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π_1-injective surfaces in graph manifolds

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Abstract. A criterion is given for an immersed horizontal π_1-injective surface in a graph manifold to be separable. Examples are constructed of such surfaces, which are not separable and do not satisfy the k-plane property, for any k. It is shown that the simple loop conjecture holds in graph manifolds and that any graph manifold with boundary has an immersed horizontal surface.

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1. Introduction

In this paper we assume that all manifolds involved are equipped with fixed Riemannian metrics and maps of surfaces are chosen to be least area in their homotopy class. We will use basic facts about least area surfaces from [FHS]. (By an abuse of notation we refer to least area maps, whereas to ensure surfaces are in general position, it is often necessary to choose a small perturbation.)

Suppose $f : S \to M$ is a map from a compact surface to a 3-manifold M. We say that f is proper if $f^{-1}(\partial M) = \partial S$. In this paper, all maps from surfaces to 3-manifolds are proper.

Suppose $f : S^1 \to S$ is a least length immersion of a circle to a surface S, which is homotopically essential. A basic fact is that f can be lifted to a finite cover to be embedded, in other words, the self-intersections can be separated in a finite cover. See Figure 1.

Let M be a compact orientable irreducible 3-manifold with $|\pi_1(M)| = \infty$, S be a compact orientable surface with Euler characteristic $\chi(S) < 0$, and $f : S \to M$

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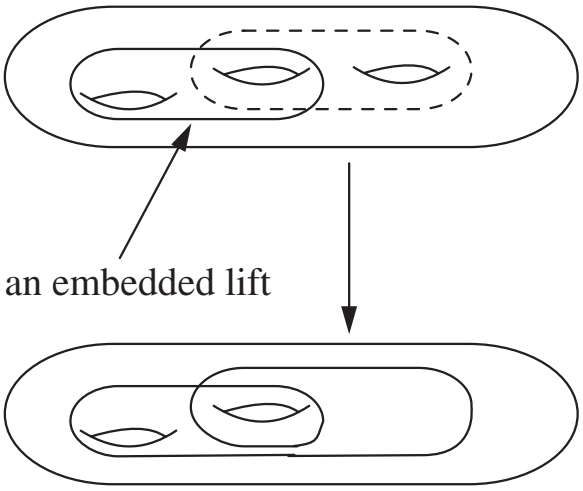

Figure 1.

be a π_1-injective (i.e. injects in $\pi_1(S)$) least area (in the smooth or PL sense) proper immersion.

Question 1. *Is there a finite cover $p : \tilde{M} \to M$ such that there is an embedding $\tilde{f} : S \to \tilde{M}$ which covers $f : S \to M$?*

A weak form of Question 1 is the following

Question 1^***.** *Is there a finite cover $p : \tilde{M} \to M$ and a finite cover $\tilde{S}$ of S such that there is an embedding $\tilde{f} : \tilde{S} \to \tilde{M}$ which covers $f : S \to M$?*

When Question 1 (Question 1^*) has a positive answer , we say that the immersed surface $f : S \to M$ can be (virtually) lifted to be embedded in a finite cover , or say it is (virtually) separable.

Question 1 has been raised by Scott and he gave a positive answer when M is a Seifert fiber space [S], and was also raised by Thurston for hyperbolic 3-manifolds [T]. For a discussion of virtually separable, see also [L]. Question 1 and Question $1^\star$ are related to a central problem of 3-manifolds (also known as the Virtual Haken Conjecture or Waldhausen Conjecture): does every closed orientable irreducible 3-manifold with $|\pi_1(M)| = \infty$ have a finite covering which contains an orientable embedded incompressible surface?

Let $p : \tilde{M} \to M$ be the universal covering. Then $p^{-1}(f(S))$ is a union of planes by [FHS]. We will say that f has the k-plane property, if any set of k planes of $p^{-1}(f(S))$ contains a disjoint pair. So an embedded surface has the 2-plane property, a surface in a non-positively curved cubed 3-manifold has the 4-plane

property [AR] and recently it has been proved that in a hyperbolic or negatively curved manifold, any immersed incompressible surface has the k-plane property for some k [RS]. The k-plane property is a useful measure of how complex are the self-intersections of the surface. When M contains an immersed surface satisfying the 3-plane property, it has been proved that homotopy equivalences are homotopic to homeomorphisms and homotopic homeomorphisms are isotopic [HS1], [HS2]. The former result has been extended to surfaces with the 4-plane property in [P1],[P2].

Question 2. *Does $f : S \to M$ have the k-plane property for some k, for any immersed incompressible surface?*

Related to Question 2 is the following

Question 2^*. *Does the preimage of $f : S \to M$ in the universal cover of M contain a disjoint pair of planes?*

If $f : S \to M$ is separable, then it is virtually separable; if it is virtually separable, then it has the k-plane property for some k; if $f : S \to M$ has the k-plane property for some k; then the preimage of $f : S \to M$ in the universal cover of M contains a disjoint pair of planes. It was expected that all of the above four questions should have positive solutions.

In this paper we provide an immersion $f : S \to M$ which provides a negative answer to all of the above four questions. (Example 2.6 and following Remark, Theorem 2.7). Our example is simply a horizontal surface in a graph manifold. Actually we give an algorithm to determine when a horizontal surface in a graph manifold has an embedded lift (Theorem 2.3). In the case of horizontal surfaces, the above four questions are equivalent (Theorem 2.7).

Note that it is still open to decide if the above questions have positive solutions for atoroidal 3-manifolds. Moreover even if the k-plane property fails for some surfaces in a 3-manifold M, one may hope to find other surfaces in M which have the k-plane property.

Inspired by Dehn's Lemma and the simple loop theorem about surface maps of D.Gabai [G], an interesting question was raised in the middle 80's (also known as the Simple Loop Conjecture).

Question 3. *Suppose a map $f : S \to M$ from a surface to a 3-manifold is a two-sided immersion and is not π_1-injective. Is there an essential simple closed curve in $\ker f_\star$?*

J. Hass gave a positive answer to Question 3 when M is a Seifert manifold [H]. In section 3 we first give a topological criterion for a surface in a graph manifold to be π_1-injective (Corollary 3.5) and then give a positive answer for Question 3 when M is a graph manifold (Theorem 3.1).

In Section 4 we prove that each graph manifold with non-empty boundary admits a horizontal immersed incompressible surface (Theorem 4.1). It is interesting to find a horizontal immersed surface which satisfies the separability criterion in Section 2. Then one can obtain a positive answer to the following problem due to Luecke and Wu [LW].

Question 4. *Is every graph manifold with non-empty boundary covered by a surface bundle over S^1?*

An early version of this paper was written in 1994. Recently certain extensions of our work in Section 2 have been made in [M]. Moreover a positive answer to Question 4 has been given in [WY].

A compact irreducible ∂-irreducible orientable 3-manifold M is a graph manifold if each component of $\overline{M - \mathcal{T}}$ is a Seifert manifold, where $\mathcal{T}$ is the canonical decomposition tori of Johannson and of Jaco-Shalen.

For a given graph manifold M, we call each component M_v of $\overline{M - \mathcal{T}}$ a vertex manifold. Define an associated graph $\Gamma(M)$ as below: Each vertex manifold M_v determines a vertex v and each decomposition torus T_e determines an edge e, vertices v and v' are connected by an edge e if and only if T_e is shared by M_v and $M_{v'}$. If we put an orientation on e for each $e \in \Gamma(M)$, then e determines a homeomorphism $g_e : T_e \to \bar{T}_e$, where T_e and $\bar{T}_e$ are tori in the boundaries of the vertex manifolds, corresponding to the beginning and the end of e respectively .

Definition. *A torus T is framed, if T is oriented and an ordered pair of oriented simple closed curves α, β, which intersect transversely exactly once, is chosen, so that the product of the orientations of the loops produces the orientation of T. Such a framed torus is denoted by $T(\alpha, \beta)$.*

Suppose $p : M \to F$ is an oriented Seifert manifold, where the orbit surface F is of genus g and has $h > 0$ boundary components and M has k singular fibers.

Definition. *M is framed, if*

(1) a section $S = F - \cup intD_i$ of $M - \cup intN_i$ is chosen and ∂S is oriented, where N is a fibered regular neighborhood of the singular fibers

(2) each torus boundary component of F is equipped with a framing $T(\alpha, \beta)$, where α is an oriented boundary component of F and β is an oriented fiber S^1

(3) the orientation of T is induced from the orientation of M.

Definition. *An orientable graph manifold M is framed, if*

(1) each vertex manifold is framed and the orientation on each M_v coincides with the restriction of the orientation of M

(2) the graph $\Gamma(M)$ is oriented.

For each (oriented) edge $e \in \Gamma(M)$, the homeomorphism $g_e : T_e(\alpha_e, \beta_e) \to \bar{T}_e(\bar{\alpha}_e, \bar{\beta}_e)$ determines uniquely a 2 by 2 matrix $\left(\begin{smallmatrix} p_e & q_e \\ r_e & s_e \end{smallmatrix}\right)$ defined as

$$g_e\left(\begin{pmatrix}\alpha_e \\ \beta_e\end{pmatrix}\right) = \left(\begin{pmatrix} p_e & q_e \\ r_e & s_e \end{pmatrix}\right) \circ \left(\begin{pmatrix}\bar{\alpha}_e \\ \bar{\beta}_e\end{pmatrix}\right) \tag{1.1}$$

where $r \neq 0$, and $qr - ps = 1$. (For convenience, the subscripts giving the edge labels will often be left out.)

For any oriented closed curve $c \in T(\alpha, \beta)$, $c = u\alpha + v\beta$. We call (u, v) the coordinates of c; if c is an oriented simple closed curve, then u and v are coprime.

In terms of coordinates, let $g(u\alpha + v\beta) = \bar{u}\bar{\alpha} + \bar{v}\bar{\beta}$. By (1.1) a simple calculation gives

$$\bar{u} = up + vr \qquad \text{and} \qquad \bar{v} = uq + vs \tag{1.2}$$

The following facts are useful.

Lemma 1.0. *Suppose F is a compact surface with non-empty boundary, which is not a disk or an annulus.*

(1) If each component c of ∂F is associated with a positive integer d_c, then there is a finite covering $q : \tilde{F} \to F$ such that $q| : \tilde{c} \to c$ is of degree d_c, where $\tilde{c}$ is a component of $q^{-1}(c)$.

(2) Each graph manifold is finitely covered by a graph manifold M which is orientable and each vertex manifold M_v is homeomorphic to $F_v \times S^1$, where F_v is a compact orientable surface and $\chi(F_v) < 0$.

(3) Moreover, each surface F_v in (2) can be chosen to be non-planar.

Proof. (1) and (2) are well-known. A nice constructive proof is given in [LW]. (1) and (2) are also corollaries of the residual finiteness of the fundamental groups of 2-dimensional orbifolds and 3-dimensional Haken manifolds [He].

Assume that we have (2), then (3) can be obtained by the fact:

Suppose F is a compact orientable surface with $\chi(F) < 0$ and with k boundary components. By the classification of compact orientable surfaces, we have $\chi(F) \leq 2 - k$. Then it is easy to see that $3\chi(F) + k$ is even and is smaller than zero. So there is a non-planar compact orientable surface $\tilde{F}$ with $\chi(\tilde{F}) = 3\chi(F)$ and with k boundary components. By [3.4.2, ZVD], there is a covering $q : \tilde{F} \to F$ of degree 3. Clearly the restriction of p to each boundary component of $\tilde{F}$ is also of degree 3.

For each vertex $v \in \Gamma(M)$, $M_v = F_v \times S^1$. Let $q_v : \tilde{F}_v \to F_v$ be the covering of degree 3 provided by the last paragraph, let $q : S^1 \to S^1$ be the covering of degree 3, and $p_v = q_v \times q : \tilde{F}_v \times S^1 \to F_v \times S^1$. Then the matrix associated to $p_v| : \tilde{T} \to T$ is $\left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right)$, where T is a component of $F_v \times S^1$, $v \in \Gamma(M)$, and $\tilde{T}$ is a component of $p^{-1}(T)$.

Now we define a new graph manifold $\tilde{M}$ as follows:

(1) $\Gamma(\tilde{M}) = \Gamma(M) = \Gamma$,

(2) for each vertex $v \in \Gamma$, the vertex manifold $\tilde{M}_v$ is $\tilde{F}_v \times S^1$,
(3) for each oriented edge $e \in \Gamma$, the associated matrix is still $\left(\begin{smallmatrix} p_e & q_e \\ r_e & s_e \end{smallmatrix}\right)$.

Then the projections $\{p_v = q_v \times q : \tilde{F}_v \times S^1 \to F_v \times S^1\}_{v\in\Gamma}$ can be matched to produce a covering $p : \tilde{M} \to M$ satisfying the conclusion (3). □

2. A criterion for the horizontal immersed surface to be separable, and the k-plane property.

If M is a Seifert manifold, we call an immersed surface $S \subset M$ vertical if S is foliated by the Seifert fibers and we say that an immersed surface $S \to M$ is horizontal, if f is transverse to the Seifert fibers everywhere. In the case M is a graph manifold, we call an immersed surface $S \to M$ horizontal, if $S \cap M_v$ is horizontal for every vertex manifold M_v.

Lemma 2.1. *Suppose M is a Seifert manifold with boundary and $f : S \to M$ is a horizontal immersed surface with c a component of the boundary of S. Then there is a finite covering $p : S \times S^1 \to M$ and an embedding $i : S \to S \times S^1$ given by $i(x) = (x, 0)$ such that $f = p \circ i$. Furthermore the degree of p is $\#t \cap f(S)$ and the degree of $p|_{c\times S^1}$ is $\#t \cap f(c)$, where t is a regular fiber of M lying on a component of ∂M containing $f(c)$.*

Proof. For simplicity we assume that both our Seifert manifold and its orbifold surface are orientable. There is always a covering of degree at most four with these properties. Since such Seifert manifolds admit S^1 actions, we define the map $p : S \times S^1 \to M$ by $p(x, t) = tf(x)$. We are going to prove that p is a covering. Clearly $p \circ \tilde{f} = f$, where $\tilde{f}(x) = (x, 1)$.

Let $q : M \to F$ be the projection to the orbit surface, $F = M/S^1$. Choose $y \in M$ and let $z = q(y)$. Let B' be a small ball centered at y and $D = q(B')$ be a small disc centered at z.

Let $(q \circ f)^{-1}(D) = D_1 \cup ... \cup D_d$. Then each D_i contains a unique point z_i mapping to z. Now $p^{-1}(y) = \{(z_1, t_1), ..., (z_d, t_d)\}$, where $t_i f(z_i) = y$. Since $f|_{D_i}$ is a homeomorphism and with image transverse to the S^1 fibering, when ϵ is small enough, $p|_{D_i\times[t_i-\epsilon,t_i+\epsilon]}$ is an embedding. Let $B_i = D_i \times [t_i - \epsilon, t_i + \epsilon]$ and $B = p(B_i)$. Then B is a regular neighborhood of y, $p^{-1}(B) = \{B_1, ..., B_d\}$ and $p| : B_i \to B$ is a homeomorphism. Hence p is a covering.

The verification of the remaining part is directly from the construction. □

Remarks on Lemma 2.1. It is important to consider the preimage $p^{-1}(f(S))$. Let S' be a component of $p^{-1}(f(S))$ other than the embedding $\tilde{f}(S)$ given in Lemma 2.1 . Let $\phi : S \times R^1 \to S \times S^1$ be the infinite cyclic covering and identify $i(S)$ as the zero section of $S \times R^1$. Suppose S' and $\tilde{f}(S)$ are in the same homotopy class. Since we assume the immersion $f : S \to M$ is least area then both $\tilde{f}(S)$

and S' are least area. Hence $\tilde{f}(S)$ and S' are disjoint [FHS]. In the infinite cyclic covering $S \times R^1$, $\phi^{-1}(S')$ is a family of parallel embeddings of S disjoint from the zero section. If S' and $\tilde{f}(S)$ are not in the same homotopy class, then the number of components of intersection of S' and $\tilde{f}(S)$ is non zero and minimal in the sense of [FHS]. This means precisely in the infinite cyclic covering $S \times R^1$, each component of $\phi^{-1}(S')$ is non-compact and runs from one end of $S \times R^1$ to the other and hence meets the zero section. Note that each such a component is invariant under the action of the infinite cyclic deck transformation group.

Lemma 2.2. *Suppose $T_i(\alpha_i, \beta_i)$, $i = 1, ..., n$ are boundary components of the framed 3-manifold $F \times S^1$, where F is a compact non-planar surface. Suppose $\{c_{ij}, j = 1, ..., k_i\}$ is a family of oriented simple closed curves on T_i, $c_{ij} = (u_{ij}, v_{ij})$, $i = 1, ..., n$. Then $\cup_{i=1}^n \{c_{ij}, j = 1, ..., k\}$ is a homological boundary of a connected immersed orientable horizontal surface S in $F \times S^1$ if and only if*

(1) $\sum_{i=1}^n \sum_{j=1}^{k_i} v_{ij} = 0$

(2) all $u_{i,j}$ have the same sign and $\sum_{j=1}^{k_i} u_{ij} = u \neq 0$.

Proof. Let $\mathcal{C} = \{c_{ij}, j = 1, ..., k_i, i = 1, ..., n\}$ be a collection of oriented simple closed curves of $\partial F \times S^1$. It is clear that if $\mathcal{C}$ bounds a connected immersed orientable surface S in $F \times S^1$, then (1) and (2) hold.

Conversely suppose (1) and (2) hold. The condition (2) implies that the set of closed curves $\{u_{ij}\alpha_i, j = 1, ..., k_i, i = 1, ..., n\}$ is homologically zero in F, where $\{\alpha_i\}$ are boundary components of F. So this family bounds a proper immersed orientable surface S' in F. Then $S'' = S' \times \{1\} \subset F \times S^1$ is a proper horizontal immersed surface. Since F contains a non-separating curve c, after connecting up the pieces of S'' by Haken summing along $c \times S^1$, we may assume that S'' is a connected horizontal immersed surface. (Here by a Haken sum, we mean a cut and paste between different sheets of a surface using a vertical torus, as in normal surface theory). By Lemma 2.1, there is a finite covering $p : S'' \times S^1 \to F \times S^1$ such that $p(S'' \times \{1\}) = S'' \subset F \times S^1$. By Condition (1) we can obtain a horizontal surface S from $S'' \times \{1\}$ by suitable Haken sums along proper vertical annuli in $S'' \times S^1$ so that $\partial p(S) = \mathcal{C}$. □

Suppose M is a graph manifold and $f : S \to M$ is a horizontal immersed surface. Recall $\mathcal{T}$ denotes the decomposition tori of M. Deform f so that $f^{-1}(\mathcal{T})$ is a family of disjoint essential simple closed curves on S, denoted by $\mathcal{C}$. Since each component of $\mathcal{T}$ is a torus, we may assume that for each component c of $\mathcal{C}$, $f(c)$ is a multiple of some simple closed curve on $\mathcal{T}$.

For a component $c \in \mathcal{C}$, c has two copies c^*, c' in $\overline{S - \mathcal{C}}$. Suppose $f(c^*) \subset \partial M_{v^*}$ and $f(c') \subset \partial M_{v'}$. Define

$$d_{c^*} = \#t_{v^*} \cap f(c^*) \qquad \text{and} \qquad d_{c'} = \#t_{v'} \cap f(c')$$

where t_v (resp. $t_{v'}$) is a regular fiber of M_v (resp. $M_{v'}$).

For each oriented simple closed curve γ on S and a base point $x \in \gamma$, suppose starting from x, γ transversely meets c_1, c_2,..., $c_n \subset \mathcal{C}$ successively. Suppose also the sup-index $'$ and $*$ have been arranged so that γ meets c_i^* first and then c_i'. Now we define

$$s_\gamma = \frac{d_{c_n^*}}{d_{c_n'}} \cdots \frac{d_{c_2^*}}{d_{c_2'}} \frac{d_{c_1^*}}{d_{c_1'}}.$$

It is easy to see that:

(1) s_γ is independent of the choice of the base point

(2) $s_{\gamma^{-1}} = s_\gamma^{-1}$.

Theorem 2.3. *Suppose M is a graph manifold and $f : S \to M$ is a horizontal immersed surface. Then there is a finite covering $p : \tilde{M} \to M$ and an embedding $\tilde{f} : S \to \tilde{M}$ such that $f = p \circ \tilde{f}$ if and only if for each simple closed curve $\gamma \subset S$, $s_\gamma = 1$.*

Proof. Let $q : M_S \to M$ be the covering corresponding to the subgroup $\pi_1(S) \subset \pi_1(M)$. Since S is a horizontal surface in a graph manifold, it is not difficult to verify that $M_S = S \times R$, where R is the real line. Also we may identify S with $S \times 0 \subset S \times R$ as a compact component of $q^{-1}(S)$. This follows for Seifert fiber spaces by Lemma 2.1 and hence easily for graph manifolds. We need the following:

Lemma 2.4. *Suppose M is a graph manifold and $f : S \to M$ is a horizontal immersed surface. Then there is a finite covering $p : \tilde{M} \to M$ and an embedding $\tilde{f} : S \to \tilde{M}$ such that $f = p \circ \tilde{f}$ if and only if $q^{-1}(S)$ has another compact component (or equivalently if and only if $q^{-1}(S)$ contains infinitely many compact components).*

Proof. Suppose there is a finite covering $p : \tilde{M} \to M$ and an embedding $\tilde{f} : S \to \tilde{M}$ such that $f = p \circ \tilde{f}$. We may assume that all the surfaces and 3-manifolds are orientable, using additional coverings if necessary. Then $\tilde{f}(S)$ is a two sided (since both S and $\tilde{M}$ are orientable) horizontal embedded surface in the graph manifold $\tilde{M}$. Since cutting along a horizontal surface in a Seifert manifold we get an I-bundle over a surface, it follows when we split $\tilde{M}$ along S, the resulting manifold is an I-bundle over a surface. Consequently $\tilde{M}$ itself is a surface S-bundle over the circle. Hence the surface covering $q : M_S \to M$ can be factored through $\tilde{M}$ and therefore $q^{-1}(S)$ contains another compact component (actually infinitely many).

On the other hand, suppose $q^{-1}(S)$ contains another compact component S'. Then S and S' are in the same homotopy class, and so are parallel and disjoint embeddings in M_S by [FHS]. Then the preimages $\tilde{S}$ and $\tilde{S}'$ of S and S' in the universal covering are two disjoint planes and the stabilizer of the first plane is $\pi_1(S) \subset \pi_1(M)$, the deck transformation group. Moreover the stabiliser of the second plane is contained in the stabiliser of the first. (This follows since the preimage of S' in the universal covering under the action of $\pi_1(S)$ becomes S'. It

also follows as in [FHS] that the stabiliser of $\tilde{S}'$ is at most a 2-fold extension of $\pi_1(S)$. However we do not need this). Let t be the element in $\pi_1(M)$ which sends $\tilde{S}$ to $\tilde{S}'$. Clearly $\pi_1(S)$ is a normal subgroup in $G =< \pi_1(S), t >$ and the universal covering modulo G is a surface S-bundle over the circle, which covers M. □

Let S_λ be a component of $\overline{S - \mathcal{C}}$. Now $q^{-1}(\mathcal{T}) = \mathcal{C} \times R \subset S \times R$ is a family of vertical infinite cylinders. Suppose $f(S_\lambda) \subset M_v$. Then the covering $q_\lambda = q|_{S_\lambda \times R} : S_\lambda \times R \to M_v$ can be factored through

$$S_\lambda \times R \overset{\phi_\lambda}{\to} S_\lambda \times S^1 \overset{p_\lambda}{\to} M_v$$

where ϕ_i is the infinite cyclic covering and p_λ is the pull back given by Lemma 2.1. Now $q_\lambda^{-1}(f(S_\lambda)) = \phi_\lambda^{-1} p_\lambda^{-1}(f(S_\lambda))$ is an infinite family of parallel copies of S_λ.

Denote the components of $\phi_\lambda^{-1}(S_\lambda \times \{0\})$ $(\subset q_\lambda^{-1}(f(S_\lambda)))$ in order of 'height'

$$\{..., S_\lambda^{(k)}, ..., S_\lambda^{(1)}, S_\lambda^{(0)}, S_\lambda^{(-1)}, ..., S_\lambda^{(-k)}, ...\},$$

where $S_\lambda^{(k+1)}$ is above $S_\lambda^{(k)}$, and all $S_\lambda^{(0)}$'s can be matched together to form $S = S \times \{0\}$. Without ambiguity, we say the component $S_\lambda^{(k)}$ is at height k.

Now we return to the proof of Theorem 2.3.

"Only if." Suppose $q^{-1}(S)$ contains another compact component S'.

The action t defined in the second half of the proof of Lemma 2.4 induces an infinite cyclic group action $\bar{t}$ on $M_S = S \times R^1$. Let m be the minimum positive integer such that $\bar{t}^m$ sends each piece $S_\lambda \times R^1$ of M_S to itself for all λ. The projection of the action $\bar{t}^m$ on S_λ is isotopic to the identity and $\bar{t}^m$ sends $S \times 0$ to a compact component of $q^{-1}(S)$. Let $\bar{S} = \bar{t}^m(S)$, then $\bar{S}$ contains a unique component of $\phi_\lambda^{-1}(S_\lambda)$ for each λ and all those components form $\bar{S}$.

The restriction of $q : S \times R \to S$ on $\bar{S}$ is a homeomorphism. We denote $q^{-1}(\gamma)$ by $\bar{\gamma}$, and $q^{-1}(c_i^\star)$ by $\bar{c}_i^*$ and so on.

Suppose $c_i^* \subset S_\lambda$ is a component $\mathcal{C}$ in $\overline{S - \mathcal{C}}$. Then $\bar{c}_i^*$ lies in a unique component of $\phi_\lambda^{-1}(S_\lambda \times \{0\})$ with height k_i^*, similarly the height of $\bar{c}_i'$ is denoted as k_i'.

Denote the annulus bounded by c_i^* and $\bar{c}_i^*$ in $c_i^* \times R$ by A_i^* and the annulus bounded by c_i' and $\bar{c}_i'$ in $c_i' \times R$ by A_i'. Let T be a component of $\mathcal{T}$ containing $f(c_i)$ and denote the two copies of T as T^* and T'. Now the map $q| : A_i^* \to T^*$ is the composition of maps $A_i^* \to c_i^* \times S^1$ and $c_i^* \times S^1 \to T^*$. Since the first one is of degree k_i^* and the second one is of degree $d_{c_i^*}$, so $deg(q|_{A_i^*}) = d_{c_i^*} k_i^*$. Similarly $deg(q|_{A_i'}) = d_{c_i'} k_i'$. It is clear we must have

$$d_{c_i^*} k_i^* = d_{c_i'} k_i' \tag{2.1}$$

Since c_i' and c_{i+1}^* (resp. c_n' and c_1^*) lie in the same component of $\phi_i^{-1}(S_\lambda)$ for some component S_λ of $\overline{S - \mathcal{C}}$, we have

$$k_i' = k_{i+1}^* \qquad \text{and} \qquad k_n' = k_1^* \tag{2.2}$$

By (2.1) and (2.2) we have

$$k'_n = k^*_n \frac{d_{c^*_n}}{d_{c'_n}} = k'_{n-1} \frac{d_{c^*_n}}{d_{c'_n}} = k^*_{n-1} \frac{d_{c^*_{n-1}}}{d_{c'_{n-1}}} \frac{d_{c^*_n}}{d_{c'_n}}$$

Inductively

$$k'_n = k^*_1 \frac{d_{c^*_1}}{d_{c'_1}} \cdots \frac{d_{c^*_{n-1}}}{d_{c'_{n-1}}} \frac{d_{c^*_n}}{d_{c'_n}}$$

Since $k'_n = k^*_1$ we have

$$\frac{d_{c^*_1}}{d_{c'_1}} \cdots \frac{d_{c^*_{n-1}}}{d_{c'_{n-1}}} \frac{d_{c^*_n}}{d_{c'_n}} = 1$$

"If." Pick a component S_λ of $\overline{S - \mathcal{C}}$ and $x \in S_\lambda$. Pick a set of oriented simple closed curves $\{\gamma\}$ such that each of them contains x and they generate $H_1(S)$. Suppose starting from x, γ transversely meets $c^*_1, c'_1, \ldots, c^*_n, c'_n$ successively. Recall c^*_1, c'_n are components of ∂S_λ. Let $k = \prod_{c \in \mathcal{C}} (d_{c^*} d_{c'})$. Then the copy of c^*_1 at height k is matched with the copy of c'_1 at height $k \frac{d_{c'_1}}{d_{c^*_1}}$, the copy of c^*_2 at height $k \frac{d_{c'_1}}{d_{c^*_1}}$ is matched with the copy of c'_2 at height $k \frac{d_{c'_1}}{d_{c^*_1}} \frac{d_{c'_2}}{d_{c^*_2}}$, ..., and finally the copy of c^*_n at height $k \frac{d_{c'_1}}{d_{c^*_1}} \cdots \frac{d_{c'_{n-1}}}{d_{c^*_{n-1}}}$ is matched with the copy of c'_n at height $k \frac{d_{c'_1}}{d_{c^*_1}} \cdots \frac{d_{c'_n}}{d_{c^*_n}} = \frac{k}{s_\gamma} = k$, since $s_\gamma = 1$. This implies during the process of extending the horizontal section of M_S from S^k_λ along γ, we will not get a spiral surface but will come back to S^k_λ. After finitely many such steps, we get a compact component of $q^{-1}(f(S))$ which differs from $S \times \{0\}$. □

Corollary 2.5. *With the same hypotheses as Theorem 2.3, if $s_\lambda \neq 1$, then each component of $q^{-1}(S)$ meets the zero section $S \times 0 \subset S \times R^1$.*

Proof. Note that $q^{-1}(S) = \cup_{\lambda \in \Lambda} q^{-1}_\lambda (f(S) \cap M_v)$, where $f(S_\lambda) \subset M_v$ and $q^{-1}_\lambda (f(S)) \subset S_\lambda \times R^1$ consists of compact components which are disjoint parallel copies of the zero section $S_\lambda \times 0$ and non-compact components which run from one end of $S_\lambda \times R^1$ to the other end. Therefore each of the latter type meets the zero section $S_\lambda \times 0$, by the Remark following Lemma 2.1.

Let $\tilde{S}$ be a non-compact component of $q^{-1}(S)$. Suppose $\tilde{S}$ is formed by only compact components of $q^{-1}_\lambda (f(S)) \subset S_\lambda \times R^1, \lambda \in \Lambda$. Then near the infinite cylinder $\gamma \times R^1, \tilde{S}$ must spiral. However in any compact set containing the zero section $S \times 0$, there are only finitely many compact components of $q^{-1}_\lambda (f(S)) \subset S_\lambda \times R^1$. So when the spirals of $\tilde{S}$ approach the zero section $S \times 0$, the spiral must cease and we reach a contradiction.

It follows that $\tilde{S}$ must contain a non-compact component of $q^{-1}_\lambda (f(S)) \subset S_\lambda \times R^1$. Hence $\tilde{S}$ meets the zero section. □

Remark on the proof of Corollary 2.5. The second paragraph can be also explained as follows;

Let $\tilde{S}$ be a non-compact component of $q^{-1}(S)$. Suppose $\tilde{S}$ is formed by only compact components of $q_\lambda^{-1}(f(S)) \subset S_\lambda \times R^1, \lambda \in \Lambda$. Then $\tilde{S} \cap (\gamma \times R^1)$ must contain lines. The reason is that if an arc $\tilde{\gamma}$ in $\tilde{S}$ with endpoints x and y covers γ once, then the heights of x and y have a ratio of s_γ. Clearly $\tilde{S} \cap (\gamma \times R^1)$ is the union of all such arcs $\tilde{\gamma}$ and so contains lines. However all the heights of the endpoints of these arcs $\tilde{\gamma}$ must be integers and this is clearly impossible.

Example 2.6. We give an example of a horizontal immersed surface $f : S \to M$ which is not separable (see Figure 2). The framed graph manifold M has the decomposition $M = M_v \cup_T M_{v'}$, where $M_v = F_v \times S^1$ and F_v is the once punctured torus, $M_{v'} = F_{v'} \times S^1$ and $F_{v'}$ is a twice punctured torus, and the gluing matrix is $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$.

S is a twice punctured surface of genus 6, $\mathcal{C} = f^{-1}(T)$ consists of two simple closed curves c_1 and c_2. Also $\overline{S - \mathcal{C}} = \{S_\lambda, S_\mu\}$ is shown in Figure 2.

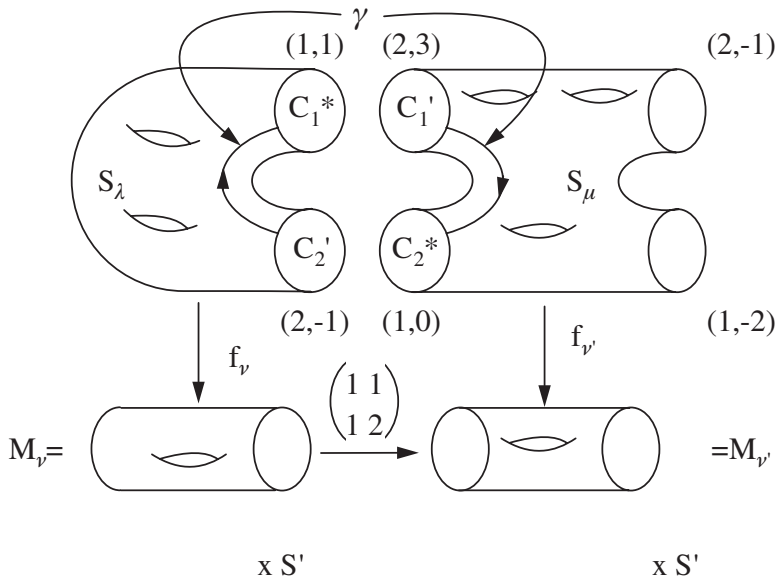

Figure 2.

By Lemma 2.2 and its proof, there is a proper immersion $f_v : S_\lambda \to M_v$ such that $f(c_1^*), f(c_2') \subset T(\alpha, \beta)$ have coordinates $(1, 1)$ and $(3, -1)$. There is also a proper immersion $f_{v'} : S_\mu \to M_{v'}$ such that $f(c_2^*), f(c_1') \subset \bar{T}(\bar{\alpha}, \bar{\beta})$ have coordinates $(3, 2)$ and $(1, 2)$; the remaining two boundary curves mapping to $\alpha^* \times$

S^1 have coordinates $(2,-3)$ and $(2,-1)$.

It is easy to see that f_v and $f_{v'}$ can be matched to give the map $f : S \to M$. (Note that the gluing matrix acts by right multiplication on row vectors of coordinates by our convention in (1.2). Note also $p_v \circ f_v : S_\lambda \to F_v$ is homotopic to a covering of degree four, hence $\chi(S_\lambda) = 4\chi(F_v) = -4$, where $p_v : F_v \times S^1 \to F_v$ is the projection. Since ∂S_λ has two components, the genus of S_λ must be two. Similarly we can determine the genus of S_μ. So the topological type of S can be determined, as claimed). Since

$$s_\gamma = \frac{d_{c_1^\star}}{d_{c'_1}} \frac{d_{c_2^\star}}{d_{c'_2}} = \frac{3}{1}\frac{3}{1} = 9,$$

it follows that $f : S \to M$ is not separable.

Remarks on Example 2.6. One can easily get a closed non-separable surface in a closed graph manifold by doubling both the surface and the manifold in Example 2.6 along their respective boundaries. One also can easily get a non-graph manifold by the following operation: pick a regular fiber t of a vertex manifold in the double of 2.6 such that $f(S) \cap \partial N(t)$ is a union of m disjoint simple closed curves, where $N(t)$ is a regular neighborhood of t. Let $E(k)$ be the exterior of a hyperbolic knot in S^3 with Seifert surface F. Now glue $E(k)$ with $M - intN(t)$ along the boundary tori so that ∂F is matched with a component of $f(S) \cap N(t)$. Then m copies of the Seifert surface F and $f(S) \cap (M - intN(t))$ form a non-separable surface in a 3-manifold which is not a graph manifold.

Theorem 2.7. *Suppose $S \to M$ is a horizontal surface in a graph manifold M. Then S is separable if and only if its preimage in the universal cover contains a disjoint pair of planes. Hence there is a horizontal surface in a graph manifold M such that any pair of preimage planes in the universal covering of M intersect.*

Proof. The easy direction is true for any immersed surface in a 3-manifold M: If S is separable, then there is a finite regular covering $q : M^* \to M$ such that the preimage $q^{-1}(S)$ consists of d embeddings of some finite cover S^* of S. Therefore $p^{-1}(S)$ consists of d families of disjoint planes, where $p : \tilde{M} \to M$ is the universal covering. Now pick any $d+1$ planes in $p^{-1}(S)$; by the pidgeon hole principle, two of them belong to the same family and therefore they are disjoint. So S has the $(d+1)$-plane property. If S has the k-plane property, then its preimage in the universal cover obviously contains a disjoint pair of planes.

Now we prove the opposite direction. Consider the horizontal immersed surface $S \subset M$ in a graph manifold M. Suppose S is not separable. Then we have $s_\gamma \neq 1$ for some simple closed curve $\gamma \subset S$ by Theorem 2.3. We are going to prove that in the universal cover of M, any two planes in the preimage must meet.

Suppose there are two such planes P_1 and P_2 which are disjoint . Dividing the universal covering by the stabilizer of P_1, the quotient is the covering space M_S

corresponding to the surface group, where the image of P_1 becomes the unique compact lift by Lemma 2.4, and the image of P_2 is a non-compact component which is disjoint from the compact lift in M_S. This contradicts Corollary 2.5.

The remaining part follows from Example 2.6. □

3. Simple loop theorem for graph manifolds

Theorem 3.1. *Suppose $f : S \to M$ is a two-sided proper map from a surface to a graph manifold. If the induced map $f_* : \pi_1(S) \to \pi_1(M)$ is not injective, then there is a simple closed curve in the kernel of f_*.*

Lemma 3.2. *Suppose $f : S \to M$ is a π_1-injective two-sided immersed surface in a Seifert manifold. Then S can be properly homotoped to be either vertical or horizontal.*

Proof. This is a lemma of Hass [H]. □

Lemma 3.3. *Suppose $f : S \to M$ is a π_1-injective two-sided immersed surface in a graph manifold. Then S can be properly homotoped so that each component of $M_v \cap S$ is either vertical or horizontal.*

Proof. Deform $f : S \to M$ so that $\mathcal{T} \cap S$ has a minimum number of components. Let S_i be a component of $M_v \cap S$. Then S_i is incompressible in S and it follows that S_i must be π_1-injective in M_v. By Lemma 3.2, S_i can be properly homotoped in M_v to be horizontal or vertical, so it is not difficult to see that f can be properly homotoped in M so that each component of $M_v \cap S$ is either horizontal or vertical, for any $v \in \Gamma(M)$. □

Lemma 3.4. *Suppose M is a graph manifold and $f : S \to M$ is an immersed surface. Assume each component of $M_v \cap S$ is either horizontal or vertical and non-boundary parallel in M_v for each vertex manifold M_v. Then $f : S \to M$ is π_1-injective.*

Proof. Suppose the immersion $S \to M$ is not π_1-injective, but satisfies the hypotheses of the Lemma. Let c be a non-trivial loop on S which is homotopically trivial in M. We may assume that c has been deformed in S so that $c \cap \mathcal{T}$ has minimal intersection number. If c misses $\mathcal{T}$, since $\mathcal{T}$ is incompressible, c is homotopically trivial in $M - \mathcal{T}$, which contradicts our assumption that each component of $M_v \cap S$ is either horizontal or vertical. So c does cross $\mathcal{T}$. Then c must have a "compressible segment", that is there is a component e of $c - \mathcal{T}$ such that e can be deformed through $M - \mathcal{T}$ into $\mathcal{T}$ relative to its endpoints. Let A be a component of $S \cap M_v$ which contains e. Then A is ∂-compressible. If A is vertical,

this contradicts the assumption that A is not boundary parallel. If A is horizontal, then A is incompressible and ∂-incompressible, which is again a contradiction. □

Corollary 3.5. *Suppose $f : S \to M$ is a non boundary parallel immersed surface in a graph manifold. Then $f : S \to M$ can be properly homotoped so that each component of $M_v \cap S$ is either vertical or horizontal if and only if f is π_1-injective.*

Proof of Theorem 3.1. Deform $f : S \to M$ so that no component of $M_v \cap S$ is boundary parallel in M_v, for each vertex manifold M_v. If each component of $M_v \cap S$ can be properly homotoped to be horizontal or vertical in the vertex manifold M_v, for any $v \in \Gamma(M)$, then it is not difficult to see that f can be properly homotoped so that each component of $M_v \cap S$ is either horizontal or vertical, for any $v \in \Gamma(M)$. Since f is not π_1-injective, by Lemma 3.4 there is a component S_i of $M_v \cap S$ which is neither horizontal nor vertical, for some vertex manifold M_v. By Lemma 3.2, S_i is not π_1-injective in M_v. By a theorem of J. Hass, there is an essential simple closed curve c on S_i lying in the kernel of f restricted to S_i. Since S_i is incompressible in S, c is an essential simple closed curve on S lying in in the kernel of f_*. □

4. The existence of horizontal surfaces in graph manifolds

Theorem 4.1. *Suppose M is a graph manifold with non-empty boundary. Then M contains a horizontal immersed surfaces.*

Proof. Suppose $p : \tilde{M} \to M$ is a finite covering . Then M contains a horizontal immersed surface if and only if $\tilde{M}$ contains a horizontal immersed surface. By passing to a finite covering, we may assume that M is framed and satisfies (2) and (3) of Lemma 1.0.

We can order the vertex manifolds as $M_1, ..., M_n$ so that

(1) $M - \Sigma_{j=1}^{l} M_j$ is connected

(2) M_n contains a component of ∂M.

Below we use $E(l)$ to denote $\overline{M - \Sigma_{j=l+1}^{n} M_j}$.

We are going to build a horizontal surface by induction:

The l-th induction step: There is a horizontal immersed surface $S(l)$ in $E(l)$ which satisfies the technical condition that $S_j = S(l) \cap M_j$ is connected and oriented.

Suppose we have finished the $(l-1)$-th step of the induction, i.e. we have built a horizontal immersed connected oriented surface $S(l-1)$ in $E(l-1)$. Assume the boundary components of $E(l-1)$, which are going to be glued with ∂-components of M_l are $\bar{T}_1, \bar{T}_2, ..., \bar{T}_h$. Also assume that $T_{h+1}, ..., T_{h+k}$ and $\bar{T}_{h+1}, ..., \bar{T}_{h+k}$ are boundary components of M_l which are to be matched together in pairs. Finally assume that $T_{h+k+1}, ..., T_{h+k+t}$, where $t > 0$, are boundary tori of M_l which are

not glued to other boundary tori at this induction step. By assumption, we can arrange that there are such tori, since otherwise $E(l)$ would be closed.

Let the gluing matrix for $g : T_i \to \bar{T}_i$ be $\begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix}$, for $i = 1, ..., h+k$.

Let C_i be the components of $\partial S(l-1)$ lying in T_i. To extend our immersed surface from $E(l-1)$ to $E(l)$, clearly the following two conditions should be satisfied:

(1) for each component c of C_i, $g(c)$ is transverse to the S^1 fiber of $M_l = F_l \times S^1$;

(2) the geometric intersection number of $g(C_i)$ with the S^1 fiber of $F_l \times S^1$ should be the same for all $i = 1, ..., h$.

Now we make some modifications to the surface $S(l-1)$ so that conditions (1) and (2) are satisfied . The modifications will be done in three steps.

Step 1. Pick $2K$ copies of $S(l-1)$, where $K = \prod_1^{k+h} |r_i|$, where r_i is the bottom-left entry of the gluing matrix.

Step 2. Since each F_j contains some non-separating simple closed curve, by making Haken sums in each vertex manifold $F_j \times S^1$, we can get a new immersed surface $S^\star(l-1)$ such that $S^*(l-1) \cap (S_j \times S^1)$ is connected.

Before we start step 3, we need to introduce the following definition and a lemma.

Definition. *Suppose $f : S \to F \times S^1$ is a horizontal immersed surface. Then there is a finite covering $p : S \times S^1 \to F \times S^1$ and an embedding $\tilde{f} : S \to S \times S^1$ such that $f = p \circ \tilde{f}$ by Lemma 1.3, where $\tilde{f}(x) = (x, 1)$, $x \in S$. Let c_1 and c_2 be two components of ∂S and a be a proper simple arc on $\tilde{f}(S)$ connecting $\tilde{f}(c_1)$ and $\tilde{f}(c_2)$. Let τ be 2π-Dehn twisting along the vertical annulus $a \times S^1$. We say that the closed curves $c_1^* = p \circ \tau^m \circ \tilde{f}(c_1)$ and $c_2^* = p \circ \tau^m \circ \tilde{f}(c_2)$ on $\partial F \times S^1$ are obtained by m-times Dehn twisting of $f(S)$ related to $c_1' = f(c_1)$ and $c_2' = f(c_2)$.*

For simplicity, sometimes we will omit the symbols of maps in the above definition. The following Lemma is derived directly from the definition.

Lemma 4.2. *Suppose S is an oriented connected horizontal immersed surface in a framed $F \times S^1$. Let c_1 and c_2 be two boundary components of S lying on the same framed boundary component $T(\alpha, \beta)$ of $F \times S^1$ with coordinates (u_1, v_1) and (u_2, v_2). If c_1^* and c_2^* are obtained by m-times Dehn twisting of S related to c_1 and c_2, then the coordinates of c_1^* and c_2^* are $(u_1, v_1 \pm m)$ and $(u_2, v_2 \mp m)$.*

For each T_i, we still use C_i to denote the boundary components of $\partial S^*(l-1)$ lying in T_i. Then each component of C_i is a simple closed curve or a multiple of a simple closed curve and C_i can be partitioned into mK pairs, where each pair is two parallel curves. Moreover if $\bar{C}_i = g(C_i)$ is the image in $\bar{T}_i$, of all curves in C_i, by g, then the geometric intersection number m_i of $\bar{C}_i$ and the S^1 fiber of $F_l \times S^1$ is a multiple of $2K$.

Step 3. Suppose $T_i \subset M_j$. By making a single Dehn twisting of S_j for every

pair in C_i whose image under g coincides with the fiber of M_l, we may assume that condition (1) on transversality is satisfied. Note that here Dehn twisting of the surface means applying a Dehn twist of the vertex manifold M_j about a vertical annulus or torus to S_j. We are going to perform more Dehn twisting so that condition (2) is also satisfied.

Now we fix a pair c_1, c_2 on T_i with coordinates (u, v). Next perform w-times Dehn twisting of $S(l-1)$, which induces w-times twisting about c_1 and c_2 and denote the resulting curves by $c_1(w), c_2(w)$ respectively. Similarly we use $C_i(w), \bar{C}_i(w)$ to denote the curves of $C_i, \bar{C}_i$ respectively after the change.

By Lemma 4.2 we have

$$c_1(w) = (u, v + w), \quad c_2 = (u, v - w).$$

By (1.2), we have

$$g(c_j(w)) = \begin{cases} (\bar{u} + wr_i, \bar{v} - ws_i), & j = 1, \\ (\bar{u} - wr_i, \bar{v} + ws_i), & j = 2. \end{cases}$$

So it is easy to see that the geometric intersection number $m_i(w)$ of $\bar{C}_i(w)$ and the S^1 fiber of $F_l \times S^1$ is changed by $2wr_i$.

Since originally m_i is a multiple of $2K$, we can choose w suitably for each i so that $m_1(w) = \cdots = m_h = 2m^*$, where $2m^*$ is a multiple of $2K$.

Now the condition (2) is satisfied. We still denote the resulting surface by $S^*(l-1)$ and we are going to extend $S^*(l-1)$ to a horizontal surface in $E(l)$.

Now we put $2m^*/|r_j|$ parallel curves of coordinates $\epsilon(r_j, -(p_j + 1))$ on T_j and $2m^*/|r_j|$ parallel curves with coordinates $\epsilon(r_j, -(s_j + 1))$ on $\bar{T}_j$, where $\epsilon r_j = |r_j|$, $j = h+1, \ldots, h+k$; then put two parallel oriented simple closed curves with coordinates $(1, v^*)$ and $2m^* - 2$ parallel oriented simple closed curves with coordinates $(1, 0)$ on T_{h+k+1}, and finally put $2m^*$ parallel curves with coordinates $(1, 0)$ on T_{h+k+j}, $j = 2, \ldots, h$.

Now the sum of the horizontal coordinates on each $T_j(\bar{T}_j)$ is $2m^*$. We can also choose v_* such that the sum of all the vertical coordinates is zero; the above coordinates determine a set of oriented closed curves which bounds a connected oriented horizontal surface S'_l in M_l, by Lemma 2.2. Clearly the curves on each T_j and on $\bar{T}_j$ can be matched under $g, j = h+1, \ldots, h+k$ by (1.2). So after the gluing, $S^*(l-1)$ and S'_l give a horizontal connected orientable surface $S(l)$ in $E(l)$. We have finished the l-th step of the induction. □

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