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Acyclic covers

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Abstract. The following question, which is directly related to the Whitehead problem of subcomplexes of acyclic 2-complexes, is studied: If $\mathfrak{P}$ is a class of groups, X is a 2-dimensional CW-complex and X' is an acyclic, infinite cyclic cover of X with $\pi_1(X')$ in $\mathfrak{P}$, must X' be contractible? A positive answer is given if X is finite and $\mathfrak{P}$ is the class of amenable groups.

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In [12], J.H.C. Whitehead posed the following problem.

Whitehead's Aspherical Complex Question. *If L is a connected subcomplex of an aspherical 2-dimensional CW-complex K, must L also be aspherical?*

Although some early work was done on this question by Cockcroft [4] this paper is concerned with the approach originating with the following theorem of Adams [1]. For a history of results concerning this problem see [2] or [9].

Theorem. *Let L be a connected subcomplex of an aspherical 2-complex K. Then there exists a characteristic subgroup N of $\pi_1(L)$ such that the Galois cover L_N of L corresponding to N is acyclic.*

Remarks.

(1) It is obvious that if L is not aspherical then L_N cannot be contractible and hence $N \neq 0$.
(2) The N in the above theorem is constructed as the intersection of all normal subgroups P of $G = \ker \pi_1(L) \to \pi_1(K)$ such that G/P is conservative.
(3) In [11] Strebel shows that the Galois covering of L corresponding to the perfect radical of G (i.e. the maximal perfect subgroup) is also acyclic.
(4) Since the cellular chain complex of the cover L_N of K is acyclic, the group $Q = \pi_1(L)/N$ and also G/N have cohomological dimension at most 2.

In the following all complexes are connected. The following exact sequence was derived by Hopf. Let Y be a 2-dimensional CW-complex with fundamental group G, then

$$0 \to H_3(G) \to \pi_2(Y)_G \xrightarrow{h} H_2(Y) \to H_2(G) \to 0$$

is exact. This shows that the Galois cover $Y' \to Y$ is acyclic if and only if $P = \pi_1(Y')$ is superperfect (i.e., $H_1(P) = H_2(P) = 0$) and the Hurewicz map $h : \pi_2(Y) \to H_2(Y')$ is zero. Hence the interest in the following result.

Theorem 1. *The following are equivalent.*

(1) Let $P \neq 1$ be a finite, superperfect normal subgroup of G and suppose $Q = G/P$ contains an element of infinite order. Then if X is any $[G, 2]$-complex, the Hurewicz map $h : \pi_2(X) \to H_2(X_P)$ is non-zero.

(2) Let $P \neq 1$ be a finite, superperfect normal subgroup of G and suppose $Q = G/P$ is free and non-trivial. Then if X is any $[G, 2]$ complex the Hurewicz map $h : \pi_2(X) \to H_2(X_P)$ is non-zero.

(3) Suppose P is a perfect normal subgroup of G with $Q = G/P$ free and non-trivial. Further suppose there exists a $[G, 2]$ complex X with $H_2(X_P) = 0$, then $P = 1$ or P is infinite.

(4) Let $X' \xrightarrow{p} X$ be a connected Galois cover with group $\mathbb{Z}$. Suppose X is 2-dimensional and X' is acyclic. Then either X' is contractible or $\pi_1(X')$ is infinite.

(5) Let $X' \xrightarrow{p} X$ be a connected Galois cover whose group contains an element of infinite order. Suppose the dimension of X is 2 and X' is acyclic, then X' is contractible or $\pi_1(X')$ is infinite.

Proof. (1) $\Rightarrow$ (2). Obvious since any non-trivial free group is torsion free.
(2) $\Rightarrow$ (3). From the exact sequence of Hopf, we see $H_2(X_P) = 0$ implies $H_2(P) = 0$ i.e. P is superperfect and $h : \pi_2(X) \to H_2(X_P)$ is zero. If $P \neq 1$ and finite this is impossible by (2).
(3) $\Rightarrow$ (4). If $P = p_*(\pi_1(X'))$ then P is perfect, normal in $G = \pi_1(X)$ and G/P is free of rank one. If X' is not contractible then $P \neq 1$ and since $H_2(X') = 0$, P is infinite by (3).
(4) $\Rightarrow$ (5). Let X be a 2-complex, $X' \to X$ a cover whose group $\pi_1(X)/\pi_1(X')$ contains an element z of infinite order. Let $H \subseteq \pi_1(X)$ be the inverse image of the subgroup generated by $\{z\}$ under the natural projection and let $X_H \to X$ be the Galois cover corresponding to H. Then $X' \to X_H$ is a Galois cover with infinite cyclic group and by (4) $\pi_1(X') = 1$ or is infinite.
(5) $\Rightarrow$ (1). Let X be a $[G, 2]$-complex and X_P the Galois cover corresponding to P. $H_1(X_P) = 0$ since P is perfect. Now $H_2(P) = 0$ and if $h : \pi_2(X) \to H_2(X_P)$ is also zero then $H_2(X_P) = 0$ by the Hopf sequence. Hence X_P is acyclic and by (5) either X_P is contractible or P is infinite. But this contradicts the fact that P is finite and non-trivial. Hence h must be non-zero. □

If one examines the above proof one sees that P being finite and non-trivial is not essential. In fact we have

Theorem 2. *Let $\mathfrak{P}$ be a class of groups, e.g. finite, torsion, trivial, etc. The following are equivalent.*

(A) Let H be a superperfect normal subgroup of G and suppose G/H contains and element of infinite order. If H belongs to $\mathfrak{P}$ and X is an arbitrary $[G,2]$-complex then the Hurewicz homomorphism $h : \pi_2(X) \to H_2(X_H)$ is non-zero.

(B) If $Y' \to Y$ is a Galois cover with group $\mathbb{Z}$ where Y is two dimensional and Y' is acyclic then $\pi_1(Y')$ does not belong to $\mathfrak{P}$.

Proof. $(A) \Rightarrow (B)$ Let $Y' \to Y$ be a Galois cover with group $\mathbb{Z}$, $\dim(Y) = 2$ and Y' acyclic. Then $H = \pi_1(Y')$ is perfect and by the Hopf sequence $H_2(H) = 0$, hence H is superperfect. If $G = \pi_1(Y)$ then $G/H \cong \mathbb{Z}$ contains an element of infinite order and so if H is in $\mathfrak{P}$ then the Hurewicz map $h : \pi_2(Y) \to H_2(Y')$ is non-zero. But this is ridiculous since $H_2(Y') = 0$.

$(B) \Rightarrow (A)$ Let H be superperfect and normal in G and suppose G/H contains an element of infinite order. Let X be a $[G,2]$-complex. There exists a subgroup $\bar{G}$, $H \subseteq \bar{G} \subseteq G$ such that $\bar{G}/H \cong \mathbb{Z}$. If $X_H, X_{\bar{G}}$ denote the covers of X corresponding to H, $\bar{G}$ respectively, then $X_H \to X_{\bar{G}}$ is a Galois cover with group $\mathbb{Z}$. If the Hurewicz map $h : \pi_2(X) \to H_2(X_H)$ is zero then $H_2(X_H) = 0$ by Hopf. Hence X_H is acyclic and by (B) $\pi_1(X_H)$ does not belong to $\mathfrak{P}$. Therefore the Hurewicz map must be zero. □

Corollary. *Let $\mathfrak{P}$ be a class of groups. If there exists a subcomplex L of a 2-dimensional aspherical complex K which is not aspherical and such that the maximal perfect subgroup of $\pi_1(L)$ belongs to $\mathfrak{P}$ then there exists an acyclic, infinite cyclic non-contractible Galois cover $X' \to X$ where X is 2-dimensional and $\pi_1(X')$ belongs to $\mathfrak{P}$.*

Proof. If there exists such a pair (K, L) then (A) is false for the class $\mathfrak{P}$ by Strebel's result [11]. Hence (B) is false for this class, i.e. there exists a Galois cover $X' \to X$ of 2-dimensional complexes, with group $\mathbb{Z}$, X' acyclic and $\pi_1(X')$ in $\mathfrak{P}$. □

Hence we see a positive answer to the following question would limit the maximal perfect subgroups that could appear in any example of a non-aspherical subcomplex of a two-dimensional apherical complex.

Acyclic Cover Question. *Let $\mathfrak{P}$ be a class of groups containing the trivial group. If X is a 2-dimensional CW-complex and X' is an acyclic, infinite cyclic Galois cover of X with $\pi_1(X')$ in $\mathfrak{P}$, is X' contractible?*

In [5], M.N. Dyer claimed to have proven part (2) of Theorem 1 (in fact, whenever Q has cohomological dimension 1 or 2) and hence to have answered the Acyclic Cover Question in the affirmative for the class of finite groups. However his proof is incorrect and not at all rectifible as he uses in an essential way the following lemma which is definitely false.

Lemma([5], 3.1). *Let $0 \to P \to G \xrightarrow{\pi} Q \to 0$ be an exact sequence of groups with P finite, then $H^i(G, \mathbb{Z}G) \cong H^i(G, \mathbb{Z}Q) \cong H^i(Q, \mathbb{Q})$ for all $i > 0$. The first isomorphism is induced by $\mathbb{Z}\pi : \mathbb{Z}G \to \mathbb{Z}Q$ and the second by $\pi : G \to Q$.*

Remarks.

(1) The first error is minor and correctible. Dyer only needs these isomorphisms for $i = 1, 2$ and when P is also superperfect. Actually in view of Theorem 1 these isomorphisms are only needed for $i = 1$ when P is perfect. In these cases the isomorphisms do exist.

(2) The serious error lies in the second sentence. Although it is true the second isomorphism is induced by π, the first is induced by $\alpha : \mathbb{Z}Q \to \mathbb{Z}G$ where $\alpha(q) = \sum_{\pi(g)=q} g$ is the isomorphism of $\mathbb{Z}Q$ and $\mathbb{Z}G^P$. Since $\pi \cdot \alpha : \mathbb{Z}Q \to \mathbb{Z}Q$ is multiplication by $|P|$ and by a theorem of Swan [10], $H^1(Q, \mathbb{Z}Q) \cong H^1(G, \mathbb{Z}Q)$ is free abelian and non-zero for Q non-trivial and free, it follows $\mathbb{Z}\pi : \mathbb{Z}G \to \mathbb{Z}Q$ can never be an isomorphism in this case.

(3) Dyer uses that $\mathbb{Z}\pi$ induces an isomorphism in step 3 of his proof to show a certain map is a split epimorphism which in turn leads to his contradiction.

There is some evidence for a positive answer to the Acyclic Cover Question for the class of finite groups. In fact if one one assumes that X is a finite, 2-dimensional CW-complex then there exists no infinite cyclic acyclic covers of X with finite fundamental group except for contractible ones. This is based on the following theorem of Eckmann [6] which in turn is an extension of a result of Cheegar and Gromov [3]. This result is a very sophisticated version of an theorem by Gottlieb [7] concerning centers and Euler characteristics. We need a definition.

Definition. A group A is called amenable if every action of A on a compact metric space has an A-invariant Borel measure.

Remarks.

(1) The class of amenable groups is closed under subgroups, quotient groups, extensions and increasing unions. It also contains all finite groups and all abelian groups and hence contains the class of elementary amenable groups, that is the class generated from finte and abelian groups by the operations of subgroups, quotient groups, extensions and increasing unions.

(2) R. Grigorchuck [8] has produced examples of finitely generated amenable groups which are not elementary amenable. These are finitely generated

subgroups of the group of based automorphisms of the based infinite triadic tree (3 edges at each vertex).

(3) It is not difficult to show the free group on two generators, F_2, is not amenable and hence any group containing F_2 is also non-amenable.

Theorem [6]. *Let X be a finite, connected complex of dimension m. Suppose $\pi_i(X) = 0$ for $1 < i < m$ and $\pi_1(X)$ is an infinite amenable group. Then $(-1)^m\chi(X) \geq 0$ and $\chi(X) = 0$ if and only if X is aspherical.*

Using this result we obtain the following evidence for a positive answer to the Acyclic Cover Question for the class of finite groups.

Theorem 3. *Let X be 2-dimensional and finite. Suppose X' is a Galois acyclic cover of X with group $\mathbb{Z}$. If $\pi_1(X')$ is amenable then X' is a $K(P, 1)$ and hence if P is finite it must be trivial and X' contractible.*

Proof. Since $\pi_1(X')$ and $\mathbb{Z}$ are amenable so is $G = \pi_1(X)$ and it is clearly infinite since it maps onto $\mathbb{Z}$. Since X' is acyclic the spectral sequence of the Galois cover $X' \to X$ shows X has the (integral) homology of a circle and hence of Euler chacteristic zero. By Eckmann's result X is aspherical and hence so is X'. If P is non-trivial and finite it has infinite cohomological dimension, so if P is finite it must be trivial. □

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