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Platonic surfaces

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Abstract. If S_O is a Riemann surface with a complete metric of finite area and constant curvature -1, let S_C denote the conformal compactification of S_O. We show that, under the assumption that the cusps of S_O are large, there is a close relationship between the hyperbolic metrics on S_O and S_C. We use this relationship to show that $\liminf_{k\to\infty} \lambda_1(P_k) \geq 5/36$, where the Platonic surface P_k is the conformal compactification of the modular surface S_k.

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Let $\Gamma = PSL(2,\mathbb{Z})$ be the group of linear fractional transformations

$$z \to \frac{az+b}{cz+d}$$

with integer coefficients with determinant 1, and let $\Gamma(k)$ denote the kth congruence subgroup

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } k) \right\}.$$

$\Gamma(k)$ then acts on the upper half plane H^2 with quotient a hyperbolic surface S_k of finite area. According to a theorem of Selberg, we have:

Theorem 0.1. ([Se]) *The first eigenvalue $\lambda_1(S_k)$ of the Laplacian acting on S_k satisfies:*

$$\lambda_1(S_k) \geq 3/16.$$

In this paper, we will consider a family of compact surfaces P_k, which we call the Platonic surfaces. They may be described conformally as being obtained

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from S_k by "filling in" the punctures of S_k. For $k = 3, 4$, and 5, the surfaces P_k correspond to the Riemann sphere with a tesselation by regular spherical k-gons. For $k > 6$, the surfaces P_k carry a similar hyperbolic tesselation, and are thus natural generalizations to hyperbolic geometry of the classical Platonic solids. See [BFK] and [SGCC] for some alternate descriptions of these surfaces in terms of graph theory.

In this paper, we will show:

Theorem 0.2. *The first eigenvalue of the Laplacian $\lambda_1(P_k)$ satisfies:*

$$\lim_{k\to\infty} \inf \lambda_1(P_k) \geq 5/36.$$

The number 5/36 arises already in the work of Huxley [Hu] and Sarnak-Xue [SX] in their geometric approach to the Selberg 3/16 Theorem, see also [TFSG]. Indeed, we will prove Theorem 0.2 by showing that the surfaces P_k are sufficiently similar to the surfaces S_k for the Huxley-Sarnak-Xue argument to apply to them as well.

More generally, we will consider the following situation: Let S_O be a Riemann surface with a complete finite-area metric of constant curvature -1. Then there is a unique compact Riemann surface S_C and finitely many points $\{p_1, \dots, p_k\}$, such that S_O is conformally equivalent to $S_C - \{p_1, \dots, p_k\}$.

A natural question is to relate the hyperbolic geometry of S_O with the hyperbolic geometry of S_C. This would seem at first glance to be problematic, since S_C need not in general carry a hyperbolic metric. Even if it does carry such a metric, S_O and S_C will still have some striking differences — for instance, S_O will be non-compact while S_C will be compact.

Nonetheless, our main technical result in §2 below will show that, in the case where all the cusps of S_O are large in a sense to be defined in §2 below, there is a close relationship between the hyperbolic metrics on S_O and S_C (and, in particular, S_C carries such a metric). Namely, there are neighborhoods $\{B_{l_i}(C_i)\}$ of the cusps C_i of S_O and $\{B(r_i, p_i)\}$ of the points p_i which depend only on the size of the cusps, such that outside these neighborhoods the metrics are close.

The main idea in establishing that these metrics are close outside of these neighborhoods is to use a variant of the Ahlfors-Schwarz Lemma [A] due to Wolpert [W], which we will describe in §2 below.

We will give two applications of this result.

The first one, in §3 below, shows that, under the assumption of large cusps, the lengths of short geodesics on S_C are bounded by the lengths of short geodesics on S_O. This is the crucial step in applying the Huxley-Sarnak-Xue machinery to the surfaces P_k.

The second application in §4 below shows that, under the assumption of large cusps, the Cheeger constants $h(S_O)$ and $h(S_C)$ are bounded in terms of one another

$$\frac{1}{C(l)} h(S_O) \leq h(S_C) \leq C(l) h(S_O)$$

by a constant $C(l)$ which tends to 1 as the size of the cusps tends to infinity. It follows from the inequalities of Cheeger [Ch] and Buser [Bu] that the first eigenvalues of S_O and S_C are bounded in terms of one another.

In [BBD], a different method was employed to compactify the surfaces S_k to obtain compact surfaces with λ_1 bounded from below. The present method contrasts with the method of [BBD] in a number of ways. First of all, the surfaces S_C obtained here can in general have large injectivity radius, as we show to be the case with the surfaces P_k, so the compact surfaces S_C which can arise from this construction can reach parts of the moduli space of surfaces not accessible by the methods of [BBD]. This point of view is developed at length in the paper [TS].

Secondly, the method of [BBD] and the present paper can be used together to construct families of surfaces of varying large genus whose Cheeger constants, and hence first eigenvalues, are bounded uniformly from below, by applying the present method to some of the cusps and the method of [BBD] to the remaining cusps. We will pursue this line of thought in detail elsewhere.

1. Some curvature calculations

We begin by considering two metrics ds^2_D and ds^2_C on the punctured hyperbolic plane $\mathbb{H}^2 - \mathrm{pt}$. The metric ds^2_D is the standard hyperbolic metric on $\mathbb{H}^2$. If we write the punctured hyperbolic plane as the punctured unit disk

$$D^* = \{z \in \mathbb{C} : 0 < |z| < 1\},$$

then the metric ds^2_D may be written as

$$ds^2_D = [\frac{2}{1-r^2}]^2[dx^2 + dy^2],$$

where we have set $r = |z|$.

The metric ds^2_C may be described as the unique metric in the standard conformal class which is complete on $\mathbb{H}^2 - \mathrm{pt}$ and has constant curvature -1. It may be realized by taking the quotient C of the standard hyperbolic metric on the upper half-plane $\mathbb{H}^2$ given by

$$ds^2 = \frac{1}{y^2}[dx^2 + dy^2]$$

by the isometry $A : z \to z + 1$, and by identifying the quotient $\mathbb{H}^2/A$ with D^* by the map

$$z \to e^{2\pi iz}.$$

From this, it is easy to write out the explicit expression for ds^2_C given by

$$ds^2_C = [\frac{1}{r\log(r)}]^2[dx^2 + dy^2].$$

The main goal of this section is the following:

Lemma 1.1. *For every ε, there exists an R and a metric ds_R^2 on D^* with the following properties:*

(i) ds_R^2 *is conformally equivalent to* ds_D^2 *(and hence also* ds_C^2*) on* D^*.
(ii) Outside a ball of radius R about 0 in the metric ds_D^2, ds_R^2 agrees with the metric ds_C^2.
(iii) The curvature of the metric ds_R^2 is everywhere between $-(\frac{1}{1+\varepsilon})$ and $-(1+\varepsilon)$.
(iv) ds_R^2 extends across $z=0$ to give a smooth metric on $D=\{z:|z|<1\}$.

We begin the proof by considering radially symmetric metrics on D^* of the form

$$ds_f^2 = f^2(r)[dx^2+dy^2] = f^2(r)[dr^2+r^2d\theta^2].$$

The curvature K_f of the metric ds_f^2 is given by the formula

$$K_f = \frac{-[(\frac{f'}{f})' + \frac{f'}{rf}]}{f^2}.$$

Setting $K_f=-1$, we have the solutions

$$f_1(r) = \frac{2}{1-r^2}$$

corresponding to ds_D^2 and

$$f_2(r) = \frac{-1}{r\log(r)} = \frac{1}{r\log(1/r)}$$

corresponding to ds_C^2.

We will need some simple facts about f_1 and f_2:

Lemma 1.2. *f_1 and f_2 satisfy the following:*

(a) $\lim_{r\to1}\frac{f_2}{f_1}=1$.
(b) $f_2 > f_1$.

Proof. We first observe that as $r\to1$, both f_1 and f_2 blow up. Hence, by L'Hospital's rule,

$$\begin{aligned}\lim_{r\to1}\frac{f_2}{f_1} &= \lim_{r\to1}\frac{(\frac{1}{f_1})}{(\frac{1}{f_2})}\\ &= \lim_{r\to1}\frac{(1-r^2)'}{-2(r\log(r))'}\\ &= \lim_{r\to1}\frac{-2r}{-2(\log(r)+1)} = 1.\end{aligned}$$

This establishes (a).

(b) amounts to the assertion that

$$1 - r^2 > -2r\log(r).$$

At 1, both sides are equal to 0, so this inequality will follow from the inequality

$$-2r < -2(\log(r) + 1),$$

or

$$r > 1 + \log(r).$$

Again, we get equality at $r = 1$, so the assertion will follow from

$$1 < 1/r,$$

which holds when $r < 1$. □

We now transform the problem of constructing the metrics ds^2_R from a conformal problem on the unit disk to a problem of metrics of the form

$$ds^2_g = g^2(r)[dr^2 + \sinh^2(r)d\theta^2].$$

The curvature of this metric is given by

$$\kappa_g = \frac{-[(\frac{g'}{g})' + 1 + \frac{g'}{g}\coth(r)]}{g^2}.$$

When $g \equiv 1$, we obtain the standard hyperbolic metric ds^2_D. It follows from our calculations above that the metric ds^2_C is given by the function

$$h(r) = \frac{f_2}{f_1}(R(r)),$$

where $R(r) = \tanh(r/2)$ is the Euclidean distance from 0 of a point whose hyperbolic distance from 0 is r. We thus have

$$h(r) = \frac{1}{\sinh(r)\log(\coth(r/2))}.$$

It follows from Lemma 1.2 that $h(r) \to 1$ as $r \to \infty$, and that $h(r) > 1$. We will need some more properties of h:

Lemma 1.3. *$h(r)$ has the following additional properties:*

(a) $h'(r)$ is negative and tends to 0 as $r \to \infty$.

(b) $h''(r)$ is positive and tends to 0 as $r \to \infty$.

Proof. It is easily seen that h' is negative if and only if the same is true of its logarithmic derivative.

We may then compute

$$\begin{aligned}\log(h(r))' &= -[\frac{\cosh(r)}{\sinh(r)} + \frac{(\coth(r/2))'}{\coth(r/2)\log(\coth(r/2)}] \\ &= -[\frac{\cosh(r)}{\sinh(r)} - \frac{1}{\sinh(r)\log(\coth(r/2))}] \\ &= -\frac{1}{\sinh(r)\log(\coth(r/2))}[\cosh(r)\log(\coth(r/2)) - 1].\end{aligned}$$

From the fact that the curvature κ_h is equal to -1, or by a direct calculation, we see that

$$(\log(h))'' = (h^2 - 1) - (\log(h))' \coth(r).$$

We now claim that assertions (a) and (b) both follow from the assertion that $\cosh(r)\log(\coth(r/2)) - 1$ is positive, and tends to 0 as $r \to \infty$. This is evident in part (a), while for part (b) we use the equation

$$(\log(h))'' = \frac{h''}{h} - \frac{(h')^2}{h^2}$$

to establish that if $(\log(h))''$ is positive and tends to 0 as $r \to \infty$, then the same is true of h.

The fact that $\cosh(r)\log(\coth(r/2)) - 1$ is positive and tends to 0 as $r \to \infty$ follows readily from L'Hospital's Rule, as above.

This proves Lemma 1.3. □

We will now prove Lemma 1.1 according to the following scheme: it is evident from the formula for curvature that, for any ε, there exists a δ with the following property: let h_ε be any function which satisfies the following conditions:

$$\begin{aligned}1 \le h_\varepsilon &\le 1 + \delta \\ |h'_\varepsilon \coth(r)| &\le \delta \\ &\text{and} \\ |h''_\varepsilon| &\le \delta,\end{aligned}$$

then the metric

$$ds^2_{h_\varepsilon} = h^2_\varepsilon[dr^2 + \sinh^2(r)d\theta^2]$$

will have curvature between $-(\frac{1}{1+\varepsilon})$ and $-(1+\varepsilon)$. We must demand as well that $h_\varepsilon \to 1$ as $r \to 0$, in order to obtain a smooth metric at $r = 0$.

Given ε, we will then construct h_ε as follows: let $k(r)$ be a smooth function which approximates the discontinuous function $k_0(r)$ defined by

$$\begin{aligned} k_0(r) &= 0 && \text{for } 0 \le r < R_0 - 3 \\ &= c_1 && \text{for } R_0 - 3 \le r < R_0 - 2 \\ &= -c_1 && \text{for } R_0 - 2 \le r < R_0 - 1 \\ &= c_2 && \text{for } R_0 - 1 \le r < R_0 \\ &= h''(r) && \text{for } r \ge R_0, \end{aligned}$$

where we will choose R_0, c_1, and c_2 later.

In order to have the anitderivative $k_1(r)$ of k_0 with $k_1(0) =$ agree with $h'(r)$ for $r \ge R_0$, we must have

$$c_2 = h'(R_0).$$

We then let $k_2(r)$ be the antiderivative of $k_1(r)$ with $k_2(0) = 1$. In order for this to equal $h(r)$ for $r \ge R_0$, we must have

$$c_1 = (h(R_0) - 1) - \frac{c_2}{2} = (h(R_0) - 1) - \frac{h'(R_0)}{2}.$$

One may then choose k to be a smooth function approximating k_0, agreeing with k_0 for $R > R_0$, and satisfying the same conditions at R_0 as k_0. Our desired function h_ε will then be the function which satisfies

$$h''_\varepsilon = k, \quad h'_\varepsilon(0) = 0, \quad h_\varepsilon(0) = 1.$$

We may then choose R_0 sufficiently large such that $\coth(R_0)$ and $h(R_0)$ are close to 1, and $h'(R_0), h''(R_0)$ are close to 0.

This then completes the proof of Lemma 1.1. □

2. A comparison theorem

Let S_O be a Riemann surface with a complete metric $ds^2_{S_O}$ of finite area and constant curvature -1. Then each cusp C_i has a neighborhood which is isometric to a neighborhood of infinity in $C = \mathbb{H}^2/(z \sim z+1)$.

For z in such a neighborhood, let $l(z)$ denote the length of the shortest closed horocycle through z. In terms of the coordinate C, we have that

$$l(z) = \frac{1}{\Im(z)}.$$

We may compactify S_O to obtain a compact Riemann surface S_C in the following way: for each cusp C_i, let

$$B_l(C_i) = \{z \in C_i : l(z) \le l\}.$$

Then $B_l(C_i)$ is conformally equivalent to a punctured disk, with the equivalence given by the map $z \to e^{2\pi iz}$.

We may then replace each neighborhood $B_l(C_i)$ with a solid disk to obtain S_C. This construction defines a unique conformal structure on S_C, and exhibits S_O conformally as

$$S_O = S_C - \{p_1, \dots, p_k\}.$$

Under the map $C \to D$ given by $z \to e^{2\pi iz}$, the distance r from $e^{2\pi iz}$ to 0 in the hyperbolic metric on D is related to $l(z)$ by

$$l(z) = \frac{2\pi}{\log(\frac{e^r+1}{e^r-1})}.$$

For each $p_i \in S_C$, let $ds^2_{S_C}$ denote the hyperbolic metric on S_C, assuming that S_C carries such a metric, and let $B(r, p_i)$ denote the ball of radius r

Definition 2.1. *The surface S_O has cusps of length $\geq l$ if, for each i, there is a simple closed horocycle h_i about the cusp C_i, such that each h_i has length $\geq l$, and such that all the h_i's are disjoint,*

In this section, we will prove:

Theorem 2.1. *For every ε, there is an l and r such that, if S_O has cusps of length $\geq l$, then outside of $\cup_i\ B_l(C_i)$ and $\cup_i\ B(r, p_i)$, we have*

$$(\frac{1}{1+\varepsilon})ds^2_{S_O} \leq ds^2_{S_C} \leq (1+\varepsilon)ds^2_{S_O}.$$

Proof. Given ε, choose R_0 as in Lemma 1.1, and assume that the cusps of S_O have length at least

$$l_0 = \frac{2\pi}{\log(\frac{e^{R_0}+1}{e^{R_0}-1})}.$$

We may then replace the hyperbolic metric on each cusp by the conformally equivalent metric given by Lemma 1.1. The resulting metric then extends across the cusps to give a new metric ds^2_{ε,R_O} on S_C with the following properties:

(i) ds^2_{ε,R_O} agrees with the hyperbolic metric on S_O outside of $\cup_i B_{l_0}(C_i)$.
(ii) ds^2_{ε,R_O} is conformally equivalent to the hyperbolic metrics on S_O and S_C.
(iii) The curvatures of ds^2_{ε,R_O} are everywhere between $-(\frac{1}{1+\varepsilon})$ and $-(1+\varepsilon)$.

We now wish to compare the metric ds^2_{ε,R_O} with the hyperbolic metric on S_C. This will be carried out using the following lemma of Wolpert [W], which is a generalization of the Ahlfors-Schwarz Lemma [A]:

Lemma 2.1. ([W]) *Let S be a compact surfacce of genus at least 2. Let ds^2 and $d\sigma^2$ determine the same conformal structures. Provided the Gauss curvatures satisfy*

$$\kappa(ds^2) \leq \kappa(d\sigma^2) < 0,$$

then $ds^2 \leq d\sigma^2$.

To prove Theorem 2.1, we apply Lemma 2.1 to the metrics $(1+\varepsilon)ds^2_{\varepsilon,R_O}$ (resp. $(\frac{1}{1+\varepsilon})ds^2_{\varepsilon,R_O}$) and $ds^2_{S_C}$. Since $(1+\varepsilon)ds^2_{\varepsilon,R_O}$ has curvature satisfying

$$\kappa((1+\varepsilon)ds^2_{\varepsilon,R_O}) \geq -1 = \kappa(ds^2_{S_C})$$

and similarly

$$\kappa((\frac{1}{1+\varepsilon})ds^2_{\varepsilon,R_O}) \leq \kappa(ds^2_{S_C}),$$

we conclude that

$$(\frac{1}{1+\varepsilon})ds^2_{\varepsilon,R_O} \leq ds^2_{S_C} \leq (1+\varepsilon)ds^2_{\varepsilon,R_O}.$$

Since ds^2_{ε,R_O} agrees with $ds^2_{S_O}$ outside the cusp neighborhoods $B_{l_0}(C_i)$, we have the same inequality with the metric ds^2_{ε,R_O} replaced by $ds^2_{S_O}$ outside these neighborhoods. Furthermore, the image of the neighborhood $B_{l_0}(C_i)$ is contained in the ball $B(R_1, p_i)$ computed in the metric ds^2_{ε,R_O}, where $R_1 = (1+\varepsilon)R_0$. But this ball is contained in the ball of radius $(1+\varepsilon)^{1/2}R_1$ computed in the metric $ds^2_{S_C}$, by the above inequality.

We may now take $r = (1+\varepsilon)^{1/2}R_1$ to complete the proof of Theorem 2.1. □

We remark that this argument shows as well that the image of $B_{l_0}(C_i)$ contains $B(\frac{1}{(1+\varepsilon)^{3/2}}R_0, p_i) - p_i$.

3. Counting short geodesics

In this section, we will relate the lengths of short geodesics on S_C with the lengths of short geodesics on S_O. We then use this to give a proof of Theorem 0.2.

We first observe that if γ is a closed geodesic on S_O, then its image in S_C is shorter, by the standard Schwarz Lemma, and hence the geodesic representing it will be still shorter. It may indeed be a great deal shorter, and even nullhomotopic.

We will, however, give a bound for lengths of geodesics on S_C in terms of lengths of geodesics on S_O of the following form:

Lemma 3.1. *For l sufficiently large, there is a constant $\delta(l)$ with the following property: Let S_O have cusps of length $\geq l$. Then, for every geodesic γ in S_C, there* *is a geodesic γ' in S_O, such that the image of γ' in S_C is homotopic to γ, and*

$$\mathit{length}(\gamma) \leq \mathit{length}(\gamma') \leq (1+\delta(l))\mathit{length}(\gamma).$$

Furthermore, $\delta(l) \to 0$ as $l \to \infty$.

The idea of the proof may be paraphrased as follows: we will choose an r_2 larger than the r of Theorem 2.1, such that any geodesic which enters $B(r_2, p_i)$ can be "pushed out of the way" to avoid $B(r, p_i)$. The increase in length of the curve will then be small compared to the legth involved in going from the boundary of $B(r_2, p_i)$ to the boundary of $B(r, p_i)$. The image of this "pushed away geodesic" in S_O will then give the homotopy class for γ'.

We will need the following elementary:

Lemma 3.2. *Given δ_1 and r_1, there is an r_2 with the following property: let γ be any curve in the ball $B(r_2, x_0)$ of radius r_2 in the hyperbolic plane $\mathbb{H}^2$, whose endpoints lie in the boundary of $B(r_2, x_0)$ Then there is a curve $\tilde{\gamma}$ homotopic to γ with a homotopy fixing the endpoints, such that $\tilde{\gamma}$ does not meet the ball $B(r_1, x_0)$, and*

$$\mathit{length}(\tilde{\gamma}) < (1+\delta_1)\mathit{length}(\gamma).$$

Proof. Indeed, we may choose $\tilde{\gamma}$ to agree with γ up to the first time γ enters $B(r_1, x_0)$ and after the last timeγ exits $B(r_1, x_0)$, and to travel around the perimeter of $B(r_1, x_0)$ from the entry point to the exit point. Choosing r_1 such that the length $l(r_1)$ of the perimeter of $B(r_1, x_0)$ satisfies

$$\frac{l(r_1)}{2(r_2 - r_1)} < \delta_1$$

certainly gives r_2 with the desired properties.

We now can complete the proof of Lemma 3.1 as follows: Given δ, let us write

$$1 + \delta = (\sqrt{1+\varepsilon_1})(1+\delta_1)$$

for some ε and δ_1. We then choose r_1 as in Lemma 2.1 and r_2 as in Lemma 3.1. Then, if the cusps of S_O have length $\geq l$, where l is sufficiently large so that the images of the $B_l(C_i)$'s all lie within the corresponding $B(r_2, p_i)$'s, then we may modify the curve γ to a curve $\tilde{\gamma}$ which does not meet any $B(r_1, p_i)$, increasing its length by a factor of at most $1 + \delta_1$. When we now measure the curve $\tilde{\gamma}$ in the metric $ds^2_{S_O}$, its length increases by a factor of at most $\sqrt{1+\varepsilon_1}$. If we denote by γ' the geodesic in the homotopy class of $\tilde{\gamma}$ in S_O, then we clearly have that

$$\text{length}(\gamma') \leq (1+\delta)\text{length}(\gamma).$$

The inequality $\text{length}(\gamma) \leq \text{length}(\gamma')$ then follows from the Ahlfors-Schwarz Lemma, as mentioned above.

This concludes the proof of Lemma 3.1. □

We will now prove Theorem 0.2. As indicated in the introduction, it will follow from the Theorem of Huxley [Hu] and Sarnak-Xue [SX], see also [TFSG] for a discussion.

Suppose that R_k is a family of Riemann surfaces, such that $PSL(2,\mathbb{Z}/k)$ acts on R_k. We then have:

Theorem 3.1. ([Hu], [Sx]) *Suppose that there are constants c_1, c_2, and c_3, and for all $\varepsilon > 0$ a constant $c_4(\varepsilon)$ such that:*

(a) $c_1 k^3 \leq vol(R_k) \leq c_3 k^3$.

(b) If f_k is an eigenfunction of the Laplacian on R_k invariant under the action of $PSL(2,\mathbb{Z}/k)$ with eigenvalue λ, then $\lambda > 5/36$.

(c) For all ε, the number of geodesics of length $\leq (6-\varepsilon)\log(k)$ on R_k is at most $c_4(\varepsilon)k^{6+\varepsilon}$.

Then

$$\liminf_{k\to\infty} \lambda_1(R_k) \geq 5/36.$$

It is argued in detail in [Hu] that the surfaces $S_k = \mathbb{H}^2/\Gamma(k)$ satisfy these conditions. The only non-trivial part is to verify (c). This is done with an explicit calculation with traces of matrices satisfying the congruence condition.

We now turn our attention to showing that (a)-(c) obtain for the surfaces P_k as well.

Observing that the quotient of P_k by $PSL(2,\mathbb{Z}/k)$ is the hyperbolic triangle T_k with angles $\pi/3, \pi/3$, and $2\pi/k$, while the quotient of S_k by $PSL(2,\mathbb{Z}/k)$ is the hyperbolic triangle with angles $\pi/3, \pi/3$, and one ideal vertex, we see that

$$\mathrm{vol}(P_k) = (1-6/k)\mathrm{vol}(S_k),$$

from which (a) follows immediately.

Furthermore, if f_k is an eigenfunction with eigenvalue λ on P_k invariant under $PSL(2,\mathbb{Z}/k)$, then f_k descends to a function on T_k whose Rayleigh quotient is λ. The lower bound $\lambda \geq 1/4$ will then follow from Cheeger's inequality and the fact that the Cheeger constant $h^N(T_k)$ with Neumann boundary conditions is ≥ 1.

But the fact that $h^N(T) \geq 1$ for any hyperbolic triangle is quite standard, see [Bu2], establishing (b).

To establish (c), we observe that the surfaces S_k have cusps of length $\geq k$. Lemma 3.1 then allows us to deduce (c) for the surfaces P_k from the analogous statement for the surfaces S_k.

This completes the proof of Theorem 0.2. □

4. The Cheeger constant

We first recall the Cheeger constant $h(S)$ of a surface. It is given by

$$h(S) = \inf_C \frac{\text{length}(C)}{\min(\text{vol}(A), \text{vol}(B))},$$

where C runs over all curves dividing S into two pieces A and B.

According to the inequalities of Cheeger [Ch] and Buser [Bu], we have that

$$(1/4)h^2 \leq \lambda_1(S) \leq c_1 h + c_2 h^2,$$

where c_1 and c_2 depend on a lower bound for the curvature of S. In particular, it follows that, in the presence of a lower bound for the curvature, a bound for below for λ_1 is equivalent to a lower bound for h.

Of course, as is discussed in [SGCC], the loss of strength in passing from an estimate for the Cheeger constant to an estimate for λ_1 is significant, so that one does not expect the constants that one obtains in Theorem 0.2 from this approach. Indeed, it is shown in [SGCC] that the Cheeger constant $h(S_k)$ is too small to give Selberg's 3/16 bound for $\lambda_1(S_k)$. On the other hand, passing through the Cheeger constant allows us to obtain spectral estimates in more general situations than are allowed for by the approach of §3.

We will show:

Theorem 4.1. *For l sufficiently large, there is a constant $C(l)$ with the following property: if S_O is a Riemann surface with cusps of length $\geq l$, then the Cheeger constants $h(S_O)$ and $h(S_C)$ satisfy*

$$(\frac{1}{C(l)})h(S_O) \leq h(S_C) \leq C(l)h(S_O).$$

Furthermore, $C(l) \to 0$ as $l \to \infty$.

Proof. Let γ be a curve in S_C dividing S_C into two pieces A and B, such that the ratio

$$\frac{\text{length}(\gamma)}{\min(\text{vol}(A), \text{vol}(B))}$$

realizes the Cheeger constant. We may assume that $\text{vol}(A) \leq \text{vol}(B)$.

As in Lemma 3.1, if l is sufficiently large, we may choose r_1 and r_2 such that γ may be pushed away from the neighborhoods $B(r_1, p_i)$ to obtain a new curve $\tilde{\gamma}$ whose length is at most $(1 + \delta(l)\text{length}(\gamma)$.

In fact, we have a choice of how to push γ. For each i, we may consider the neighborhoods $B(r_2, p_i)$ and the sets

$$A_i = A \cap B(r_2, p_i), \quad B_i = B \cap B(r_2, p_i).$$

If γ does not meet $B(r_1, p_i)$, then we do not change γ in $B(r_2, p_i)$. Otherwise, we may push γ so that, for each i, if $\mathrm{vol}(A_i) \le \mathrm{vol}(B_i)$, then $\tilde{\gamma}$ divides $B(r_2, p_i)$ into two pieces A_i', B_i' with

$$A_i' = A_i \cup B(r_1, p_i), \quad B_i' = B_i - B(r_1, p_i).$$

Similarly, if $\mathrm{vol}(B_i) \le \mathrm{vol}(A_i)$, then we choose $\tilde{\gamma}$ so that

$$B_i' = B_i \cup B(r_1, p_i), \quad A_i' = A_i - B(r_1, p_i).$$

We now claim that $\tilde{\gamma}$ divides S_C into two pieces A' and B' with

$$\mathrm{vol}(A') \ge (1 - \varepsilon')\mathrm{vol}(A), \quad \mathrm{vol}(B') \ge (1 - \varepsilon')\mathrm{vol}(B),$$

with

$$\varepsilon' = 2\frac{\mathrm{vol}(B(r_1, p_i))}{\mathrm{vol}(B(r_2, p_i))}.$$

This is clear, since the only times a piece is taken from A_i (resp. B_i) is when $\mathrm{vol}(A_i)$ is larger than $(1/2)\mathrm{vol}(B(r_2, p_i)$.

We now regard $\tilde{\gamma}$ as a curve in S_O, and compute

$$\frac{\mathrm{length}(\tilde{\gamma})}{\min(\mathrm{vol}(A'), \mathrm{vol}(B'))}$$

in the metric $ds^2_{S_O}$.

But in passing from the metric $ds^2_{S_C}$ to the metric $ds^2_{S_O}$, the length of $\tilde{\gamma}$ is multiplied by a factor of at most $\sqrt{1+\varepsilon}$, while the volumes of the parts of A' and B' not meeting $B(r_1, p_i)$ are divided by at most $1 + \varepsilon$. Also, the balls $B(r_1, p_i)$ have larger volume in the metric $ds^2_{S_O}$ than in the metric $ds^2_{S_C}$, as follows from the Schwarz inequality, or can be seen directly.

We thus have that

$$\frac{\mathrm{length}(\tilde{\gamma})}{\min(\mathrm{vol}(A'), \mathrm{vol}(B'))} \le \frac{(1+\varepsilon)^{3/2}(1+\delta)}{1-\varepsilon'} h(S_C).$$

$h(S_O)$ is less than the left-hand side, so we thus have

$$h(S_O) \le (C_1(l))h(S_C),$$

with

$$C_1(l) = \frac{(1+\varepsilon)^{3/2}(1+\delta)}{(1-\varepsilon')}.$$

To obtain an inequality in the opposite direction, we proceed in the identical manner, switching the roles of S_O and S_C. We must make the following changes in the argument: first of all, we must reprove Lemma 3.2 in the case of a punctured disk rather than a disk. The proof is identical, except we no longer demand that the resulting curve $\tilde{\gamma}$ is homotopic to γ. This allows us to retain the option of pushing γ in either direction around the puncture.

Secondly, we need an estimate of the form

$$\mathrm{vol}(B_{l_1})_{ds^2_{S_C}} \geq (\mathrm{const}(l_1))(\mathrm{vol}(B_{l_1})_{ds^2_{S_O}}).$$

But the volume of B_{l_1} in the metric $ds^2_{S_O}$ is precisely l_1, while the metric of a ball of radius r_1 in the hyperbolic plane is $2\pi(\cosh(r_1) - 1)$. Choosing r_1 so that

$$l_1 = \frac{2\pi}{\log(\frac{e^{r_1}+1}{e^{r_1}-1})}$$

and using L'Hospital's rule, we see that

$$\frac{\mathrm{vol}(B_{l_1})}{\mathrm{vol}(B(r_1, p_i))} \to 1 \text{ as } l_1 \to \infty.$$

Passing from the metric $ds^2_{S_O}$ to the metric ds^2_{ε,R_O} and then to the metric $ds^2_{S_C}$ introduces some factors of $1+\varepsilon$ into this calculation to give us the desired estimate.

Putting these together, we find a constant $C_2(l)$ such that

$$h(S_C) \leq C_2(l)h(S_O),$$

with $C_2(l) \to 1$ as $l \to \infty$.

This then concludes the proof of Theorem 4.1. □

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