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Spherical minimal immersions of the 3-sphere

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Abstract. In 1966 Takahashi [11] proved that a minimal isometric immersion $f : S^m(1) \to S^N(r)$ of round spheres exists iff $r = \sqrt{m/\lambda_p}$, where λ_p is the p-th eigenvalue of the Laplacian on S^m; in this case, the components of f are spherical harmonics on S^m of order p. This immersion is unique up to congruence on the range and agrees with the generalized Veronese map if $m = 2$ as was shown in 1967 by Calabi [1]. In 1971 DoCarmo and Wallach [3] proved that the same rigidity holds for $p = 2, 3$. The main aim of their work, however, was to show that, for $m \geq 3$ and $p \geq 4$, unicity fails, and, indeed, the set of (congruence classes of) minimal isometric immersions $f : S^m \to S^N(\sqrt{m/\lambda_p})$ can be parametrized by a moduli space $\mathcal{M}_m^p$, a compact convex body in a representation space $\mathcal{F}_m^p$ of $SO(m+1)$ of dimension ≥ 18. In 1994, the first author [14] determined the exact dimension of the moduli, and with Gauchman [5] in 1996, revealed intricate connections beween the irreducible components of $\mathcal{F}_m^p$ and the geometry of the immersions these components represent. The purpose of the present paper is to provide a complete geometric description of the fine details of the (boundary of the) 18-dimensional space $\mathcal{M}_3^4$, the first nontrivial moduli. This is made possible by several reductions that make use of the splitting $SO(4) = SU(2) \cdot SU(2)'$ as well as rely on the structure of $SU(2)$ equivariant minimal isometric immersions treated in the work of DeTurck and the second author [2] in 1992. The equivariant imbedding theorem [14] asserts that the structure of $\mathcal{M}_3^4$ reappears in the moduli $\mathcal{M}_m^p$ for $m \geq 3$ and $p \geq 4$.

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Minimal isometric immersions of round spheres into round spheres form an interesting subject that has been studied by a number of authors, see e.g. [2,3,11,12]. We can write such maps either as isometric minimal immersions $f : S^m(1) \to S^N(r)$ as was done in [2], or as we do here and as was done in [12], as minimal immersions $f : S^m(1) \to S^N(1)$ with homothety $1/r^2$ (which we call spherical minimal immersions). The components of such an immersion must be eigenfunctions of the Laplacian on $S^m(1)$ which are hence harmonic homogeneous polynomials of degree p with eigenvalue $\lambda_p = p(p+m-1)$, in which case the homothety is equal to λ_p/m. As is well known, if $m = 2$, or if $m \geq 3$ and $p = 2, 3$, the immersion is unique up to congruence and agrees with the generalized Veronese immersion. But for $m \geq 3$ and $p \geq 4$, there are many such immersions, and for a fixed degree p the

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congruence classes of such minimal immersions form a convex compact body in a vector space of high dimension. The dimension of this convex body was computed explicitly in [14]. In particular, if $m = 3$ and $p = 4$, it is equal to 18 (see also [9] for this special case) and the dimension quickly grows with m and p.

The main purpose of the present paper is to examine this convex body in detail in the special case of $m = 3$, $p = 4$, although some of the results also apply to the general case. In this special case we are able to describe the structure of the convex body and its boundary completely with a number of interesting consequences. This is due to the fact that this low degree is the only case where the full convex body can be reconstructed in a certain way from the special subclass of equivariant minimal isometric immersions studied in more detail in [2,5,6]. This structure is also important for higher degrees and higher domain dimensions, since, according to the equivariant embedding theorem [14], the moduli space for degree four and domain dimension three can be equivariantly embedded into that for any degree ≥ 4 and domain dimension ≥ 3. In particular, using the examples developed here and in [2], a variety of spherical minimal immersions can be explicitly constructed from any domain of dimension ≥ 4 and any degree ≥ 3.

Before we describe our results, we set up some notation. If f is a spherical minimal immersion $f : S^m(1) \to S^N(1) \subset \mathbf{R}^{N+1} = V$ which uses degree p homogeneous harmonic polynomials, we denote by $\mathcal{M}_f$ the set of all spherical minimal immersions $f' = A \circ f$ with $A : \mathbf{R}^{N+1} \to \mathbf{R}^{N'+1}$ any linear map (in other words, the components of $f' \in \mathcal{M}_f$ are linear combinations of the components of f). $\mathcal{M}_f$ is a compact convex body in a linear subspace $\mathcal{F}_f \subset S^2V$ parametrized by $A^\top A - I$. The points on the boundary of $\mathcal{M}_f$ consist of spherical minimal immersions with ambient dimension less than N. If f_p is the standard spherical minimal immersion of S^m, consisting, up to homothety, of an orthonormal basis of the set $\mathcal{H}^p$ of all homogeneous harmonic polynomials of degree p, then $\mathcal{M}_{f_p} = \mathcal{M}^p$ is by definition the set of all spherical minimal immersions of degree p. For $f \in \mathcal{M}^p$, $\mathcal{M}_f$ forms a linear slice in $\mathcal{M}^p$. If f lies in the interior of $\mathcal{M}^p$, $\mathcal{M}_f$ is of course equal to $\mathcal{M}^p$. But if f lies on the boundary of $\mathcal{M}^p$, then $\mathcal{M}_f$ is a linear slice contained in the boundary of $\mathcal{M}^p$. We call f linearly rigid if $\mathcal{M}_f = \{f\}$. The linearly rigid spherical minimal immersions are precisely the extremal points of $\mathcal{M}^p$ in the sense of convex geometry (by the connecting lemma in Section 1.1). By the Krein-Milman theorem, a convex set is the convex hull of its subset of extremal points, so that $\mathcal{M}^p$ is the convex hull of the linearly rigid spherical minimal immersions.

$SO(m+1)$ acts on $\mathcal{M}^p$ via precomposition $f \to f \circ A$, $f \in \mathcal{M}^p$, $A \in SO(m+1)$, which makes $\mathcal{F}^p$ a linear $SO(m+1)$ representation space. A group G is the isotropy group of that action at a point f iff f is equivariant under G. We will say that f is full if the image of f spans all of V.

What is special about $m = 3$ is that $SO(4) = SU(2) \cdot SU(2)'$ with each $SU(2)$ acting transitively on S^3. Hence we can consider the $SU(2)$ and $SU(2)'$ equivariant spherical minimal immersions, which can also be viewed as the fixed point set $(\mathcal{M}^p)^{SU(2)}$ or $(\mathcal{M}^p)^{SU(2)'}$. We will show that for $p = 4$, and this is what makes

$m = 3, p = 4$ so special, that $\mathcal{F}^4 = (\mathcal{F}^4)^{SU(2)} \oplus (\mathcal{F}^4)^{SU(2)'}$, each one being a 9 dimensional linear subspace of $\mathcal{F}^4$. As an $SO(4)$ representation, this splitting is $R_8' \oplus R_8$ where $SU(2)$ acts trivially on R_8' and via the unique 9 dimensional irreducible representation on R_8 and in a reversed role for $SU(2)'$. Furthermore, we show that each point in $\partial\mathcal{M}^4$ lies on a straight line in $\mathcal{F}^4$ connecting a point $f_1 \in \partial(\mathcal{M}^4)^{SU(2)}$ with a point $f_2 \in \partial(\mathcal{M}^4)^{SU(2)'}$. This straight line consists of the immersions $(\sqrt{c_1}f_1, \sqrt{c_2}f_2)$ with $c_1 > 0, c_2 > 0, c_1 + c_2 = 1$. An orientation reversing isometry of S^3 will interchange R_8' and R_8 and hence $\partial(\mathcal{M}^4)^{SU(2)}$ with $\partial(\mathcal{M}^4)^{SU(2)'}$. Thus the boundary immersions $\partial\mathcal{M}^4$ are completely determined by the equivariant boundary immersions in $\partial(\mathcal{M}^4)^{SU(2)}$. Equivariant immersions in $(\mathcal{M}^4)^{SU(2)}$ have ambient dimension $N + 1 = 5, 10, 15, 20$ or 25, and one can easily exclude $N + 1 = 5$. This will enable us to completely determine all ambient dimensions:

Theorem A. *If $f : S^3 \to S^N$ is a full degree 4 spherical minimal immersion, then the only possible ambient dimensions are $N + 1 = 10, 15, 16$, or $19 - 25$, and each one occurs.*

Furthermore, it will follow that the spherical minimal immersions with $N+1 = 10, 15$, or 20 consist only of $SU(2)$ or $SU(2)'$ equivariant ones. Combining this with Proposition 1 in [2], p.449, yields the following uniqueness result for the lowest possible ambient dimension:

Theorem B. *There exists a degree 4 spherical minimal immersion $\mathcal{I} : S^3 \to S^9$ such that if $f : S^3 \to S^9$ is any degree 4 spherical minimal immersion, then there exist isometries $A \in O(10)$ and $B \in O(4)$ such that $f = A \circ \mathcal{I} \circ B$. Furthermore $\mathcal{I}$ is $SU(2)$ equivariant with image an embedded space form S^3/D_2^*, where D_2^* is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$.*

One can easily describe $\mathcal{I} : S^3 \to S^9$ explicitly:

$$\mathcal{I}(z,w) = \Big((1/\sqrt{2})(z^4 - \bar{w}^4), \sqrt{6}z^2\bar{w}^2, \sqrt{2}(z^3w + \bar{z}\bar{w}^3), \sqrt{6}(z\bar{z}^2w - \bar{z}w^2\bar{w}),$$
$$\sqrt{3/2}(z^2w^2 - \bar{z}^2\bar{w}^2), (1/\sqrt{2})(|z|^4 - 4|z|^2|w|^2 + |w|^4) \Big).$$

Here z and w are complex with $|z|^2+|w|^2 = 1$. Notice that the first four coordinates are complex, the fifth is purely imaginary and the sixth is real so that $\mathcal{I}$ maps into $\mathbf{R}^{10}$ and one easily checks that it maps into the unit sphere in $\mathbf{R}^{10}$. Also notice that the map is invariant under $(z, w) \to (iz, iw)$ and $(z, w) \to (\sqrt{i}\bar{w}, -\sqrt{i}\bar{z})$ which generate a group isomorphic to D_2^*. It will follow that $\mathcal{I}$ is an embedding of S^3/D_2^* into S^9.

Finally we describe the equivariant boundary immersions in $(\mathcal{M}^4)^{SU(2)} \subset \mathcal{F}^4$. They naturally divide into 3 subsets, denoted by **I**, **II**, and **III**, depending on if the ambient dimension is $N+1 = 10, 15$, or 20. Notice that the action of $SO(4)$ reduces to an $SU(2)'$ action on $(\mathcal{M}^4)^{SU(2)}$ and hence each set is the union of $SU(2)'$ orbits. We will show:

Theorem C. *$\partial(\mathcal{M}^4)^{SU(2)} = \mathbf{I} \cup \mathbf{II} \cup \mathbf{III}$ and one has:*

a) $\mathbf{I}$ is a single $SU(2)'$ orbit (the $SU(2)'$ orbit of $\mathcal{I}$) and is an octahedral manifold $S^3/\mathbf{O}^*$ embedded minimally in a sphere in $R'_8 = (\mathcal{F}^4)^{SU(2)}$, the embedding given by a degree 8 equivariant minimal isometric immersion.

b) $\mathbf{II} = \mathbf{II}_0 \cup \mathbf{II}_1$ where $\mathbf{II}_0$ is a 6 dimensional connected set consisting of linearly rigid immersions, and $\mathbf{II}_1$ is the 4 dimensional set $SU(2)' \cdot \operatorname{int} \mathcal{D}$ where $\mathcal{D}$ is a flat 2-dimensional disk with boundary circle on the octahedral manifold $\mathbf{I}$. Furthermore $\mathcal{D} = \mathcal{M}_{\mathcal{J}}$, where $\mathcal{J}$ is the unique $U(2)$ equivariant minimal immersion in $\mathbf{II}$.

c) $\mathbf{III}$ is dense, open and connected in the 8 dimensional boundary $\partial(\mathcal{M}^4)^{SU(2)}$. The opposite $\mathcal{J}^o$ of $\mathcal{J}$ on the boundary $\partial(\mathcal{M}^4)^{SU(2)}$ is of type III. $\mathcal{M}_{\mathcal{J}_o}$ is 6 dimensional and

$$\mathbf{III} = SU(2)' \cdot \operatorname{int} \mathcal{M}_{\mathcal{J}_o}.$$

The $U(2)$ equivariant map $\mathcal{J} : S^3 \to S^{14}$, is given explicitly as follows:

$$\begin{aligned}\mathcal{J}(z,w) = (1/\sqrt{2})\big(&z^4, w^4, 2\sqrt{3}z^2\bar{w}^2, 2z^3w, 2zw^3,\\ &2\sqrt{3}(z\bar{z}^2w - \bar{z}w^2\bar{w}), \sqrt{6}z^2w^2, |z|^4 - 4|z|^2|w|^2 + |w|^4\big).\end{aligned}$$

$\mathcal{J}$ is invariant under $(z,w) \to (iz, iw)$ and the image is an embedded lens space $S^3/\mathbf{Z}_4$. $\mathcal{J}$ is at the center of the disk $\mathcal{D} = \mathcal{M}_{\mathcal{J}}$ and the center of $U(2)$ acts as a rotation on this disk and hence $SU(2)' \cdot \mathcal{D} = SU(2)' \cdot \mathcal{M}_{\mathcal{J}}$ is 4-dimensional. This gives an explicit description to all elements in $\mathbf{II}_1$.

There exists only a 1-dimensional space of $U(2)$ equivariant immersions in $(\mathcal{M}^4)^{SU(2)}$ and hence only two $U(2)$ equivariant elements in $\partial(\mathcal{M}^4)^{SU(2)}$, one of which is $\mathcal{J}$ and the other one the antipodal point $\mathcal{J}_o$ (which lies in $\mathbf{III}$). The orbit $SU(2)'\mathcal{J}$ (and hence also $SU(2)'\mathcal{J}_o$) is a minimally embedded $\mathbf{R}P^2$ in a sphere in R'_8 which is in fact the standard rigid minimal isometric immersion for $m = 2$ and $p = 4$.

It is more difficult to find explicit examples in the 6-dimensional set of linearly rigid elements in $\mathbf{II}_0$. One explicit example in $\mathbf{II}_0$ is $\mathcal{I}_o$, the antipodal to the immersion $\mathcal{I}$ in Theorem B. Since the components of $\mathcal{I}$ are orthogonal of equal length, one can easily write down $\mathcal{I}_o$ explicitly by choosing such a basis for the orthogonal complement of the subspace spanned by the components of $\mathcal{I}$ in $\mathcal{H}^4$. In particular $\mathcal{I}_o : S^3 \to S^{14}$ and we will show that $\mathcal{I}_o$ is linearly rigid.

By Theorem C, the only exremal points in $(\mathcal{M}^4)^{SU(2)}$ are the elements in $\mathbf{I}$ and $\mathbf{II}_0$. Hence $(\mathcal{M}^4)^{SU(2)}$ is the convex hull of such immersions. As a consequence, all of $\mathcal{M}^4$ is the convex hull of $\mathbf{I}$, $\mathbf{II}_0$ and the images $\mathbf{I}'$, $\mathbf{II}_0'$ in $\partial(\mathcal{M}^4)^{SU(2)'}$

Also, notice that if we connect a linear slice $\mathcal{M}_f$ in $\mathbf{III}$ with a corresponding linear slice in $\partial(\mathcal{M}^4)^{SU(2)'}$, we obtain a 13 dimensional linear slice in $\partial\mathcal{M}^4$, which is the largest dimensional linear slice in the 17-dimensional boundary.

Another interesting question that one can ask, is about the image $f(S^3)$. Are they embeddings or immersions of a quotient S^3/G and what quotients can arise? For interior points of $\mathcal{M}^p$, it was observed in [18] that they are always embedded spheres or projective spaces, depending on if p is odd or even. But on $\partial\mathcal{M}^p$ many other images can occur. It is still an open problem whether all space forms can arise. In [2] it was shown that every homogeneous space form admits an equivariant minimal isometric embedding and in [4] an example was constructed of an inhomogeneous lens space with a minimal isometric immersion into S^{190} with degree 32 polynomials. It follows from [2] that for the equivariant maps in $\mathcal{M}^4$, the images are embedded homogeneous space forms S^3/D_2^*, lens spaces S^3/Z_4, or projective spaces (the generic case). It also follows that there exist many non equivariant minimal isometric embeddings of the homogeneous space forms S^3/D_2^* and S^3/Z_4.

For $m = 3$ and $p > 4$ there are corresponding results, but they only describe a portion of the moduli space which are not sufficient to prove analogues of Theorems A and B. See Theorems 4 and 5 for details. This is due to the fact that $\mathcal{F}^p$ splits into many irreducible summands under $SO(4)$ (see Theorem 3) and our methods apply to only some of those summands.

It is interesting to compare the results in this paper with some of the results in [15]. One can formulate the above mentioned connecting lemma, by saying that $\mathcal{M}^4$ is the convex hull of its slices by the two irreducible components in $\mathcal{F}^4$. For $p = 6$ (and $m = 3$) the moduli space of $SU(2)$ equivariant immersions $(\mathcal{M}^6)^{SU(2)}$ has two irreducible components and in [15] it is shown that a minimal immersion in each component has ambient dimension $N + 1 \geq 14$. The degree 6 minimal immersion $f : S^3 \to S^6$ constructed in [2] cannot lie in either component and not in the convex hull of the linear slices with each component either. Thus for larger degrees, the moduli space is not any more the convex hull of its linear irreducible slices, which shows that the structure of the moduli space gets much more complicated for $p > 4$.

In section 1 we collect several preparatory results that hold in the general case. Section 2 describes some results about the $SU(2)$ equivariant minimal maps of S^3 but any degree p and in section 3 we prove the above results for $m = 3$ and $p = 4$.

1. General theory

1.1. Eigenmaps

A *spherical harmonic* of order p on S^m is, by definition, an eigenfunction of the Laplacian $\triangle$ on S^m with eigenvalue $\lambda_p = p(p+m-1)$, or equivalently, the restriction (to S^m) of a harmonic homogeneous polynomial in the variables $(x_0, \dots, x_m) \in \mathbf{R}^{m+1}$. The linear space of spherical harmonics is denoted by $\mathcal{H}^p = \mathcal{H}^p_m$. (Unless important, we suppress the source dimension m.) In a similar vein, a *p-eigenmap* $f : S^m \to S_V$, where V is a Euclidean vector space with S_V, the unit sphere in V, is a map whose components $\alpha \circ f$, $\alpha \in V^*$, are spherical harmonics of order p on S^m, or equivalently, a harmonic p-homogeneous spherical polynomial map $f : \mathbf{R}^{m+1} \to V$. *Spherical* means that f maps S^m into S_V and, in this case, we identify $f : \mathbf{R}^{m+1} \to V$ with its restriction $f : S^m \to S_V$.

$f : S^m \to S_V$ is *full* if it has no zero component. In this case, precomposition with f, $\alpha \to \alpha \circ f$, $\alpha \in V^*$, gives a linear embedding $V^* \to \mathcal{H}^p$ whose image; the space of components of f, is denoted by V_f. Since V is Euclidean, we have the isomorphisms

$$V \cong V^* \cong V_f \subset \mathcal{H}^p.$$

In what follows, V, V^* and V_f will be identified under these isomorphisms.

Two p-eigenmaps $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ are said to be *congruent* if $f_2 = U \circ f_1$ for some isometry $U : V_1 \to V_2$.

Let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ be full p-eigenmaps. f' is said to be *derived* from f, written as $f' \leftharpoonup f$, if there exists a linear map $A : V \to V'$ such that $A \circ f = f'$. Since f is full, A is uniquely determined. Since f' is also full, A is onto.

Let $f : S^m \to S_V$ be a full p-eigenmap. Let $\mathcal{E}_f \subset S^2V$ denote the orthogonal complement of

$$\mathcal{W}_f = \operatorname{span}\{f(x) \odot f(x) \,|\, x \in S^m\}$$

in S^2V. Let

$$\mathcal{L}_f = \{C \in \mathcal{E}_f \,|\, C + I \geq 0\},$$

where $I = I_V$ = identity of V and '$\geq$' means 'positive semidefinite'. Clearly, $\mathcal{L}_f$ is a convex body in $\mathcal{E}_f$ and the origin of $\mathcal{E}_f$ is contained in the interior of $\mathcal{L}_f$. A DoCarmo-Wallach type argument gives the following:

Theorem 1. *Given a full p-eigenmap $f : S^m \to S_V$, the set of congruence classes of full eigenmaps $f' : S^m \to S_{V'}$ that are derived from f can be parametrized by the convex body $\mathcal{L}_f$. The parametrization is given by associating to the congruence class of f' the endomorphism*

$$\langle f' \rangle_f = A^\top \cdot A - I \in S^2V,$$

where $f' = A \circ f$ and $^\top$ denotes transpose.

Remark. For (all) spherical minimal immersions this is due to DoCarmo-Wallach [3,17]. The present relative version is taken from [12].

The convex body $\mathcal{L}_f$ of $\mathcal{E}_f(\subset S^2V)$ is called the *moduli space* associated to the full p-eigenmap $f : S^m \to S_V$. f corresponds to the origin $0\,(= \langle f\rangle_f)$. The interior $\operatorname{int}\mathcal{L}_f$ parametrizes those full p-eigenmaps $f' : S^m \to S_{V'}$ for which $f \leftharpoonup f'$, or equivalently, $f' = A \circ f$ with $A : V \to V'$ invertible. As A is onto, this holds iff $\dim V = \dim V'$. Thus the boundary $\partial\mathcal{L}_f$ corresponds to those p-eigenmaps $f' : S^m \to S_{V'}$ for which $f' \leftharpoonup f$ and $\dim V' < \dim V$.

Since $\leftharpoonup$ is a transitive relation, for $f' \leftharpoonup f$, $\mathcal{L}_{f'}$ can be embedded into $\mathcal{L}_f$. The image is an affine slice (that is, the intersection of $\mathcal{L}_f$ with an affine subspace). More precisely, let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ with $f' \leftharpoonup f$, i.e. $f' = A \circ f$, and define

$$\iota : S^2V' \to S^2V$$

by

$$\iota(C') = A^\top \cdot C' \cdot A + \langle f'\rangle_f = A^\top \cdot (C' + I_{V'}) \cdot A - I_V.$$

Then, as easy computation shows, ι maps $\mathcal{E}_{f'}$ injectively into $\mathcal{E}_f$ and

$$\iota(\mathcal{L}_{f'}) = \iota(\mathcal{E}_{f'}) \cap \mathcal{L}_f.$$

From now on, we identify $\mathcal{L}_{f'}$ with its image in $\mathcal{L}_f$.

Up to scaling the components of an orthonormal basis in $\mathcal{H}^p$ with respect to the L^2-scalar product give rise to the standard p-eigenmap $f_p : S^m \to S_{\mathcal{H}^p}$. For $f = f_p$, we denote $\mathcal{L}^p = \mathcal{L}^p_m = \mathcal{L}_{f_p}$, $\mathcal{E}^p = \mathcal{E}^p_m = \mathcal{E}_{f_p}$ etc. We call $\mathcal{L}^p$ the *standard moduli space*. Since all p-eigenmaps are derived from f_p, $\mathcal{L}^p$ parametrizes the congruence classes of all full p-eigenmaps $f : S^m \to S_V$. For simplicity, we set $\langle f\rangle_{f_p} = \langle f\rangle$. We say that a full p-eigenmap $f : S^m \to S_V$ is of *boundary type* if $\dim V < \dim \mathcal{H}^p$, or equivalently, if $\langle f\rangle \in \partial\mathcal{L}^p$. Also, for any full p-eigenmap $f : S^m \to S_V$, $\mathcal{L}_f$ is an affine slice of $\mathcal{L}^p$. Integrating the condition of sphericality for f, we see [3,17] that $\mathcal{E}^p$ consists of traceless endomorphisms of $S^2(\mathcal{H}^p)$. We thus have:

Corollary. *$\mathcal{L}^p$ and (hence) $\mathcal{L}_f$ are compact.*

Let $f : S^m \to S_V$ and $f' : S^m \to S_{V'}$ be full p-eigenmaps and assume that $f' \leftharpoonup f$ with $f' = A \circ f$, where $A : V \to V'$ is linear and onto. Then $V_{f'}$ is contained in V_f and the inclusion $V_{f'} \subset V_f$ is given by sending $\alpha' \circ f'$, $\alpha' \in (V')^*$, to $(\alpha' \circ A) \circ f$.

Connecting Lemma. *Let $f_1 : S^m \to S_{V_1}$ and $f_2 : S^m \to S_{V_2}$ be full p-eigenmaps and assume that they are incongruent. Let $c_1, c_2 > 0$ with $c_1 + c_2 = 1$. Then the point*

$$c_1\langle f_1\rangle + c_2\langle f_2\rangle \in \mathcal{L}^p$$

on the segment connecting $\langle f_1\rangle$ and $\langle f_2\rangle$ is represented by the p-eigenmap $f : S^m \to S_V$, $V = V_1 \times V_2$, defined by $f = (\sqrt{c_1}f_1, \sqrt{c_2}f_2)$ and made full. In particular,

$f \rightharpoonup f_1, f_2$ and

$$V_f = V_{f_1} + V_{f_2}$$

so that

$$\dim V_f = \dim V_{f_1} + \dim V_{f_2} - \dim(V_{f_1} \bigcap V_{f_2}).$$

Proof. Setting $f_1 = A_1 \circ f_p$ and $f_2 = A_2 \circ f_p$ with $A_1 : \mathcal{H}^p \to V_1$ and $A_2 : \mathcal{H}^p \to V_2$ linear and onto, we have

$$\langle f_1 \rangle = A_1^\top A_1 - I \quad \text{and} \quad \langle f_2 \rangle = A_2^\top A_2 - I.$$

By definition, $f = (\sqrt{c_1}A_1, \sqrt{c_2}A_2) \circ f_p$ so that

$$\begin{aligned} \langle f \rangle &= (c_1 A_1^\top A_1 + c_2 A_2^\top A_2) - I \\ &= c_1(A_1^\top A_1 - I) + c_2(A_2^\top A_2 - I) \\ &= c_1\langle f_1 \rangle + c_2\langle f_2 \rangle \end{aligned}$$

since $c_1 + c_2 = 1$. The rest is clear.

Given a full p-eigenmap $f : S^m \to S_V$ of boundary type, $\langle f \rangle \in \partial\mathcal{L}^p$, the line $\mathbf{R} \cdot \langle f \rangle$ intersects $\partial\mathcal{L}^p$ in $\langle f \rangle$ and another point called the antipodal of $\langle f \rangle$. A representative $f_o : S^m \to S_{V_o}$ of the antipodal of $\langle f \rangle$ is called the *antipodal* p-eigenmap of f. (f_o is unique up to congruence.)

The connection between f and f_o is subtle. A related (and again subtle) problem has been posed by R.T.Smith in his Thesis [10]: Given a p-eigenmap $f : S^m \to S_V$, does there exist a p-eigenmap $f' : S^m \to S_{V'}$ such that $V_{f'}$ is the orthogonal complement of V_f in $\mathcal{H}^p$? The following observation will be useful:

Antipodal Lemma. *Let $f : S^m \to S_V$ be a full p-eigenmap of boundary type and assume that, relative to an orthonormal basis in V, the components of f are orthogonal in $\mathcal{H}^p$ and have the same norm. Then the antipodal $f_o : S^m \to S_{V_o}$ of f has the same property and*

$$V_f \oplus V_{f_o} = \mathcal{H}^p$$

is an orthogonal direct sum.

Proof. Relative to an orthonormal basis in V and up to a constant multiple, the components of f give an orthonormal basis $\mathcal{V}$ in $V_f \subset \mathcal{H}^p$. Select an orthonormal basis $\mathcal{V}_o$ from $V_f^\perp \subset \mathcal{H}^p$. Let $f_o : \mathbf{R}^{m+1} \to V_o$ be a full harmonic p-homogeneous polynomial map whose components, relative to an orthonormal basis in V_o, are the elements of $\mathcal{V}_o$. Since $\mathcal{V} \bigcup \mathcal{V}_o$ (suitably normalized) gives the components of f_p, up to a constant multiple, f_o is spherical so that it restricts to a full p-eigenmap $f_o : S^m \to S_{V_o}$. By construction, $f_p = (cf, c_o f_o)$ for some constants $c, c_o > 0$. Taking norms, we have $c^2 + c_o^2 = 1$ so that the connecting lemma applies. We obtain that the origin is on the segment connecting $\langle f \rangle$ and $\langle f_o \rangle$. Since both f and f_o are of boundary type, f_o is the antipodal of f.

Let $f : S^m \to S_V$ be a p-eigenmap and assume that f is equivariant with respect to a homomorphism $\rho_f : G \to SO(V)$, where $G \subset SO(m+1)$ is a closed subgroup. Equivariance means that

$$f \circ g = \rho_f(g) \circ f,$$

for all $g \in G$. ρ_f defines an orthogonal G-module structure on V and therefore on V^*. Under the isomorphism $V^* \cong V_f \subset \mathcal{H}^p$, V becomes a G-submodule of $\mathcal{H}^p|_G$. Indeed, since $g \in G$ acts on V^* as $g \cdot \alpha = \alpha \circ \rho_f(g)^{-1}$, $\alpha \in V^*$, we have

$$(g \cdot \alpha) \circ f = (\alpha \circ \rho_f(g)^{-1}) \circ f = (\alpha \circ f) \circ g^{-1}.$$

The G-module structure on V extends to that of S^2V given by

$$g \cdot C = \rho_f(g) \cdot C \cdot \rho_f(g)^{-1}.$$

$\mathcal{E}_f$ is clearly a G-submodule of S^2V. In fact, $\mathcal{L}_f$ is G-invariant since, for $f' \backsim f$, we have

$$g \cdot \langle f' \rangle_f = \langle f' \circ g^{-1} \rangle_f.$$

Given a closed subgroup $G' \subset G$, a p-eigenmap $f' : S^m \to S_{V'}$, $f' \backsim f$, is equivariant with respect to a homomorphism $\rho_{f'} : G' \to SO(V')$ iff $\langle f' \rangle_f$ is left fixed by G'. Thus the congruence classes of full p-eigenmaps that are equivariant with group G' are parametrized by the linear slice

$$(\mathcal{L}_f)^{G'} = \mathcal{L}_f \cap (\mathcal{E}_f)^{G'}.$$

Remark. Let $f : S^m \to S_V$ be a full p-eigenmap and assume that, relative to an orthonormal basis of V, the components of f in $\mathcal{H}^p$ form an orthogonal basis $\mathcal{V}$ with the same norm. Then, the isotropy group at $\langle f \rangle \in \mathcal{L}^p$ can be written as

$$SO(m+1)_{\langle f \rangle} = \{g \in SO(m+1) \,|\, V_f = V_{f \circ g}\}. \tag{1}$$

Indeed, for $g \in SO(m+1)$, $U \in SO(V)$ with $U \circ f = f \circ g$ exists iff $V_f = V_{f \circ g}$ and is the transfer matrix between $\mathcal{V}$ and $\mathcal{V} \circ g$ obtained by precomposing the elements of $\mathcal{V}$ with g.

The following lemma is contained in [12] (pp.24-25).

Transversality Lemma. *Let $f : S^m \to S_V$ be a full p-eigenmap that is equivariant with respect to a homomorphism $\rho_f : G \to SO(V)$, where $G \subset SO(m+1)$ is a closed subgroup. Let $f' : S^m \to S_{V'}$ be a full p-eigenmap such that $\langle f' \rangle_f \in \operatorname{int} \mathcal{L}_f$. If $a : \mathbf{R} \to G$ is a 1-parameter subgroup such that the orbit $t \mapsto a(t) \cdot \langle f' \rangle_f$, $t \in \mathbf{R}$, is tangent to $\operatorname{int} \mathcal{L}_f$ at $t = 0$ then it is entirely contained in $\operatorname{int} \mathcal{L}_f$.*

The standard p-eigenmap $f_p : S^m \to S_{\mathcal{H}^p}$ is equivariant with respect to the homomorphism $\rho_p : SO(m+1) \to SO(\mathcal{H}^p)$ that is just the orthogonal $SO(m+1)$-module structure on $\mathcal{H}^p$ given by precomposing spherical harmonics by the inverse of linear isometries on S^m. Thus $\mathcal{E}^p$ is an $SO(m+1)$-submodule of $S^2(\mathcal{H}^p)$. A result of Calabi [1] asserts that $\mathcal{E}^p_2$ is trivial for any $p \geq 2$. As we will see below $\mathcal{E}^p_m$ is nontrivial for $m \geq 3$ and $p \geq 2$.

1.2. Spherical minimal immersions

A *spherical minimal immersion* is a conformal p-eigenmap $f : S^m \to S_V$. The conformality factor is then λ_p/m. We say that $f : S^m \to S_V$ is a *minimal immersion with homothety* λ_p/m. The condition of homothety can be written as

$$\langle f_*(X), f_*(Y)\rangle = (\lambda_p/m)\langle X, Y\rangle,$$

for all vector fields X and Y on S^m.
Let $f : S^m \to S_V$ be a full homothetic minimal immersion with homothety λ_p/m. We define

$$\mathcal{F}_f = \text{span}\,\{\, f_*(X)\check{}\odot f_*(Y)\check{}\,|\, X, Y \in T(S^m)\}^{\perp} \subset S^2V,$$

where $\check{}$ denotes translation of vectors to the origin. (Here and in what follows it is understood that X and Y belong to the same tangent space of S^m.)
Let

$$\mathcal{M}_f = \{C \in \mathcal{F}_f |\, C + I \geq 0\}.$$

A result of Takahashi [11] implies that

$$\mathcal{F}_f \subset \mathcal{E}_f$$

(cf. [12]). The defining relation for $\mathcal{L}_f$ in $\mathcal{E}_f$ is the same as for $\mathcal{M}_f$ in $\mathcal{F}_f$. Thus, the inclusion above gives

$$\mathcal{M}_f = \mathcal{F}_f \cap \mathcal{L}_f,$$

as a linear slice of $\mathcal{L}_f$. In particular, $\mathcal{M}_f$ is a compact convex body in $\mathcal{F}_f$.

Theorem 2. *Given a full homothetic minimal immersion $f : S^m \to S_V$, the set of congruence classes of full homothetic minimal immersions $f' : S^m \to S_{V'}$ that are derived from f can be parametrized by the convex body $\mathcal{M}_f$.*

The convex body $\mathcal{M}_f$ is said to be the *moduli space* associated to the full minimal immersion $f : S^m \to S_V$. We say that f is *linearly rigid* [17] if $\mathcal{M}_f$ reduces to a point. Note that the connecting, antipodal and transversality lemmas remain valid in the context of minimal immersions.

Since $SO(m+1)$ acts transitively on the unit sphere bundle of S^m, the standard p-eigenmap $f_p : S^m \to S_{\mathcal{H}^p}$ is conformal and thereby a minimal immersion with homothety λ_p/m. The *standard moduli space* $\mathcal{M}^p = \mathcal{M}^p_m = \mathcal{M}_{f_p}$ is the linear slice of $\mathcal{L}^p$ by the $SO(m+1)$-submodule $\mathcal{F}^p = \mathcal{F}^p_m = \mathcal{F}_{f_p} \subset S^2(\mathcal{H}^p)$ and it parametrizes the congruence classes of all full minimal immersions with homothety λ_p/m. The ultimate goal is to describe $\mathcal{M}^p$. As a first step, we need to determine the $SO(m+1)$-module structure of (its linear span) $\mathcal{F}^p$, that is, the decomposition of $\mathcal{F}^p$ into irreducible submodules. By DoCarmo-Wallach [3,17], $\mathcal{F}^p$ is nontrivial iff $m \geq 3$ and $p \geq 4$.

1.3. Some representation theory

$V^{(u_1,\dots,u_d)} = V_{m+1}^{(u_1,\dots,u_d)}$, $d = [(m+1)/2]$, denotes the (unique) complex irreducible $SO(m+1)$-module with highest weight vector $(u_1, \dots, u_d)$ relative to the standard maximal torus in $SO(m+1)$. In particular, $V^{(p,0,\dots,0)} = \mathcal{H}^p$ as complex $SO(m+1)$-modules. (Here and in what follows we denote $\mathcal{H}^p$ and its complexification by the same symbol.) For $m = 3$ and $v > 0$, $V_4^{(u,v)}$ means $V_4^{(u,v)} \oplus V_4^{(u,-v)}$.

By DoCarmo-Wallach [3]:

$$\mathcal{H}^p \otimes \mathcal{H}^q = \sum_{(u,v)\in\triangle_0^{p,q};\, u+v\equiv p+q \,(\mathrm{mod}\, 2)} V^{(u,v,0,\dots,0)}, \;\; p \geq q \geq 1, \;\; m \geq 3,$$

where $\triangle_0^{p,q}$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(p-q, 0)$, (p, q) and $(p+q, 0)$.

Setting $p = q$ and deleting the components that belong to the skew-symmetric part of $\mathcal{H}^p \otimes \mathcal{H}^p$, we have

$$S^2(\mathcal{H}^p) = \sum_{(u,v)\in\triangle_0^p;\, u,v \text{ even}} V^{(u,v,0,\dots,0)}, \tag{2}$$

where we simplified the notation by setting $\triangle_0^p = \triangle_0^{p,p}$.
We have

$$\mathcal{E}^p \otimes \mathbf{C} = \sum_{(u,v)\in\triangle_1^p;\, u,v \text{ even}} V^{(u,v,0,\dots,0)}, \tag{3}$$

where $\triangle_1^p$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(2, 2)$, (p, p) and $(2p-2, 2)$. For a quick proof cf. [5].

Remark. Let $f : S^m \to S_V$ be a full p-eigenmap and assume that

$$\langle f \rangle \in V^{(2p-2l,2l,0,\dots,0)}, \;\; l = 1, \dots, [p/2].$$

These components correspond to the northeast side of the triangle $\triangle_1^p$. Then [14] we have

$$\dim V \geq \dim \mathcal{H}^{p-1}/(m+1). \tag{4}$$

(This is because the condition guarantees that the partial derivatives of the components of f span $\mathcal{H}^{p-1}$.)

The main result of DoCarmo-Wallach [3,17] asserts that $\mathcal{F}_m^p$ is nontrivial iff $m \geq 3$ and $p \geq 4$.

The following result is proved in [14] and gives the positive resolution of the so-called DoCarmo-Wallach conjecture:

Theorem 3. *For $m \geq 3$ and $p \geq 4$,*

$$V^{(2,2,0,\dots,0)}, \dots, V^{(2p-2,2,0,\dots,0)}$$

are not components of $\mathcal{F}^p$ so that we have

$$\mathcal{F}^p \otimes \mathbf{C} = \sum_{(u,v)\in\triangle_2^p; u,v \text{ even}} V^{(u,v,0,\dots,0)}, \tag{5}$$

where $\triangle_2^p$ is the closed convex triangle in $\mathbf{R}^2$ with vertices $(4,4)$, (p,p) and $(2p-4,4)$.

Remark. In the lowest nonrigid range $m = 3$ and $p = 4$, Theorem 3 has been proved by Muto [9].

1.4. Isotropic minimal immersions

Let $f : S^m \to S_V$ be a minimal immersion with homothety λ_p/m. We denote by $\beta_k(f)$ and $\mathcal{O}_f^k$, $k \le p$, the *k-th fundamental form* and the *k-th osculating bundle* of f. We say that f is *isotropic of order* k, $2 \le k \le p$, if, for $2 \le l \le k$, we have

$$\begin{aligned}&\langle \beta_l(f)(X_1,\dots,X_l), \beta_l(f)(X_{l+1},\dots,X_{2l})\rangle \\ &\quad = \langle \beta_l(f_p)(X_1,\dots,X_l), \beta_l(f_p)(X_{l+1},\dots,X_{2l})\rangle\end{aligned}$$

where $X_1,\dots,X_{2l}$ are vector fields on S^m [5]. In this case, for $2 \le l \le k$, the osculating bundles $\mathcal{O}_f^l$ and $\mathcal{O}_{f_p}^l$ are isomorphic with a fibrewise isometry. Restricting to the base point $o = (1,0,\dots,0) \in S^m$, as $SO(m)$-modules, we thus have

$$\mathcal{O}_{f;o}^l = \mathcal{O}_{f_p;o}^l = \mathcal{H}_{m-1}^l, \quad 2 \le l \le k.$$

In particular, since the osculating bundles are in the normal bundle, we obtain that, for a full minimal immersion $f : S^m \to S_V$ with homothety λ_p/m, isotropy of order k implies that

$$\dim V \ge \sum_{l=0}^{k} \dim \mathcal{H}_{m-1}^l = \dim \mathcal{H}_m^k.$$

(The last equality is because of branching over $SO(m) \subset SO(m+1)$.)

According to a result of [5], the space of congruence classes of full minimal immersions $f : S^m \to S_V$ with homothety λ_p/m that are isotropic of order k, $2 \le k \le p$, is parametrized by a linear slice $\mathcal{M}^{p;k}$ of $\mathcal{M}^p$ whose linear span $\mathcal{F}^{p;k}$ is an $SO(m+1)$-submodule of $S^2(\mathcal{H}^p)$. We have [5]

$$\mathcal{F}^{p;k} \otimes \mathbf{C} \supset \sum_{(u,v)\in\triangle_k^p;\, u,v \text{ even}} V^{(u,v,0,\dots,0)},$$

where $\triangle_k^p \subset \triangle_2^p$ is the subtriangle with vertices $(2(k+1), 2(k+1))$, (p,p) and $(2(p-k-1), 2(k+1))$. Moreover (and this is more difficult), for $m \ge 4$, equality holds.

Combining this with the above, we see that given a full minimal immersion $f : S^m \to S_V$ with homothety λ_p/m, if $\langle f\rangle \in V^{(2l,2k)}$, $l = k,\dots,p-k$, then (f is isotropic of order $k-1$) $\dim V \ge \dim \mathcal{H}_m^{k-1}$.

1.5. Representations of $SU(2) = S^3$

From now on we specialize the source dimension to $m = 3$. The case of the 3-sphere S^3 as a source deserves a special attention since it is itself a Lie group; the Lie group of quaternions of unit length. We write an element of S^3 as $a + bj$, where $|a|^2 + |b|^2 = 1$, $a, b \in \mathbf{C}$. Associating to $a + bj \in S^3$ the special unitary matrix

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2)$$

gives an isomorphism $S^3 \cong SU(2)$. Viewing a complex 2-vector $(z, w) \in \mathbf{C}^2$ as a real 4-vector $(x, u, y, v) \in \mathbf{R}^4$, $z = x + iy$, $w = u + iv$, gives an embedding of $SU(2)$ into $SO(4)$ as a normal subgroup. The orthogonal transformation $\gamma = \operatorname{diag}(1, 1, 1, -1) \in O(4)$ (or equivalently, $\gamma : z \mapsto z$, $w \mapsto \bar{w}$) conjugates $SU(2)$ to the subgroup

$$SU(2)' = \gamma SU(2)\gamma, \quad \gamma^{-1} = \gamma, \tag{6}$$

of $SO(4)$ and (as simple computation shows), we have

$$SU(2) \bigcap SU(2)' = \{\pm I\} \tag{7}$$

and (for reasons of dimension)

$$SU(2) \cdot SU(2)' = SO(4). \tag{8}$$

The complex irreducible $SU(2)$-modules are parametrized by the dimension of the module. More concretely, let W_p, $p \geq 0$, be the linear space of complex homogeneous polynomials of degree p in z and w. The standard basis in W_p is $\{z^{p-q}w^q\}_{q=0}^{p}$. W_p is a complex irreducible $SU(2)$-module with $\dim W_p = p + 1$ and each complex irreducible $SU(2)$-module is equivalent to W_p for some p. As $SU(2)$-modules, we have

$$W_r \otimes W_s = \sum_{t=0}^{s} W_{r+s-2t}, \;\; r \geq s \geq 0. \tag{9}$$

Restricting from $SO(4)$ to $U(2)$, the $SO(4)$-module $\mathcal{H}^p$ of complex spherical harmonics on S^3 of order p splits as

$$\mathcal{H}^p|_{U(2)} = \sum_{a+b=p;\, a,b\geq 0} \mathcal{H}^{a,b}, \tag{10}$$

where $\mathcal{H}^{a,b}$ is the complex irreducible $U(2)$-module of harmonic polynomials of degree a in z, w and degree b in $\bar{z}, \bar{w}$. (This is easily seen by writing a harmonic p-homogeneous polynomial in terms of the variables $z, \bar{z}, w, \bar{w}$.) The center $\Gamma \subset U(2)$

acts on each $\mathcal{H}^{a,b}$ as a character. Restricting further to $SU(2) \subset U(2)$, we thus obtain

$$\mathcal{H}^p|_{SU(2)} = (p+1)W_p$$

as complex $SU(2)$-modules. More generally, by the local product structure (8), the $SO(4)$-module $\mathcal{H}^p$ splits into the tensor product

$$\mathcal{H}^p = W_p \otimes W_p'. \tag{11}$$

Here we use the following notation. If W is an $SU(2)$-module then W' denotes the $SU(2)'$-module obtained from W by conjugating first $SU(2)'$ back to $SU(2)$ within $SO(4)$. Moreover, if $-I$ acts on W' trivially then W' is also considered as an $SO(4)$-module with trivial action of $SU(2)$ on W'. Similarly, if $-I$ acts on W trivially then W is also an $SO(4)$-module with $W|_{SU(2)'}$ being trivial.

In the next lemma recall that we write $V^{(u,v)}$, $v > 0$, for $V_4^{(u,v)} \oplus V_4^{(u,-v)}$.

Lemma. *Let $u \geq v \geq 1$ and $u+v$ even. Then*

$$V^{(u,v)} = W_{u-v} \otimes W'_{u+v} \oplus W_{u+v} \otimes W'_{u-v}$$

and hence

$$V^{(u,v)}|_{SU(2)} = (u+v+1)W_{u-v} \oplus (u-v+1)W_{u+v}. \tag{12}$$

Proof. The northern vertex (u,v) in $\triangle_0^{u,v}$ is missed by the subtriangles $\triangle_0^{u-1,v-1}$ and $\triangle_0^{u+1,v-1}$ overlapping in $\triangle_0^{u,v-2}$. We thus obtain

$$V^{(u,v)} \oplus (\mathcal{H}^{u-1} \otimes \mathcal{H}^{v-1}) \oplus (\mathcal{H}^{u+1} \otimes \mathcal{H}^{v-1}) = (\mathcal{H}^u \otimes \mathcal{H}^v) \oplus (\mathcal{H}^u \otimes \mathcal{H}^{v-2}).$$

We work out each tensor product using

$$\begin{aligned} \mathcal{H}^r \otimes \mathcal{H}^s &= (W_r \otimes W_r') \otimes (W_s \otimes W_s') \\ &= (W_{r+s} \oplus W_{r+s-2} \oplus \ldots \oplus W_{r-s}) \\ &\otimes (W'_{r+s} \oplus W'_{r+s-2} \oplus \ldots \oplus W'_{r-s}),\ r \geq s, \end{aligned}$$

and arrive at the stated equality.

Corollary. *Let $u \geq v \geq 1$ and $u+v$ even. Then $V^{(u,v)}|_{SU(2)}$ contains the trivial $SU(2)$-module iff $u = v$. The multiplicity of the trivial $SU(2)$-module in $V^{(u,u)}$ is $2u+1$.*

With the notation introduced above, we have

$$V^{(u,u)} = W_{2u} \oplus W'_{2u} \tag{13}$$

as (complex) $SO(4)$-modules.

Finally, we make some comments on real irreducible $SU(2)$-modules. For p even, W_p is the complexification of an irreducible real $SU(2)$-submodule R_p. In fact, R_p is the real subspace of the complex antilinear map of W_p that sends $z^q w^{p-q}$ to $(-1)^q z^{p-q} w^q$, $q = 0, \dots, p$. The standard basis of R_p is given by

$$z^p + w^p,\ i(z^p - w^p),\ z^{p-1}w - zw^{p-1},\ i(z^{p-1}w + zw^{p-1}), \dots, i^{p/2} z^{p/2} w^{p/2}.$$

Using (11) and (13), we obtain the following real $SO(4)$ modules for p even

$$\mathcal{H}^p = R_p \otimes R'_p \quad , \quad \mathcal{H}^p|_{SU(2)} = (p+1)R_p \quad , \quad V^{(u,u)} = R_{2u} \oplus R'_{2u}. \tag{14}$$

For p odd, W_p, considered as a real $SU(2)$-module, is irreducible. We denote this real representation by $[W_p]_{\mathbf{R}}$. Hence we obtain the following real $SO(4)$-modules for p odd

$$\mathcal{H}^p = [W_p \otimes W'_p]_{\mathbf{R}} \quad , \quad \mathcal{H}^p|_{SU(2)} = \frac{p+1}{2}[W_p]_{\mathbf{R}} \quad , \quad V^{(u,u)} = R_{2u} \oplus R'_{2u}. \tag{15}$$

2. $SU(2)$ equivariant eigenmaps and minimal immersions

Since S^3 acts on itself by left quaternionic multiplication, a p-eigenmap $f : S^3 \to S_V$ is $SU(2)$ *equivariant* if there exists a homomorphism $\rho_f : SU(2) \to SO(V)$ such that

$$f \circ L_g = \rho_f(g) \circ f, \quad g \in SU(2),$$

where L_g is left quaternionic multiplication on S^3 by g as a quaternion. Clearly, a p-eigenmap f is $SU(2)$ equivariant iff $f \circ \gamma$ is $SU(2)'$ equivariant. The spaces of congruence classes of full $SU(2)$ (resp. $SU(2)'$) equivariant p-eigenmaps are parametrized by the linear slices

$$(\mathcal{L}^p)^{SU(2)} = \mathcal{L}^p \cap (\mathcal{E}^p)^{SU(2)} \ \ (\text{resp. } (\mathcal{L}^p)^{SU(2)'} = \mathcal{L}^p \cap (\mathcal{E}^p)^{SU(2)'}).$$

We can find $(\mathcal{E}^p)^{SU(2)}$ from the decomposition formula (3) by setting $m = 3$, restricting both sides to $SU(2)$ and counting the trivial components. In fact, according to (14):

$$(V^{(u,u)})^{SU(2)} = R'_{2u} \otimes \mathbf{C}$$

as $SU(2)'$-modules. In view of this, (3) and Corollary in 1.5, we have

$$\begin{aligned}(\mathcal{E}^p \otimes \mathbf{C})^{SU(2)} &= \sum_{k=1}^{[p/2]} (V^{(2k,2k)})^{SU(2)} \\ &= \sum_{k=1}^{[p/2]} (R'_{4k} \otimes \mathbf{C})\end{aligned}$$

so that, as real $SU(2)'$-modules:

$$(\mathcal{E}^p)^{SU(2)} = \sum_{k=1}^{[p/2]} R'_{4k}.$$

Similarly

$$(\mathcal{E}^p)^{SU(2)'} = \sum_{k=1}^{[p/2]} R_{4k}$$

as real $SU(2)$-modules. In the lowest nonrigid range $p = 2$ (for eigenmaps), $\triangle_1^p$ in (3) reduces to the single point $(2, 2)$ so that we have

$$\mathcal{E}^2 = (\mathcal{E}^2)^{SU(2)} \oplus (\mathcal{E}^2)^{SU(2)'} = R'_4 \oplus R_4$$

as $SO(4)$-modules. Thus the moduli space $\mathcal{L}^2$ is 'split' by two 5-dimensional orthogonal slices $(\mathcal{L}^2)^{SU(2)} = \mathcal{L}^2 \cap R'_4$ and $(\mathcal{L}^2)^{SU(2)'} = \mathcal{L}^2 \cap R_4$ parametrizing $SU(2)$ and $SU(2)'$ equivariant quadratic eigenmaps. It is now a crucial observation to be generalized below that $\mathcal{L}^2$ is the convex hull of these slices [13].

Remark. As a convex set $(\mathcal{L}^2)^{SU(2)} \cong (\mathcal{L}^2)^{SU(2)'}$ is the convex hull of a projective plane embedded into S^4 as the Veronese surface [14]. More generally, for p even,

$$(\mathcal{L}^p)^{SU(2)} = \{C_0 \in S^2(R_p) \,|\, \operatorname{trace} C_0 = 0 \operatorname{and} C_0 + I \geq 0\}.$$

This follows from $\mathcal{H}^p|_{SU(2)} = (p+1)R_p$ (cf. (14)) and Schur's lemma applied to symmetric endomorphisms of $\mathcal{H}^p$ in $(\mathcal{E}^p)^{SU(2)}$ that commute with the action of $SU(2)$.

The situation is analogous for minimal immersions. We have

$$(\mathcal{M}^p)^{SU(2)} = \mathcal{M}^p \cap (\mathcal{F}^p)^{SU(2)},$$

and similarly for $SU(2)'$. Moreover, as $SU(2)'$-modules

$$\begin{aligned}(\mathcal{F}^p \otimes \mathbf{C})^{SU(2)} &= \sum_{k=2}^{[p/2]} (V^{(2k,2k)})^{SU(2)} \\ &= \sum_{k=2}^{[p/2]} (R'_{4k} \otimes \mathbf{C}).\end{aligned}$$

As for eigenmaps, we see that the $SU(2)$ and $SU(2)'$ equivariant minimal immersions correspond to the northwest side of $\triangle_2^p$ in (5). As real $SU(2)'$-modules

$$(\mathcal{F}^p)^{SU(2)} = \sum_{k=2}^{[p/2]} R'_{4k}.$$

In particular, counting dimensions

$$\begin{aligned}\dim(\mathcal{M}^p)^{SU(2)} &= \dim(\mathcal{F}^p)^{SU(2)} \\ &= \sum_{k=2}^{[p/2]}(4k+1) \\ &= (2[p/2]+5)([p/2]-1).\end{aligned}$$

This formula has been derived in [2] using a 'heuristic argument'.

In the lowest nonrigid range $p = 4$ (for minimal immersions):

$$\mathcal{F}^4 = (\mathcal{F}^4)^{SU(2)} \oplus (\mathcal{F}^4)^{SU(2)'} = R_8' \oplus R_8$$

as real $SO(4)$-modules. Intersecting with the moduli space we see that $\mathcal{M}^4$ is 'split' by two 9-dimensional orthogonal slices $(\mathcal{M}^4)^{SU(2)} = \mathcal{M}^4 \cap R_8'$ and $(\mathcal{M}^4)^{SU(2)'} = \mathcal{M}^4 \cap R_8$ corresponding to $SU(2)$ and $SU(2)'$ equivariant minimal immersions. It is also clear that precomposing quartic minimal immersions with γ has the effect of interchanging R_8 and R_8'.

Let $\mathcal{B}_1^p = (\partial\mathcal{M}^p)^{SU(2)}$ and $\mathcal{B}_2^p = (\partial\mathcal{M}^p)^{SU(2)'}$. Thus $\mathcal{B}_1^p$ (resp. $\mathcal{B}_2^p$) parametrize the boundary type $SU(2)$ (resp. $SU(2)'$) equivariant minimal immersions $f : S^3 \to S_V$ with homothety $\lambda_p/3$.

Theorem 4. *For $p \geq 4$, $\partial\mathcal{M}^p \cap V^{(2k,2k)}$ is the union of segments with one endpoint on $\mathcal{B}_1^p \cap V^{(2k,2k)}$ and the other on $\mathcal{B}_2^p \cap V^{(2k,2k)}$. Equivalently, every full (boundary) minimal immersion $f : S^3 \to S_V$ of degree p such that $\langle f\rangle \in V^{(2k,2k)}$ is congruent to one of the form $(\sqrt{\lambda_1}f_1, \sqrt{\lambda_2}f_2 \circ \gamma) : S^3 \to S_{V_1\times V_2}$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$, where $f_1 : S^3 \to S_{V_1}$ and $f_2 : S^3 \to S_{V_2}$ are full $SU(2)$ equivariant (boundary) minimal immersions with $\langle f_1\rangle, \langle f_2\rangle \in R_{4k}' \subset V^{(2k,2k)}$.*

Proof. The connecting lemma establishes the equivalence of the two statements. To prove the first it is enough to show that any segment connecting $\mathcal{B}_1^p \cap V^{(2k,2k)}$ and $\mathcal{B}_2^p \cap V^{(2k,2k)}$ is entirely contained in the boundary of $\mathcal{M}^p$. Let f_1 and f_2 as in the second statement and let $\langle f\rangle$ be in the interior of the segment connecting $\langle f_1\rangle$ and $\langle f_2 \circ \gamma\rangle$. We need to show that $\langle f\rangle \in \partial\mathcal{M}^p$. By the connecting lemma cited above, we have

$$V_f = V_{f_1} + V_{f_2\circ\gamma}.$$

Since f_1 and f_2 are $SU(2)$ equivariant and of boundary type, V_{f_1} is a proper $SU(2)$-submodule of $\mathcal{H}^p$ and $V_{f_2\circ\gamma}$ is a proper $SU(2)'$-submodule of $\mathcal{H}^p = W_p \otimes W_p'$ (for simplicity we complexify again). We need the following elementary statement whose proof is an easy application of Schur's lemma.

Lemma. *Let G be a compact Lie group, R an irreducible G-module and W a trivial G-module. Then any G-submodule Z of $R \otimes W$ is of the form $Z = R \otimes W_0$, where $W_0 \subset W$ is a linear subspace.*

Returning to our previous setting, we have

$$V_{f_1} = W_p \otimes W_0' \quad \text{and} \quad V_{f_2\circ\gamma} = W_0 \otimes W_p'$$

where $W_0 \subset W_p$ and $W_0' \subset W_p'$ are proper linear subspaces. Hence

$$V_f = V_{f_1} + V_{f_2\circ\gamma} = W_p \otimes W_0' + W_0 \otimes W_p'.$$

But $W_0 \otimes W_0'$ is equal to the overlap of both subspaces and hence V_f can never have the full dimension. Thus f is of boundary type.

Theorem 4 gives a complete description of full minimal immersions $f : S^3 \to S_V$ with homothety $\lambda_p/3$ whose parameter point $\langle f\rangle$ is in one of the irreducible components $V^{(4,4)}, V^{(6,6)}, \ldots, V^{(2[p/2],2[p/2])}$ corresponding to the northwest side of the triangle $\triangle_2^p$ in (5). It states that such minimal immersions are obtained from $SU(2)$ equivariant ones by the prescription given in the connecting lemma. In particular, in the lowest nonrigid range $p = 4$, $\triangle_2^4$ collapses to the single point $(4, 4)$ so that Theorem 4 completely describes all full quartic minimal immersions in terms of $SU(2)$ equivariant ones. The proof of Theorem 4 also gives all possible range dimensions of such minimal immersions for $p \geq 4$. This gives a partial answer to a problem posed by DoCarmo: What are the possible (in particular, minimum) range dimensions of all spherical minimal immersions?

Theorem 5. *Let $f : S^3 \to S_V$ be a full minimal immersion of degree $p \geq 4$ and assume that $\langle f\rangle \in V^{(2k,2k)}$. Then the possible range dimensions of f (plus one) are*

a) For p even

$$\dim V = (p+1)^2 - rs, \quad \textit{with} \quad 1 \leq r, s \leq p - [(p+1)/4] \quad \textit{or}$$
$$\dim V = (p+1)r, \quad \textit{with} \quad [(p+1)/4] + 1 \leq r \leq p+1 \ \textit{(for the equivariant ones).}$$

b) For p odd

$$\dim V = (p+1)^2 - rs, \quad \textit{with} \quad 1 \leq r, s \leq p - [(p+1)/8] \quad \textit{or}$$
$$\dim V = (p+1)r, \quad \textit{with} \quad [(p+1)/8] + 1 \leq r \leq p+1 \ \textit{(for the equivariant ones).}$$

Proof. It follows from the proof of Theorem 4 that $\dim V_f = (p+1)\dim W_0' + \dim W_0(p+1) - \dim W_0' \dim W_0$. Furthermore, notice that it follows from (15) that for p odd, the dimension of W_0 and W_0' must be even. But there are further restrictions on the possible dimensions of equivariant embeddings. In [15] it was proved that if $f : S^3 \to S_V$ is a full $SU(2)$ equivariant minimal immersion of degree p and $\langle f\rangle \in V^{(2k,2k)}$ then $\dim V \geq (p+1)^2/4$. Of course for the equivariant ones we also need that $\dim V$ is divisible by $p+1$ if p is even and by $2(p+1)$ if p is odd. This easily implies the claim in the theorem.

Remark. For $p = 4$ and $p = 5$, $(\mathcal{F}^p)^{SU(2)}$ consists only of a single irreducible summand. But starting with $p = 6$, it has at least two irreducible summands, in particular $(\mathcal{F}^6)^{SU(2)} = R'_8 \oplus R'_{12}$. In [15] it was shown that if $p = 6$ and if $\langle f \rangle$ lies in R'_8 or in R'_{12}, then $\dim V \geq 14$. The connecting lemma hence shows that any point in the convex hull of $\mathcal{M}^6 \cap R'_8$ with $\mathcal{M}^6 \cap R'_{12}$ also has $\dim V \geq 14$. But in [2] an example was constructed of an equivariant minimal isometric immersion with $p = 6$ and $\dim V = 7$ (and was shown to be unique among such equivariant isometric immersions). It follows that it cannot be in the convex hull of $\mathcal{M}^6 \cap R'_8$ with $\mathcal{M}^6 \cap R'_{12}$, i.e this immersion must "bulge out" in the moduli space. Thus the moduli space for $p \geq 6$ must have a much more complicated structure than for $p = 4$. It was conjectured in [2] that this degree 6 immersion into S^6 is the only degree 6 immersion with $\dim V = 7$. The above remarks show that a proof of this fact must be much more complicated than the proof of theorem B.

3. Quartic minimal immersions

From now on we let $p = 4$ (and $m = 3$). Theorem 4 asserts that $\mathcal{M}^4$ is the convex hull of $(\mathcal{M}^4)^{SU(2)}$ and $(\mathcal{M}^4)^{SU(2)'}$ corresponding to $SU(2)$ and $SU(2)'$ equivariant quartic minimal immersions. The possible range dimensions for a full $SU(2)$-equivariant quartic minimal immersion $f : S^3 \to S_V$ are

$$\dim V = 5, 10, 15, 20, 25.$$

This is because V is an $SU(2)$-submodule of $\mathcal{H}^4|_{SU(2)} = 5R_4$ so that it must be a multiple of R_4. The range dimension 5 is not realized. This follows from a general theorem of Moore [8] or by easy computation in the use of the equivariant construction below. Thus Theorem 5 gives all possible range dimensions of full quartic minimal immersions $f : S^3 \to S_V$:

$$\dim V = 10, 15, 16, 19, 20, 21, 22, 23, 24, 25.$$

We will see later that the range dimensions $10, 15$, and 20 actually occur for full $SU(2)$ equivariant quartic minimal immersions so that, without equivariance, all the rest of the range dimensions above are realized. Notice that it also follows from the proof of Theorem 5 that $\dim V = 10, 15$ and 20 can only occur if the minimal immersion is $SU(2)$ equivariant or $SU(2)'$ equivariant. In [2] it was shown that there exists a unique $SU(2)$ equivariant minimal immersion with $\dim V = 10$ (up to precomposition with an isometry of the domain) and hence Theorems A and B in the introduction follow.

We say that an equivariant minimal immersion is of type I, II, or III if the ambient dimension is $\dim V = 10, 15$, or 20. We denote by **I**, **II**, and **III** the set of all minimal immersions of type I, II, and III. Similarly for $\mathbf{I}'$, $\mathbf{II}'$, and $\mathbf{III}'$ in $\partial(\mathcal{M}^4)^{SU(2)'}$. It follows also, that for example the immersions with $\dim V = 16$ in $\partial\mathcal{M}^4$ can only be obtained by connecting an immersion in **I** with one in $\mathbf{I}'$.

Furthermore, if we connect a linear slice $\mathcal{M}_f$ in **III** with another linear slice $\mathcal{M}_{f'}$ in **III**$'$, we obtain a 13 dimensional linear slice in $\partial\mathcal{M}^4$, which is the largest dimensional linear slice in the boundary of $\mathcal{M}^4$. The equivariant immersions must have an embedded image since they are given by orbits under the action of $SU(2)$ and $SU(2)'$, in fact the orbit of a polynomial $p \in \mathcal{H}^4$ is diffeomorphic to $SU(2)/G$ where G is the isotropy group of p. From [2] it follows that the possible isotropy groups are D_2^*, Z_4 and Z_2, Z_2 being the principal isotropy group. Hence in the equivariant case the images are embedded S^3/D_2^*, lens spaces S^3/Z_4, or projective spaces $\mathbf{R}P^3$. We will see that all three cases actually occur.

Since $(\mathcal{M}^4)^{SU(2)'}$ is the γ-copy of $(\mathcal{M}^4)^{SU(2)}$, it remains to describe the latter and this is our main purpose in this section. For simplicity we denote $\mathcal{B} = (\mathcal{M}^4)^{SU(2)}$ and from now on all minimal immersions will be $SU(2)$ equivariant. The $SO(4)$ action on $(\mathcal{M}^4)^{SU(2)}$ reduces to an action of $SU(2)'$ and hence **I**, **II**, and **III** are each union of $SU(2)'$ orbits. The action of $SU(2)'$ on $R_8' = (\mathcal{F}^4)^{SU(2)}$ is the action on polynomials in z, w of degree 8. The orbits are all 3-dimensional, except one orbit, the one through z^4w^4, which gives rise to a minimal isometric embedding of $\mathbf{R}P^2$ into a sphere in R_8'. It follows from [2] that the possible finite isotropy groups of $SU(2)'$ acting on R_8' and hence also on $\partial(\mathcal{M}^4)^{SU(2)}$ consist of the cyclic group Z_8 (the principal isotropy group), the binary dihedral groups D_2^*, D_3^* and D_4^*, and the binary octahedral group O^*. As was observed in [2], many of the orbits are again minimal isometric immersions in their respective spheres.

We now turn to the 'equivariant construction' for $SU(2)$ that provides an explicit description of all $SU(2)$ equivariant minimal immersions as (constant curvature) $SU(2)$ orbits of polynomials in $SU(2)$-submodules of $\mathcal{H}^p$. We first summarize some of the results in [2]. Each equivariant construction used here is based on an $SU(2)$-submodule of $\mathcal{H}^p$, where we now (briefly) return to the general case $p \geq 4$. As a first example we take this to be W_p. Consider a polynomial $\xi \in W_p$, of unit length. In terms of the standard basis in W_p, we write

$$\xi(z,w) = c_0 z^p + c_1 z^{p-1} w + \ldots + c_p w^p, \quad c_0, \ldots, c_p \in \mathbf{C}. \tag{16}$$

Let $f_\xi : S^3 \to S_{W_p}$ be the orbit map:

$$f_\xi(g) = g \cdot \xi = \xi \circ L_{g^{-1}}, \quad g \in SU(2) = S^3.$$

More explicitly, setting $g = a + bj \in S^3$, we have

$$f_\xi(a+bj)(z,w) = \xi(\bar{a}z - bw, \bar{b}z + aw), \quad z, w \in \mathbf{C}.$$

It is important to note here that f_ξ is equivariant with respect to the conjugate subgroup $SU(2)'$ (and not with respect to $SU(2)$). More precisely, by the identifications we made, we have

$$f_\xi\left(\gamma \begin{bmatrix} \alpha & -\beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \gamma \cdot (a+bj)\right) = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \cdot f_\xi(a+bj).$$

Note that the matrices differ by conjugation with $\operatorname{diag}(1,1,-1,-1)$ (that preserves the local product structure (8)). For this reason, we will usually make f_ξ $SU(2)$ equivariant by precomposing it with γ.

f_ξ is minimal iff

$$\sum_{q=0}^{p}(2q-p)^2(p-q)!q!|c_q|^2 = \frac{p(p+2)}{3},$$
$$\sum_{q=0}^{p}(p-q)!q!|c_q|^2 = 1,$$
$$\sum_{q=0}^{p-2}(q+2)!(p-q)!c_q\bar{c}_{q+2} = 0,$$
$$\sum_{q=0}^{p-1}(p-2q-1)(q+1)!(p-q)!c_q\bar{c}_{q+1} = 0.$$

The first two equations are real and the last two are complex so that we have 6 constraints on the $2(p+1)$ real variables $\Re(c_q)$ and $\Im(c_q)$, $q=0,\dots,p$.

For p even, we can take R_p instead of W_p by requiring $\xi \in R_p$ (so that f_ξ will actually map into S_{R_p}). Using the standard basis in R_p, we obtain that this additional requirement translates into

$$c_p = \bar{c}_0,\ \ c_{p-1} = -\bar{c}_1,\dots,c_{(p/2)+1} = (-1)^{(p/2)+1}\bar{c}_{(p/2)-1},\ \ c_{p/2} = i^{p/2}t,\ t\in\mathbf{R}.$$

We incorporate these $p+1$ additional constraints by writing

$$\xi(z,w) = c_0z^p + \bar{c}_0w^p + c_1z^{p-1}w - \bar{c}_1zw^{p-1} + \ldots + i^{p/2}tz^{p/2}w^{p/2}.$$

The system of equations for minimality of f_ξ thus reduces to

$$\sum_{q=0}^{(p/2)-1} 2(2q-p)^2(p-q)!q!|c_q|^2 = \frac{p(p+2)}{3},$$
$$\sum_{q=0}^{(p/2)-1} 2(p-q)!q!|c_q|^2 + ((p/2)!)^2t^2 = 1,$$
$$\sum_{q=0}^{(p/2)-3} 2(q+2)!(p-q)!c_q\bar{c}_{q+2} + (-1)^{(p/2)+1}(((p/2)+1)!)^2c_{(p/2)-1}^2$$
$$+ (-i)^{p/2}2(p/2)!((p/2)+2)!c_{(p/2)-2}t = 0,$$
$$\sum_{q=0}^{(p/2)-2} (p-2q-1)(q+1)!(p-q)!c_q\bar{c}_{q+1} + (-i)^{p/2}(p/2)!((p/2)+1)!c_{(p/2)-1}t = 0.$$

As a first explicit example we now show that there is no $SU(2)$ equivariant quartic minimal immersion with range $V = R_4$. Indeed any such should come from the equivariant construction above for $p = 4$. The equations for minimality reduce to

$$\begin{aligned} 96|c_0|^2 + 6|c_1|^2 &= 1, \\ 48|c_0|^2 + 12|c_1|^2 + 4t^2 &= 1, \\ 3c_1^2 + 8c_0 t &= 0, \\ 6c_0\bar{c}_1 - c_1 t &= 0. \end{aligned}$$

These equations are inconsistent and we are done.

Next we turn to the description of type I minimal immersions. These are obtained from the equivariant construction applied to $W_4 = 2R_4$. Setting $p = 4$ the constraints for W_4 specialize to

$$\begin{aligned} &48|c_0|^2 + 3|c_1|^2 + 3|c_3|^2 + 48|c_4|^2 = 1, \\ &24|c_0|^2 + 6|c_1|^2 + 4|c_2|^2 + 6|c_3|^2 + 24|c_4|^2 = 1, \\ &4c_0\bar{c}_2 + 3c_1\bar{c}_3 + 4c_2\bar{c}_4 = 0, \\ &6c_0\bar{c}_1 + c_1\bar{c}_2 - c_2\bar{c}_3 - 6c_3\bar{c}_4 = 0. \end{aligned}$$

This system has solutions, for example

$$c_0 = \sqrt{6}/24,\ c_1 = 0,\ c_2 = \sqrt{2}/4,\ c_3 = 0,\ c_4 = -\sqrt{6}/24.$$

To work out the orbit map $f_\xi : S^3 \to S_{W_4}$, we identify W_4 with $\mathbf{C}^5$ by the orthonormal basis

$$z^4/\sqrt{24},\ z^3w/\sqrt{6},\ z^2w^2/2,\ zw^3/\sqrt{6},\ w^4/\sqrt{24}.$$

We obtain (replacing the variable $a+bj$ with $z+wj$ in S^3, precomposing with γ and up to an isometry on the range) the full quartic minimal immersion $\mathcal{I} : S^3 \to S^9$ of type I, given by

$$\begin{aligned} \mathcal{I}(z,w) = \Big(&(1/\sqrt{2})(z^4 - \bar{w}^4), \sqrt{6}z^2\bar{w}^2, \\ &\sqrt{2}(z^3w + \bar{z}\bar{w}^3), \sqrt{6}(z\bar{z}^2w - \bar{z}w^2\bar{w}), \\ &\sqrt{3/2}(z^2w^2 - \bar{z}^2\bar{w}^2), (1/\sqrt{2})(|z|^4 - 4|z|^2|w|^2 + |w|^4) \Big). \end{aligned} \tag{17}$$

(The first four coordinates are complex, the fifth is purely imaginary and the sixth is real so that $\mathcal{I}$ maps into $\mathbf{C}^4 \times (i\mathbf{R}) \times \mathbf{R} = \mathbf{R}^{10}$.) An important property of $\mathcal{I}$ is that its components are orthogonal with the same norm.

According to a rigidity result in [2], up to isometries on the source and the range, this is the only type I minimal immersion and the image of the immersion is an embedded S^3/D_2^*. Hence

$$\mathbf{I} = SU(2)' \cdot \langle \mathcal{I} \rangle,$$

where we used $SU(2)$ equivariance of $\mathcal{I}$. For the next result we recall that the octahedral manifold [2] is the quotient $S^3/\mathbf{O}^*$, where $\mathbf{O}^*$ is the binary octahedral group (that is the twofold cover of the group of symmetries of the regular octahedron along $S^3 \to SO(3)$).

Theorem 6. **I** *is a single* $SU(2)'$*-orbit. Furthermore, this orbit is an octahedral manifold* $S^3/\mathbf{O}^*$ *embedded minimally in an 8-sphere of* R_8'*. This embedding is given by a degree 8 equivariant immersion of* S^3*.*

To prove the second part, we will have to show that $\mathbf{O}^*$ is the isotropy group of the $SU(2)'$ action at $\langle \mathcal{I} \rangle$, a computation which we postpone for the moment. Once this is done, it follows that the orbit $S^3/\mathbf{O}^*$ must be minimally embedded in a sphere since according to [2] there exists only one orbit with isotropy $\mathbf{O}^*$ which is hence an isolated exceptional orbit and hence must be minimal. Or, as was first observed in [18], $S^3/\mathbf{O}^*$ is isotropy irreducible and hence for every invariant polynomial the orbit construction must give rise to an isometric embedding up to scaling.

We will first consider the type II immersions:

Theorem 7. *We have*

$$\dim \mathbf{II} \leq 6. \tag{18}$$

The set **II** *splits into the disjoint union*

$$\mathbf{II} = \mathbf{II}_0 \cup \mathbf{II}_1 \tag{19}$$

corresponding to linearly rigid and nonrigid quartic minimal immersions. We have

$$\dim \mathbf{II}_0 = 6. \tag{20}$$

and

$$\mathbf{II}_1 = SU(2)' \cdot \mathcal{D}, \tag{21}$$

where $\mathcal{D}$ *is a flat* 2*-dimensional disk with boundary circle on the octahedral manifold* **I***.*

Proof. We show (18) by a careful dimension computation. For type II minimal immersions the $SU(2)$-module is $3R_4$. Since R_4 can be thought of as the $SU(2)$-module of quartic polynomials

$$\xi(z,w) = c_0 z^4 + \bar{c}_0 w^4 + c_1 z^3 w - \bar{c}_1 z w^3 - t z^2 w^2,\ c_0, c_1 \in \mathbf{C},\ t \in \mathbf{R},$$

we look upon a general element of $3R_4$ as a triple:

$$\begin{pmatrix} a_0z^4 + \bar{a}_0w^4 + a_1z^3w - \bar{a}_1zw^3 - rz^2w^2 \\ b_0z^4 + \bar{b}_0w^4 + b_1z^3w - \bar{b}_1zw^3 - sz^2w^2 \\ c_0z^4 + \bar{c}_0w^4 + c_1z^3w - \bar{c}_1zw^3 - tz^2w^2 \end{pmatrix}$$

where $a_0, a_1, b_0, b_1, c_0, c_1 \in \mathbf{C}$ and $r, s, t \in \mathbf{R}$. The decomposition $3R_4 (= V)$ is not unique, in fact, $SO(3)$ acts on $3R_4$ in a natural way. Thus, rotating $(r, s, t) \in \mathbf{R}^3$, we may assume that $r = s = 0$ and $t \geq 0$. We still have the freedom to rotate along the third axis. This amounts to the change

$$\begin{aligned} a_0 &\mapsto \cos\alpha \cdot a_0 - \sin\alpha \cdot b_0, \\ b_0 &\mapsto \sin\alpha \cdot a_0 + \cos\alpha \cdot b_0, \end{aligned} \tag{22}$$

and similarly for a_1 and b_1. The equations for minimality are

$$\begin{aligned} 96(|a_0|^2 + |b_0|^2 + |c_0|^2) + 6(|a_1|^2 + |b_1|^2 + |c_1|^2) &= 1, \\ 48(|a_0|^2 + |b_0|^2 + |c_0|^2) + 12(|a_1|^2 + |b_1|^2 + |c_1|^2) + 4t^2 &= 1, \\ 3(a_1^2 + b_1^2 + c_1^2) + 8c_0t &= 0, \\ 6(a_0\bar{a}_1 + b_0\bar{b}_1 + c_0\bar{c}_1) - c_1t &= 0. \end{aligned} \tag{23}$$

Note that these equations are invariant under the action (22) of $SO(2)$. For fixed $t \in \mathbf{R}$ we can solve the first two equations and obtain

$$\begin{aligned} |a_0|^2 + |b_0|^2 + |c_0|^2 &= r_0(t)^2 \\ |a_1|^2 + |b_1|^2 + |c_1|^2 &= r_1(t)^2, \end{aligned}$$

where

$$\begin{aligned} r_0(t)^2 &= \frac{1}{144}(1 + 4t^2) \\ r_1(t)^2 &= \frac{1}{18}(1 - 8t^2). \end{aligned}$$

The second equation reduces the range of t to

$$0 \leq t \leq 1/\sqrt{8}.$$

If $t = 1/\sqrt{8}$ then $a_1 = b_1 = c_1 = 0$. The third equation in (23) gives $c_0 = 0$ (the fourth is automatically satisfied) so that we have

$$|a_0|^2 + |b_0|^2 = 1/96.$$

We obtain that, for $t = 1/\sqrt{8}$, the solution set is the 3-sphere (of radius $1/\sqrt{96}$). The action of $SO(2)$ on (a_0, b_0) (whose orbits are essentially given by the Hopf fibration) reduces this to a 2-dimensional solution set.

Now let $0 \le t < 1/\sqrt{8}$. Since both radii $r_0(t)$ and $r_1(t)$ are positive, the first two equations above say that

$$(a_0, b_0, c_0) \in S^5_{r_0(t)} \quad \text{and} \quad (a_1, b_1, c_1) \in S^5_{r_1(t)}$$

in two copies of $\mathbf{C}^3$. If $t = 0$ then the third and fourth equations in (23) reduce to

$$a_1^2 + b_1^2 + c_1^2 = 0$$

and

$$a_0\bar{a}_1 + b_0\bar{b}_1 + c_0\bar{c}_1 = 0.$$

The first of these is a complex quadric that intersected with $S^5_{1/\sqrt{18}}$ gives a smooth 3-dimensional manifold for (a_1, b_1, c_1). (In fact, this is the real projective space.) For fixed (a_1, b_1, c_1), the second equation is a complex plane that intersected with $S^5_{1/12}$ gives a great 3-sphere. Putting these together, the product is a 6-dimensional manifold on which $SO(2)$ acts without fixed points. The quotient gives a 5-dimensional solution set.

Finally, let $0 < t < 1/\sqrt{8}$. Given $(a_1, b_1, c_1) \in S^5_{r_1(t)}$, we use the third equation in (23) to get

$$c_0 = -\frac{3}{8t}(a_1^2 + b_1^2 + c_1^2).$$

The fourth equation in (23) is an affine complex plane

$$a_0\bar{a}_1 + b_0\bar{b}_1 + c_0\bar{c}_1 = \frac{1}{6}c_1 t$$

that, intersected with $S^5_{r_0(t)}$ and knowing the value of c_0, reduces the solution set for (a_0, b_0, c_0) to at most one dimension. This is because a_1 and b_1 cannot vanish simultaneously. (Indeed, if $a_1 = b_1 = 0$ then $c_0\bar{c}_1 = -3c_1^2\bar{c}_1/(8t) = -3|c_1|^2c_1/(8t)$. On the other hand, $c_0\bar{c}_1 = c_1 t/6$. Combining these we obtain $t = 0$; a contradiction.) This, combined with the 5-dimensional solution set for (a_1, b_1, c_1) gives a 6-dimensional solution set. As before, the action of $SO(2)$ reduces this to 5-dimensions.

Summarizing, for fixed $0 \le t \le 1/\sqrt{8}$, the solution set is always at most 5-dimensional. Varying t now gives (18).

Next we consider $\mathbf{II}_1$ in the splitting (19). Given a full minimal immersion $f : S^3 \to S_V$ of type II, if f is linearly nonrigid, that is $\dim \mathcal{M}_f \ge 1$, then the points on $\partial\mathcal{M}_f$ correspond to type I minimal immersions so that $\partial\mathcal{M}_f \subset \mathbf{I}$. Thus, to describe $\mathbf{II}_1$ we consider line segments connecting pairs of points in $\mathbf{I}$ and use the connecting lemma to make sure that the points in the interior of the segment

correspond to type II quartic minimal immersions. Since **I** is a single orbit, we may assume that one endpoint of the segment is $\langle \mathcal{I} \rangle$. We now choose $g = a + bj \in SU(2)$ so that $\gamma g \gamma \in SU(2)'$ and let the other endpoint be $\langle \mathcal{I} \circ (\gamma g \gamma) \rangle$. By the connecting lemma, the space of components of any quartic minimal immersion corresponding to the interior of the segment connecting these two points is the $SU(2)$-module

$$V_{\mathcal{I}\circ(\gamma g\gamma)} + V_{\mathcal{I}}$$

and, assuming that the endpoints are distinct, the interior points correspond to type II or type III according as this is $3R_4$ or $4R_4$. To simplify the computations, we consider the quotient

$$(V_{\mathcal{I}\circ(\gamma g\gamma)} + V_{\mathcal{I}})/V_{\mathcal{I}} = V_{\mathcal{I}\circ(\gamma g\gamma)}/(V_{\mathcal{I}\circ(\gamma g\gamma)} \bigcap V_{\mathcal{I}}).$$

This quotient is trivial iff

$$\langle \mathcal{I} \circ (\gamma g \gamma) \rangle = \langle \mathcal{I} \rangle,$$

a task we also have to carry out to prove Theorem 6 since $\gamma g \gamma$ then belongs to the isotropy group of $SU(2)'$. The quotient is equal to R_4 or $2R_4$ according to whether we have type II or type III in the aforementioned segment. Technically speaking, we need to make the substitution $z \mapsto az + b\bar{w}$ and $w \mapsto -b\bar{z} + aw$ corresponding to $\gamma g \gamma$, $g = a + bj$, in each of the polynomials in

$$\begin{aligned} V_{\mathcal{I}} = \text{ span } \{ & z^4 - \bar{w}^4, z^2\bar{w}^2, z^3 w + \bar{z}\bar{w}^3, z\bar{z}^2 w - \bar{z}w^2\bar{w}, \\ & \Im(z^2 w^2), |z|^4 - 4|z|^2|w|^2 + |w|^4 \} \end{aligned}$$

and work out the components modulo $V_{\mathcal{I}}$. Elementary computations now give that $V_{\mathcal{I}\circ(\gamma g\gamma)}$ modulo $V_{\mathcal{I}}$ is spanned by the following polynomials:

$$\mu z^4 + 4\beta z^3 \bar{w} + 4\bar{\beta} z \bar{w}^3 \tag{24}$$

$$\nu z^4 - 2\alpha z^3 \bar{w} + 2\bar{\alpha} z \bar{w}^3 \tag{25}$$

$$\mu z^3 w - \beta z^3 \bar{z} + \bar{\beta} w \bar{w}^3 + 3\beta z^2 w \bar{w} - 3\bar{\beta} z \bar{z} \bar{w}^2 \tag{26}$$

$$-2\nu z^3 w - \alpha z^3 \bar{z} - \bar{\alpha} w \bar{w}^3 + 3\alpha z^2 w \bar{w} + 3\bar{\alpha} z \bar{z} \bar{w}^2 \tag{27}$$

$$\Im(\mu z^2 w^2) + 4\Im(\beta(z w^2 \bar{w} - z^2 \bar{z} w)) \tag{28}$$

$$\nu \Re(z^2 w^2) + 2\Re(\alpha(z^2 \bar{z} w - z w^2 \bar{w})), \tag{29}$$

where

$$\begin{aligned} \alpha &= a\bar{b}(|a|^2 - |b|^2) \\ \beta &= a^3 b + \bar{a}\bar{b}^3 \\ \mu &= a^4 - \bar{a}^4 + b^4 - \bar{b}^4 \\ \nu &= a^2\bar{b}^2 + \bar{a}^2 b^2. \end{aligned}$$

Lemma. (24) – (29) *are linearly dependent iff*

$$\Re(\alpha\bar{\beta}) = 0 \quad \text{and} \quad \alpha\mu + 2\beta\nu = 0. \tag{30}$$

Proof. We first observe that (24)-(25), (26)-(27) and (28)-(29) are mutually orthogonal. Thus, we need to study linear dependence of each pair of polynomials. The proposition now follows by case-by-case verification by splitting the first two pairs of polynomials into real and imaginary parts and evaluating each 4×4-subdeterminant of the corresponding 4×6-matrices. The last pair gives only 2×2-subdeterminants of a 2×4-matrix.

The remaining task is to work out (30) in terms of a and b. The first equation in (30) gives

$$(|a|^2 - |b|^2)\Re(a^2b^2) = 0. \tag{31}$$

It is convenient to use 'isoparametric' coordinates on S^3, that is to set

$$a = \cos t\, e^{i\theta} \quad \text{and} \quad b = \sin t\, e^{i\phi}. \tag{32}$$

$t = 0, \pi/2$ correspond to the two great orthogonal circles cut out from S^3 by the span of the first and last two coordinate axes; a fixed $0 < t < \pi/2$ corresponds to the Clifford torus T_t parametrized by θ and ϕ.

Case I Let $|a|^2 = |b|^2$. We are on the 'middle' Clifford torus $T_{\pi/4}$. We have $\alpha = 0$ so that, the second equation in (30) reduces to $\beta\nu = 0$.

If $\beta = 0$ then, substituting (32) into the expression of β we obtain

$$\phi = -\theta + (2k+1)\pi/4, \;\; k \in \mathbf{Z},$$

or equivalently,

$$a = (1/\sqrt{2})e^{i\theta}, \quad \text{and} \quad b = (1/\sqrt{2})e^{-i\theta}\epsilon^{2k+1}, \;\; k \in \mathbf{Z},$$

where $\epsilon = e^{i\pi/4}$.

If $\nu = 0$, we get

$$\phi = \theta + (2k+1)\pi/4, \;\; k \in \mathbf{Z},$$

so that

$$a = (1/\sqrt{2})e^{i\theta} \quad \text{and} \quad b = (1/\sqrt{2})e^{i\theta}\epsilon^{2k+1}, \;\; k \in \mathbf{Z}.$$

Summarizing Case I, the solution set is the union of 8 closed curves in $T_{\pi/4}$ and they lift to $[0, 2\pi]^2$ to give straight segments with slope ± 1 and θ- and ϕ-intercepts being any odd multiples of $\pi/4$.

Case II We assume that $t \neq \pi/4$. If $t = 0$ then

$$a = e^{i\theta} \quad \text{and} \quad b = 0.$$

We obtain the entire great circle T_0.

If $t = \pi/2$ then

$$a = 0 \quad \text{and} \quad b = e^{i\phi}$$

and the solution set is $T_{\pi/2}$.

Finally, let $0 < t < \pi/2$ and $t \neq \pi/4$. Working out each coefficient α, β, μ, ν and substituting them into the second equation in (30), we finally arrive at the solution set

$$a = \cos t\, \epsilon^l \quad \text{and} \quad b = \sin t\, \epsilon^l \epsilon^{2k+1}, \;\; k, l \in \mathbf{Z}. \tag{33}$$

For fixed t as above, this is the union of 32 points and on $[0, 2\pi]^2$ they correspond to the intersection points of the straight segments obtained above. As t moves, these points sweep 32 curves that, on $T_{\pi/4}$, meet the existing solution set in triple intersection points and on T_0 and $T_{\pi/2}$ they also produce 8 triple intersection points distributed equidistantly. Summarizing, the solution set consists of 48 closed curves meeting in 48 triple intersection points. Looking at each case separately, we see that the triple intersection points are given (as quaternions) by

$$(1/\sqrt{2})(\epsilon^k + \epsilon^l j), \;\; k \not\equiv l \,(\mathrm{mod}\, 2); \;\; \epsilon^k, \;\; \epsilon^l j, \;\; k, l \in \mathbf{Z}. \tag{34}$$

These form a group of order 48 and is conjugate in S^3 to the binary octahedral group $\mathbf{O}^*$. By abuse of notation, we denote this conjugate by the same symbol. We obtain that the orbit $\mathbf{I}$ is the 'octahedral manifold' $S^3/\mathbf{O}^*$ in $\mathcal{B}$ and the second part of Theorem 6 follows.

Looking now back at the 48 curves above, we see that on the quotient $\mathbf{I} = S^3/\mathbf{O}^*$ they give exactly 3 closed curves intersecting at $\langle \mathcal{I} \rangle$. After conjugation with γ, they are orbits of the (mutually orthogonal) 1-parameter subgroups corresponding to Z, $(1/\sqrt{2})(Y + X)$ and $(1/\sqrt{2})(Y - X)$ in $su(2)$, where

$$Z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

form the standard orthonormal basis in $su(2)$. Denote these orbits by σ, σ' and σ''. More explicitly, σ is parametrized by

$$\theta \mapsto \gamma e^{i\theta} \gamma \cdot \langle \mathcal{I} \rangle, \;\; \theta \in \mathbf{R}$$

(corresponding to $t = 0$ in Case II) and σ' (resp. σ'') are parametrized by (33) with $k = l = 0$ (resp. $k = 1$ and $l = 0$). Note that they intersect orthogonally at $\langle \mathcal{I} \rangle$.

We now take a closer look at σ. A quick check of Case II reveals that $V_{\mathcal{I} \circ (\gamma e^{i\theta} \gamma)}$ modulo $V_{\mathcal{I}}$ does not depend on θ. The same is true for

$$V_{\mathcal{I} \circ (\gamma e^{i\theta} \gamma)} + V_{\mathcal{I}}$$

so that the connecting lemma implies that σ is on the boundary of the relative moduli space corresponding to any interior point of any segment connecting two distinct points of σ. We choose the midpoint of the segment connecting $\langle \mathcal{I} \rangle$ and $e^{i\pi/8} \cdot \langle \mathcal{I} \rangle$ that has the type II representative $\mathcal{J} : S^3 \to S^{14}$ given by

$$\mathcal{J}(z,w) = (1/\sqrt{2})\big(z^4, w^4, 2\sqrt{3}z^2\bar{w}^2, 2z^3w, 2zw^3, \\ 2\sqrt{3}(z\bar{z}^2w - \bar{z}w^2\bar{w}), \sqrt{6}z^2w^2, |z|^4 - 4|z|^2|w|^2 + |w|^4\big).$$

Thus, we have $\sigma \subset \partial\mathcal{M}_{\mathcal{J}}$. Our immediate purpose is to show that $\sigma = \partial\mathcal{M}_{\mathcal{J}}$ holds. To prove this, we make a slight detour and, observing that $\mathcal{J}$ is $U(2)$ equivariant, we claim that the segment

$$\mathbf{R} \cdot \langle \mathcal{J} \rangle \bigcap (\mathcal{M}^4)^{SU(2)} \tag{35}$$

parametrizes all full quartic $U(2)$ equivariant minimal immersions $f : S^3 \to S_V$. Indeed, the $U(2)$ equivariant quartic minimal immersions are parametrized by the fixed point set $(\mathcal{M}^4)^{U(2)}$ so that the claim follows once we show that $(R_8')^{U(2)}$ is 1-dimensional. Since R_8' is $SU(2)$ fixed, we have $(R_8')^{U(2)} = (R_8')^{\Gamma}$, where

$$\Gamma = \{ \operatorname{diag}(e^{i\theta}, e^{i\theta}) \,|\, \theta \in \mathbf{R}\} \subset SU(2)' \tag{36}$$

is the center of $U(2)$. As noted above, $\gamma \in O(4)$ switches R_8' and R_8 and Γ to

$$\Gamma' = \{ \operatorname{diag}(e^{i\theta}, e^{-i\theta}) \,|\, \theta \in \mathbf{R}\}; \tag{37}$$

the standard (1-dimensional) maximal torus in $SU(2)$. Thus, $(R_8')^{\Gamma}$ corresponds to $(R_8)^{\Gamma'}$. On the other hand, Γ' acts on the standard basis in R_8 diagonally with a unique Γ'-fixed polynomial $-z^2w^2$ and the claim follows.

Remark. For p even, $W_p = \mathcal{H}_2^{p/2}$, where the $SU(2)$-module structure on the space of spherical harmonics on S^2 is given by the projection $SU(2) \to SO(3)$. Thus we also have $R_p = \mathcal{H}_2^{p/2}$ as real modules. The $SU(2)'$ orbit of $\langle J \rangle$ is $\mathbf{R}P^2$ embedded minimally into its respective 8-sphere as the image of the standard minimal immersion $f_2 : S^2 \to S^8$. Indeed, $(\mathcal{F}^4)^{SU(2)} = R_8' = \mathcal{H}_2^4$ and $(R_8')^{U(2)}$ corresponds to the zonals $(\mathcal{H}_2^4)^{SO(2)}$ whose $SO(3)$ orbit on the unit sphere gives the image of f_2.

We are now ready to prove that $\sigma = \partial\mathcal{M}_{\mathcal{J}}$. By the above, $\sigma \subset \partial\mathcal{M}_{\mathcal{J}}$ so that $\mathcal{M}_{\mathcal{J}}$ is at least 2-dimensional. Since $\langle \mathcal{J} \rangle$ is Γ-fixed, Γ leaves $\mathcal{M}_{\mathcal{J}}$ and its boundary invariant. Γ acts on $\partial\mathcal{M}_{\mathcal{J}}$ without fixed points since a fixed point is automatically $U(2)$ fixed and there are only two of these on the entire boundary. Thus, $\dim \mathcal{M}_{\mathcal{J}}$ must be even, therefore 2 or 4. Finally, it cannot be 4 since, in that case, $\partial\mathcal{M}_{\mathcal{J}}$

would be a topological S^3 (by convexity) and it would have to coincide with $\mathbf{I}$ (for reasons of dimension). The latter is $S^3/\mathbf{O}^*$ that is topologically distinct from S^3. We obtain that $\mathcal{M}_{\mathcal{J}}$ is a flat 2-disk $\mathcal{D}$ with center $\langle\mathcal{J}\rangle$, in particular, $\partial\mathcal{M}_{\mathcal{J}}$ is 1-dimensional, and thus it must coincide with σ.

The argument is entirely analogous for σ' and σ'' so that they are the boundary circles of 2-disks $\mathcal{D}'$ and $\mathcal{D}''$. Note that $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ are orthogonal to each other at the common boundary point $\langle\mathcal{I}\rangle$. We now let $SU(2)'$ act on this configuration and realize that $\mathcal{D}'$ and $\mathcal{D}''$ are on the $SU(2)'$ orbit of $\mathcal{D}$. We thus arrive at (21).

At this point, without having a detailed study of the type III quartic minimal immersions, we can only assert that $\mathbf{II}_0$ is of dimension at least 3 and postpone the proof of (20). More specifically, we claim now that the antipodal orbit

$$SU(2)' \cdot \langle\mathcal{I}_o\rangle$$

consists of type II linearly rigid quartic minimal immersions. To do this, we first determine the antipodal of $\mathcal{I}$. Recall that $\mathcal{I}$ has orthogonal components with the same norm so that the antipodal lemma applies. It immediately gives that the antipodal $\mathcal{I}_o$ is of type II. The $SU(2)'$-orbit through $\langle\mathcal{I}_o\rangle$, being the antipodal of $\mathbf{I}$, is again an octahedral manifold. It remains to show that $\mathcal{I}_o$ is linearly rigid. Assume that $\mathcal{M}_{\mathcal{I}_o}$ is nontrivial and consider a line segment through $\langle\mathcal{I}_o\rangle$ with endpoints $\langle f_1\rangle$ and $\langle f_2\rangle$ on $\partial\mathcal{M}_{\mathcal{I}_o}$. Clearly, f_1 and f_2 are of type I. Consequently, the antipodals f_1^o and f_2^o are of type II. Let $\langle f\rangle$ be the intersection of the segment connecting $\langle f_1^o\rangle$ and $\langle f_2^o\rangle$ with the line $\mathbf{R}\cdot\langle\mathcal{I}_o\rangle$. We claim that f is of type III which is a contradiction since, in this case, it should be congruent to $\mathcal{I}$ (the antipodal of $\mathcal{I}_o$) that is of type I. To prove the claim we use the connecting lemma and compute

$$\begin{aligned} V_f &= V_{f_1^o} + V_{f_2^o} \\ &= V_{f_1}^{\perp} + V_{f_2}^{\perp} \\ &= (V_{f_1} \bigcap V_{f_2})^{\perp}. \end{aligned}$$

On the other hand

$$\begin{aligned} \dim(V_{f_1} \bigcap V_{f_2}) &= \dim V_{f_1} + \dim V_{f_2} \\ &\quad - \dim(V_{f_1} + V_{f_2}) \\ &= 10 + 10 - \dim V_{\mathcal{I}_o} \\ &= 10 + 10 - (25 - 10) = 5. \end{aligned}$$

Theorem 7 follows (with the proof of (20) postponed).

We now consider type III quartic minimal immersions, and claim that the antipodal $\mathcal{J}_o$ of $\mathcal{J}$ is of type III. These are actually the two endpoints of the segment (35) parametrizing the $U(2)$ equivariant quartic minimal immersions. Recall that

$\langle \mathcal{J} \rangle$ is the midpoint of the segment connecting $\langle \mathcal{I} \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma) \rangle$ both of type I. Thus, the antipodal of $\langle \mathcal{J} \rangle$ must be on the segment connecting $\langle \mathcal{I}_o \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)_o \rangle$ provided that this segment is on the boundary $\mathcal{B}$. Thus, by the connecting lemma again, we need to work out

$$V_{\mathcal{I}_o} + V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)_o}.$$

By the antipodal lemma, this is equal to

$$V_{\mathcal{I}}^{\perp} + V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)}^{\perp} = (V_{\mathcal{I}} \bigcap V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)})^{\perp}.$$

On the other hand

$$\dim(V_{\mathcal{I}} \bigcap V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)}) = \dim V_{\mathcal{I}} + \dim V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)} - \dim(V_{\mathcal{I}} + V_{\mathcal{I}\circ(\gamma e^{i\pi/8}\gamma)})$$

and this is $10 + 10 - 15 = 5$-dimensional and the claim follows.

Lemma. *Let $f : S^3 \to S_V$ be a full $SU(2)$ equivariant quartic minimal immersion of type III. Then, we have*

$$\dim \mathcal{M}_f \geq 4.$$

Proof. As before, the Lie algebra $su(2)$ is considered as the tangent space of S^3 at the identity. For $U \in su(2)$, we denote by $\tilde{U}$, the right invariant extension of U on S^3. Given $C \in S^2V$, we define the linear map

$$\Psi(C) : su(2) \times su(2) \to \mathcal{P}^8,$$

by

$$\Psi(C)(U, U') = \langle C f_*(\tilde{U})\check{}, f_*(\tilde{U}')\check{} \rangle = \langle C\tilde{U}(f), \tilde{U}'(f) \rangle,$$

where $\mathcal{P}^q$ is the space of homogeneous polynomials of degree q on $\mathbf{R}^4$. Evaluating $\tilde{U}(f)$ on the basis Z, X, Y it follows easily that this function belongs to $\mathcal{P}^4$ so that $\Psi(C)$ maps into $\mathcal{P}^8$. Since $\Psi(C)$ is symmetric in the arguments U and U', it can be considered as a linear map

$$\Psi(C) : S^2(su(2)) \to \mathcal{P}^8,$$

or equivalently, an element $\Psi(C) \in \mathcal{P}^8 \otimes S^2(su(2))$. We now vary C in S^2V and obtain the linear map

$$\Psi : S^2V \to \mathcal{P}^8 \otimes S^2(su(2)).$$

Since the right invariant vector fields (pointwise) span each tangent space in S^3, we have

$$\ker \Psi = \mathcal{F}_f.$$

To estimate this kernel we first observe that Ψ is a homomorphism of $SU(2)$-modules, where the module structure on V is given by $SU(2)$ equivariance of f. Explicitly, for $g \in SU(2)$, we have

$$\Psi(g \cdot C)(Ad(g)(U), Ad(g)(U')) = \Psi(C)(U, U') \circ L_{g^{-1}}.$$

Since f is of type III, we have $V = 4R_4$ as $SU(2)$-modules. Thus

$$\begin{aligned} S^2V = S^2(4R_4) &= 4S^2(R_4) \oplus 6(R_4 \otimes R_4) \\ &= 10R_8 \oplus 6R_6 \oplus 10R_4 \oplus 6R_2 \oplus 10R_0 \end{aligned}$$

(cf. (9)). On the other hand,

$$\begin{aligned} \mathcal{P}^8 &= \mathcal{H}^8 \oplus \mathcal{H}^6 \oplus \mathcal{H}^4 \oplus \mathcal{H}^2 \oplus \mathcal{H}^0 \\ &= 9R_8 \oplus 7R_6 \oplus 5R_4 \oplus 3R_2 \oplus R_0, \end{aligned}$$

where the first row is isomorphism as $SO(4)$-modules, the second as $SU(2)$-modules. Finally, $su(2) = R_2$ so that

$$S^2(su(2)) = R_4 \oplus R_0.$$

Putting all these together, we obtain

$$\mathcal{P}^8 \otimes S^2(su(2)) = 9R_{12} \oplus 16R_{10} \oplus 30R_8 \oplus 31R_6 \oplus 30R_4 \oplus 18R_2 \oplus 6R_0.$$

Comparing this with the domain of Ψ we see that $4R_0$ must be in the kernel.

Let **III** denote the set of points that correspond to type III quartic minimal immersions. Recall also that $\mathcal{J}_o$ is the unique full $U(2)$ equivariant quartic boundary minimal immersion of type III.

Theorem 8. III *is everywhere dense, open and connected in the* 8*-dimensional boundary* $\mathcal{B}$. $\mathcal{M}_{\mathcal{J}_o} \subset \mathcal{B}$ *is* 6*-dimensional and*

$$\mathbf{III} = SU(2)' \cdot \operatorname{int} \mathcal{M}_{\mathcal{J}_o}.$$

Proof. First of all, by (18), the complement of **III** in $\mathcal{B}$ is of codimension 2 so that **III** is everywhere dense, open and connected in $\mathcal{B}$. We now claim that

$$\dim \mathcal{M}_{\mathcal{J}_o} = 6.$$

$U(2)$ leaves $\mathcal{M}_{\mathcal{J}_o}$ invariant since it fixes $\langle \mathcal{J}_o \rangle$. By local unicity of the $U(2)$ fixed points, the center Γ of $U(2)$ acts on $\partial \mathcal{M}_{\mathcal{J}_o}$ without fixed points. It follows that $\dim \mathcal{M}_{\mathcal{J}_o}$ is even. By the previous lemma, $\dim \mathcal{M}_{\mathcal{J}_o}$ is either 4 or 6. The claim will follow if we show that $\dim \mathcal{M}_{\mathcal{J}_o} \geq 6$.

Consider the $SU(2)$-module homomorphism $\Psi : S^2V \to \mathcal{P}^8 \otimes S^2(su(2))$ introduced in the previous lemma for $f = \mathcal{J}_o : S^3 \to S_V$. Since $\mathcal{J}_o$ is $U(2)$ equivariant, Ψ is a homomorphism of $U(2)$-modules, where the $U(2)$-module structure on V is given by the equivariance of $\mathcal{J}_o$, and the center $\Gamma \subset U(2)$ acts on $su(2)$ trivially. Being a $U(2)$-submodule of $\mathcal{H}^4$, V (complexified) decomposes according to (10). We have

$$V \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{H}^{4,0} \oplus \mathcal{H}^{3,1} \oplus \mathcal{H}^{1,3} \oplus \mathcal{H}^{0,4}. \tag{38}$$

Indeed, since $\dim V = 20$, we need only to show that $\mathcal{H}^{2,2}$ is not a component of V. This, however, follows from the antipodal and connecting lemmas. Indeed, $\mathcal{I}$ and $\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)$ do have components in $\mathcal{H}^{2,2}$, and so does $\mathcal{J}$ since $\langle \mathcal{J} \rangle$ is the midpoint of the segment connecting $\langle \mathcal{I} \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma) \rangle$. Since $\mathcal{I}$ has orthogonal components with the same norm the antipodal lemma applies, and we see that $\mathcal{I}_o$ and $\mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)_o$ do not have components in $\mathcal{H}^{2,2}$. The same holds for $\mathcal{J}_o$ as $\langle \mathcal{J}_o \rangle$ is the midpoint of the segment connecting $\langle \mathcal{I}_o \rangle$ and $\langle \mathcal{I} \circ (\gamma e^{i\pi/8}\gamma)_o \rangle$.

As in the previous lemma, we now count the trivial components in the $U(2)$-modules that contribute to the domain and range of the (complexified) Ψ. Since Γ acts on these components as a character, we will also keep track of the corresponding weights. $S^2(V)$ contains 6 trivial components with zero weight; 4 coming from the symmetric squares of the components in (38) (e.g. $S^2(\mathcal{H}^{3,1})$ is contained in $\mathcal{H}^{3,1} \otimes \mathcal{H}^{1,3}$ that has weight $(3-1)+(1-3) = 0$), and 2 coming from $\mathcal{H}^{4,0} \otimes \mathcal{H}^{0,4}$ and $\mathcal{H}^{3,1} \otimes \mathcal{H}^{1,3}$. In addition, $S^2(V)$ contains one trivial component for each weight $6, 2, -2, -6$, (e.g. $\mathcal{H}^{4,0} \otimes \mathcal{H}^{3,1}$ has weight $(4-0)+(3-1) = 6$). As $U(2)$-modules, $S^2(su(2)) = \mathcal{H}^{2,2} \oplus \mathcal{H}^0$ since Γ acts on $su(2)$ trivially. All the trivial components in $\mathcal{P}^8 \otimes S^2(su(2))$ are in $\mathcal{H}^4 \otimes (\mathcal{H}^{2,2} \oplus \mathcal{H}^0)$, and, by (10), we have 2 trivial components with zero weight, and one trivial component for each weight $4, 2, -2, -4$. Comparing, we see that the kernel of Ψ must contain at least 4 trivial components with zero weight, and two trivial components with weights ± 6. We see that $\ker \Psi$ is at least 6-dimensional. The claim follows.

By the transversality lemma, the $SU(2)'$ orbit of $\operatorname{int} \mathcal{M}_{\mathcal{J}_o}$ is an 8-dimensional smooth manifold since Γ leaves $\mathcal{M}_{\mathcal{J}_o}$ invariant, and $SU(2)'$ does not have 2-dimensional subgroups. This orbit must then be open in $\mathcal{B}$. Its boundary is contained in $\mathbf{I} \cup \mathbf{II}$. It follows that $SU(2)' \cdot \operatorname{int} \mathcal{M}_{\mathcal{J}_o}$ is a connected component of $\mathbf{III}$ (and its boundary is equal to $\mathbf{I} \cup \mathbf{II}$). Since $\mathbf{III}$ is connected, Theorem 8 follows. As a byproduct, we obtain that $\mathbf{II}$ and $\mathbf{II}_0$ are connected (and 6-dimensional) since $\mathbf{I} \subset \mathbf{II}$ and $\mathbf{II}_1 \subset \mathbf{II}$ are of codimension ≥ 2.

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