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Open manifolds with nonnegative Ricci curvature and large volume growth

Changyu Xia

Abstract. In this paper, we study complete open n-dimensional Riemannian manifolds with nonnegative Ricci curvature and large volume growth. We prove among other things that such a manifold is diffeomorphic to a Euclidean n-space R^n if its sectional curvature is bounded from below and the volume growth of geodesic balls around some point is not too far from that of the balls in R^n.

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1. Introduction

Let (M, g) be an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature. The relative volume comparison theorem [BC, GLP] says that the function $r \to \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n}$ is monotone decreasing, where $B(p, r)$ denotes the geodesic ball around $p \in M$ with radius r and ω_n is the volume of the unit ball in the Euclidean space R^n. Define α_M by

$$\alpha_M = \lim_{r\to\infty} \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n}.$$

It is easy to show that α_M is independent of $p \in M$, hence it is a global geometric invariant of M. We always have

$$\alpha_M \omega_n r^n \le \mathrm{vol}[B(x,r)] \le \omega_n r^n, \quad \forall r > 0, \quad \forall x \in M. \tag{1.1}$$

We say (M, g) has *large volume growth* if $\alpha_M > 0$. It should be noticed that, in this case, $0 < \alpha_M \le 1$ and when $\alpha_M = 1$, M is isometric to R^n by Bishop-Gromov comparison theorem [BC, GLP].

A manifold M is said to have finite topological type if there is a compact domain Ω whose boundary $\partial\Omega$ is a topological manifold such that $M \setminus \Omega$ is homeomorphic to $\partial\Omega \times [0, \infty)$. Abresch-Gromoll [AG] first obtain the finiteness of topological type for complete n-manifolds (M, g) with $\mathrm{Ric}_M \geq 0$ and small diameter growth $\mathrm{diam}(p, r) = o(\frac{1}{r^n})$, provided that the sectional curvature $K_M \geq K_0 > -\infty$.

Let (M, g) be an n-dimensional complete manifold with $\mathrm{Ric}_M \geq 0$ and $\alpha_M > 0$. It has been proved by Li [L] that M has finite fundamental group. Anderson [A] has showed that the order of the fundamental group of M is bounded from above by $\frac{1}{\alpha_M}$. Perelman [P] has proved that there is a small constant $\epsilon(n) > 0$ depending only on n such that if $\alpha_M > 1 - \epsilon(n)$, then M is contractible. It has been shown by Shen [S2] that M has finite topological type, provided that $\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} = \alpha_M + o\left(\frac{1}{r^{n-1}}\right)$ and, either the conjugate radius $conj_M \geq c > 0$ or the sectional curvature $K_M \geq K_0 > -\infty$. Petersen [Pe] conjectured that if $\alpha_M > \frac{1}{2}$ then M is diffeomorphic to R^n. Recently, Cheeger and Colding [CC] gave a partial answer to Petersen's conjecture. In fact, they proved that there exists a small constant $\delta(n) > 0$ such that if $\alpha_M \geq 1 - \delta(n)$, then M is diffeomorphic to R^n. Another result which supports stongly Petersen's conjecture has been obtained by do Carmo and the author recently in [CX].

In the present paper, we study complete manifolds with nonnegative Ricci curvature and large volume growth. Let M be a complete manifold and $p \in M$ be fixed; we say that $K_p^{\min} \geq c$ if for any minimal geodesic γ issuing from p all sectional curvatures of the planes which are tangent to γ are greater than or equal to c. This notion was first introduced by Klingenberg [K].

Theorem 1.1. *Let (M, g) be a complete Riemannian n-manifold with Ricci curvature* $\mathrm{Ric}_M \geq 0$, $\alpha_M > 0$. *Suppose that* $K_p^{min} \geq -C$ *for some point* $p \in M$ *and some positive constant* C. *If for all* $r > 0$, *we have*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} < \left\{ 1 + 2^{-n} \left(\frac{1}{8\sqrt{C}r} \log \left(\frac{2}{1 + e^{-2\sqrt{C}r}} \right) \right)^{n-1} \right\} \alpha_M, \tag{1.2}$$

then M *is diffeomorphic to* R^n.

The following result is a generalization of Shen's theorem mentioned above.

Theorem 1.2. *Let (M, g) be a complete Riemannian n-manifold with Ricci curvature* $\mathrm{Ric}_M \geq 0$, $\alpha_M > 0$. *Suppose that* $K_p^{min} \geq -C$ *for some* $p \in M$ *and* $C > 0$. *If*

$$\limsup_{r \to +\infty} \left\{ \left(\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M \right) r^{n-1} \right\} < 2^{-n} \left(\frac{\log 2}{8\sqrt{C}} \right)^{n-1} \alpha_M, \tag{1.3}$$

then M *has finite topological type.*

Let (M, g) be an n-dimensional complete noncompact Riemannian manifold. Fix a point $p \in M$. For any $r > 0$, let

$$k_p(r) := \inf_{M \setminus B(p,r)} K$$

where $B(p,r)$ is the open geodesic ball around p with radius r, K denotes the sectional curvature of M, and the infimum is taken over all the sections at all points on $M \backslash B(p,r)$. It is easy to see that $k_p(r) \leq 0$ and that $k_p(r)$ is a monotone function of r.

U. Abresch [A] proved that if $\int_0^\infty r k_p(r) dr > -\infty$, then M is of finite topological type. Recently, Sha and Shen [SS] showed that a complete open Riemannian manifold M has finite topological type if $\mathrm{Ric}_M \geq 0,\ \alpha_M > 0$ and

$$k_p(r) \geq -\frac{C}{1+r^2} \tag{1.4}$$

for some constant $C > 0$ and all $r > 0$.

In this paper we then prove the

Theorem 1.3. *Given $C > 0$, and an integer $n \geq 2$, there is a positive constant $\epsilon = \epsilon(n, C)$ such that any complete Riemannian n-manifold M with Ricci curvature $\mathrm{Ric}_M \geq 0,\ \alpha_M > 0,\ k_p(r) \geq -\frac{C}{1+r^2}$ and*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} \leq (1+\epsilon)\alpha_M \tag{1.5}$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to R^n.

Now we list the following Toponogov-type comparison theorem for complete manifolds with $K_p^{\min} \geq c$ obtained by Machigashira which will be used in this paper. Let $M^2(c)$ be the complete simply connected surface of constant curvature c. Throughout this paper, all geodesics are assumed to have unit speed.

Lemma 1.1 ([M1], [M2]) *Let M be a complete Riemannian manifold and p be a point of M with $K_p^{\min} \geq c$.*

(i) Let $\gamma_i : [0, l_i] \to M,\ i = 0, 1, 2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p,\ \gamma_0(0) = \gamma_1(l_1)$ and $\gamma_0(l_0) = \gamma_2(0)$. Then, there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \to M^2(c),\ i = 0, 1, 2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2),\ \tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ which are such that

$$L(\gamma_i) = L(\tilde{\gamma}_i) \quad \textit{for } i = 0,\ 1,\ 2$$

and

$$\angle(-\gamma_1'(l_1), \gamma_0'(0)) \geq \angle(-\tilde{\gamma}_1{}'(l_1), \tilde{\gamma}_0{}'(0)),$$
$$\angle(-\gamma_0'(l_0), \gamma_2'(0)) \geq \angle(-\tilde{\gamma}_0{}'(l_0), \tilde{\gamma}_2{}'(0)).$$

(ii) Let $\gamma_i : [0, l_i] \to M,\ i = 1, 2$ be two minimizing geodesics starting from p. Let $\tilde{\gamma}_i : [0, l_i] \to M^2(c)$ for $i = 1, 2$ be minimizing geodesics starting from same point such that $\angle(\gamma_1'(0), \gamma_2'(0)) = \angle(\tilde{\gamma}_1{}'(0), \tilde{\gamma}_2{}'(0))$. Then $d(\gamma_1(l_1), \gamma_2(l_2)) \leq d_c(\tilde{\gamma}_1(l_1), \tilde{\gamma}_2(l_2))$, where d_c denotes the distance function in $M^2(c)$.

2. Proof of Theorem 1.1 and Theorem 1.2

Let M be an n-dimensional Riemannian manifold and $1 \leq k \leq n-1$. If for any point $x \in M$ and any $(k+1)$-mutually orthogonal unit tangent vectors $e, e_1, ..., e_k \in T_xM$, we have $\sum_{i=1}^{k} K(e \wedge e_i) \geq 0$, we say that the k-th Ricci curvature of M is nonnegative and denote this fact by $\mathrm{Ric}_M^{(k)} \geq 0$. Here, $K(e \wedge e_i)$ denote the sectional curvature of the plane spanned by e and $e_i (1 \leq i \leq k)$. Notice that if $\mathrm{Ric}_M^{(k)} \geq 0$ then $\mathrm{Ric}_M \geq 0$.

We shall prove the following more general theorem than Theorem 1.1.

Theorem 2.1. *Let (M, g) be a complete Riemannian n-manifold with* $\mathrm{Ric}_M^{(k)} \geq 0$, $\alpha_M > 0$. *Suppose that* $K_p^{min} \geq -C$ *for some* $C > 0$ *and* $p \in M$. *If for all* $r > 0$, *we have*

$$\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} < \left\{ 1 + 2^{-n} \left(\frac{1}{8\sqrt{C}r} \log \left(\frac{2}{1+e^{-2\sqrt{C}r}} \right) \right)^{\frac{kn}{k+1}} \right\} \alpha_M, \tag{2.1}$$

then M *is diffeomorphic to* R^n.

For a point $p \in M$; we set $d_p(x) = d(p,x)$. Notice that the distance function d_p is not a smooth function (on the cut locus of p). Hence the critical points of d_p are not defined in a usual sense. The notion of critical points of d_p was introduced by Grove-Shiohama [GS].

A point $q (\neq p) \in M$ is called a critical point of d_p if there is, for any non-zero vector $v \in T_qM$, a minimal geodesic γ from q to p making an angle $\angle(v, \gamma'(0)) \leq \frac{\pi}{2}$ with v. We simply say that q is a critical point of p. It is now well-known that a complete noncompact Riemannian n-manifold M is diffeomorphic to R^n if there is a $p \in M$ such that p has no critical points other than p.

Let Σ be a closed subset of the unit tangent sphere S_pM at $p \in M$. Let $B_\Sigma(p,r)$ denote the set of points $x \in B(p,r)$ such that there is a minimizing geodesic γ from p to x with $\frac{d\gamma}{dt}(0) \in \Sigma$. For $0 < r \leq \infty$, let $\Sigma_p(r)$ denote the set of unit vectors $v \in \Sigma$ such that the geodesic $\gamma(t) = \exp_p(tv)$ is minimizing on [0, r). Notice that

$$\Sigma_p(r_2) \subset \Sigma_p(r_1),\ 0 < r_1 < r_2;\ \Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r). \tag{2.2}$$

The following generalized Bishop-Gromov volume comparison theorem was observed in [S2].

Lemma 2.1. ([S2]) *Let (M,g) be a complete n-manifold with* $\mathrm{Ric}_M \geq 0$. *Let* $\Sigma \subset S_pM$ *be a closed subset. Then the function* $r \to \frac{\mathrm{vol}[B_\Sigma(p,r)]}{\omega_n r^n}$ *is monotone decreasing.*

Lemma 2.2. ([S2]) *Let (M,g) be a complete n-manifold with* $\mathrm{Ric}_M \geq 0$. *The function*

$$r \to \frac{\mathrm{vol}[B_{\Sigma_p(r)}(p,r)]}{\omega_n r^n}$$

is monotone decreasing. If in addtion that M has large volume growth, then

$$\frac{\mathrm{vol}[B_{\Sigma_p(r)}(p,r)]}{\omega_n r^n} \geq \alpha_M, \quad \forall r > 0. \tag{2.3}$$

Lemma 2.3. *Let (M,g) be a complete n-manifold with* $\mathrm{Ric}_M \geq 0$ *and* $\alpha_M > 0$. *Then*

$$\frac{\mathrm{vol}[B_{\Sigma_p(\infty)}(p,r)]}{\omega_n r^n} \geq \alpha_M, \quad \forall r > 0. \tag{2.4}$$

Proof. Observe that

$$\frac{\mathrm{vol}[B_{\Sigma_p(r)}(p,r)]}{\omega_n r^n} = \frac{\mathrm{vol}[B_{\Sigma_p(\infty)}(p,r)] + \mathrm{vol}[B_{\Sigma_p(r)\setminus\Sigma_p(\infty)}(p,r)]}{\omega_n r^n}. \tag{2.5}$$

By the standard argument, we have

$$\mathrm{vol}[B_{\Sigma_p(r)\setminus\Sigma_p(\infty)}(p,r)] \leq \frac{r^n}{n} \cdot \mathrm{vol}(\Sigma_p(r) \setminus \Sigma_p(\infty)) \tag{2.6}$$

It follows from (2.2) that

$$\lim_{r\to\infty} \mathrm{vol}(\Sigma_p(r) \setminus \Sigma_p(\infty)) = 0. \tag{2.7}$$

Substituting (2.6) into (2.5) and letting $r \to \infty$, one obtains by virtue of (2.7) and (2.3)

$$\begin{aligned} \lim_{r\to\infty} \frac{\mathrm{vol}[B_{\Sigma_p(\infty)}(p,r)]}{\omega_n r^n} &\geq \lim_{r\to\infty} \frac{\mathrm{Vol}[B_{\Sigma_p(r)}(p,r)]}{\omega_n r^n} \\ &\geq \alpha_M. \end{aligned}$$

Using Lemma 2.1, one obtains (2.4). □

Lemma 2.4. *Let* (M, g) *be a complete n-manifold with* $\mathrm{Ric}_M \geq 0$ *and* $\alpha_M > 0$. *Let* R_p *denote the(point set) union of rays issuing from p. Then for any* $r > 0$ *and any* $x \in \partial B(p,r)$,

$$d(x, R_p) \leq 2\alpha_M^{-\frac{1}{n}} \left\{ \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M \right\}^{\frac{1}{n}} r. \tag{2.9}$$

Proof. Let $s = d(x, R_p)$; then $s \le r$ and

$$B(x,s) \cup B_{\Sigma_p(\infty)}(p,2r) \subset B(p,2r). \tag{2.10}$$

The left hand side of (2.10) is a disjoint union. By (1.1), we have

$$\mathrm{vol}(B(x,s)) \ge \alpha_M \omega_n s^n.$$

From Lemma 2.1 and Lemma (2.3), one obtains

$$\begin{aligned} 2^n \mathrm{vol}[B(p,r)] &\ge \mathrm{vol}[B(p,2r)] \\ &\ge \mathrm{vol}[B(x,s)] + \mathrm{vol}[B_{\Sigma_p(\infty)}(p,2r)] \\ &\ge \alpha_M \omega_n s^n + \alpha_M \omega_n (2r)^n. \end{aligned} \tag{2.11}$$

thus

$$s^n \le 2^n r^n \alpha_M^{-1} \left\{ \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M \right\}.$$

This proves (2.9). □

Let $p, q \in M$. The *excess function* $e_{pq}(x)$ is defined by

$$e_{pq}(x) := d(p,x) + d(q,x) - d(p,q)$$

Lemma 2.5. ([AG, S1]) *Let (M,g) be a complete n-manifold with $\mathrm{Ric}_M^{(k)} \ge 0$ for some $1 \le k \le n-1$. Let $\gamma : [0,a] \to M$ be a minimal geodesic from p to q. Then for any $x \in M$,*

$$e_{pq}(x) \le 8 \left(\frac{s^{k+1}}{r} \right)^{\frac{1}{k}}, \tag{2.12}$$

where $s = d(x,\gamma)$, $r = \min(d(p,x), d(q,x))$.

Let $\gamma : [0,\infty) \to M$ be a ray issuing from p and let $x \in M$. It is easy to see that $e_{p,\gamma(t)}(x) = d(p,x) + d(\gamma(t),x) - t$ is decreasing in t and that $e_{p,\gamma(t)}(x) \ge 0$. We define the excess function $e_{p,\gamma}$ associated to p and γ as

$$e_{p,\gamma}(x) = \lim_{t \to +\infty} e_{p,\gamma(t)}(x). \tag{2.13}$$

Then

$$e_{p,\gamma}(x) \le e_{p,\gamma(t)}(x), \quad \forall t > 0. \tag{2.14}$$

Lemma 2.6. *Let (M,g) be a complete open Riemannian manifold with $K_p^{min} \ge -C$ for some $C > 0$ and $p \in M$. Suppose that $x \ne p$ is a critical point of p. Then for any ray $\gamma : [0,\infty) \to M$ issuing from p*

$$e_{p,\gamma}(x) \ge \frac{1}{\sqrt{C}} \log \left(\frac{2}{1 + e^{-2\sqrt{C} d(p,x)}} \right). \tag{2.15}$$

Proof. For any $t > 0$, take a minimal geodesic $\sigma_t : [0, d(x, \gamma(t))] \to M$ from x to $\gamma(t)$. Since x is a critical point of p, there exists a minimal geodesic τ from x to p such that $\sigma_t'(0)$ and $\tau'(0)$ make an angle at most $\frac{\pi}{2}$. Applying Lemma 1.1 to the geodesic triangle $(\gamma|_{[0,t]}, \sigma_t, \tau)$, we obtain

$$\cosh(\sqrt{C}t) \leq \cosh\left(\sqrt{C}d(x, \gamma(t))\right) \cosh\left(\sqrt{C}d(p, x)\right). \tag{2.16}$$

Multiplying the above inequality by $2\exp\left(\sqrt{C}(d(p,x) - t)\right)$ and letting $t \to +\infty$, we obtain

$$\exp\left(\sqrt{C}d(p, x)\right) \leq \exp\left(\sqrt{C}e_{p,\gamma}(x)\right) \cosh\left(\sqrt{C}d(p, x)\right). \tag{2.17}$$

Then Lemma 2.6 follows from (2.17).

Proof of Theorem 2.1. We shall prove that M contains no critical points of p(other than p) and therefore it is diffeomorphic to R^n. To do this, take an arbitrary point $x(\neq p) \in M$ and set $r = d(p, x)$. It follows from (2.1) and (2.9) that

$$d(x, R_p) < \left(\frac{1}{8\sqrt{C}}\log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right)\right)^{\frac{k}{k+1}} \cdot r^{\frac{1}{k+1}}.$$

Thus we can find a ray $\gamma : [0, +\infty) \to M$ issuing from p and satisfying

$$s := d(x, \gamma) < \left(\frac{1}{8\sqrt{C}}\log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right)\right)^{\frac{k}{k+1}} \cdot r^{\frac{1}{k+1}}. \tag{2.18}$$

Take $q \in \gamma$ such that $d(x, q) = d(x, \gamma)$. By (2.18), $d(x, q) < r$. Also one can easily deduce from triangle inequality that

$$\min\left(d(p, x), d(\gamma(t), x)\right) = r, \quad \forall\, t \geq 2r.$$

Thus $q \in \gamma((0, 2r))$ and so

$$d(x, \gamma|_{[0,2r]}) = s.$$

Using (2.12), (2.14) and (2.18), we obtain

$$\begin{aligned} e_{p,\gamma}(x) &\leq e_{p,\gamma(2r)}(x) \\ &\leq 8\left(\frac{s^{k+1}}{r}\right)^{\frac{1}{k}} \\ &< \frac{1}{\sqrt{C}}\log\left(\frac{2}{1 + e^{-2\sqrt{C}r}}\right). \end{aligned} \tag{2.19}$$

By (2.15) and (2.19), x is not a critical point of p. Thus M is diffeomorphic to R^n. This completes the proof of Theorem 2.1. □

Theorem 1.2 is a consequence of the following more general result.

Theorem 2.2. *Let (M, g) be a complete Riemannian n-manifold with* $\mathrm{Ric}_M^{(k)} \geq 0$, $\alpha_M > 0$. *Suppose that* $K_p^{min} \geq -C$ *for some* $p \in M$ *and* $C > 0$. *If*

$$\limsup_{r\to+\infty}\left\{\left(\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M\right) r^{\frac{kn}{k+1}}\right\} < 2^{-n}\left(\frac{\log 2}{8\sqrt{C}}\right)^{\frac{kn}{k+1}} \cdot \alpha_M, \tag{2.20}$$

then M has finite topological type.

Proof of Theorem 2.2. By the Isotopy Lemma [C, G, GS], it suffices to show that for any $x \in M$, if $d(p,x)$ is large enough then x is not a critical point of p. Our assumption (2.20) enables us to find a small number $\epsilon > 0$ and a sufficiently large r_1 such that

$$\left(\frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} - \alpha_M\right) r^{\frac{kn}{k+1}} < 2^{-n}\left(\frac{\log 2}{8\sqrt{C}} - \epsilon\right)^{\frac{kn}{k+1}} \alpha_M, \quad \forall r \geq r_1. \tag{2.21}$$

Since

$$\lim_{r\to+\infty} \log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right) = \log 2,$$

there is a sufficiently large r_2 such that

$$\frac{\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right)}{8\sqrt{C}} > \frac{\log 2}{8\sqrt{C}} - \epsilon, \quad \forall r \geq r_2. \tag{2.22}$$

Let $r_0 = \max(r_1, r_2)$; then for any $r \geq r_0$ we have from (2.21) and (2.22) that

$$\begin{aligned} \frac{\mathrm{vol}[B(p,r)]}{\omega_n r^n} &< \left\{1 + 2^{-n}\left(\frac{\frac{\log 2}{8\sqrt{C}} - \epsilon}{r}\right)^{\frac{kn}{k+1}}\right\} \cdot \alpha_M \\ &< \left\{1 + 2^{-n}\left(\frac{1}{8\sqrt{C}r}\log\left(\frac{2}{1+e^{-2\sqrt{C}r}}\right)\right)^{\frac{kn}{k+1}}\right\} \cdot \alpha_M \end{aligned} \tag{2.23}$$

Now one can repeat the arguments as in the proof of Theorem 2.1 to prove that $M \setminus B(p, r_0)$ contains no critical points of p. Therefore M has finite topological type. This completes the proof of Theorem 2.2. □

Proof of Theorem 1.3. Let $\delta = \delta(C) < \frac{1}{20}$ be a solution of the following inequality

$$\cosh^2(4\sqrt{C}\delta) - \cosh\left(6\sqrt{C}\delta\right) < 0. \tag{2.24}$$

We take our $\epsilon = \epsilon(n, C)$ in Theorem 1.3 to be

$$\epsilon = \left(\frac{\delta}{8}\right)^n \tag{2.25}$$

Take an arbityary point $x(\neq p) \in M$ and let $r = d(p, x)$. It suffices to prove that x is not a critical point of p. Let $\gamma : [0, 2r] \to M$ be a minimizing geodesic from p to $q = \gamma(2r)$ such that $s := d(x, \gamma) = d(x, B_{\Sigma_p(\infty)}(p, 2r))$. Using the same arguments as in the proof of (2.9), we obtain

$$d(x, B_{\Sigma_p(\infty)}(p, 2r)) \leq 2\alpha_M^{-\frac{1}{n}} \left\{\frac{\mathrm{vol}[B(p, r)]}{\omega_n r^n} - \alpha_M\right\}^{\frac{1}{n}} \cdot r. \tag{2.26}$$

Take a minimizing geodesic σ from x to q. For any minimal geodesic σ_1 from x to p, let $\tilde{p} = \sigma_1(\delta r)$ and $\tilde{q} = \sigma(\delta r)$. Applying the Toponogov comparison theorem to the hinge $(\sigma|_{[0,\delta r]},\ \sigma_1|_{[0,\delta r]})$ in $M - B_{\frac{r}{4}}(p)$, we have

$$\cosh\left(\frac{4\sqrt{C}}{r(x)} d(\tilde{p}, \tilde{q})\right) \leq \cosh^2(4\sqrt{C}\delta) - \sinh^2(4\sqrt{C}\delta)\, \cos\theta \tag{2.27}$$

where $\theta = \angle(\sigma'(0), \sigma_1'(0))$ be the angle of σ and σ_1 at x and we have used the fact that the sectional curvature of M satisfies $K_M \geq -\frac{4^2 C}{r^2}$ on $M - B_{\frac{r}{4}}(p)$. Let $m \in \gamma$ such that $d(x, m) = d(x, \gamma)$; it then follows from the triangle inequality that

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &\geq d(p, q) - d(p, \tilde{p}) - d(q, \tilde{q}) \\ &= d(p, m) + d(q, m) - [d(p, x) - d(\tilde{p}, x)] \\ &\qquad - [d(x, q) - d(x, \tilde{q})] \\ &= 2\delta r + [d(p, m) - d(p, x)] + [d(q, m) - d(q, x)] \\ &\geq 2\delta r - 2d(x, m). \end{aligned} \tag{2.28}$$

From (2.25), (2.26) and our assumption (1.5), we have

$$\begin{aligned} d(x, m) &= d(x, B_{\Sigma_p(\infty)}(p, 2r)) \\ &\leq 2\epsilon^{\frac{1}{n}} r \\ &\leq \frac{\delta r}{4}. \end{aligned} \tag{2.29}$$

Thus we have

$$d(\tilde{p}, \tilde{q}) \geq \frac{3}{2}\delta r. \tag{2.30}$$

Substituting (2.30) into (2.27) and using (2.24), we find that

$$\begin{aligned}\sinh^2(4\sqrt{C}\delta)\cos\theta &\leq \cosh^2(4\sqrt{C}\delta) - \cosh\left(\frac{4\sqrt{C}}{r(x)}d(\tilde{p}, \tilde{q})\right) \\ &\leq \cosh^2(4\sqrt{C}\delta) - \cosh\left(6\sqrt{C}\delta)\right) \\ &< 0,\end{aligned} \tag{2.31}$$

or

$$\theta > \frac{\pi}{2}. \tag{2.32}$$

Hence x is not a critical point of p. Thus M is diffoemorphic to R^n. The theorem follows.

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Changyu Xia
Departamento de Matemática-IE
Universidade de Brasília
Campus Universitário
70910-900-Brasília-DF
Brasil
e-mail: xia@ipe.mat.unb.br

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