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The conjugacy problem for Dehn twist automorphisms of free groups

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Abstract. A *Dehn twist automorphism* of a group G is an automorphism which can be given (as specified below) in terms of a graph-of-groups decomposition of G with infinite cyclic edge groups. The classic example is that of an automorphism of the fundamental group of a surface which is induced by a Dehn twist homeomorphism of the surface. For $G = F_n$, a non-abelian free group of finite rank n, a normal form for Dehn twist is developed, and it is shown that this can be used to solve the conjugacy problem for Dehn twist automorphisms of F_n.

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1. Introduction

If ϕ_1 and ϕ_2 are automorphisms of groups G_1 and G_2 respectively, then we say that ϕ_1 and ϕ_2 are *conjugate* if there is an isomorphism $\alpha : G_1 \longrightarrow G_2$ such that $\phi_2 = \alpha\phi_1\alpha^{-1}$. They are *conjugate up to inner automorphism* if there is an isomorphism $\alpha : G_1 \longrightarrow G_2$ and an element $x \in G_2$ such that $\phi_2 = ad_x\alpha\phi_1\alpha^{-1}$. (Here ad_x denotes the inner automorphism of G_2 given by $ad_x(g) = xgx^{-1}$ for all $g \in G_2$.) If $G_1 = G_2 = G$, then ϕ_1 and ϕ_2 are conjugate up to inner automorphism precisely when they represent conjugate elements $\widehat{\phi_1}, \widehat{\phi_2}$ of the outer automorphism group $\mathrm{Out}(G)$.

This paper is concerned with the determination of whether two given Dehn twist automorphisms (defined below) of the free group F_n are conjugate or conjugate up to inner automorphism. The results here will be extended in the forthcoming paper [KLV] to roots of Dehn twist automorphisms (and hence to all au-

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tomorphisms of F_n of linear growth) and will be a key part of the second author's complete solution of the conjugacy problem for $\mathrm{Out}(F_n)$ as announced in [L1].

A *Dehn twist* $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ consists of a graph of groups $\mathcal{G}$ and, for every edge e of $\mathcal{G}$, a specified *twistor* z_e in the center of the edge group G_e. (See 5.) Every Dehn twist determines a *Dehn twist automorphism* D_v of the fundamental group $\pi_1(\mathcal{G}, v)$ for each vertex v of $\mathcal{G}$ and hence an automorphism of the abstract group $\pi_1(\mathcal{G})$ which is well defined up to inner automorphism. Thus D determines an outer automorphism $\widehat{D} \in \mathrm{Out}(\pi_1(\mathcal{G}))$.

A *Dehn twist automorphism of* F_n is an automorphism which is conjugate to such a Dehn twist automorphism D_v of $\pi_1(\mathcal{G}, v)$ for some graph of groups $\mathcal{G}$. Dehn twists on F_n are the natural analogues of geometric Dehn twists (i.e. multiple Dehn twists along sets of disjoint simple closed curves on surfaces); indeed, the automorphisms induced by the geometric Dehn twists on surfaces with boundary are special cases of Dehn twist automorphisms of F_n. On the other hand, an example of a Dehn twist automorphism of $F(a, b, c)$ which is not geometric is given by the automorphism $a \to a,\ b \to b,\ c \to w\,c\,w^{-1}$, if $w \in F(a, b)$ is not a power of x, $xyx^{-1}y^{-1}$, or x^2y^2 for any basis x, y of $F(a, b)$.

Dehn twists can be given in clearly inefficient ways. We define *efficient Dehn twists* in 6. The main result of this paper is the following classification of automorphisms determined by efficient Dehn twists. (For background on graph of groups isomorphisms and the induced isomorphisms of their fundamental groups, see 4.)

1.1. Theorem. *Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are graphs of groups with $\pi_1(\mathcal{G}_1) \cong \pi_1(\mathcal{G}_2) \cong F_n$ and that v and w are vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Let $D_1 = D(\mathcal{G}_1, (z_e)_{e\in E(\mathcal{G}_1)})$ and $D_2 = D(\mathcal{G}_2, (z_e)_{e\in E(\mathcal{G}_2)})$ be efficient Dehn twists inducing automorphisms D_v and D_w of $\pi_1(\mathcal{G}_1, v)$ and $\pi_1(\mathcal{G}_2, w)$ respectively. Let $h : \pi_1(\mathcal{G}_1, v) \longrightarrow \pi_1(\mathcal{G}_2, w)$ be an isomorphism.*
(a) *$\widehat{D_2} = \widehat{hD_1h^{-1}} \in Out(\pi_1(\mathcal{G}_2))$ if and only if there is a graph of groups isomorphism $H : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ which induces the isomorphism h up to inner automorphism (i.e., $\widehat{H} = \widehat{h}$) and which takes twistors to twistors (i.e., $H_e(z_e) = z_{H(e)}$ for all $e \in E(\mathcal{G}_1)$).*
(b) *$D_w = hD_vh^{-1} \in Aut(\pi_1(\mathcal{G}_2, w))$ if and only if there is a graph of groups isomorphism $H : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ which takes v to w, with induced isomorphism $H_{*v} = h : \pi_1(\mathcal{G}_1, v) \longrightarrow \pi_1(\mathcal{G}_2, w)$, and which takes twistors to twistors.*

The material developed to prove Theorem 1.1 allows us to determine along the way the *centralizer, fixed subgroup, index and infinite attracting fixed words* (there aren't any) of a Dehn twist automorphism of a free group. These results are given in 7.

In 8. we give an algorithm for transforming an arbitrary Dehn twist D of a graph of groups $\mathcal{G}$ to an efficient Dehn twist D' of a graph of groups $\mathcal{G}'$ in such a way that the induced isomorphisms of $\pi_1(\mathcal{G})$ and $\pi_1(\mathcal{G}')$ are conjugate up to inner automorphism. We further point out that one can use the Whitehead algorithm to decide whether two graphs of groups are isomorphic in such a way as to preserve the data in part (a) or (b) of Theorem 1.1. This leads to our solution of the conjugacy problem for Dehn twist outer automorphisms:

1.2. Theorem. *There exists an algorithm which, given two Dehn twists D_1 and D_2 based on graphs of groups $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\pi_1(\mathcal{G}_1) \cong \pi_1(\mathcal{G}_2) \cong F_n$, decides in finitely many steps whether the induced outer automorphisms $\widehat{D_1}$ and $\widehat{D_2}$ are conjugate. If D_1 and D_2 are efficient and if v and w are vertices of the graph of $\mathcal{G}_1$ and the graph of $\mathcal{G}_2$ respectively, this algorithm also decides whether the induced automorphisms D_v and D_w of $\pi_1(\mathcal{G}_1, v)$ and $\pi_1(\mathcal{G}_2, w)$ are conjugate.*

Remarks: (1) This paper is written totally within the genre of graphs of groups and their associated actions on R-trees. If an automorphism $\varphi \in \text{Aut}(F_n)$ is given in terms of the image of some basis of F_n, it is possible [L2] to decide whether φ is a Dehn twist automorphism — whether it is conjugate to some automorphism D_v of $\pi_1(\mathcal{G}, v)$ given by a Dehn twist D of a graph of groups $\mathcal{G}$ — and, if so, to derive from the data for φ the data for D and for $\mathcal{G}$, namely $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$. However, in this paper – and in particular in Theorem 1.2 – we always assume that a Dehn twist automorphism is given in terms of the graph of groups data (see 8.1).
(2) The algorithm in Theorem 1.2 is purely combinatorial and operates entirely in terms of the graph of groups data of D_1 and D_2. It is noteworthy (see 6.7) that the underlying justification of this *combinatorial* algorithm comes from the study of the *dynamics* of Dehn twist automorphisms acting on the closure of Culler-Vogtmann's "Outer Space"; we use a result of our previous paper [CL2] concerning these dynamics to prove Theorem 1.1, which in turn implies Theorem 1.2.

2. Outer homomorphisms of groups

Much of this paper concerns outer automorphisms rather than ordinary automorphisms. In this context, the following notion turns out to be natural and useful:

2.1. Definition. Let $f : G \to H$ be a group homomorphism. Then we denote by $\widehat{f} : G \to H$ the *outer homomorphism* induced by f. This is the equivalence class

$$\hat{f} = \{ad_h f : G \to H \mid h \in H\}$$

of homomorphisms from G to H.

Notice that for any homomorphisms $f_1 : G_1 \to G_2$ and $f_2 : G_2 \to G_3$ one has $\widehat{f_2 f_1} = \widehat{f_2}\widehat{f_1}$. Furthermore, for any automorphism $f : G \to G$ the set $\hat{f} = \text{Inn}(G)\cdot f$ is precisely the induced outer automorphism in the usual sense. The notion of outer homomorphism, though it does not seem to be standard, is the natural morphism induced on the level of fundamental groups by continuous maps between path-connected topological spaces without specified base point.

If $\alpha : G \to G'$ and $\beta : H \to H'$ are fixed isomorphisms, then every $\hat{f} : G \to H$ induces $\widehat{f'} = \widehat{\beta f \alpha^{-1}} : G' \to H'$, and $\widehat{f'}$ does not depend on α and β but only on their induced outer isomorphisms. This observation can be applied as follows to *malnormal* subgroups. (Recall that $U \subset G$ is a malnormal subgroup if $g \in G$ and $g \notin U$ implies that $gUg^{-1} \cap U = \{1\}$).

2.2. Lemma. *Let $U_i \subset G$ and $V_i \subset H$ $(i = 1, 2)$ be malnormal subgroups of G and H with U_1 conjugate to U_2 and V_1 conjugate to V_2; say $U_2 = gU_1g^{-1}$ and $V_2 = hV_1h^{-1}$ for some $g \in G$ and $h \in H$.*
(a) *Every outer homomorphism $\widehat{f_1} : U_1 \to V_1$ with representative f_1 determines an outer homomorphism $\widehat{F_1} : U_2 \to V_2$ with representative $F_1 = ad_h \circ f_1 \circ {ad_g}^{-1}$. This outer homomorphism is independent of the choice of representative f_1 and of the conjugators g and h. In particular, if U_1 and U_2 are conjugate malnormal subgroups of a group G then there is a canonical identification between $Out(U_1)$ and $Out(U_2)$.*
(b) *Let $f : G \longrightarrow H$ be a homomorphism with $f(U_1) \subset V_1$ and $f(U_2) \subset V_2$. Let $f_i : U_i \longrightarrow V_i$ $(i = 1, 2)$ be the maps induced by restricting f. Then $\widehat{f_2} = \widehat{F_1}$ where $F_1 = ad_h \circ f_1 \circ {ad_g}^{-1}$, as in (a).*

Proof. **(a)** We first note that, if $x \in V_1$ then the outer homomorphism class of $\mathrm{ad}_h f_1 \mathrm{ad}_g^{-1}$ equals that of $\mathrm{ad}_h \mathrm{ad}_x f_1 \mathrm{ad}_g^{-1}$. This is because $hxh^{-1} \in V_2$ and

$$\mathrm{ad}_h\, \mathrm{ad}_x\, f_1\, \mathrm{ad}_g^{-1} \;=\; \mathrm{ad}_{hxh^{-1}}(\mathrm{ad}_h\; f_1\, \mathrm{ad}_{g^{-1}})\;.$$

Now suppose that $g' \in G,\ h' \in H$ are elements such that $U_2 = g'U_1{g'}^{-1}$ and $V_2 = h'V_1{h'}^{-1}$. Then the outer homomorphism class of $\mathrm{ad}_h f_1 \mathrm{ad}_g^{-1}$ equals that of $\mathrm{ad}_{h'} f_1 \mathrm{ad}_{g'}^{-1}$ because malnormality implies that $g^{-1}g' \in U_1$ and $h^{-1}h' \in V_1$ and because

$$ad_{h'}\, f_1 \cdot \mathrm{ad}_{g'}^{-1} = \mathrm{ad}_h\, \mathrm{ad}_{(h^{-1}h')}\, f_1\, \mathrm{ad}_{g'^{-1}g}\, \mathrm{ad}_{g^{-1}} = \mathrm{ad}_h\, \mathrm{ad}_{(h^{-1}h')\cdot f_1(g^{-1}g')}\, f_1\, \mathrm{ad}_{g^{-1}}$$

Since $(h^{-1}h') \cdot f_1\,(g^{-1}g') \in V_1$ the first paragraph of the proof applies.

The preceding two paragraphs prove (a).

(b) We have $f = ad_{f(g)}\, f\, ad_{g^{-1}}$ and since $U_2 = gU_1g^{-1}$ we can write this as $f_2 = ad_{f(g)}\, f_1\, ad_{g^{-1}}|U_2$. If $f(U_1) = \{1\}$ the result claimed is trivial. In case $f(U_1) \neq \{1\}$, the given element h with $h^{-1}V_2h = V_1$ satisfies

$$f(g)\, h^{-1}\biggl(h\, f(U_1)\, h^{-1}\biggr)\, h\, f(g)^{-1} = f(U_2)$$

and hence

$$f(g)\, h^{-1}V_2\, h\, f(g)^{-1} \cap V_2 \neq \{1\}.$$

Thus malnormality implies that $x \equiv f(g)\, h^{-1} \in V_2$ so that $f_2 = ad_x ad_h\, f_1\, ad_{g^{-1}}$ $= ad_x\, F_1$ with $x \in V_2$. Thus $\widehat{f_2} = \widehat{F_1}$ as claimed. □

3. Graphs of groups

For the convenience of the reader we recall in this and the following section some standard definitions and facts concerning graphs of groups. For general background see [S], [B], [C] or[D–D]. We mainly follow the notation of [CL2]. We include some basic results which we will need which do not seem to have appeared before. (See (3.9) for normal forms for representatives of conjugacy classes and (3.10) for the fact that vertex groups are malnormal in path groups.)

3.1. A *graph of groups* is given by

$$\mathcal{G} = (\Gamma(\mathcal{G}),\, \{G_v\}_{v\in V(\mathcal{G})},\, \{G_e\}_{e\in E(\mathcal{G})},\, \{f_e : G_e \to G_{\tau(e)}\}_{e\in E(\mathcal{G})})$$

where we use the following notation:
- $\Gamma(\mathcal{G})$ is a finite connected graph,
- $V(\mathcal{G})$ is the set of vertices of $\Gamma(\mathcal{G})$,
- $E(\mathcal{G})$ is the set of oriented edges of $\Gamma(\mathcal{G})$.

For any $v \in V(\mathcal{G})$ and $e \in E(\mathcal{G})$ we denote:
- $\bar{e}$ is the edge oppositely oriented to e,
- $\tau(e)$ is the terminal vertex of e (so that $\tau(\bar{e})$ is the initial vertex of e),
- G_v is the vertex group at v,
- $G_e = G_{\bar{e}}$ is the edge group at e,
- $f_e : G_e \to G_{\tau(e)}$ is an injective homomorphism.

3.2. We denote by $\Pi(\mathcal{G})$ the *path group* of $\mathcal{G}$ (called the *Bass group* $\beta(\mathcal{G})$ in [CL2]), which is generated by the *stable letters* t_e $(e \in E(\mathcal{G}))$ and the elements $r \in G_v$ $(v \in V(\mathcal{G}))$, subject to the relations in the G_v , and to
(i) $t_{\bar{e}} = t_e^{-1}$ and
(ii) $t_e\, f_e(a)\, t_e^{-1} = f_{\bar{e}}(a) \in G_{\tau(\bar{e})}$ for all $a \in G_e,\ e \in E(\mathcal{G})$.

3.3. Every element of $\Pi(\mathcal{G})$ is given by a *word*

$$W = r_0 t_1 r_1 \dots t_q r_q$$

where $t_i = t_{e_i}$ and each r_i is an element of the free product $*\{G_v \,|\, v \in V(\mathcal{G})\}$. We say that W is *connected*, with *initial* vertex $\tau(\bar{e}_1)$ and *terminal* vertex $\tau(e_1)$, if $r_0 \in G_{\tau(\bar{e}_q)}$ $r_q \in G_{\tau(e_q)}$ and $\tau(e_i) = \tau(\bar{e}_{i+1})$ with $r_i \in G_{\tau(e_i)}$ for all $i = 1, \dots, q-1$. Note that connected words are transformed to connected words when they are shortened by the operations of 3.2, but that a trivial word $f_{\bar{e}}(a)^{-1}\, t_e\, f_e(a) t_{\bar{e}}$ can be inserted into a connected word so as to make it non-connected. Finally, W is *a closed, connected word based at* v if it is connected and $v = \tau(\bar{e}_1) = \tau(e_q)$. This includes all $W \in G_v$.

3.4. We denote by $\pi_1(\mathcal{G}, v) \subset \Pi(\mathcal{G})$ the *fundamental group* of $\mathcal{G}$ based at the vertex v. It consists precisely of those elements of $\Pi(\mathcal{G})$ which are represented by a closed connected word based at v. For distinct vertices $v_1, v_2 \in V(\mathcal{G})$ the subgroups $\pi_1(\mathcal{G}, v_1)$ and $\pi_1(\mathcal{G}, v_2)$ are conjugate in $\Pi(\mathcal{G})$. Notice that $\Pi(\mathcal{G})$ is canonically isomorphic to $\pi_1(\mathcal{G}_*, v_*) = \Pi(\mathcal{G}_*)$ where $\mathcal{G}_*$ is the graph of groups obtained from $\mathcal{G}$ by identifying all vertices $v \in V(\mathcal{G})$ to a unique vertex v_* and defining its vertex group to be the free product of all G_v (and replacing each f_e by [inclusion $\circ f_e$]).

3.5. A word $W \in \Pi(\mathcal{G})$ as in 3.3 is *reduced* if $q = 0$ or if $t_i = t_{i+1}^{-1}$ implies that $r_i \notin f_{e_i}(G_{e_i})$ ($i = 1, \dots, q-1$). A non-trivial reduced word need not be connected, but can be identified with a (necessarily connected) reduced word in $\Pi(\mathcal{G}_*)$. So classical results of Bass and Serre on connected words in a path group apply to arbitrary words in $\Pi(\mathcal{G})$. By applying the relations 3.2 above sufficiently often any word $W \in \Pi(\mathcal{G})$ can be transformed into a reduced word. Also, reduced words have the following uniqueness property:

3.6. Proposition. *If* $V = r_0 t_1 r_1 \dots t_q r_q$ *and* $W = s_0 t'_1 s_1 \dots t'_{q'} s_{q'}$ *are reduced words representing the same element* $g \in \Pi(\mathcal{G})$ *then:*

(a) $t_i = t'_i$ *for all* $i = 1, \dots, q$. *In particular,* $q = q'$ ($\equiv$ *length*(g)).

(b) *For all* $i = 1, \dots, q$ *there exist elements* $h_i \in G_{e_i}$ *such that*

$$s_0 = r_0 f_{\bar{e}_1}(h_1^{-1}),$$
$$s_i = f_{e_i}(h_i) r_i f_{\bar{e}_i+1}(h_{i+1}^{-1}) \quad \text{for} \quad i = 1, \dots, q-1, \text{ and}$$
$$s_q = f_{e_q}(h_q) r_q.$$

(c) V *is connected if and only if* W *is connected.*

Proof. (a) and (b) follow from [S, p. 50] or [B, 1.10], while (c) follows from (a) and (b). □

3.7. **(a)** A product VW of two reduced words in $\Pi(\mathcal{G})$ is called *reduced* if the concatenation of the two words is reduced. This is equivalent to " length $(VW) =$ length(V) + length(W)", and in this case we write $V * W$ for VW. For example, if length $(V) = 0$, then $VW = V * W$ for any W. One must be careful concerning connectivity. A reduced connected word V may be factored as the reduced product of reduced non-connected words: $V = A * B = (A * r) * (r^{-1} * B)$, where $r \in G_v$ for some vertex v far away from the path carrying $V = A * B$. However, we do have:

(b) If $V = A * B * C$ with V and B connected, then A and C are also connected and the terminal vertex of A equals the initial vertex of B and the terminal vertex of B equals the intial vertex of C.

3.8. A word $W \in \Pi(\mathcal{G})$ is *cyclically reduced* if it is reduced and if, furthermore, $\tau(e_q) = \tau(\bar{e}_1)$ and $t_1 = t_q^{-1}$ imply $r_q r_0 \notin f_{e_q}(G_{e_q})$. Through cyclic permutations (which can be effected by conjugation in $\Pi(\mathcal{G})$) and the relations of 3.2 one can transform any word W into a cyclically reduced word.

3.9. Proposition. *Closed, connected, cyclically reduced words V and W represent conjugate elements of $\Pi(\mathcal{G})$ if and only if there is a cyclic permutation $W_2 * W_1$ of $W = W_1 * W_2$ and an element $r \in G_v$ (where V is based at $v \in V(\mathcal{G})$) such that $V = r * W_2 * W_1 * r^{-1}$.*

Proof. If there exists such a cyclic permutation then clearly the elements are conjugate. Now suppose that V and W represent conjugate elements.

Let $U \in \Pi(\mathcal{G})$ be a reduced word with $V = UWU^{-1}$. As W is cyclically reduced, one has either $UW = U * W$ or $WU^{-1} = W * U^{-1}$. We assume the first case (the second works similarly) and obtain

$$\text{length}(V) = \text{length}(UWU^{-1}) \geq \text{length}(UW) - \text{length}(U^{-1}) =$$
$$\text{length}(U) + \text{length}(W) - \text{length}(U^{-1}) = \text{length}(W).$$

By symmetry between V and W we get length (V) = length (W). We can assume w.l.g. that U is not a reduced product $U = U' * W^k$ for any $k \geq 1$. Thus U^{-1} will have all stable letters cancelled against W when reducing the product UWU^{-1}. It follows that W is a reduced product of connected subwords $W = W_1 * W_2$ with length $(W_2U^{-1}) = 0$. Since W_1 and $V = U * W_1 * (W_2U^{-1})$ are connected, it follows by 3.7(b) that W_2U^{-1} is connected. Then $W_2U^{-1} \equiv r \in G_v$ where v is the terminal vertex of W_1 and the initial vertex of W_2. Hence $V = UWU^{-1} = (UW_2^{-1}) * W_2 * W_1 * (UW_2^{-1})^{-1} = r * W_2 * W_1 * r^{-1}$, as claimed. □

3.10. Lemma. *The subgroup $\pi_1(\mathcal{G}, v)$ is malnormal in $\Pi(\mathcal{G})$.*

Proof. Let V, $W \in \pi_1(\mathcal{G}, v)$ and $U \in \Pi(\mathcal{G})$ be reduced words with $V = U\,W\,U^{-1}$. Since malnormality of subgroups is invariant with respect to conjugation, we can assume that W is cyclically reduced. Following the proof of 3.9 we may write $W = W_1 * W_2$ and $V = U * W_1 * (W_2U^{-1})$ where W_1 is connected. It follows from 3.7(b) that U is a connected word which begins and ends at v. Thus $U \in \pi_1(\mathcal{G}, v)$. □

3.11. The malnormality given by 3.10, combined with Lemma 2.2, tells us that for any graphs of groups $\mathcal{G}_1$, $\mathcal{G}_2$ and vertices v, $v' \in V(\mathcal{G}_1)$ and w, $w' \in V(\mathcal{G}_2)$ there is a canonical identification between the outer homomorphisms from $\pi_1(\mathcal{G}_1, v)$ to $\pi_1(\mathcal{G}_2, w)$ and those from $\pi_1(\mathcal{G}_1, v')$ to $\pi_1(\mathcal{G}_2, w')$. To be precise, $\pi_1(\mathcal{G}_1, v') = \text{ad}_W(\pi_1(\mathcal{G}_1, v))$ if and only if $W \in \Pi(\mathcal{G}_1)$ is a connected word with initial vertex v' and terminal vertex v, and a similar statement holds for $V \in \Pi(\mathcal{G}_2)$. With such W and V, the outer homomorphism with representative $f : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, w)$

is identified with the outer homomorphism with representative $ad_V \circ f \circ ad_W^{-1}$: $\pi_1(\mathcal{G}_1, v') \to \pi_1(\mathcal{G}_2, w')$.

For any representative homomorphism $f : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, w)$, we thus suppress basepoints and denote the corresponding outer homomorphism simply by $\widehat{f} : \pi_1\mathcal{G}_1 \to \pi_1\mathcal{G}_2$. If $\mathcal{G} = \mathcal{G}_1 = \mathcal{G}_2,\ v = w$ and $v' = w'$ in the discussion above, the groups $\mathrm{Out}(\pi_1(\mathcal{G}, v))$ and $\mathrm{Out}(\pi_1(\mathcal{G}, v'))$ are thus canonically identified, and we denote this group by $\mathrm{Out}(\pi_1\mathcal{G})$.

4. Isomorphisms of graphs of groups

4.1. Definition. A *graph of groups isomorphism* $H : \mathcal{G}_1 \to \mathcal{G}_2$ is a quadruple of the form

$$H = (H_\Gamma, (H_v)_{v\in V(\mathcal{G}_1)},\ (H_e)_{e\in E(\mathcal{G}_1)},\ (\delta(e))_{e\in E(\mathcal{G}_1)})$$

where $H_\Gamma : \Gamma(\mathcal{G}_1) \to \Gamma(\mathcal{G}_2)$ is a graph isomorphism, and each $H_v : G_v \to G_{H_\Gamma(v)}$ and $H_e = H_{\bar{e}} : G_e \to G_{H_\Gamma(e)}$ is a group isomorphism. (In order to avoid double indices we will often write $H(e)$ and $H(v)$ instead of $H_\Gamma(e)$ or $H_\Gamma(v)$.) Moreover, $\delta(e) \in G_{\tau(H(e))}$ and (with ad_x as defined in §1),

$$H_{\tau(e)} f_e = ad_{\delta(e)}\ f_{H(e)} H_e\ . \qquad (*)$$

Note. This definition agrees with the restriction to 1-dimensional complexes of the definition by Haefliger [H] for isomorphisms of complexes of groups. It is a special case of the more general definition of morphism of graph of groups in Bass [B].

4.2. A graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ induces isomorphisms $H_* : \Pi(\mathcal{G}_1) \to \Pi(\mathcal{G}_2)$ and $H_{*v} : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, H_\Gamma(v))$, defined on generators by

$$H_*(r) = H_w(r) \quad \text{for} \quad w \in V(\mathcal{G}),\ r \in G_w, \quad \text{and}$$
$$H_*(t_e) = \delta(\bar{e})\, t_{H(e)} \delta(e)^{-1}\ .$$

We denote by $\widehat{H} : \pi_1\mathcal{G}_1 \to \pi_1\mathcal{G}_2$ the outer isomorphism induced by H_{*v}, where $\widehat{H}$ does not depend on the choice of $v \in V(\mathcal{G}_1)$, see 3.11 .

4.3. The composition of two graph of groups isomorphisms $H' : \mathcal{G}_1 \to \mathcal{G}_2$ and $H'' : \mathcal{G}_2 \to \mathcal{G}_3$ is a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_3$ which satisfies $\widehat{H} = \widehat{H''}\widehat{H'}$ and $H_* = H''_* H'_*$. To be precise, H is given by $H_\Gamma = H''_\Gamma H'_\Gamma$, $H_v = H''_{H'_\Gamma(v)} H'_v$, $H_e = H''_{H'_\Gamma(e)} H'_e$ and $\delta(e) = H''_{\tau(H'(e))}(\delta'(e))\delta''(H'_\Gamma(e))$ if $v \in V(\mathcal{G}_1),\ e \in E(\mathcal{G}_1)$.

In particular, for any $H : \mathcal{G}_1 \to \mathcal{G}_2$ there is an *inverse* isomorphism $H^{-1} : \mathcal{G}_2 \to \mathcal{G}_1$ which satisfies $\widehat{H}^{-1} = \widehat{H^{-1}}$ and $H_*^{-1} = (H^{-1})_*$. Moreover $(H^{-1})_{H(v)} = H_v^{-1}$ and $H_*^{-1}(t_{H(e)}) = H^{-1}_{\tau(\bar{e})}(\delta(\bar{e})^{-1})\, t_e H^{-1}_{\tau(e)}(\delta(e))$ for all $v \in V(\mathcal{G}_1),\ e \in E(\mathcal{G}_1)$.

4.4. Every graph of groups $\mathcal{G}$ gives rise to a tree $\mathcal{T}_{\mathcal{G}}$ on which $\pi_1(\mathcal{G}, v)$ acts (see [B,1.16], [CL2, §5]).

Bass-Serre theory is built so that the notions of "equivariantly isomorphic tree actions" and "isomorphic graphs of groups" are essentially equivalent. This is stated precisely in the following Lemmas 4.5, 4.6. (See Bass [B], Corollary 4.5 and Proposition 2.4 for detailed proofs.)

4.5. Lemma. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two graphs of groups with an isomorphism $h : \pi_1(\mathcal{G}_1, v_1) \to \pi_1(\mathcal{G}_2, v_2)$ and a simplicial homeomorphism $\widetilde{H} : \mathcal{T}_{\mathcal{G}_1} \to \mathcal{T}_{\mathcal{G}_2}$ which is h-equivariant (i.e. $\widetilde{H}(g \cdot x) = h(g) \cdot \widetilde{H}(x)$ for all $g \in \pi_1(\mathcal{G}_1, v_1)$ and all $x \in \mathcal{T}_{\mathcal{G}_1}$). Then there is a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ with $\widehat{H} = \hat{h} : \pi_1\mathcal{G}_1 \to \pi_1\mathcal{G}_2$.*

4.6. Lemma. *If $H : \mathcal{G}_1 \to \mathcal{G}_2$ is a graph of groups isomorphism, with induced isomorphism $h = H_{*_v} : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, H_\Gamma(v))$, then there exists an h-equivariant simplicial homeomorphism $\widetilde{H} : \mathcal{T}_{\mathcal{G}_1} \to \mathcal{T}_{\mathcal{G}_2}$.*

5. Dehn twists

5.1. Definition. [CL2, §6] A *Dehn twist* $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ consists of a graph of groups $\mathcal{G}$ and a family of elements $(z_e)_{e \in E(\mathcal{G})}$ with $z_e \in \mathrm{Center}(G_e)$ and $z_{\bar{e}} = z_e^{-1}$. ("D is based on $\mathcal{G}$ with *twistors* z_e"). This determines an automorphism $D_* : \Pi(\mathcal{G}) \to \Pi(\mathcal{G})$ given on the generators as follows:

$$D_*|_{G_v} = \mathrm{id} \ , \qquad D_*(t_e) = t_e f_e(z_e) \quad (v \in V(\mathcal{G}), \quad e \in E(\mathcal{G})).$$

The automorphism D_* restricts to an automorphism $D_v : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v)$ for every $v \in V(\mathcal{G})$ and hence defines (see 3.11) an outer automorphism $\widehat{D} \in \mathrm{Out}(\pi_1\mathcal{G})$.

5.2. For any Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ and any connected word $W \in \Pi(\mathcal{G})$ with initial vertex v' and terminal vertex v Definition 5.1 gives

$$D_{v'}\, ad_W = ad_{D_*(W)}\, D_v : \pi_1(\mathcal{G}, v) \to \pi_1(\mathcal{G}, v') \ .$$

5.3. Lemma. *Let $D_1 = D(\mathcal{G}_1, (z_e)_{e \in E(\mathcal{G}_1)})$ and $D_2 = D(\mathcal{G}_2, (z_e)_{e \in E(\mathcal{G}_2)})$ be Dehn twists and $H : \mathcal{G}_1 \to \mathcal{G}_2$ a graph of groups isomorphism which preserves twistors in that*

$$H_e(z_e) = z_{H(e)} \quad \text{for all} \quad e \in E(\mathcal{G}_1).$$

Then: **(a)** $H_* D_{1*} H_*^{-1} = D_{2*} : \Pi(\mathcal{G}_2) \to \Pi(\mathcal{G}_2)$.
(b) $\widehat{D_2} = \widehat{H}\widehat{D_1}\widehat{H}^{-1} \in \mathrm{Out}(\pi_1 \mathcal{G}_2)$.

Proof. The claim (a) follows from straightforward calculation on the generators of $\Pi(\mathcal{G}_2)$ using the formulas of 4.2 and 4.3. Then (b) follows from (3.11). □

5.4. Proposition. *Let $\mathcal{G}$ be a graph of groups with the property that (*) for every edge e there is an element $r_e \in G_{\tau(e)}$ with*

$$f_e(G_e) \cap r_e f_e(G_e) r_e^{-1} = \{1\} \ .$$

Then two Dehn twists $D' = D(\mathcal{G}, (z'_e)_{e \in E(\mathcal{G})})$, $D'' = D(\mathcal{G}, (z''_e)_{e \in E(\mathcal{G})})$ based on $\mathcal{G}$ determine the same outer automorphisms $\widehat{D'} = \widehat{D''} \in \mathrm{Out}(\pi_1 \mathcal{G})$ if and only if $z'_e = z''_e$ for all $e \in E(\mathcal{G})$.

Proof. Suppose $\widehat{D'} = \widehat{D''}$. For every edge $e \in E(\mathcal{G})$ we consider the element $w_e = t_e r_e t_{\bar{e}} r_{\bar{e}} \in \pi_1(\mathcal{G}, v)$, where $v = \tau(\bar{e})$. We compute

$$D'_*(w_e) = t_e f_e(z'_e) r_e f_e(z'_e)^{-1} t_{\bar{e}} r_{\bar{e}} \quad \text{and}$$
$$D''_*(w_e) = t_e f_e(z''_e) r_e f_e(z''_e)^{-1} t_{\bar{e}} r_{\bar{e}}.$$

Since $\widehat{D'} = \widehat{D''}$, these words represent conjugate elements in $\pi_1(\mathcal{G}, v)$. From 3.9 it follows that there is an element $s \in G_v$ such that $D'_*(w_e) = s D''_*(w_e) s^{-1}$. Then 3.6 implies that there exist elements $h_1, h_2 \in G_e$ with

(a) $f_{\bar{e}}(h_1)^{-1} = s$,
(b) $f_e(h_1) f_e(z'_e) r_e f_e(z'_e)^{-1} f_e(h_2)^{-1} = f_e(z''_e) r_e f_e(z''_e)^{-1}$, and
(c) $f_{\bar{e}}(h_2) r_{\bar{e}} = r_{\bar{e}} s^{-1}$.

From (a) and (c), $r_{\bar{e}} f_{\bar{e}}(h_1) r_{\bar{e}}^{-1} = f_{\bar{e}}(h_2)$. So (*) implies that $h_1 = h_2 = 1$. Then (b) and (*) imply that $z_{e'}^{-1} z''_e = 1$.
The converse implication is obvious. □

5.5. An alternative viewpoint, which we will not adopt in this paper, is to consider a Dehn twist $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ as a graph of groups automorphism $\mathcal{D} : \mathcal{G} \to \mathcal{G}$, with identity map for the graph automorphism $\mathcal{D}_\Gamma$ and identity maps for all the group automorphisms $\mathcal{D}_v$ and $\mathcal{D}_e$. Furthermore the elements $\delta(e)$ are chosen to satisfy $\delta(\bar{e}) t_e \delta(e)^{-1} = t_e f_e(z_e)$ for all $e \in E(\mathcal{G})$. (This can be achieved, for example, if one chooses a representative e^+ from each set $\{e, \bar{e}\}$ and defines $\delta(e^+) = f_{e^+}(z_{e^+})^{-1}$ and $\delta(\bar{e}^+) = 1$). It follows directly from the definitions in 4.2 and 5.1 that this graph of groups automorphism induces the same automorphisms as D on $\Pi(\mathcal{G})$ and on $\pi_1(\mathcal{G}, v)$, for any $v \in V(\mathcal{G})$.

6. Efficient Dehn twists

6.1. General assumption. For the rest of the paper we always assume that for any graph of groups $\mathcal{G}$ the fundamental group $\pi_1\mathcal{G}$ is a free group of finite rank $n \geq 2$. We remind the reader that the graph $\Gamma(\mathcal{G})$ is always finite and connected.

6.2. Definition. A Dehn twist $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ is called *efficient* (called *proper with all twistors non-trivial* in [CL2, §13]) if the following conditions hold:

(i) $\mathcal{G}$ is *minimal*: There is no vertex v of valence 1 with $v = \tau(e)$ and surjective edge map $f_e : G_e \to G_v$.

(ii) There is no *invisible vertex*: There is no vertex v of valence 2 with $v = \tau(e_1) = \tau(e_2)$ $(e_1 \neq e_2)$ such that both edge maps $f_{e_i} : G_{e_i} \to G_v,\ i = 1, 2$, are surjective.

(iii) There are *no unused edges*: For every edge e the twistor is non-trivial: $1 \neq z_e \in \text{center}(G_e)$. In particular one has $G_e \cong \mathbb{Z}$ for all edges $e \in E(\mathcal{G})$.

(iv) There are no *proper powers*: If $r^p \in f_e(G_e)$ and $p \neq 0$ then $r \in f_e(G_e)$.

(v) If $v = \tau(e_1) = \tau(e_2)$ then the edges e_1 and e_2 are not *positively bonded*: There are no positive powers m, n such that $f_{e_1}(z_{e_1}^m)$ is conjugate to $f_{e_2}(z_{e_2}^n)$ in G_v.

6.3. Remark. Conditions (iii) and (v) imply that:

(vi) There are *no conjugate triples*: There is no vertex $v = \tau(e_1) = \tau(e_2) = \tau(e_3)$, where e_1, e_2, e_3 are distinct edges, such that there exist non-trivial $y_i \in G_{e_i}$ $(i = 1, 2, 3)$ with $f_{e_1}(y_1), f_{e_2}(y_2)$ and $f_{e_3}(y_3)$ all conjugate in G_v.

Conditions (iii), (iv) and (vi) assert that $\mathcal{G}$ is a *very small* graph of groups, as defined in [CL2].

6.4. *Lemma. Let $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ be an efficient Dehn twist. Then every vertex group G_v of $\mathcal{G}$ has rank at least 2. In particular $\mathcal{G}$ satisfies the condition (*) in Proposition 5.4.*

Proof. From 6.2 (iii) and the assumption in 3.1 that $\Gamma(\mathcal{G})$ is connected it follows that every vertex group has rank at least 1. If for some vertex v of $\Gamma(\mathcal{G})$ one has $G_v \cong \mathbb{Z}$, then it follows from 6.2 (iv) that f_e is surjective for all edges e with $\tau(e) = v$. Hence, by 6.2 (i) and (ii), there are at least 3 distinct such edges. By 6.2 (iii) each has non-trivial twistor. But then at least 2 of them are positively bonded, in contradiction to 6.2 (v). Hence every G_v has rank at least 2.

By 6.1 and 6.2 (iii), (iv) the group $f_e(G_e)$ is a maximal cyclic subgroup in $G_{\tau(e)}$. But then we can pick any $r_e \in G_{\tau(e)} - f_e(G_e)$ to satisfy the condition (*) in Proposition 5.4., since maximal cyclic subgroups of free groups are malnormal. This follows easily from considering the standard free action of the free group on a simplicial tree and the minimal subtree (a line !) fixed by the maximal cyclic subgroup. □

6.5. Lemma. *Let $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ be an efficient Dehn twist. Let $W = r_0 t_1 r_1 \ldots\ t_q r_q \in \Pi(\mathcal{G})$ be a reduced word of length q. Then $D_*(W) = W$ if and only if $q = 0$.*

Proof. By definition, words of length 0 are fixed by D_*. To see that words of greater length are not fixed, start by choosing as generator of the cyclic group G_e, for each edge e, the element a_e such that $a_e^{n(e)} = z_e$ with $n(e) > 0$. Since $z_{\bar{e}} = {z_e}^{-1}$ this choice dictates that $n(e) = n(\bar{e})$ and $a_{\bar{e}} = {a_e}^{-1}$. In the word W let each $t_j = t_{e_j}$, and set $n(e_j) = n_j$ and $a_j = a_{e_j}$. Let $x_j = f_{e_j}(z_{e_j}) = f_{e_j}(a_j^{n_j})$

We **claim** that: $W = D_*(W)$ with $q > 0$ would imply

$$1 \ = \ {r_q}^{-1}\, f_{e_q}(a_q^{n_1+n_2+\cdots+n_q})\, r_q$$

This is impossible since in an efficient graph of groups there are no unused edges, so that $a_q \neq 1$ and each $n_j > 0$.

When $q = 1$, $W^{-1}D_*(W) = (r_0t_1r_1)^{-1}(r_0t_1x_1r_1)$, and the claim is immediate. When $q > 1$, note that $D_*(W)$ is reduced, since W is reduced. So in the following equation with non-reduced right-hand side,

$$1 = W^{-1}D_*(W) = r_q^{-1}t_q^{-1}\ldots r_2^{-1}(t_2^{-1}r_1^{-1}x_1r_1t_2)x_2r_2\ldots t_qx_qr_q,$$

we must have ${r_1}^{-1}x_1r_1 = r_1^{-1}f_{e_1}(a_{e_1}^{n_1})r_1 \in f_{\bar{e}_2}(G_{\bar{e}_2})$ and $\tau(e_1) = \tau(\bar{e}_2)$, $r_1 \in G_{\tau(e_2)}$. Since there are no proper powers by 6.2 (iv), $r_1^{-1}f_{e_1}(a_{e_1})r_1$ itself belongs to $f_{\bar{e}_2}(G_{\bar{e}_2})$. Thus it is a is a power of $f_{\bar{e}_2}(a_{\bar{e}_2})$. But since $r_1^{-1}f_{e_1}(G_{e_1})r_1$ also contains no proper powers, a symmetric argument gives

$$r_1^{-1}f_{e_1}(a_{e_1})r_1 = \Big(f_{\bar{e}_2}(a_{\bar{e}_2})\Big)^{\pm 1}$$

By 6.2 (v), the edges e_1 and $\bar{e}_2$ are not positively bonded, so the exponent sign above is negative. Since $a_{\bar{e}_2} = a_{e_2}^{-1}$ we conclude (using 3.2) that

$$\begin{aligned} 1 = W^{-1}D_*(W) &= r_q^{-1}t_q^{-1}\ldots r_2^{-1}f_{e_2}(a_{e_2}^{n_1})x_2r_2\ldots t_qx_qr_q \\ &= r_q^{-1}t_q^{-1}\ldots r_2^{-1}f_{e_2}(a_{e_2}^{n_1+n_2})r_2\ldots t_qx_qr_q \end{aligned}$$

Continuing in this fashion, the claim is proved. □

6.6. Lemma. *Let $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ be an efficient Dehn twist, and consider vertices $v, v' \in V(\mathcal{G})$. Suppose that $W \in \Pi(\mathcal{G})$ is a connected word with initial vertex v' and terminal vertex v such that*

$$D_{v'} = ad_W\, D_v\, ad_{W^{-1}} : \pi_1(\mathcal{G}, v') \to \pi_1(\mathcal{G}, v')\ .$$

Then $v = v'$ and $W \in G_v$.

Proof. Straightforward calculation using 5.2 gives for any $U \in \pi_1(\mathcal{G}, v')$:

$$D_{v'}(U) \ = \ ad_W \, (ad_{W^{-1}D_*(W)} \, D_v) \; ad_{W^{-1}}(U) \ .$$

Thus our hypothesis implies that $ad_{W^{-1}D_*(W)} = 1 \in \mathrm{Aut}(\pi_1(\mathcal{G}, v))$. Since the Dehn twist is efficient, $\pi_1(\mathcal{G}, v)$ is a free group of rank at least 2, by 6.3. Since $W^{-1} \, D_*(W)$ lies in the center of this free group, and hence is trivial, this lemma follows from 6.5. □

Key Observation

The key observation of this paper, stated in Theorem 6.9 below (roughly the converse of Lemma 5.3), is that if two efficient Dehn twists determine conjugate outer automorphisms, then their graph of groups data are necessarily equal. Thus in an algorithm for conjugacy it is sufficient to check the graph of groups data. The main ingredient in justifying this observation is the Parabolic Orbits Theorem [CL2, 13.2]. For the convenience of the reader, we now describe its content, in a weakened form and with the terminology adapted to the conventions in this paper.

6.7. Consider the Culler–Vogtmann space CV_n (also called "outer space") of free actions of F_n on metric simplicial $\mathbb{R}$-trees (i.e. on trees $\mathcal{T}_{\mathcal{G}'}$ as in 4.4 with $\pi_1(\mathcal{G}', v) \cong F_n$ and all edge groups and vertex groups trivial, provided with a $\pi_1(\mathcal{G}', v)$-equivariant length on the edges). An efficient Dehn twist $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ defines [CL2, §9] (after having chosen a "marking", i.e. an identification $\pi_1(\mathcal{G}, v) = F_n$) a simplex $\sigma(\mathcal{G})$ on the boundary of CV_n, given by all possible lengths on the edges of $\Gamma(\mathcal{G})$. The Parabolic Orbits Theorem then states that under both forward or backward iteration of the induced action of $\widehat{D}$ every point $[\mathcal{T}_{\mathcal{G}'}]$ of CV_n converges to a point in $\sigma(\mathcal{G})$ (which can be precisely determined in terms of the translation lengths on $\mathcal{T}_{\mathcal{G}'}$ of the twistors of D). As $\sigma(\mathcal{G})$ determines $\mathcal{T}_{\mathcal{G}}$ (up to $\pi_1(\mathcal{G}, v)$-equivariant isomorphisms), one derives as a consequence:

6.8. Corollary. ([CL2, 13.4], adapted version). *Let $D_1 = D(\mathcal{G}_1, (z_e)_{e\in E(\mathcal{G}_1)})$, $D_2 = D(\mathcal{G}_2, (z_e)_{e\in E(\mathcal{G}_2)})$ be efficient Dehn twists and let $h : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, w)$ be an isomorphism with $\widehat{D_2} = \hat{h}\widehat{D_1}\hat{h}^{-1} \in \mathrm{Out}(\pi_1\mathcal{G}_2)$. Then there exists an h-equivariant simplicial homeomorphism $\widetilde{H} : \mathcal{T}_{\mathcal{G}_1} \to \mathcal{T}_{\mathcal{G}_2}$.*

We have now assembled all ingredients necessary to prove Theorem 1.1 of the Introduction:

6.9. Theorem. *Let $D_1 = D(\mathcal{G}_1, (z_e)_{e\in E(\mathcal{G}_1)})$ and $D_2 = D(\mathcal{G}_2, (z_e)_{e\in E(\mathcal{G}_2)})$ be efficient Dehn twists inducing automorphisms D_v and D_w of $\pi_1(\mathcal{G}_1, v)$ and $\pi_1(\mathcal{G}_2, w)$ respectively. Let $h : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, w)$ be an isomorphism.*

(a) *If* $\widehat{D_2} = \hat{h}\widehat{D_1}\hat{h}^{-1} \in \mathrm{Out}(\pi_1\mathcal{G}_2)$ *then there exists a graph of groups isomorphism* $H : \mathcal{G}_1 \to \mathcal{G}_2$ *with* $\widehat{H} = \hat{h}$ *and* $H_e(z_e) = z_{H(e)}$ *for all* $e \in E(\mathcal{G}_1)$.
(b)*If* $D_w = h\ D_v h^{-1} \in \mathrm{Aut}(\pi_1(\mathcal{G}_2, w)$ *then there exists a graph of groups isomorphism* $H : \mathcal{G}_1 \to \mathcal{G}_2$ *with* $H_\Gamma(v) = w$, $H_{*v} = h$ *and* $H_e(z_e) = z_{H(e)}$ *for all* $e \in E(\mathcal{G}_1)$.

Proof. **(a)** From Corollary 6.8 and Lemma 4.5 it follows that there is a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ with $\widehat{H} = \hat{h}$. We consider the Dehn twist $D' = D(\mathcal{G}_2, (H_e(z_e))_{H(e)\in E(\mathcal{G}_2)})$. By Lemma 5.3 we have $\widehat{D'} = \widehat{H}\widehat{D_1}\widehat{H}^{-1} \in \mathrm{Out}(\pi_1\mathcal{G}_2)$. Hence, since $\widehat{H} = \hat{h}$, the hypothesis $\widehat{D_2} = \hat{h}\widehat{D_1}\hat{h}^{-1}$ implies $\widehat{D'} = \widehat{D_2} \in \mathrm{Out}(\pi_1\mathcal{G}_2)$. Thus Lemma 6.4 and Proposition 5.4 prove $H_e(z_e) = z_{H(e)}$ for all $e \in E(\mathcal{G}_1)$.
(b) Consider the outer automorphisms $\widehat{D_1} \in \mathrm{Out}(\pi_1(\mathcal{G}_1))$ and $\widehat{D_2} \in \mathrm{Out}(\pi_1(\mathcal{G}_2))$ which are determined by D_v and D_w. By Part (a) there exists a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ with $\widehat{H} = \hat{h}$ and $H_e(z_e) = z_{H(e)}$ for all $e \in E(\mathcal{G}_1)$. By definition 4.2, $\widehat{H} = \widehat{H_{*v}}$ where $H_{*v} : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, H_{\Gamma(v)})$. Thus $\widehat{H_{*v}} = \hat{h}$, and by 3.11 there is a connected word $W \in \Pi(\mathcal{G}_2)$ with initial vertex $H_{\Gamma(v)}$ and terminal vertex w such that

$$H_{*v} \ = \ \mathrm{ad}_W\, h\ .$$

Again, let $D' = D\big(\mathcal{G}_2, (H_e(z_e))_{H(e)\in E(\mathcal{G}_2)}\big)$. By 5.3 we then have

$$D'_* = H_* D_* H_*^{-1} : \ \Pi(\mathcal{G}_2) \to \Pi(\mathcal{G}_2)$$

and hence

$$\begin{aligned} D'_{H_\Gamma(v)} &= H_{*v}\, D_v\, H_{*v}^{-1} \\ &= \left(H_{*v} h^{-1}\right) D_w \left(h H_{*v}^{-1}\right) \\ &= \mathrm{ad}_W\, D_w\, \mathrm{ad}_{W^{-1}} \colon\ \pi_1\big(\mathcal{G}_2, H_\Gamma(v)\big) \to \pi_1\big(\mathcal{G}_2, H_\Gamma(v)\big) \end{aligned}$$

Thus by Lemma 6.6, $H_\Gamma(v) = w$ and $W \in G_w$. Since $H_{*v} = ad_W\, h : \pi_1(\mathcal{G}_1, v) \to \pi_1(\mathcal{G}_2, w)$ we may now define a new graph of groups isomorphism $K : \mathcal{G}_1 \to \mathcal{G}_2$ with the desired properties by letting K be identical with H, except that, for edges $e \in E(\mathcal{G}_1)$ with $\tau(\bar{e}) = v$, we have $\delta_K(e) = \mathrm{ad}_{W^{-1}}\, \delta_H(e)$. □

7. Centralizer, fixed subgroup and index of an efficient Dehn twist

In this section we derive some properties of efficient Dehn twists which follow easily from the material presented in the previous sections. Notice that the subsequent §8 can be read independently from this section.

7.1. Proposition. **(a)** *Let $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ be an efficient Dehn twist with induced outer automorphism $\widehat{D} \in Out(\pi_1(\mathcal{G}))$. Then the centralizer $C(\widehat{D})$ of $\widehat{D}$ in $Out(\pi_1(\mathcal{G}))$ is given by*

$$C(\widehat{D}) = \{\widehat{H} \mid H : \mathcal{G} \xrightarrow{\cong} \mathcal{G}, \quad H_e(z_e) = z_{H(e)} \;\; \textit{for all } e \in E(\mathcal{G})\}.$$

(b) *If $v \in V(\mathcal{G})$ the centralizer of the automorphism D_v in $Aut(\pi_1(\mathcal{G}, v))$ is given by*

$$C(D_v) = \{H_{*v} \mid H : \mathcal{G} \xrightarrow{\cong} \mathcal{G}\,,\; H_e(z_e) = z_{H(e)} \;\; \textit{for all} \;\; e \in E(\mathcal{G})\,,\; H_\Gamma(v) = v\}.$$

Proof. In both (a) and (b) it follows from the special case $D_1 = D_2$ in Lemma 5.3 that the right hand side is contained in the left hand side. The opposite inclusions follow similarily from Theorem 6.9 (with $v = w$ in case (b)). □

7.2. Proposition. *If $D = D(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})})$ is an efficient Dehn twist and v is a vertex of $\mathcal{G}$, then $Fix(D_v) = G_v$.*

Proof. This is an immediate corollary of 6.5 . □

7.3. For any graph of groups $\mathcal{G}$, with finitely generated fundamental group isomorphic to a free group F_n and finitely generated vertex and edge groups, an elementary Mayer-Vietoris argument in group homology (pointed out to us by M. Bridson) gives the formula

$$rk(\pi_1(\mathcal{G})) \;=\; \sum_{v\in V(\mathcal{G})} rk(G_v) \;-\; \sum_{e\in E^+(\mathcal{G})} rk(G_e) \;+\; rk(\pi_1(\Gamma(\mathcal{G}))\;,$$

where $E^+(\mathcal{G})$ consists of one "positively oriented" edge from each pair of oppositely oriented edges in $E(\mathcal{G})$ — so that $\#E^+(\mathcal{G})$ is the number of geometric edges in the graph. But note that $rk(\pi_1(\Gamma(\mathcal{G})) = 1 - \chi(\Gamma(\mathcal{G})) = 1 + \#E^+(\mathcal{G}) - \#V(\mathcal{G})$. Thus, if every group G_e is isomorphic to $\mathbb{Z}$, we obtain:

$$\begin{aligned} rk(\pi_1(\mathcal{G})) - 1 \;&=\; \sum_{v\in V(\mathcal{G})} rk(G_v) \;-\; \#E^+(\mathcal{G}) \;+\; (\;1 + \;\#E^+(\mathcal{G}) - \#V(\mathcal{G})\;) \;-1 \\ &=\; \sum_{v\in V(\mathcal{G})} (rk(G_v) \;-\; 1) \end{aligned}$$

7.4. In [GJLL] the *index* of a free group automorphism Φ is defined by

$$ind(\Phi) \;=\; rk(\mathrm{Fix}(\Phi)) \;+\; a(\Phi)\,/\,2 \;-\; 1$$

where $a(\Phi)$ denotes the number of equivalence classes of attracting infinite fixed words of Φ , see [GJLL] or [CL1]. The main theorem of [GJLL] asserts that if $\Phi_1, \Phi_2, \cdots \Phi_q$ are automorphisms of a free group F_n of rank n which determine the same outer automorphism class ϕ and are pairwise *non-similar* (i.e., $\Phi_i \neq ad_u \, \Phi_j \, ad_{u^{-1}}$ for any $u \in F_n$ if $i \neq j$), then one has

$$\sum_{i=1,\ldots q} ind(\Phi_i) \ \leq \ n \ - \ 1 \ .$$

The index $ind(\phi)$ of the outer automorphism ϕ is defined to be the maximum of the left hand side, taken over all possible sets of pairwise non-similar automorphisms $\{\Phi_1, \Phi_2, \ldots , \Phi_q\}$.

7.5. Consider vertices $v, w \in V(\mathcal{G})$ and a connected word $W \in \Pi(\mathcal{G})$ with initial vertex w and terminal vertex v. Then 5.2 gives

$$ad_W D_v \, ad_{W^{-1}} = ad_{W D_*(W^{-1})} D_w \colon \ \pi_1(\mathcal{G}, w) \to \pi_1(\mathcal{G}, w) \ ,$$

and, as $W D_*(W^{-1}) \in \pi_1(\mathcal{G}, w)$, this automorphism also represents the outer automorphism $\hat{D}$. Furthermore, by 7.2, one has

$$\text{Fix } (ad_W D_v ad_{W^{-1}}) = ad_W(\text{Fix } (D_v)) = ad_W(G_v) \ ,$$

and hence, by Lemma 6.4 , $rk(\text{Fix } (ad_W D_v ad_{W^{-1}}) \geq 2$.

7.6. Lemma. *Let $W, W' \in \Pi(\mathcal{G})$ be connected words with initial vertex w and terminal vertices v and v' respectively. Then the automorphism $ad_W D_v ad_{W^{-1}}$ and $ad_{W'} D_{v'} ad_{W'^{-1}}$ of $\pi_1(\mathcal{G}, w)$ are similar if and only if $v = v'$.*

Proof. If $v = v'$ then $W'W^{-1} \in \pi_1(\mathcal{G}, w)$ and $ad_{W'W^{-1}} \ (ad_W D_v \, ad_{W^{-1}}) \ ad_{WW'^{-1}}$ $= ad_{W'} D_{v'} ad_{W'^{-1}}$. Hence the two automorphisms are similar.

On the other hand, for any $W'' \in \pi_1(\mathcal{G}, w)$ the equation

$$ad_{W''}(ad_W D_v ad_{W^{-1}}) ad_{W''^{-1}} = ad_{W'} D_{v'} ad_{W'^{-1}}$$

is equivalent to

$$ad_{W'^{-1} W'' W} D_v ad_{W^{-1} W''^{-1} W'} = D_{v'} \ .$$

Thus Lemma 6.6 implies $v = v'$. □

As a direct consequence of 7.2 - 7.6 we obtain:

7.7. Corollary. *Let $D = D(\mathcal{G}, (z_e)_{e \in E(\mathcal{G})})$ be an efficient Dehn twist and let $w \in V(\mathcal{G})$ be any vertex. We have:*

(a) *$ind(\hat{D}) = rk(\pi_1(\mathcal{G})) - 1$.*
(b) *Every representative $\Phi \in Aut(\pi_1(\mathcal{G}, w))$ of $\hat{D}$ with $ind(\Phi) > 0$ is similar to $ad_W D_v ad_{W^{-1}}$ for some vertex $v \in V(\mathcal{G})$ and any connected word W with initial vertex w and terminal vertex v.*
(c) *None of the D_v has attracting infinite fixed words: $a(D_v) = 0$ for all $v \in V(\mathcal{G})$.* □

Notice also that a particular interesting case arises if $\Gamma(\mathcal{G})$ has only one vertex v, as then rank $\mathrm{Fix}(D_v) = n$. Automorphisms $\Phi \in Aut(F_n)$ with $\mathrm{rank}(\mathrm{Fix}(\Phi)) = n$ have been investigated in [CT]; it follows from [L2] or [CT] that they are in fact all Dehn twists with a single vertex. Hence our solution to the conjugacy problem applies also to this class of automorphisms.

We conclude this section by restating our structural result in Theorem 6.9 in a language which comes close to a "normal form" for Dehn twists:

7.8. Assume we are given a finite connected graph Γ and two finite families of integers $(r(v))_{v \in V(\Gamma)}$ and $(n(e))_{e \in E(\Gamma)}$, all of which elements satisfy $r(v) \geq 2$, $n(e) \geq 1$ and $n(e) = n(\bar{e})$. Furthermore let $(\alpha_e)_{e \in E(\Gamma)}$ be a family of elements $\alpha_e \in F_{r(\tau(e))}$, where F_n denotes the (standard copy of a) free group of rank $n \in \mathbb{N}$.

We consider the graph of groups $\mathcal{G}$ given by $\Gamma(\mathcal{G}) = \Gamma$, $\gamma_v\colon G_v \xrightarrow{\cong} F_{r(v)}$, $\gamma_e\colon G_e \xrightarrow{\cong} \mathbb{Z}$ and $f_e(a_e) = \gamma_v^{-1}(\alpha_e)$, with a_e defined through $a_e = \gamma_e^{-1}(1)$ (for all $v \in V(\Gamma)$, $e \in E(\Gamma)$). We assume that $\gamma_{\bar{e}} = -\gamma_e$.

Let D be the Dehn twist based on $\mathcal{G}$ with twistors $a_e^{n(e)}$. If D is efficient, then we call the *data set*

$$\Delta = (\Gamma, (r(v))_{v \in V(\Gamma)}, (n(e))_{e \in E(\Gamma)}, (\alpha_e)_{e \in E(\Gamma)})$$

efficient and write $D = D(\Delta)$, $\mathcal{G} = \mathcal{G}(\Delta)$. Obviously we can associate to every efficient Dehn twist D an efficient data set Δ with $D(\Delta) = D$.

7.9. Corollary. *(a) Two Dehn twists $D = D(\Delta)$ and $D' = D(\Delta')$, defined by efficient data sets $\Delta = (\Gamma(r(v))_{v \in V(\Gamma)}, (n(e))_{e \in E(\Gamma)}, (\alpha_e)_{e \in E(\Gamma)})$ and $\Delta' = (\Gamma', (r(v))_{v \in V(\Gamma')}, (n(e))_{e \in E(\Gamma')}, (\alpha_e)_{e \in E(\Gamma'})$, determine conjugate outer automorphisms if and only if there is a graph isomorphism $H\colon \Gamma \to \Gamma'$ and automorphisms $\Phi_v\colon F_{r(v)} \to F_{r(v)}$ for each $v \in V(\Gamma)$, which satisfy*
(i) *$r(v) = r(H(v))$ for all $v \in V(\Gamma)$,*
(ii) *$n(e) = n(H(e))$ for all $e \in E(\Gamma)$, and*
(iii) *$[\alpha_{H(e)}] = \Phi_{r(e)}[\alpha_e]$ for all $e \in E(\Gamma)$ (where brackets denote the conjugacy class).*
(b) For vertices $w \in V(\Gamma), w' \in V(\Gamma')$ the automorphisms D_w and $D'_{w'}$ determined by D and D' as in (a) are conjugate if and only if there are H and Φ_v as above, which satisfy the conditions (i) - (iii), and furthermore

(iv) $H(w) = w'$.

Proof. This is a direct consequence of Theorem 6.9. □

8. The algorithm

In this section we describe a complete algorithm which decides whether or not two given Dehn twists define conjugate outer automorphisms. The algorithm consists of two parts: The first part (Algorithm I) is an algorithm which, given an arbitrary Dehn twist D, constructs an efficient Dehn twist D' with induced outer automorphism $\widehat{D'}$ conjugate to $\widehat{D}$. The second part (Algorithm II) checks whether two given efficient Dehn twists D_1, D_2 define conjugate outer automorphisms.

8.1. Throughout this section we assume that the data for $D = D\big(\mathcal{G}, (z_e)_{e\in E(\mathcal{G})}\big)$ are given combinatorially as follows: For every vertex group G_v and edge group G_e one has specified some basis and has given the injections f_e in terms of these bases. If the twistor $z_e \neq 1$ then G_e is infinite cyclic, since it has non-trivial center, and we choose the generator a_e so that $z_e = {a_e}^{n(e)}$ with $n(e) > 0$. We call the exponent $n(e) \in \mathbb{Z}$ the *twist exponent*. If $z_e = 1$ we set $n(e) = 0$. This convention implies that $n(e) = n(\bar{e})$ and $a_{\bar{e}} = a_e^{-1}$, since $z_{\bar{e}} = z_e^{-1}$.

8.2. The Moves. We describe 5 operations by which a given Dehn twist $D = D(\mathcal{G},\, (a_e^{n(e)})_{e\in E(\mathcal{G})})$ may be changed to a Dehn twist $D' = D(\mathcal{G}',\, (a_e^{n(e)})_{e\in E(\mathcal{G}')})$ with induced automorphism $\widehat{D'}$ conjugate to $\widehat{D}$:

(1) *Transition to a proper subgraph:* If $\Gamma(\mathcal{G})$ contains a vertex v of valence 1 and if the adjacent edge e has $f_e : G_e \to G_v$ an isomorphism, then $\mathcal{G}'$ is obtained from $\mathcal{G}$ by deleting both v and e, with all other data unchanged.

(2) *Delete an invisible vertex with negatively bonded edges:* If a vertex v of $\Gamma(\mathcal{G})$ is adjacent to precisely 2 edges e' and e'' (both oriented towards v), and if both edge injections $f_{e'}, f_{e''}$ are isomorphisms such that $f_{e'}(a_{e'}) = f_{e''}(a_{e''}^{-1})$, then $\mathcal{G}'$ is derived from $\mathcal{G}$ by replacing e', e'' and v by a single edge e which runs from $\tau(\bar{e}')$ to $\tau(\bar{e}'')$. Define $G_e = G_v$, $f_e = f_{\bar{e}''} f_{e''}^{-1}$, $f_{\bar{e}} = f_{\bar{e}'} f_{e'}^{-1}$, $a_e = f_{e'}(a_{e'})$ and $n(e) = n(e') + n(\bar{e}'')$.

(3) *Fold positively bonded edges:* Consider two edges e, e' with twist exponents $0 < n(e) \le n(e')$ and common end point $\tau(e) = \tau(e') = v$, such that $f_e(a_e) = r\, f_{e'}(a_{e'})\, r^{-1}$ for some $r \in G_v$. Replace e' by a new edge e'' which joins $\tau(\bar{e}')$ to $\tau(\bar{e})$ and define $G_{e''} = G_{e'}$, $a_{e''} = a_{e'}$, $f_{\bar{e}''} = f_{\bar{e}'}$, $f_{e''}(a_{e''}) = f_{\bar{e}}(a_e)$ and $n(e'') = n(e') - n(e)$.

(4) *Contract unused edges:* If e is an edge with $a_e^{n(e)} = 1$, then replace e together with its endpoints $\tau(e)$, $\tau(\bar{e})$ by a single vertex v with vertex group an amalgamated product

$$G_v = G_{\tau(\bar{e})} \underset{G_e}{*} G_{\tau(e)}, \quad \text{if} \quad \tau(e) \neq \tau(\bar{e}) \ ,$$

or an HNN–extension

$$G_v = \langle G_{\tau(e)}, \theta \mid \theta^{-1} f_{\bar{e}}(G_e)\, \theta = f_e(G_e) \rangle \ , \quad \text{if} \quad \tau(e) = \tau(\bar{e}) \ .$$

For any edge e' pointing towards $\tau(e)$ or $\tau(\bar{e})$ replace $f_{e'}$ by the composition of $f_{e'}$ with the canonical injection of $G_{\tau(e)}$ or $G_{\tau(\bar{e})}$ into G_v.

(5) *Get rid of proper powers:* If the generator a_e of an edge group G_e is mapped by f_e to a proper power $g^p \in G_{\tau(e)}$, then we change $\mathcal{G}$ as follows: We replace e by a new edge e' with same endpoints as e and $G_{e'} = \langle g \rangle, \ a_{e'} = g, \ n(e') = pn(e)$. We replace the vertex group $G_{\tau(\bar{e})}$ by a new group

$$G_{\tau(\bar{e}')} = G_{\tau(\bar{e})} \underset{G_e}{*} \langle g \rangle$$

and define $f_{e'}$ as well as $f_{\bar{e}'}$ to be the canonical injections. For any edge e'' pointing towards $\tau(\bar{e})$ replace $f_{e''}$ by the composition of $f_{e''}$ with the canonical injection of $G_{\tau(\bar{e})}$ into $G_{\tau(\bar{e}')}$.

8.3. Lemma. *For any of the operations (1) – (5) in Definition 8.2 and proper choices of vertices $w \in V(\mathcal{G}), \ w' \in V(\mathcal{G}')$ there is an isomorphism between fundamental groups $\rho : \pi_1(\mathcal{G}, w) \to \pi_1(\mathcal{G}', w')$ which satisfies $\widehat{D'}\hat{\rho} = \hat{\rho}\widehat{D}$.*

Proof. For operation (1) we choose $w \neq v$ and define w' to be the corresponding vertex of $\Gamma(\mathcal{G}')$. The isomorphism ρ is given through replacing every letter in a reduced word of $\pi_1(\mathcal{G}, w)$ by the corresponding letter of $\pi_1(\mathcal{G}', w')$. For the other operations we proceed similarily: For (2) we choose $w \neq v$ and define ρ through deleting in every word any occurence of $t_{e'}$ or $t_{e'}^{-1}$, and through replacing any $t_{e''}$ by t_e^{-1} (and $t_{e''}^{-1}$ by t_e) as well as any $r \in G_v$ by $f_{\bar{e}}(r)$, while leaving all other symbols unchanged. For (3) choose w arbitrary and define ρ through replacing any $t_{e'}$ by the product $t_{e''} t_e r$ (and any $t_{\bar{e}'}$ by $r^{-1} t_{\bar{e}} t_{\bar{e}''}$). For (4) let $w \neq \tau(\bar{e}), \tau(e)$ (and thus $w' \neq v$), and in the amalgamated product case simply delete any t_e , whereas in the HNN case replace t_e by θ (and similarily with the inverses). For operation (5) we chose w arbitrary and replace t_e by $t_{e'}$ (and $t_{\bar{e}}$ by $t_{\bar{e}'}$) .
The equation $\widehat{D'}\hat{\rho} = \hat{\rho}\widehat{D}$ can be verified directly from these definitions for ρ and from the Definition 5.1. □

8.4. Proposition. *For any Dehn twist one can iteratively apply the operations (1) – (5) from 8.2 only a finite number of times.*

Proof. Notice that the following facts are true for each of the operations (1) – (5):

(a) The number of edges does not increase when passing from $\mathcal{G}$ to $\mathcal{G}'$.
(b) In the free group $\pi_1(\mathcal{G}', v')$ the set of conjugacy classes of edge group generators $\{[f_e(a_e)] \mid e \in E(\mathcal{G}')\}$ differs from the set $\{\rho[f_e(a_e)] \mid e \in E(\mathcal{G})\}$ (for ρ as in 8.3) only in that some classes of the latter may have been deleted or replaced by proper roots.

Hence (1), (2), (4) and (5) can be applied only a finite number of times: The operations (1), (2) and (4) strictly decrease the number of edges in $\mathcal{G}$, whereas (5) replaces some of the $\rho[f_e(a_e)]$ by a proper root. But every conjugacy class in a free group has only finitely many proper roots.

Operation (3) strictly decreases the total number of twist exponents and hence can be applied only finitely many times before, after or between the operations of type (1), (2), (4) or (5). □

8.5. Proposition. *If none of the operations (1) – (5) can be performed on a given Dehn twist D, then D is efficient.*

Proof. This follows directly from the definitions. □

8.6. Algorithm I. Given any Dehn twist D, check whether any of the operations (1) – (5) in 8.2 can be performed. (As D is given as in 8.1 this can be done in finitely many steps.) If so, do the operation; in case more than one is possible, chose any one at random. Then rename the obtained Dehn twist D' to D and repeat the procedure. If none of the operations can be performed, stop.

Summarizing 8.3 – 8.6 we obtain:

8.7. Corollary. *Algorithm I transforms any Dehn twist in finitely many steps into an efficient Dehn twist without changing the conjugacy class of the induced outer automorphism.* □

Next we present the algorithm for deciding the conjugacy problem for efficient Dehn twists:

8.8. Algorithm II. Given two efficient Dehn twists $D_1 = D(\mathcal{G}_1, (a_e^{n(e)})_{e \in E(\mathcal{G}_1)})$ and $D_2 = D(\mathcal{G}_2, (a_e^{n(e)})_{e \in E(\mathcal{G}_2)})$, proceed as follows:

(1) Check whether there is a graph isomorphism $H_\Gamma : \Gamma(\mathcal{G}_1) \to \Gamma(\mathcal{G}_2)$ with $\operatorname{rank}(G_v) = \operatorname{rank}(G_{H(v)})$ and $n(e) = n(H_\Gamma(e))$ for all $v \in V(\mathcal{G}_1),\ e \in E(\mathcal{G}_1)$. If so, list all (finitely many) such graph isomorphisms.

(2) For each H_Γ listed in (1) and every $v \in V(\mathcal{G}_1)$ check whether there is an isomorphism $H_v : G_v \to G_{H(v)}$ such that $H_v(f_e(a_e))$ is conjugate to $f_{H(e)}(a_{H(e)})$ for all $e \in E(\mathcal{G}_1)$ with $\tau(e) = v$. This can be done by applying the Whitehead algorithm to the two families of conjugacy classes,

$$([f_e(a_e)] \mid e \in E(\mathcal{G}_1),\ \tau(e) = v)$$

in G_v , and

$$([f_{H(e)}(a_{H(e)})] \mid e \in E(\mathcal{G}_1),\ \tau(e) = v)$$

in $G_{H(v)}$, see for example [H] .

8.9. Proposition. *Algorithm II decides whether, given two efficient Dehn twists* $D_1 = D(\mathcal{G}_1, (a_e^{n(e)})_{e\in E(\mathcal{G}_1)})$ *and* $D_2 = D(\mathcal{G}_2, (a_e^{n(e)})_{e\in E(\mathcal{G}_2)})$ *, there exists a graph of groups isomorphism* $H : \mathcal{G}_1 \to \mathcal{G}_2$ *with* $H_e(a_e^{n(e)}) = a_{H(e)}^{n(H(e))}$ *for all* $e \in E(\mathcal{G}_1)$*.*

Proof. If the algorithm finds in step (1) a graph isomorphism H_Γ and in step (2) isomorphisms H_v for all $v \in V(\mathcal{G}_1)$ with the desired properties, then we can complete these data to a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ as follows: We define H_e by $H_e(a_e) = a_{H(e)}$ and define the elements $\delta(e)$ to be the elements in $G_{\tau(H(e))}$ which conjugate $H_{\tau(e)}(f_e(a_e))$ to $f_{H(e)}(a_{H(e)})$, which exist by step (2) . The equation (*) in 4.1 follows then directly from these definitions. Thus H is a graph of groups isomorphism and from $n(e) = n(H_\Gamma(e))$ as given through step (1) we obtain $H_e(a_e^{n(e)}) = a_{H(e)}^{n(H(e))}$ for all $e \in E(\mathcal{G}_1)$.

Conversely, if a graph of groups isomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ with $H_e(a_e^{n(e)}) = a_{H(e)}^{n(H(e))}$ for all $e \in E(\mathcal{G}_1)$ exists, then it explicitly gives a graph isomorphism H_Γ as in (1) and vertex group isomorphisms H_v as in (2). □

8.10. Corollary. *Algorithm I and Algorithm II together give a solution to the conjugacy problem for outer automorphisms of free groups defined by Dehn twists.*

Proof. Let D_1 and D_2 be two (not necessarily efficient) Dehn twists, and let D_1' and D_2' be the efficient Dehn twists based on, say, $\mathcal{G}_1'$ and $\mathcal{G}_2'$, which are obtained from D_1 and D_2 by Algorithm I . By Lemma 8.3 the automorphisms $\widehat{D_1}$ and $\widehat{D_2}$ are conjugate if and only $\widehat{D_1'}$ and $\widehat{D_2'}$ are. Algorithm II decides whether there exists a graph of groups isomorphism $H : \mathcal{G}_1' \to \mathcal{G}_2'$ which preserves twistors. If so, $\widehat{D_1}$ and $\widehat{D_2}$ and hence $\widehat{D_1'}$ and $\widehat{D_2'}$ are conjugate, and otherwise they are not: This is precisely the content of Theorem 6.9 (a) . □

8.11. Notice that Algorithm II also solves the conjugacy problem for (non-outer) automorphisms of free groups which are given by efficient Dehn twists: By Theorem 6.9 (b) two such automorphisms D_v and $D'_{v'}$ are conjugate if and only if there exists a corresponding graph of groups isomorphism H which preserves twistors and maps v to v' . But the existence of such an H is detected precisely by Algorithm II.

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