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On nonpositively curved Euclidean submanifolds: splitting results

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Abstract. In this article, we prove that a n-dimensional, non-positively curved Euclidean submanifold with codimension p and with minimal index of relative nullity $\nu = n-2p$ is (in an open dense subset) locally the product of p hypersurfaces.

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Let $f : M^n \to \mathbb{Q}_c^{n+p}$ be an isometric immersion from a Riemannian manifold into a complete simply connected Riemannian manifold of constant sectional curvature c (superscripts will always denote dimensions). Denote by ν the *index of relative nullity* of f,

$$\nu(x) = \dim\{X \in T_xM : \alpha_f(X,Y) = 0, \forall\, Y \in T_xM\},$$

where α_f stands for the vector valued second fundamental form of f. It is well known that having $\nu > 0$ imposes strong restrictions on the manifold M^n and on its isometric immersion f. In [F1], the first author proved the inequality $\nu \geq n-2p$ when the sectional curvature of M^n satisfies $K_M \leq c$ and gave several applications of this result. First let us show that this estimate is sharp.

Example. For each $i = 1, \dots, p$, let $S_i \subseteq \mathbb{R}^3$ be a negatively curved surface. Then the product $M^{2p} = S_1 \times \cdots \times S_p \subseteq \mathbb{R}^{3p}$ satisfies the equality $\nu = n-2p = 0$.

More generically, let $M_i^{n_i} \subseteq \mathbb{R}^{n_i+1}$ be nowhere flat nonpositively curved hypersurfaces, $i = 1, \dots, p$. The Gauss equation tells us that the relative nullity ν_i of $M_i^{n_i}$ is $\nu_i = n_i - 2$. Then, the product manifold $M^n = M_1^{n_1} \times \cdots \times M_p^{n_p} \subseteq \mathbb{R}^{n+p}$ also have $\nu = n-2p$.

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The first author proved in [F2] a general splitting theorem for Euclidean submanifolds f of nonpositive sectional curvature, under the additional assumption that the normal bundle of f is flat. The main purpose of this paper is to drop that assumption in the borderline case $\nu = n - 2p$ to prove that the above example is essentially the unique one with minimal relative nullity index.

Theorem 1. *Let $f : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion into Euclidean space of a Riemannian manifold with nonpositive sectional curvature. Assume that $\nu = n - 2p$ everywhere. Then there exists an open dense subset $\mathcal{U} \subset M^n$ such that $f|_{\mathcal{U}}$ splits locally as a product of p Euclidean hypersurfaces, that is, for any $x \in \mathcal{U}$, there exist a neighborhood $x \in \mathcal{V} \subseteq \mathcal{U}$ and p nowhere flat Euclidean hypersurfaces $f_i : M_i^{n_i} \to \mathbb{R}^{n_i+1}$ of nonpositive sectional curvature, such that*

$$\mathcal{V} = M_1 \times \cdots \times M_p \qquad \textit{and} \qquad f|_{\mathcal{V}} = f_1 \times \cdots \times f_p$$

split.

First of all, note that when f is analytic, the splitting occurs on the entire M. In the general case, each n_i is constant in a connected components of $\mathcal{U}$, in fact, the universal covering space of any component of of $\mathcal{U}$ is the product of p Euclidean hypersurfaces. However, there are examples in which the n_i's are not constant in the entire $\mathcal{U}$. Secondly, it is interesting to observe that, from Theorem 1 of [M] we have that $f|_{\mathcal{V}}$ in the above is isometrically rigid if and only if each factor is rigid.

Corollary 2. *Let $f : M^n \to \mathbb{Q}_c^{n+p}$, $2p \leq n$, be an isometric immersion of a connected Riemannian manifold M^n with $K_M \leq c$ and Ricci curvature $Ric_M < c$. Then $c = 0$, $n = 2p$ and f splits locally as a product of p negatively curved surfaces of $\mathbb{R}^3$. Moreover, the splitting is global provided that M^n is a Hadamard manifold.*

The assumption on the Ricci curvature in the above can be replaced by the weaker one $\nu = 0$. Also, the Hadamard condition can probably be relaxed a bit. Combining our results and [Z], we can state the complex analogue of the above:

Theorem 3. *Let X^n be an immersed complex submanifold of $\mathbb{C}\mathbb{Q}_c^{n+p}$, the complex space form of constant holomorphic sectional curvature c. Assume that X^n has nonpositive extrinsic sectional curvature. Then the index of relative nullity of X^n satisfies $\nu \geq n - p$ and:*

(1) when $\nu = n - p = 0$, we must have $c = 0$;

(2) when $c = 0$ and $\nu = n - p$, X^n is locally holomorphically isometric to a product

$$\mathbb{C}^k \times X^{n_1} \times \cdots \times X^{n_p} \subseteq \mathbf{X}^{n+p}, \qquad n = k + \sum_{i=1}^{p} n_i \,,$$

for some $0 \leq k \leq \nu$, where each $X^{n_i} \subseteq \mathbb{C}^{n_i+1}$ is a nowhere flat nonpositively curved hypersurface.

Moreover, if X^n is complete, then its universal covering is holomorphically isometric to the product $\mathbb{C}^{\nu} \times \Sigma_1 \times \cdots \times \Sigma_p$, where each $\Sigma_i \hookrightarrow \mathbb{C}^2$ is a complete immersion of the unit disc. All dimensions here are the complex ones.

Notice that the real analyticity of X^n prevented k from jumping around. The last part of Theorem 3 is because, by a theorem of Abe in [A], any complete immersed complex submanifold of $\mathbb{C}^m$ with one dimensional Gauss image must be a cylinder.

Remark. Any Euclidean hypersurface $g : H^m \to \mathbb{R}^{m+1}$ of nonpositive sectional curvature without flat points can be described locally by means of the Gauss parametrization in the following way (see [DG] for details). Take a surface $\xi : V^2 \to \mathbb{S}^m$ in the Euclidean unit sphere and a smooth function γ on V^2. The map $\Psi : T_{\xi}^{\perp} V \to \mathbb{R}^{m+1}$ given by

$$\Psi(v) = \gamma\xi + \text{grad }\gamma + v$$

parametrizes g over the normal bundle of ξ, in the open set of normal vectors v which satisfies $\det(\gamma \text{Id} + \text{Hess}_{\gamma} - B_v) < 0$. Here, B_v denotes the second fundamental operator of ξ in the direction v. In this parametrization, ξ is the Gauss map of g and $\gamma = \langle g, \xi \rangle$ its support function. For a discussion on the isometric deformations of those hypersurfaces see [DFT]. Observe that any isometric immersion f as in Theorem 1 can now be explicitly parametrized locally along $\mathcal{U}$ using the Gauss parametrization for each factor.

The flatness of the normal bundle

Let $\alpha : V^n \times V^n \to W^p$ be a symmetric bilinear map, where V and W are real vector spaces of dimension n and p, respectively, and W is equipped with an inner product $\langle\, , \rangle$. Assume α is *nonpositive* as defined in [F1], i.e.,

$$K_{\alpha}(X, Y) = \langle \alpha(X, X), \alpha(Y, Y) \rangle - \| \alpha(X, Y) \|^2 \leq 0,$$

for all X, $Y \in V$. Denote by ν the dimension of the null space N of α:

$$N = \{X \in V \ \mid \ \alpha(X, Y) = 0, \ \forall \ Y \in V\}.$$

Recall that a subspace $T \subseteq V$ is said to be *asymptotic*, if $\alpha(X, Y) = 0$ for all $X, Y \in T$. We know from [F1] that, for the above α, $\nu \geq n - 2p$. The main technical part of this article is the following diagonalization result for the borderline case $\nu = n - 2p$.

Proposition 4. *Let $\alpha : V^n \times V^n \to W^p$ be a symmetric, nonpositive bilinear map. If $\nu = n - 2p$, then there exist a basis $\{e_1, \ldots, e_n\}$ of V and an orthonormal* *basis* $\{w_1, \dots, w_p\}$ *of* W *such that* $\{e_{2p+1}, \dots, e_n\}$ *is a basis of the null space* N, *and for each* $i, j \leq 2p$,

$$\alpha(e_i, e_j) = \delta_{ij}(-1)^i w_{[\frac{i+1}{2}]}\,.$$

*Proof.*We will carry out the induction on p. When $p = 1$, α is just a symmetric bilinear form, so it can always be diagonalized. The nonpositivity condition will force the rank of α to be less or equal than 2, and when it equals 2, the two nonzero eigenvalues must be of opposite sign. Now assume that the result holds when dim $W < p$, and consider the case dim $W = p$.

By restricting α to a subspace $\widetilde{V}^{2p}$ such that $V = N \oplus \widetilde{V}$, we may assume that $n = 2p$ and $\nu = 0$. Denote by α_X the endomorphism $\alpha_X(Y) = \alpha(X, Y)$. By Proposition 6 of [F1] we know that there exists an asymptotic subspace $T^p \subseteq \widetilde{V}^{2p}$ of α. Set

$$r = \min\{\text{rank } \alpha_X : 0 \neq X \in T\} > 0.$$

Fix a vector $X \in T$ with rank $\alpha_X = r$ and let $V' = Ker(\alpha_X) \supseteq T$. Thus, by the first claim in the proof of Proposition 6 of [F1], we know that the image $\alpha(V' \times V')$ is perpendicular to the image subspace $\text{Im}(\alpha_X)$, that is, we have the restriction map

$$\alpha\,|_{V' \times V'} \colon V' \times V' \to \text{Im}(\alpha_X)^{\perp}.$$

Let N' be its null space. If there is $Y \in N' \setminus T$, then span$(T \cup \{Y\})$ would be an asymptotic subspace of α of dimension $p + 1$. By Proposition 8 of [F1], we get $\nu \geq 1$, a contradiction to our assumption. Therefore, $N' \subseteq T$.

For each $Y \in N' \subseteq T$, we have $\text{Ker}(\alpha_Y) \supseteq V' = \text{Ker}(\alpha_X)$, so rank $\alpha_Y = r$. Therefore,

$$V' = \text{Ker}(\alpha_Y), \quad \forall\, 0 \neq Y \in N'. \tag{1}$$

Put $W_0 = \text{span}\{\text{Im}(\alpha_Y) : Y \in N'\}$ which has dimension $r + s$, for some $s \geq 0$. Again from the proof of Proposition 6 of [F1], we know that $\alpha(V' \times V')$ is perpendicular to W_0, that is,

$$\beta = \alpha\,|_{V' \times V'} \colon V' \times V' \to W_0^{\perp}$$

is itself a symmetric, nonpositive bilinear map, with dim $V' = 2p - r$, dim $W_0^{\perp} = p - r - s$. Write $q = \dim N'$. Then by Proposition 9 of [F1] we have

$$q \geq (2p - r) - 2(p - r - s) = r + 2s. \tag{2}$$

On the other hand, if $\{Y_1, \dots, Y_q\}$ is a basis of N' and $Z \in V \setminus V'$, from (1) we obtain that the set of vectors $\{\alpha(Y_1, Z), \cdots, \alpha(Y_q, Z)\}$ in W_0 must be linearly independent. Thus

$$q \leq r + s. \tag{3}$$

We conclude from (2) and (3) that $s = 0$ and $q = r$. So we can apply the induction hypothesis on β. However, we want to show first that $r = 1$.

Assume the contrary, that is, $q > 1$. Take a subspace V_1^r such that $V_1 \oplus V' = V$. Choose any $Y \in N'$ not collinear with X. Since $s = 0$, (the restriction of) both α_X and α_Y give isomorphisms between V_1 and $W_0^\perp$. Fix an orthonormal basis $\{w_1, \dots, w_r\}$ of $W_0^\perp$. Let $\{v_1, \dots, v_r\}$ be the basis of V_1 such that $\alpha_X(v_i) = w_i$ and write $\alpha_Y(v_i) = \sum_{j=1}^r B_{ij} w_j$. That is, we identify V_1 and $W_0^\perp$ through α_X, and use the matrix B to represent α_Y.

If B has a real eigenvalue λ, then $\alpha_{Y-\lambda X}$ would have rank less than r, which contradicts (1). So the matrix B has no real eigenvalues. By considering a complex eigenvector which corresponds to a complex eigenvalue of B, we obtain two 2-planes $P \subseteq V_1$, $Q \subseteq W_0^\perp$, such that both α_X and α_Y give isomorphisms between P and Q.

Now let us fix an orthonormal basis $\{w_1, w_2\}$ of Q, and let $\{e_3, e_4\}$ be the basis of P such that $\alpha_X(e_3) = w_1$, $\alpha_X(e_4) = w_2$. Write

$$\alpha_Y(e_3) = aw_1 + bw_2, \quad \alpha_Y(e_4) = cw_1 + dw_2.$$

Replacing Y by $Y - dX$, we may assume that

$$d = 0.$$

We know that the 2×2 real matrix with entries $a, b, c, 0$ can not have any real eigenvalue, or equivalently,

$$4bc + a^2 < 0.$$

Set $e_1 = X$, $e_2 = Y$. For arbitrary real constants x and y, let us consider the vectors $Z = xe_1 + xye_2 + xe_3 - e_4$ and $Z' = ye_2 + e_3$. We have

$$Z \wedge Z' = xye_1 \wedge e_2 + xe_1 \wedge e_3 + ye_2 \wedge e_4 + e_3 \wedge e_4.$$

Define the symmetric bilinear form R on $\Lambda^2 V$, the curvature of α, as

$$R(Z_1 \wedge Z_2, Z_3 \wedge Z_4) = \langle \alpha(Z_1, Z_3), \alpha(Z_2, Z_4) \rangle - \langle \alpha(Z_1, Z_4), \alpha(Z_2, Z_3) \rangle. \tag{4}$$

Hence, the matrix of R under the partial basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ is

$$R = \begin{bmatrix} 0 & 0 & 0 & c-b \\ 0 & -1 & -b & -f \\ 0 & -b & -c^2 & -g \\ c-b & -f & -g & -h \end{bmatrix}.$$

Therefore $-R(Z \wedge Z', Z \wedge Z') = x^2 + c^2y^2 + h + 2(2b - c)xy + 2fx + 2gy$. Thus, the nonpositivity of α gives us

$$c^2y^2 + 2((2b - c)x + g)y + (x^2 + 2fx + h) \geq 0.$$

Hence, the discriminant with respect to y must be nonpositive, that is,

$$0 \leq c^2(x^2+2fx+h)-((2b-c)x+g))^2 = (4bc-4b^2)x^2+2(c^2f+cg-2bg)x+(c^2h-g^2).$$

Since $a^2+4bc<0$, the leading coefficient is negative, which is a contradiction for x sufficiently large. This completes the proof of the claim that $q=r=1$.

Now applying the induction hypothesis on the restriction map β, we obtain an orthonormal basis $\{w_1,\ldots,w_p\}$ of W and a basis $\{e_1',e_2,e_2',\ldots,e_p,e_p'\}$ of $V'=Ker(\alpha_X)$ such that $X=e_1'$, $\mathrm{Im}(\alpha_X)=\mathrm{span}\{w_1\}$,

$$\alpha(e_i,e_j)=\delta_{ij}w_i,\quad \alpha(e_i',e_j')=-\delta_{ij}w_i,\quad \alpha(e_i,e_j')=0,\quad \forall\ 2\leq i,j\leq p,$$

and of course $\alpha(e_1',e_1')=\alpha(e_1',e_i)=\alpha(e_1',e_i')=0$, for all $2\leq i\leq p$.

Choose a vector $e_1\in V\backslash V'$ such that $\alpha(e_1,e_1')=w_1$. Write $\alpha=(A^1,\ldots,A^p)$, where each $A^k_{ab}=\langle\alpha(e_a,e_b),w_k\rangle$ is a symmetric $2p\times 2p$ matrix. Here for convenience we adopt the notations $e_i'=e_{p+i}$ and $i'=i+p$, for $i\leq p$. Under the basis $\{e_a\wedge e_b;\ 1\leq a<b\leq 2p\}$ of Λ^2V, the coordinate matrix of the bilinear form R becomes

$$R_{ab,cd}=\sum_{k=1}^{p}(A^k_{ac}A^k_{bd}-A^k_{ad}A^k_{bc}).$$

The nonpositivity of α simply says that $R(Z_1\wedge Z_2,Z_1\wedge Z_2)\leq 0$. For any three vectors Z_i, $i=1,2,3$, by considering the nonpositivity at $Z_1\wedge(Z_2+xZ_3)$ for arbitrary x, we have

$$R(Z_1\wedge Z_2,Z_1\wedge Z_2)\cdot R(Z_1\wedge Z_3,Z_1\wedge Z_3)\geq (R(Z_1\wedge Z_2,Z_1\wedge Z_3))^2. \tag{5}$$

For all $2\leq i\leq p$ and $2\leq a\neq i,i'$, from the above and $R_{ia,ia}=0$ we have $R_{1i,ia}=-A^i_{1a}=0$. That is, $A^i_{1j}=A^i_{1j'}=0$, for all $2\leq i\neq j\leq p$. Replacing e_1 by $e_1-\sum_{i=2}^p(A^i_{1i}e_i-A^i_{1i'}e_i')$, we may assume that

$$A^i_{1j}\equiv 0,\quad \forall\ i,j\geq 2. \tag{6}$$

For $2\leq i\leq p$, set

$$b_i=A^i_{11},\ a_i=A^1_{1i},\ c_i=A^1_{1i'}.$$

Thus,

$$R_{11',11'}=-1,$$

$$R_{1i,1i}=b_i-a_i^2,\quad R_{11',1i}=-a_i,$$

$$R_{1i',1i'}=b_i-c_i^2,\quad R_{11',1i'}=-c_i,$$

since $A^1_{11'}=1$. From (5) and $R_{11',11'}R_{1i,1i}\geq(R_{11',1i})^2$ we get $b_i\leq 0$. Similarly, replacing i by i', we have $b_i\geq 0$. Therefore, all $b_i=0$.

Now we take any nonsingular 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} A_{11}^1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and set

$$\tilde{e}_1 = ae_1 + ce_1', \quad \tilde{e}_1' = be_1 + de_1', \quad \tilde{e}_i = e_i - a_i e_1', \quad \tilde{e}_i' = e_i' - c_i e_1', \quad 2 \le i \le p.$$

Then under the new basis $\{\tilde{e}_a\}$ of V, we have $\alpha(\tilde{e}_a, \tilde{e}_b) = 0$, if $a \ne b, b'$, and

$$\alpha(\tilde{e}_i, \tilde{e}_i) = w_i, \quad \alpha(\tilde{e}_i', \tilde{e}_i') = -w_i \ , \quad \forall\, 1 \le i \le p.$$

This completes the proof of Proposition 4. □

Let us examine the diagonalizing frame $\{w_i\}$ of Proposition 4. Set

$$\mathcal{D} = \{X \in V \,:\, \operatorname{rank}(\alpha_X) \le 1\}.$$

This set of course depends only on α. By Proposition 4, we know that $\mathcal{D}$ is the union of p subspaces of dimension $\nu + 2$, denoted by $\mathcal{D}_i$, $i = 1, \dots, p$, with $\mathcal{D}_i \cap \mathcal{D}_j = N$ for all $i \ne j$. If we choose a plane $V_i \subseteq \mathcal{D}_i$ which has trivial intersection with N, then V is the direct sum

$$V = N \oplus V_1 \oplus \cdots \oplus V_p$$

and $\alpha(\mathcal{D}_i \times \mathcal{D}_j) = 0$ if $i \ne j$, while all $\alpha(\mathcal{D}_i \times \mathcal{D}_i)$ are one dimensional and mutually perpendicular. So the orthonormal frame $\{w_i\}$ is uniquely determined up to permutations.

It is interesting to note that $K \le 0$ does not implies in general that the symmetric curvature operator R is negative semidefinite. However, it is easy to see using Proposition 4 that, in our case, we really have $R \le 0$. In fact, $\{e_i \wedge e_{i+p} : 1 \le i \le p\}$ is a basis of the orthogonal complement F of the nullity space of R in $\Lambda^2 V$ formed by the unique (up to scaling) decomposable elements in F. Indeed, $e_i \wedge e_{i+p}$ is eigenvector of R of eigenvalue $K(e_i, e_{i+p}) \ne 0$.

We are now in position to give the remaining proofs.

Proofs of Theorem 1 and Corollary 2. For each $x \in M^n$, consider $\alpha_f(x)$ the vector valued second fundamental form of f at x. Since $K_M \le 0$, the Gauss equation tells us that $\alpha_f(x)$ is nonpositive. Thus, we apply Proposition 4 to it to obtain the special (smooth) orthonormal frame $\{w_i, 1 \le i \le p\}$. By Theorem 1 and Corollary 2 of [F2], we only need to prove that the normal bundle of f is flat. We will show indeed that this frame is normal parallel.

For each $1 \le i \le p$, consider the shape tensor A_{w_i} on M^n defined by $\langle A_{w_i} X, Y\rangle = \langle \alpha_f(X,Y), w_i\rangle$. By Proposition 4, $V_i = \mathrm{Im}\, A_{w_i}$ are two dimensional distributions on M^n such that

$$V_1 \oplus \cdots \oplus V_p = \Delta^\perp, \tag{7}$$

where Δ stands for the relative nullity distribution of f. Let ψ_{ij} be the 1–forms defined by $\psi_{ij}(X) = \langle \nabla^\perp_X w_i, w_j\rangle$. We only need to show that $\psi_{ij} = 0$, for all i, j.

Recall that the Codazzi equation for A_{w_i} is

$$\nabla_X(A_{w_i}Y) - A_{w_i}\nabla_X Y - A_{\nabla^\perp_X w_i} Y = \nabla_Y(A_{w_i}X) - A_{w_i}\nabla_Y X - A_{\nabla^\perp_Y w_i} X. \tag{8}$$

Taking in (8) $X, Y \in V_i^\perp = \mathrm{Ker}\ A_{w_i}$ we easily obtain using (7) that

$$A_{w_j}(\psi_{ij}(X)Y - \psi_{ij}(Y)X) = 0, \ \ \forall\ X, Y \in V_i^\perp, \ \ 1 \le j \le p.$$

Suppose that there is $X_0 \in V_i^\perp$, and $j \ne i$ such that $\psi_{ij}(X_0) \ne 0$. The above equation implies that $V_i^\perp \subset V_j^\perp \oplus$ span $\{X_0\}$, that is,

$$T_x M \ne V_i^\perp + V_j^\perp = (V_i \cap V_j)^\perp,$$

which is a contradiction by (7). Thus $V_i^\perp \subset \mathrm{Ker}\ \psi_{ij}$, for all i, j. By the orthonormality of $\{w_i\}$ we have $\psi_{ij} = -\psi_{ji}$. Therefore, $T_x M = V_i^\perp + V_j^\perp \subset \mathrm{Ker}\ \psi_{ij}$. Notice that the Ricci equations imply that the V_i's are orthogonal. This concludes our proof. □

The proof of Theorem 3 can be obtained by combining the diagonalization theorem of [Z] (together with the similar argument of the orthogonality of the special frame) and the proof of the Theorem 1 of [F2]. So we shall omit it here.

Final comments

i) Let us explain Theorem 1 a little bit. We have everywhere on M^n the orthogonal decomposition $TM = N \oplus V_1 \oplus \cdots \oplus V_p$ of the tangent bundle into distributions. Let $\widetilde{V}_i$ be the distribution spanned by all vector fields in V_i and all $\nabla_{X_1} \cdots \nabla_{X_s} X_{s+1}$, where all $X_j \in V_i$. It is shown in [F2] that $\widetilde{V}_i \perp \widetilde{V}_j$ whenever $i \ne j$, and all $\widetilde{V}_i$ are parallel distributions (in the neighborhood where they have constant dimensions). Let $n_i(x)$ be the dimension of $\widetilde{V}_i$ at x. Each n_i is a lower semicontinuous integer-valued function. If $k = n - \sum_{i=1}^p n_i$, then $0 \le k \le \nu$. Let $\mathcal{U}$ be the open dense subset of M^n which is the disjoint union of open subsets $\mathcal{U}_j$ in which $k(x)$ takes constant value j. All n_i are necessarily constant in $\mathcal{U}_j$, and we have the desired local splitting on $\mathcal{U}_j$. Observe that, using the Gauss parametrization, it is easy to construct examples of submanifolds with the functions n_i nonconstant. Therefore, for $\nu > 0$ we can only obtain the local splitting along an open dense subset. With this is mind, the same argument as in Corollary 2 of [F2] proves the following

Theorem 5. *Let* $f : M^n \to \mathbb{Q}_c^{2n-r}$, $2 \le r \le n/2$, *be an isometric immersion with flat normal bundle of a connected Riemannian manifold with* $K_M \le c$ *and* $Ric_M < c$. *Then* $c = 0$ *and* f *splits locally as a product of* r *nonpositively curved Euclidean submanifolds, that is,* $f = f_1 \times \cdots \times f_r$ *locally, with* $f_i : M_i^{n_i} \to \mathbb{R}^{2n_i - 1}$. *The splitting is global provided* M^n *is a Hadamard manifold.*

Again, the assumption on the Ricci curvature can be replaced by $\nu = 0$.

ii) We believe that the case $\nu = n - 2p > 0$ for an isometric immersion $f : M^n \to \mathbb{Q}_c^{n+p}$, with $c \neq 0$, cannot occur. It would be interesting either to prove its nonexistence or to construct such an example. The complex case should be similar.

iii) Taking the curvature tensor R as a 4–tensor on M^n, it is defined the *nullity space* of M^n at x as the subspace $\Gamma(x) = \{X \in T_xM : R(X, Y, Z, W) = 0,\ \forall\, Y, Z, W \in T_xM\}$. This is an intrinsic subspace, so its dimension $\mu(x)$ called the *nullity index* of M^n is an intrinsic function. For an isometric immersion f of M^n into Euclidean space we always have that the relative nullity distribution Δ of f satisfies $\Delta \subset \Gamma$. Thus, our assumption on the relative nullity distribution in Theorem 1 can be replaced by the intrinsic one $\mu = n - 2p$. The same holds for Corollary 2.

iv) Now let us consider the more general situation discussed in Theorem 1 of [F2], namely, $\nu = n - p - r$, for some $2 \le r \le p$. It is natural to ask if it can be generalized by dropping the flatness of the normal bundle assumption as we did for the case $r = p$. The answer to this question seems to be negative, since the algebraic decomposition Proposition 4 does not generalizes, even for the case $r = p - 1$, as the following example shows. Take A_i defined as

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The bilinear form $\alpha = (A_1, A_2, A_3) : \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}^3$ is nonpositive, has $\nu = n - p - r = 0$ for $r = p - 1 = 2$ but is not decomposable. It is easy to generalize this example for all p. Thus the analogous result to Proposition 4 is false for $\nu = n - p - r$ and $2 \le r \le p - 1$.

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