s} + \rho' - \rho_2 = \sum n_\sigma \sigma + \rho^{'} - \rho_2 = \sum n^{'}_\sigma \sigma,$$

where it is easy to check that $d(s^{'}) = d(\tilde{s}),\ m(s^{'}) = m(\tilde{s}),\ n'_{\rho_1} > 0$ and $n'_{\rho_2} < 0$.

Now write

$$s^{'} = \sum m_\sigma \sigma + \rho_1 - \rho_2 = t + \rho_1 - \rho_2,$$

so $m_\sigma = n'_\sigma$ unless σ is ρ_1 or ρ_2, and $m_{\rho_1} = n^{'}_{\rho_1} - 1 \geq 0,\ m_{\rho_2} = n^{'}_{\rho_2} + 1 \leq 0$. Thus $m(t) = m(\tilde{s}) - 2$. Let R be a Sylow p-subgroup of $\mathrm{stab}_G(\mu)$ containing R_1. Then $R_2 \leq R^g$ for some $g \in \mathrm{stab}_G(\mu)$, so a change of lift alters $s^{'}$ to $s^{''} = t + \rho_1 - g\rho_2$, where $g\rho_2 = (\mu, {}^gR_2)$.

Suppose, for the moment, that $R_1 \neq R \neq R_2^g$. Then we can find sequences of subgroups

$$R_1 \lhd S_1 \lhd \cdots \lhd S_s \lhd R$$

and

$${}^gR_2 \lhd T_1 \lhd \cdots \lhd T_t \lhd R.$$

Let

$$v_1 = (\mu, R_1, S_1) + (\mu, S_1, S_2) + \cdots + (\mu, S_s, R),$$

and

$$v_2 = (\mu, {}^gR_2, T_1) + (\mu, T_1, T_2) + \cdots + (\mu, T_t, R).$$

Then for $i = 1, 2$,

$$(-1)^n \partial v_i = g^{i-1}\rho_i - (\mu, R) + X_i,$$

where X_i is a sum of cells which do not contain μ, but their vertices which are not in μ contain (as groups) all the vertices of μ. It follows that X_i involves only cells of depth strictly greater than d, and therefore that

$$s^{''} = t + \rho_1 - g\rho_2 \equiv t + (-1)^n \partial(v_1 - v_2),\ \text{modulo cells of depth greater than } d,$$

and a change by homology alters s''' to $s''' = s'' - (-1)^n \partial(v_1 - v_2)$ and yields

$$s^{'''} \equiv t,\ \text{modulo cells of depth greater than } d.$$

But $m(s^{'''}) = m(t) = m(\tilde{s}) - 2$, a contradiction.

As for the remaining cases, if $R_1 = R = {}^gR_2$ then $s^{'} = t$. If $R_1 \neq R = {}^gR_2$, then $(-1)^n \partial v_1 \equiv \rho_1 - g\rho_2$, modulo cells of depth greater than d. The case $R_1 = R \neq {}^gR_2$ is similar.

Remark. A relative version of this theorem also holds. Let Y be a set of subgroups of G that is closed under subgroups and conjugation. Let Δ_Y be the subcomplex of Δ in which we only allow chains of p-subgroups not in Y. Then if Δ_Y is not empty, Δ_Y/G is contractible.

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Peter Symonds
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
USA
e-mail: symonds@ms.uky.edu

Current address:
Department of Mathematics
U.M.I.S.T.
P.O. Box 88
Manchester M6O 1QD
England

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