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A Lagrangian camel

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Abstract. We prove the Lagrangian analogue of the symplectic camel theorem: there are compact Lagrangian submanifolds of $\mathbb{R}^{2n}$ that cannot be moved through a small hole by a global Hamiltonian isotopy with compact support.

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1. Introduction

In [19], Claude Viterbo constructed a symplectic capacity $c_{\mathrm{gf}}(V)$ for V an open set of $\mathbb{R}^{2n}$, and used it to prove several interesting results in symplectic geometry, including the following Symplectic Camel Theorem. Here the subscript "gf" stands for "generating functions", because this is the tool used to define $c_{\mathrm{gf}}(V)$; we summarize in Appendix A the definition and basic properties of this symplectic capacity.

Let us recall what the Symplectic Camel Theorem states. We consider the space $\mathbb{C}^n = \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, endowed with the coordinates

$$z = x + iy = (x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$$

with the standard symplectic form

$$\Omega = \Omega_{\mathbb{R}^{2n}} = -d\lambda_{\mathbb{R}^{2n}} = dx \wedge dy = \sum_{j=1}^{n} dx_j \wedge dy_j$$

and with the Euclidean scalar product and norm

$$< z, z' > = z\bar{z}' \qquad \|z\| = \sqrt{< z, z >}$$

Let us define $\mathbb{R}^{2n}_+ = \{z \in \mathbb{R}^{2n}; y_n > 0\}$ and $\mathbb{R}^{2n}_- = \{z \in \mathbb{R}^{2n}; y_n < 0\}$, and, for $\eta > 0$, the holed hyperplane $\Sigma_\eta = \{z \in \mathbb{R}^{2n}; y_n = 0 \text{ and } \|z\| \geq \eta\}$.

The Camel Theorem says that if $n \geq 2$ and V is a (bounded) open set with $\overline{V} \subset \mathbb{R}^{2n}_-$ and $c_{\mathrm{gf}}(V) > \pi\eta^2$, then it is impossible to find a Hamiltonian isotopy $(\Phi_t)_{t\in[0,1]}$ of $\mathbb{R}^{2n}$ with compact support in $\mathbb{R}^{2n} - \Sigma_\eta$, such that $\Phi_1(V) \subset \mathbb{R}^{2n}_+$.

Remark 1.1. In [8], Y. Eliashberg and M. Gromov showed, using pseudo-holomorphic curves, that this is impossible if V is a Euclidean ball of radius $r > \eta$ (see also [11, 12]). As the gf-capacity of a ball of radius r is πr^2, Viterbo's theorem is more general.

When trying to study the flux of Lagrangian isotopies, the Lagrangian Camel problem comes as a natural question. Instead of looking at an open set $V \subset \mathbb{R}^{2n}$, we consider a closed Lagrangian embedding $j : L \hookrightarrow \mathbb{R}^{2n}$. We are primarily interested in the quantity $c_{\mathrm{gf}}(L, j)$ that we define now.

Definition 1.2. The *gf-capacity* $c_{\mathrm{gf}}(L, j)$ of the embedding is the infimum of all $c_{\mathrm{gf}}(V)$, V being any open neighborhood of $j(L)$ in $\mathbb{R}^{2n}$.

Since $j(L)$ has empty interior and does not even bound an open set, we could expect its capacity to vanish. However, we will prove the following result, where $w(L, j)$ is defined as follows, following Viterbo.

A theorem of Weinstein [20] says that the embedding j can be extended to a symplectic embedding $J : U \to \mathbb{R}^{2n}$, where U is a neighborhood of L in T^*L. We will call (U, J) a *Weinstein neighborhood* of the embedding j. Let μ be a closed 1-form on L, representing the Maslov class $\mu(j)$ of the embedding. Then the (negative) *μ-width* of U is defined as $\|U\|_\mu = \sup\big\{s \geq 0\,;\, -s\mu(L) \subset U\big\}$. The number $\|U\|_\mu$ depends of course on the representative μ chosen for the Maslov class $\mu(j)$: the "smaller" the form μ, the greater the μ-width of U.

Definition 1.3. Let $w(L, j)$ denote the supremum of all possible $\|U\|_\mu$, where U is a Weinstein neighborhood and μ represents the Maslov class of the embedding.

The basic result of this paper is the following.

Theorem 1.4. *We suppose $n \geq 2$.*

1. *If $j : T^n \hookrightarrow \mathbb{R}^{2n}$ is a Lagrangian embedding, then the gf-capacity of $j(T^n)$ satisfies*
$$c_{\mathrm{gf}}(T^n, j) \geq 2w(T^n, j) > 0.$$
2. *If $j : L \hookrightarrow \mathbb{R}^{2n}$ is a Lagrangian embedding and L admits a Riemannian metric with strictly negative sectional curvature (for instance all non-orientable surfaces L with $\chi(L)$ strictly negative and divisible by 4: see* [9] *and also* [2]*), then*
$$c_{\mathrm{gf}}(L, j) \geq (n-1)w(L, j) > 0.$$

More generally, if (L_1, j_1), ... ,(L_m, j_m) *are* m *Lagrangian embeddings of this type and* (L, j) *is the product embedding, then*

$$c_{\mathrm{gf}}(L, j) \geq (n - m)w(L, j) > 0.$$

Remark 1.5. In particular, if $L = S(r_1) \times \cdots \times S(r_n)$ is a split torus in $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$, each $S(r_k)$ being a Euclidean circle of radius r_k, then

$$c_{\mathrm{gf}}(L) = \pi \min(r_1^2, \ldots, r_n^2).$$

Indeed, using polar coordinates in each factor $\mathbb{C}$, it is easy to construct a Weinstein neighborhood whose width is precisely $\pi r^2 = \pi \min(r_1^2, \ldots, r_n^2)$. On the other hand, the capacity of such a split torus is clearly less than that of the cylinder $B^2\big(0, r\big) \times \mathbb{R}^{2n-2}$, which is again πr^2.

Corollary 1.6. (Lagrangian Camel Theorem). *Let* $j : L \hookrightarrow \mathbb{R}^{2n}_-$ *be one of the above embeddings. Then for* $0 < \eta < c(L, j)$ *it is impossible to find a Hamiltonian isotopy* $(\Phi_t)_{t\in[0,1]}$ *of* $\mathbb{R}^{2n}$ *with compact support in* $\mathbb{R}^{2n} - \Sigma_\eta$*, such that* $\Phi_1\big(j(L)\big) \subset \mathbb{R}^{2n}_+$.

Indeed, any isotopy moving $j(L)$ into $\mathbb{R}^{2n}_+$ will also move a neighborhood of $j(L)$ from $\mathbb{R}^{2n}_-$ into $\mathbb{R}^{2n}_+$, which is impossible by the Symplectic Camel Theorem.

Remark 1.7. There are several results like Theorem 1.4 that are already proved, see for instance Viterbo [18] and Polterovich [13]. The problem is that, to our knowledge, there is no corresponding symplectic camel theorem that can be applied to the capacities they use. So the alternative was either to prove the corresponding symplectic camel theorem, or to establish Theorem 1.4. Because of our greater familiarity with generating functions, we chose the second option. Basically, we will follow the arguments developed in [17, 18] and adapt them to the theory of generating functions, but the reader will notice some slight restrictions in comparison to these references. The reason for this is that we could not use the natural S^1-invariance of the action functional: generating functions are a kind of discretization of this functional, and it is still unclear whether one can recover this natural action or not.

Let us now briefly explain the relation between the camel problem and the mean property of the flux of Lagrangian isotopies.

Most generally, let (M, ω) be a symplectic manifold. Any symplectic isotopy $(\phi_t)_{t\in[0,1]}$ determines a closed 1-form α on M, whose cohomology class is the *flux* of the isotopy, see [3] (the easiest way to define α is to say that its integral over a smooth loop in M is the symplectic area swept out by this loop under the isotopy). This cohomology class $[\alpha]$ depends only on the homotopy class of the isotopy $(\phi_t)_{t\in[0,1]}$ with endpoints fixed. Two basic and very important properties are: (i) an isotopy $(\phi_t)_{t\in[0,1]}$ is Hamiltonian if and only if the flux of $(\phi_t)_{t\in[0,\tau]}$ vanishes for each $\tau \in [0,1]$, and (ii) the flux of an isotopy vanishes if and only if it is homotopic (with endpoints fixed) to a Hamiltonian isotopy (for this last statement, we assume either that M is compact or that the isotopy is compactly supported).

Let us now turn to the Lagrangian case. Similarly, let $(j_t)_{t\in[0,1]}$ be a Lagrangian isotopy of a closed manifold L into M, that is $j_t : L \hookrightarrow M$ is a smooth family of Lagrangian embeddings. We can define in the same way a closed 1-form on L, whose cohomology class is (by definition) the flux of the isotopy. We ask whether this flux has the following mean property, as in the case of symplectic isotopies:

Given a Lagrangian isotopy $(j_t)_{t\in[0,1]}$ with vanishing flux, is it homotopic, with endpoints fixed, to a Lagrangian isotopy $(k_t)_{t\in[0,1]}$ such that the flux of each $(k_t)_{t\in[0,\tau]}$ vanishes for $\tau \in [0,1]$?

It is immediate to see that such an isotopy $(k_t)_{t\in[0,1]}$ would in fact be induced by a global Hamiltonian isotopy. We now show that our Lagrangian Camel Theorem gives an example (in a non-compact symplectic manifold) where this property does not hold.

Indeed, let $M = \mathbb{R}^{2n} - \Sigma_\eta$ with the symplectic structure induced from that of $\mathbb{R}^{2n}$, and $j : L \hookrightarrow \mathbb{R}^{2n}_- \subset M$ be as in Theorem 1.4. Using the presence (in $\mathbb{R}^{2n}$) of a contracting Liouville vector field, we can isotop L to an arbitrarily small Lagrangian L' (but this cannot be done by a global Hamiltonian isotopy); then we move L' to $L'' \subset \mathbb{R}^{2n}_+ \subset M$ through the hole of Σ_η (by a Hamiltonian isotopy), we expand L'' to L''' in such a way that L''' is just the translate (in $\mathbb{R}^{2n}$) of L. It is easy to see that this Lagrangian isotopy from L to L''' has zero flux. Now, if it were homotopic (with endpoints fixed) to a Lagrangian isotopy with flux vanishing at every intermediate time, this last isotopy would be induced by a global Hamiltonian isotopy of $\mathbb{R}^{2n} - \Sigma_\eta$ that could be assumed to have compact support (remember that L is compact), thus contradicting the Lagrangian Camel theorem.

Remark 1.8. While working on this subject, we discovered that Y. Chekanov [5] found a more surprising counterexample to the mean property for the flux of Lagrangian isotopies: it happens in $\mathbb{R}^{2n}$ that some Lagrangian submanifolds can be connected by Lagrangian isotopies with zero flux, but not through Hamiltonian isotopies.

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2. A Hamiltonian system in T^*L

Let $j : L \hookrightarrow \mathbb{R}^{2n}$ be a Lagrangian embedding, L being a closed n-manifold. Here T^*L is endowed with (local) cotangent coordinates (q, p) and with the symplectic form $\omega_L = dq \wedge dp$. Let $V = J(U)$: it is a bounded open set in $\mathbb{R}^{2n}$ with finite gf-capacity $c_{\mathrm{gf}}(V)$, see Appendix A. Here (V, J) is a Weinstein neighborhood of j.

We consider a fixed Riemannian metric on L. It induces a bundle isomorphism $TL \cong T^*L$ and a metric on the vector bundle T^*L. If $v \in T_qL$ and $p \in T_q^*L$ are corresponding elements for that isomorphism, we write $v = p^\flat$ and $p = v^\sharp$. In particular, $\|v\|_q = \|v^\sharp\|_q$.

Let $\rho > 0$ be small enough so that $B_\rho = \big\{(q, p) \in T^*L; \|p\|_q \le \rho\big\}$ is contained in U. We consider a smooth function $h : [0, +\infty] \to \mathbb{R}^-$ such that:

1. $h \equiv -a$ on $[0, \varepsilon/2]$
2. h is increasing, strictly convex on $[\varepsilon/2, \varepsilon]$
3. $h' \equiv c$ on $[\varepsilon, \rho - \varepsilon]$
4. h is increasing, strictly concave on $[\rho - \varepsilon, \rho - \varepsilon/2]$
5. $h \equiv 0$ on $[\rho - \varepsilon/2, +\infty]$

where $\varepsilon > 0$ is very small with respect to ρ, $c > 0$ is not the length of a closed geodesic of L, and $a > c_{\mathrm{gf}}(V)$. See Figure 1.

Then we define a compactly supported Hamiltonian function $H : T^*L \to \mathbb{R}$ by

$$H(q, p) = h\big(\|p\|\big) \tag{1}$$

Let $\phi = (\phi_t)_{t\in[0,1]}$ be the Hamiltonian isotopy of T^*L it generates: it is obtained by integrating the Hamiltonian vector field X associated to H, defined by $i_X\omega_L = dH$.

The isotopy ϕ is easily proved to be a reparametrization of the cogeodesic flow. Indeed, let $K : T^*L \to \mathbb{R}$ be the standard Hamiltonian

$$K(q, p) = \frac{r^2}{2} = \frac{\|p\|^2}{2} \tag{2}$$

It generates the *cogeodesic flow*, denoted by $(g_t)_{t\in\mathbb{R}}$: if $z = (q, p)$ is a point in T^*L and $v = p^\flat \in T_qL$, then there is on L a unique geodesic $(q_t)_{t\in\mathbb{R}}$ such that $q_0 = q$ and $\dot q_0 = v$, and we have $g_t(z) = \big(q_t, (\dot q_t)^\sharp\big)$.

Since $H(q, p) = h(\|p\|)$, we can write $H(z) = a \circ K(z)$, with

$$a(s) = h\big(\sqrt{2s}\big) \tag{3}$$

Hence $X_H(z) = c(z)X_K(z)$, where

$$c(z) = a' \circ K(z) = \frac{h'(\|p\|)}{\|p\|} \tag{4}$$

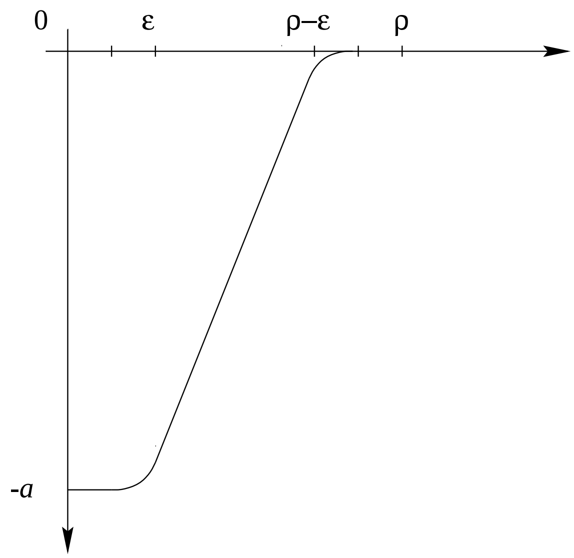

Figure 1.
Graph of h.

Consequently, since H and K are constant along both g_t- and ϕ_t-orbits, we have

$$\phi_t(z) = g_{c(z)t}(z) \tag{5}$$

ie. the isotopy ϕ is a reparametrization of the cogeodesic flow.

Let $z = (q, p) \in B_\rho$ be a fixed point of ϕ_1. Then, according to (5), the projection on L of its ϕ-orbit is a closed geodesic γ with length $\ell(\gamma) = c(z)\|p\| = h'(\|p\|)$.

Let us consider the symplectic vector bundle $E = \cup_{t\in S^1} E_t$ over S^1 (seen as $[0, 1]$ with endpoints identified), where the fiber

$$E_t = \overline{T_z T^*L} \times T_{\phi_t(z)} T^*L \qquad t \in [0, 1] \tag{6}$$

is endowed with the symplectic form $\big(-\omega_L(z)\big) \oplus \omega_L\big(\phi_t(z)\big)$. It has a canonical Lagrangian subbundle $V = \cup_{t\in S^1} V_t$, namely

$$V_t = \mathrm{Vert}(z) \oplus \mathrm{Vert}\big(\phi_t(z)\big) \tag{7}$$

where $\mathrm{Vert}(z)$ is the vertical subspace at $z \in T^*L$ of the bundle $T^*L \to L$. The graphs of the differentials $d\phi_t(z) : T_z T^*L \to T_{\phi_t(z)} T^*L$ define a continuous path $\Gamma : [0, 1] \to \Lambda(E)$ of Lagrangian subspaces $\Gamma_t \subset E_t$. We may therefore consider the Maslov-Duistermaat index

$$\mathrm{ind}_\phi(z) := \mathrm{ind}_V(\Gamma)$$

as defined in Appendix B.2.

In this setting, J. J. Duistermaat [7] has proved the following result for *convex* Hamiltonians:

Proposition 2.1. *Let $z = (q, p) \in T^*L$ be a fixed point of ϕ_1, and γ be the underlying geodesic on L. Then, $i(\gamma)$ denoting the Morse index of γ as a closed geodesic, we have*

$$\begin{cases} \mathrm{ind}_\phi(z) = i(\gamma) + n & \text{if } h \text{ is strictly convex at } \|p\| \\ \mathrm{ind}_\phi(z) = i(\gamma) + n - 1 & \text{if } h \text{ is strictly concave at } \|p\| \end{cases} \tag{8}$$

We will deduce the formula in the concave case from that in the convex case. The idea is the following. We can express E as the sum $E' \oplus E''$ of two symplectic subbundles, and we also have Lagrangian splittings $V = V' \oplus V''$, $\Gamma = \Gamma' \oplus \Gamma''$. Thus $\mathrm{ind}_\phi(z) = \mathrm{ind}_{V'}(\Gamma') + \mathrm{ind}_{V''}(\Gamma'')$. We will see that $\mathrm{ind}_{V''}(\Gamma'')$ does not depend on the convexity/concavity of h, and for the other term $\mathrm{ind}_{V'}(\Gamma')$ we will have explicit simple formulas enabling us to conclude. To do so, we will need a few facts about the (co)geodesic flow $(g_t)_{t\in\mathbb{R}}$, that we recall now (see [10] for details).

If the cotangent bundle is endowed with the Levi-Civita connection corresponding to the metric, then we have a splitting

$$T_z(T^*L) = \mathrm{Hor}(z) \oplus \mathrm{Vert}(z) \qquad (z \in T^*L) \tag{9}$$

into horizontal and vertical subbundles. Given $z = (q, p) \in T^*L$, both $\mathrm{Hor}(z)$ and $\mathrm{Vert}(z)$ are canonically isomorphic to T_qL, hence they carry a well-defined scalar product. In that setting, the symplectic form ω_L has the expression:

$$\omega_L(z)(\delta z, \delta z') = \, < \delta_h z, \delta_v z' >_q - < \delta_h z', \delta_v z >_q \tag{10}$$

where δ_h and δ_v denote the horizontal and vertical parts of a vector, identified to their images in T_qL. In particular, (9) is a *Lagrangian splitting*. We also note that the Hamiltonian vector field associated with $K(q,p) = 1/2\,\|p\|^2$ has the form $X_K(q,p) = (p, 0)$.

Let $\gamma = (\gamma_t)_{t\in[0,T]}$ be a geodesic on L, and $z = {\gamma_0}^\sharp \in T^*L$. Then the Jacobi vector fields $(Y_t)_{t\in[0,T]}$ along γ are in one-to-one correspondence with the g-invariant vector fields $(Z_t)_{t\in[0,T]}$ along the orbit of z. This correspondence is given by $Y_t \mapsto Z_t = (Y_t, \nabla Y_t)$, using the splitting (9).

Since $\mathrm{Hor}(z)$ and $\mathrm{Vert}(z)$ are isomorphic to $T_qL \cong T_q^*L = (\mathbb{R}p) \oplus p^\perp$, we have associated splittings $\mathrm{Hor}(z) = \mathrm{Hor}'(z) \oplus \mathrm{Hor}''(z)$ and $\mathrm{Vert}(z) = \mathrm{Vert}'(z) \oplus \mathrm{Vert}''(z)$, and then $T_zT^*L = T_z'T^*L \oplus T_z''T^*L$, where $T_z'T^*L = \mathrm{Hor}'(z) \oplus \mathrm{Vert}'(z)$ is 2-dimensional and $T_z''T^*L = \mathrm{Hor}''(z) \oplus \mathrm{Vert}''(z)$ is $(2n-2)$-dimensional. Now $TT^*L = T'T^*L \oplus T''T^*L$ is a splitting into *symplectic* orthogonal subbundles, and the (co)geodesic flow preserves that decomposition.

The subbundle $T'T^*L$ is obviously trivial. Given $z \in T^*L$ and $t \in \mathbb{R}$, $dg_t(z)$ induces an isomorphism $T'_zT^*L \to T'_{g_t(z)}T^*L$ whose matrix in the obvious bases is $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. This comes from the fact that the Jacobi field $(Y_t)_{t\in[0,T]}$ along a geodesic $\gamma = (\gamma_t)_{t\in[0,T]}$ such that $Y_0 = \alpha\dot\gamma_0$ and $\nabla Y_0 = \beta\dot\gamma_0$ is given by $Y_t = (\alpha + \beta t)\dot\gamma_t$.

Proof of Proposition 2.1. Differentiating (5), we obtain:

$$d\phi_t(z).\delta z = dg_{c(z)t}(z).\delta z + t\big[dc(z)\delta z\big] X_K\big(\phi_t(z)\big) \tag{11}$$

It follows that the flow $(\phi_t)_{t\in\mathbb{R}}$ also preserves the decomposition $TT^*L = T'T^*L \oplus T''T^*L$. Indeed, if $\delta z \in T''_zT^*L$ then $dc(z)\delta z = 0$, hence $d\phi_t(z)\delta z = dg_{tc(z)}(z)\delta z$— in particular, $d\phi_t(z)\delta z$ does not depend on the concavity/convexity of h at $\|p\|$.

Thus $E = E' \oplus E''$ splits into two symplectic vector subbundles, and both $V = V' \oplus V''$ and $\Gamma = \Gamma' \oplus \Gamma''$ split into Lagrangian subbundles of E' and E'' respectively. Hence $\mathrm{ind}_V(\Gamma) = \mathrm{ind}_{V'}(\Gamma') + \mathrm{ind}_{V''}(\Gamma'')$ by additivity of the Maslov-Duistermaat index under direct sums. We have just seen that $\mathrm{ind}_{V''}(\Gamma'')$ does not depend on the concavity/convexity of h at $\|p\|$, so it only remains to see how $\mathrm{ind}_{V'}(\Gamma')$ depends on it.

If $\delta z = (\delta_h z, \delta_v z) = (\alpha, \beta) \in T'_zT^*L = \mathrm{Hor}'(z) \oplus \mathrm{Vert}'(z) \cong \mathbb{R}^2$, then a straightforward computation shows that $d\phi_t(z)\delta z = \big(\alpha + t\beta h''(r), \beta\big)$. We thus see that the matrix of the induced isomorphism from $T'zT^*L$ to $T'_{\phi_t(z)}T^*L$ is

$$\begin{pmatrix} 1 & t\,h''(r) \\ 0 & 1 \end{pmatrix} \tag{12}$$

We have $E' \cong \overline{\mathbb{R}}^2 \times \mathbb{R}^2$, $V' \cong (0 \times \mathbb{R}) \times (0 \times \mathbb{R})$ and Γ_t is the graph of the linear symplectomorphism A_t of $\mathbb{R}^2$ whose matrix is (12). To compute $\mathrm{ind}_{V'}(\Gamma')$ according to Appendix B, we choose the Lagrangian subspace $\alpha = (\mathbb{R} \times 0) \times (0 \times \mathbb{R}) \subset \overline{\mathbb{R}}^2 \times \mathbb{R}^2$: we have $\alpha \cap \Gamma_t = 0$ for all $t \in [0, 1]$. Hence the Maslov-Duistermaat index of Γ is given by $\mathrm{ind}(\Gamma) = \mathrm{ind}\, Q(\Gamma_1, \alpha; \Gamma_0)$. It is easy to see from the definitions that the index of $Q(\Gamma_1, \alpha; \Gamma_0)$ is also the coindex of $Q(\Gamma_0, \alpha; \Gamma_1)$, that we now evaluate.

Let us consider the linear map $C : \Gamma_0 \to \alpha$ such that $u + Cu \in \Gamma_1$ for all $u \in \Gamma_0 = \Delta$. We write

$$\begin{aligned} u &= (u_1, u_2; u_1, u_2) \in \Delta \\ Cu &= (v_1, 0; 0, v_2) \in \alpha \\ u + Cu &= (w_1, w_2; w_1 + h''(r)w_2, w_2) \in \Gamma_1 \end{aligned} \tag{13}$$

since $d\phi_1(z)(w_1, w_2) = (w_1 + h''(r)w_2, w_2)$.

Then, by definition, see (28):

$$\begin{aligned} Q(\Gamma_0, W_0; \Gamma_1)(u) &= \big(-\Omega_{\mathbb{R}^{2n}} \oplus \Omega_{\mathbb{R}^{2n}}\big)(Cu, u) \\ &= -\Omega_{\mathbb{R}^{2n}}\big((v_1, 0), (u_1, u_2)\big) + \Omega_{\mathbb{R}^{2n}}\big((0, v_2), (u_1, u_2)\big) \\ &= -v_1u_2 - v_2u_1 = h''(r)(u_2)^2 \end{aligned} \tag{14}$$

Since $\operatorname{coind} Q$ is the number of strictly positive eigenvalues of Q, we see that $\operatorname{ind}_{V'}(\Gamma') = 1$ if $h''(r) > 0$, and $\operatorname{ind}_{V'}(\Gamma') = 0$ if $h''(r) < 0$. Consequently,

$$\operatorname{ind}_\phi^{\text{concav}}(z) = \operatorname{ind}_\phi^{\text{convex}}(z) - 1$$

which finishes the proof of Proposition 2.1.

3. The Hamiltonian system viewed from $\mathbb{R}^{2n}$

We define the compactly supported Hamiltonian $\mathbf{H} : \mathbb{R}^{2n} \to \mathbb{R}$ and its associated Hamiltonian isotopy $(\Phi_t)_{t\in[0,1]}$ in the obvious way:

$$\begin{cases} \mathbf{H} = H \circ J^{-1} & \text{on } V \\ \mathbf{H} = 0 & \text{on } \mathbb{R}^{2n} - V \end{cases} \tag{15}$$

We will apply Viterbo's theory of symplectic capacities, as summarized in Appendix A. According to Theorem A.4, we have $c_-(\mathbf{H}) = 0$ (since $\mathbf{H} \leq 0$) and $c_+(\mathbf{H}) > 0$ (since Φ_1 is not the identity map). Thus Φ_1 has a fixed point $z = z_+$ such that $0 < A_{\mathbf{H}}(z) = c_+(\mathbf{H}) \leq c_{\text{gf}}(V)$. This implies $z \in V$, since $A_{\mathbf{H}} = 0$ outside V. Similarly, $A_{\mathbf{H}} = a$ on the set $\{\mathbf{H} = -a\}$, which is ruled out by the hypothesis $a > c_{\text{gf}}(V)$. Consequently, we may define $(q,p) = J^{-1}(z)$: this is a fixed point of ϕ_1 satisfying $\|p\| \in]\varepsilon/2, \rho - \varepsilon/2[$. But, as we have seen, $h'(\|p\|)$ is now the length of a closed geodesic on L, so by assumption we cannot have $h'(\|p\|) = c$. We have thus proved the following result, that will allow us to apply Proposition 2.1.

Lemma 3.1. *If a is strictly greater than $c_{\text{gf}}(V)$ and c is distinct from the length of any closed geodesic on L, then ϕ_1 has a fixed point $z = (q,p)$ such that h is strictly convex or strictly concave at $\|p\|$.*

In the setting of Appendix A, let $S_1 : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ be a generating function for Φ_1 such that $S_1(w,\xi) = Q_\infty(\xi)$ outside a compact set of $\mathbb{R}^{2n} \times \mathbb{R}^k$, where Q_∞ is a non-degenerate quadratic form on $\mathbb{R}^k$.

Definition 3.2. Let $z \in \mathbb{R}^{2n}$ be a fixed point of Φ_1, and (z,ξ) be the corresponding critical point of S_1. From Viterbo's uniqueness theorem [19, 15], it follows that the integer $\operatorname{ind} d^2 S_1(z,\xi) - \operatorname{ind} Q_\infty$ does not depend on S_1, but only on Φ_1. We call it the *gf-index of* z, denoted by $\operatorname{ind}_{\text{gf}}(z)$. The *nullity of* z, denoted by $\nu(z)$, will be the dimension of $\operatorname{Ker}\big(d\Phi_1(z) - \operatorname{Id}\big) \cong \operatorname{Ker}\big(d\phi_1(z) - \operatorname{Id}\big)$

Note that if z is as in Lemma 3.1 and γ is the corresponding closed geodesic on L, then the (equivariany) nullity of γ is $\nu(\gamma) = \nu(z) - 1$.

Proposition 3.3. *The fixed point z of Lemma 3.1 can be chosen so that*

$$2n - \nu(z) \leq \operatorname{ind}_{\text{gf}}(z) \leq 2n$$

Proof. We know from Appendix A that $c = c_+(H) = S_1(z,\xi)$ is a critical value of S_1 obtained by minimax, so that $H^{2n+\operatorname{ind} Q_\infty}(S_1^{c+\eta}, S_1^{c-\eta}) \neq 0$ for $\eta > 0$ small enough. But the set of critical points of S_1 at the level c is a non-degenerate critical manifold, so that, by standard Morse theory, there must be on the level c a critical point (z,ξ) such that $\operatorname{ind} d^2S_1(z,\xi) \leq 2n + \operatorname{ind} Q_\infty \leq \operatorname{ind} d^2S_1(z,\xi) + \dim\operatorname{Ker} d^2S_1(z,\xi)]$. It follows from the very definition of generating functions that $\operatorname{Ker} d^2S_1(z,\xi)$ is isomorphic to $\operatorname{Ker}\big(d\Phi_1(z) - \mathrm{Id}\big)$, so that its dimension is $\nu(z)$. □

Next, we relate the Maslov class $\mu(j)$ of the embedding j with the two indices defined above.

Proposition 3.4. *Let z be a fixed point of ϕ_1 as in Lemma 3.1, and γ be the corresponding closed geodesic on L. Then*

$$\operatorname{ind}_{\mathrm{gf}}(z) = \operatorname{ind}_\phi(z) + \big(\mu(j),\gamma\big)$$

Proof. To relate the Maslov-Duistermaat index and the gf-index, we define still another Lagrangian subbundle $C = \cup_{t\in S^1} C_t$ of the symplectic vector bundle E – see (6) – this time connected to the embedding J: we consider a fixed Lagrangian subspace in $\mathbb{R}^{2n}$, say $\mathbb{R}^n \times 0$, and then define

$$C_t = dJ(z)^{-1}(\mathbb{R}^n \times 0) \times dJ\big(\phi_t(z)\big)^{-1}(\mathbb{R}^n \times 0)$$

Recall that $\Gamma_t \subset E_t$ is the graph of $d\phi_t(z)$; now, if Γ'_t is the graph of $d\Phi_t(z)$ in $\overline{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$, then it follows from the definition of the Maslov-Duistermaat index that $\operatorname{ind}_C(\Gamma) = \operatorname{ind}(\Gamma')$. Since $d\Phi_0(z) = \mathrm{Id}$, it follows from Propositions B.7 and B.8 that $\operatorname{ind}(\Gamma') = \operatorname{ind}_{\mathrm{gf}}(z)$. Then, according to (32), we have $\operatorname{ind}_{\mathrm{gf}}(z) = \operatorname{ind}_C(\Gamma) = \operatorname{ind}_V(\Gamma) + \operatorname{ind}_C(V)$. But it is clear that $\operatorname{ind}_C(V) = \big(\mu(j),\gamma\big)$. □

Corollary 3.5. *To the Hamiltonian H of (1) there corresponds a real number $c(H) \in\,]0, c_{\mathrm{gf}}(V)]$. A fixed point $z = (q,p)$ of the associated Hamiltonian isotopy can be chosen so that, γ denoting the projected closed geodesic on L,*

$$c(H) = \|p\|h'(\|p\|) - h(\|p\|) + \oint_\gamma j^*\lambda_{\mathbb{R}^{2n}} \tag{16}$$

and

$$\begin{cases} \big(\mu(j),\gamma\big) \in [n - i(\gamma) - \nu(z), n - i(\gamma)] & \textit{in the convex case} \\ \big(\mu(j),\gamma\big) \in [n - i(\gamma) - \nu(z) + 1, n - i(\gamma) + 1] & \textit{in the concave case} \end{cases} \tag{17}$$

Proof. Formula (16) is just a reformulation of the relation $c(H) = c(\mathbf{H}) = A_{\mathbf{H}}(z) = \oint_{t\mapsto\Phi_t(z), t\in[0,1]} \lambda_{\mathbb{R}^{2n}} - \mathbf{H}dt$ Along the Φ-orbit of z, the Hamiltonian $\mathbf{H}$ is constant: $\mathbf{H}\big(\Phi_t(z)\big) = h(\|p\|)$. And $\oint_{t\mapsto\Phi_t(z), t\in[0,1]} \lambda_{\mathbb{R}^{2n}} - \oint_\gamma j^*\lambda_{\mathbb{R}^{2n}} = \oint_{t\mapsto\phi_t(z), t\in[0,1]} \lambda_L = \|p\|h'(\|p\|)$. Finally, (17) follows from Propositions 2.1, 3.3 and 3.4. □

4. A limit process

The functions h and H that we considered so far depend on ρ, c and ε. Now we fix the numbers ρ and c, and we consider ε as a parameter converging to 0. Hence we have a family of functions h_ε and Hamiltonians H_ε.

The limit of $c(H_\varepsilon)$ as $\varepsilon \to 0$ does exist: this is because $\varepsilon \leq \varepsilon'$ implies $H_\varepsilon \leq H_{\varepsilon'}$ by construction, and then $c(H_{\varepsilon'}) \leq c(H_\varepsilon)$ by Theorem A.4; as $c(H)$ is bounded from above by $c_{\mathrm{gf}}(V)$, we conclude. Let us write

$$K(\rho, c) = \lim_{\varepsilon \to 0} c(H_\varepsilon)$$

Now let ε_m be a real sequence converging to 0. For each m, we find a closed geodesic γ_m, a real number $r_m \in \,]\varepsilon_m/2, \varepsilon_m[\, \cup \,]\rho - \varepsilon_m, \rho - \varepsilon_m/2[$ such that

$$c(H_{\varepsilon_m}) = r_m h'(r_m) - h(r_m) + \int_{\gamma_m} j^* \lambda_{\mathbb{R}^{2n}}$$

We may suppose that we are in one of two cases: $r_m \in \,]\varepsilon_m/2, \varepsilon_m[$ for all m (convex case), or $r_m \in \,]\rho - \varepsilon_m, \rho - \varepsilon_m/2[$ for all m (concave case).

In both cases, we have $\ell(\gamma_m) = h'(r_m) \leq c$. Due to the compactness of the set of closed geodesics of length bounded by c, we may suppose that γ_m converges to a closed geodesic γ.

Corollary 4.1. *The number $K(\rho, c) \in \,]0, c_{\mathrm{gf}}(V)]$ satisfies*

$$K(\rho, c) = \begin{cases} \rho c + \oint_\gamma j^* \lambda_{\mathbb{R}^{2n}} & \textit{in the convex case} \\ \rho \ell(\gamma) + \oint_\gamma j^* \lambda_{\mathbb{R}^{2n}} & \textit{in the concave case} \end{cases} \tag{18}$$

for a closed geodesic γ on L satisfying (17).

5. Proof of Theorem 1.4

Let $J : U \hookrightarrow \mathbb{R}^{2n}$ be a Weinstein neighborhood of the embedding j, and μ be a closed 1-form on L, representing the Maslov class $\mu(j) \in H^1(L; \mathbb{R})$. We will also denote by σ the Liouville class of the embedding: $\sigma(\gamma) = \oint_\gamma j^* \lambda_{\mathbb{R}^{2n}}$.

Following [18], we define a continuous family of Lagrangian embeddings. Let $\rho > 0$ be small enough so that $B_\rho \subset U$, and define

$$\|U\|_{\mu,\rho} = \sup\bigl\{s \geq 0\,;\, -s\mu(L) + B_\rho \subset U\bigr\}$$

For $s \in [0, \|U\|_{\mu,\rho}]$, we consider the symplectic transformation

$$\begin{array}{rcl} T_s : T^*L & \to & T^*L \\ (q,p) & \mapsto & \bigl(q, p - s\mu(q)\bigr) \end{array}$$

and then the Lagrangian embedding

$$j_s = J \circ (T_s)_{|L} : L \hookrightarrow \mathbb{R}^{2n} \tag{19}$$

that can be extended to $J_s : B_\rho \hookrightarrow \mathbb{R}^{2n}$.

Applying Corollary 4.1 for each parameter s, we obtain a map $s \in [0, \|U\|_{\mu,\rho}] \mapsto K_s(\rho, c) \in]0, c_{\mathrm{gf}}(V)]$. Because of property 6 in Theorem A.4, it is *continuous*.

Furthermore, for each such s, there exists on L a closed geodesic γ_s with length $\ell(\gamma_s) \le c$, such that

$$K_s(\rho, c) = \begin{cases} \rho\ell(\gamma_s) + \sigma(\gamma_s) - s\mu(\gamma_s) & \text{in the concave case} \\ \rho c + \sigma(\gamma_s) - s\mu(\gamma_s) & \text{in the convex case} \end{cases} \tag{20}$$

(this is because $\int_{\gamma_s} j_s^* \lambda_{\mathbb{R}^{2n}} - \oint_{\gamma_s} j^* \lambda_{\mathbb{R}^{2n}} = -s(\mu(j), \gamma_s))$.

5.1. The negative curvature case

If L admits a metric with strictly negative sectional curvature, then $i(\gamma) = 0$ and $\nu(\gamma) = 0$ for any closed geodesic. Hence $\nu(z) = 1$ for our fixed point, and

$$\begin{cases} \mu(\gamma) \in [n-1, n] & \text{in the convex case} \\ \mu(\gamma) \in [n, n+1] & \text{in the concave case} \end{cases}$$

Since $n \ge 2$, we obtain $\mu(\gamma) \ge n - 1 > 0$ in any case. Again, the set of closed geodesics of length bounded by c being compact, the quantities $\ell(\gamma_s)$ and $\sigma(\gamma_s)$ that appear in (20) can take only a *finite* number of values. This implies that, when s grows from 0 to $\|U\|_{\mu,\rho}$, the point $\big(s, K_s(\rho, c)\big)$ moves on a finite set of straight lines of $\mathbb{R}^2$, with slopes $\le -(n-1)$. Accordingly, we must have

$$0 < K_s(\rho, c) \le K_0(\rho, c) - (n-1)s \qquad \forall\, s \in [0, \|U\|_{\mu,\rho}[$$

In particular,

$$K(\rho, c) = K_0(\rho, c) \ge (n-1)\|U\|_{\mu,\rho}$$

and then, since $\|U\|_{\mu,\rho} \to \|U\|_\mu$ as $\rho \to 0$,

$$K(j) := \lim_{\rho \to 0} \lim_{c \to \infty} K(\rho, c) \ge (n-1)\|U\|_\mu \tag{21}$$

We are now ready to finish the proof of Theorem 1.4 in this case. We may obviously assume that $V = J(U)$, where U and J are as before. Now (21) shows that, for any $\delta > 0$ arbitrarily small, we can find $\rho > 0$ and $c > 0$ such that $J(B_\rho) \subset V$ and $K(\rho, c) \ge (n-1)\|U\|_\mu - \delta$. This means that, for all $\delta > 0$, there is a Hamiltonian $\mathbf{H}$ with compact support in V, such that $c(\mathbf{H}) \ge (n-1)\|U\|_\mu - 2\delta$. Hence

$$c_{\mathrm{gf}}(V) \ge (n-1)\|U\|_\mu$$

by the very definition of $c_{\mathrm{gf}}(V)$.

If $L = L_1 \times \cdots \times L_m$ is the product of m manifolds, each having a metric with strictly negative curvature, then $i(\gamma) = 0$ and $\nu(\gamma) = m - 1$ for any closed geodesic. Hence

$$\begin{cases} \mu(\gamma) \in [n-m, n] & \text{in the convex case} \\ \mu(\gamma) \in [n-m+1, n+1] & \text{in the concave case} \end{cases}$$

Since $m < n$ and $n \geq 4$, we may proceed as above, whence

$$c_{\mathrm{gf}}(V) \geq (n-m)\|U\|_\mu.$$

5.2. The torus case

The torus case is handled with in the same spirit, with some slight complications. With the flat (product) metric, the closed geodesics of T^n satisfy $i(\gamma) = 0$ and $\nu(\gamma) = n - 1$, hence $\nu(z) = n$ for our fixed points. We thus get the estimates

$$\begin{cases} \mu(\gamma) \in [0, n] & \text{in the convex case} \\ \mu(\gamma) \in [1, n+1] & \text{in the concave case} \end{cases}$$

and the arguments used for the negative curvature case fail because $\mu(\gamma) = 0$ is now possible.

Remark 5.1. Since the torus is orientable, the Maslov index of any loop will be *even*. Hence $\mu(\gamma) \geq 2$ in the concave case.

First, we will study how $K(\rho, c)$ grows with c, ρ being fixed throughout the entire discussion.

Let $\mathcal{C}'$ be the set of those $c > 0$ such that $K(\rho, c)$ can *only* be realized as $K(\rho, c) = \rho c + \sigma(\gamma)$, with $\mu(\gamma) \geq 0$. It is an *open* set (its complement is easily seen to be closed). Similarly, the set $\mathcal{C}''$ of those $c > 0$ such that $K(\rho, c)$ can *only* be realized as $K(\rho, c) = \rho\ell(\gamma) + \sigma(\gamma)$, with $\mu(\gamma) \geq 2$ (remember that $\mu(\gamma)$ is even)is open.

The complement $\mathcal{C}'''$ of $\mathcal{C}' \cup \mathcal{C}''$ consists of *isolated* points: this is because for such a $c > 0$, $K(\rho, c)$ can be expressed in both ways:

$$K(\rho, c) = \rho c + \sigma(\gamma_1) = \rho\ell(\gamma_2) + \sigma(\gamma_2)$$

where $\ell(\gamma_1)$ and $\ell(\gamma_2)$ are bounded by c, and there is only a finite number of such possibilities.

On each connected component of $\mathcal{C}'$, we have $K(\rho, c) = \rho c + \text{constant}$. On each connected component of $\mathcal{C}''$, we have $K(\rho, c) = \text{constant}$. Thus, the total measure of $\mathcal{C}'$ is not greater than $c_{\mathrm{gf}}(V)/\rho$.

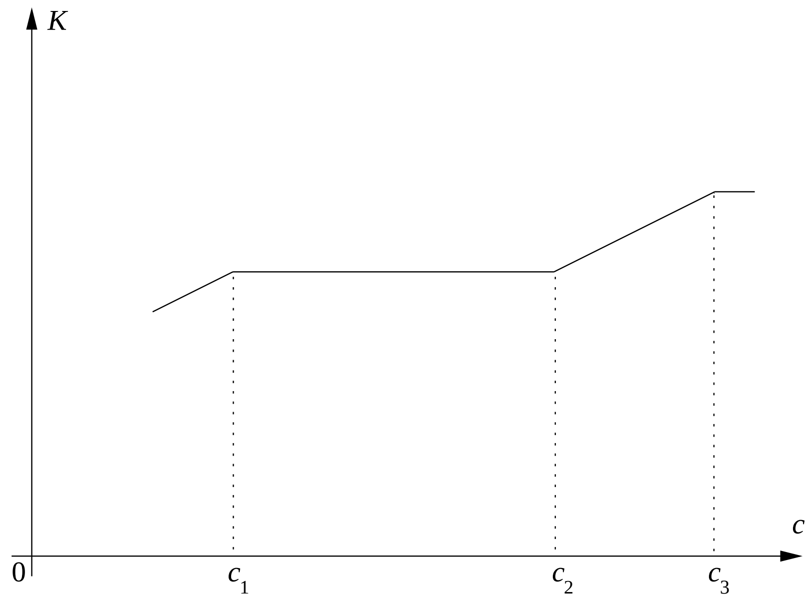

Figure 2.
Graph of $c \mapsto K(\rho, c)$

Hence the graph of $c \mapsto K(\rho, c)$ looks like Figure 2.

Next, we study the dependence of $c \mapsto K_s(\rho, c)$ with respect to the parameter s, with obvious notations.

Note that when we move s, we change the "breakpoints" where $c \mapsto K_s(\rho, c)$ might have a discontinuous derivative. However, they can be followed continuously: a point $c_s \in \mathcal{C}_s'''$ can be written as $c_0 + s\big(\mu(\gamma_1) - \mu(\gamma_2)\big)$ for some $c_0 \in \mathcal{C}_0'''$ and γ_1, γ_2 closed geodesics of length $\leq c$.

For the same reason as before, if $c \in \mathcal{C}_{s_0}'$, then $c \in \mathcal{C}_s'$ for s close enough to s_0, and similarly for $\mathcal{C}''$.

If $]c_1, c_2[$ is a component of $\mathcal{C}_{s_0}''$, then for s close enough to s_0 we have continuous functions $c_1(s)$ and $c_2(s)$ such that $c_1(s_0) = c_1$, $c_2(s_0) = c_2$, and $]c_1(s), c_2(s)[$ is a component of $\mathcal{C}_s''$. A similar statement holds for the components of $\mathcal{C}_s'$. Thus, we can follow their components, although "flat" ones may disappear as in Figure 3.

It follows easily that on any component of $\mathcal{C}_s' \cup \mathcal{C}_s''$, the numbers $K_s(\rho, c)$ can be realized by geodesics of the same Maslov index (see equation (20)). In particular, a "flat" component, as long as it does not disappear, goes down with s at a speed greater or equal to 2.

We do not conclude that there exists some $c > 0$ such that $K_s(\rho, c) \leq K_0(\rho, c) - 2s$ as in the negative curvature case, since whole components of $\mathcal{C}_s'$ might be realized by geodesics of zero index and components of $\mathcal{C}_s''$ may disappear. However, it is easy to see that there exists a continuous $s \mapsto c(s)$ such that $K_s\big(\rho, c(s)\big) \leq$

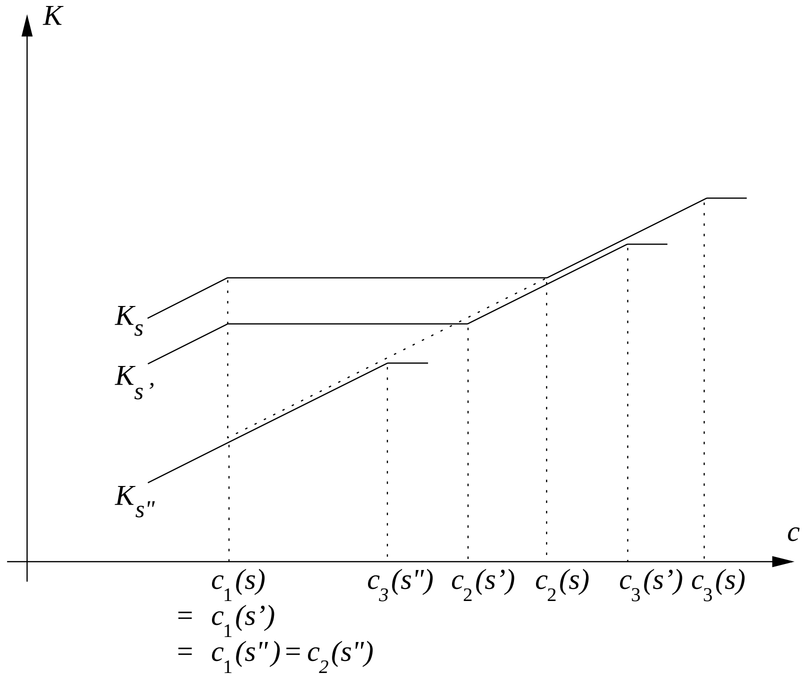

Figure 3.
Cancellation of flat component ($0 \leq s < s < s' < s''$)

$K_0\big(\rho, c(0)\big) - 2s$, and we conclude as before:

$$c_{\mathrm{gf}}(V) \geq 2\|U\|_\mu.$$

Remark 5.2. The referee has suggested the following construction for such a continuous $s \mapsto c(s)$ as above. Let us consider the function $k(s,c) := K_s(\rho, c)$, defined on $[0, \|U\|_{\mu,\rho}] \times]\rho^{-1}c_{\mathrm{gf}}(V), +\infty[$. Let $\overline{\mathcal{C}'_s} := \cup_s \{s\} \times \mathcal{C}'_s$ and $\overline{\mathcal{C}''_s} := \cup_s \{s\} \times \mathcal{C}''_s$: they are disjoint subsets of $[0, \|U\|_{\mu,\rho}] \times]\rho^{-1}c_{\mathrm{gf}}(V)[$, whose complement is a discrete union of segments. We have

$$\begin{cases} \frac{\partial k}{\partial s} \leq 0, & \frac{\partial k}{\partial c} = \rho \quad \text{on } \overline{\mathcal{C}''} \\ \frac{\partial k}{\partial s} \leq -2, & \frac{\partial k}{\partial c} = 0 \quad \text{on } \overline{\mathcal{C}''} \end{cases}$$

Then we set $c(s) = \rho^{-1}c_{\mathrm{gf}}(V) + \frac{\alpha}{\rho}\big(\|U\|_{\mu,\rho} - s\big)$, where $\alpha \geq 2$ is not the slope of any of the segments in the complement of $\overline{\mathcal{C}'} \cup \overline{\mathcal{C}''}$. We see that the continuous function $s \mapsto k\big(s, c(s)\big)$ always has a right derivative, which is less than or equal to -2.

Appendix A. Gf-capacity

We recall some basic facts from Viterbo's theory of capacities on the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$. The reader is referred to [19] for proofs (and more results).

Remark A.1. A general warning must be made about sign conventions, which are not always the same from one paper to the other.

Let V be a bounded open set in $\mathbb{R}^{2n}$. A (time-dependent) Hamiltonian function $\mathbf{H} = \mathbf{H}_t(z) : [0,1]\times\mathbb{R}^{2n} \to \mathbb{R}$ is V*-admissible* if there is a compact set C of such that $\operatorname{supp}(\mathbf{H}_t) \subset C$ for each $t \in [0,1]$. The set of smooth V-admissible Hamiltonians will be denoted by $\mathcal{H}_V$.

To each $\mathbf{H} \in \mathcal{H}_V$ there corresponds a complete Hamiltonian vector field $X = (X_t)_{t\in[0,1]}$ defined by the relation

$$i_{X_t}\Omega = d\mathbf{H}_t \qquad \forall\, t \in [0,1] \tag{22}$$

This vector field generates a Hamiltonian isotopy $\Phi = (\Phi_t)_{t\in[0,1]}$ of $\mathbb{R}^{2n}$. If $z \in \mathbb{R}^{2n}$ is a fixed point of Φ_1, then its *action* $A_{\mathbf{H}}(z)$ is the real number

$$A_{\mathbf{H}}(z) = \int_{t\mapsto\Phi_t(z), t\in[0,1]} \lambda_{\mathbb{R}^{2n}} - \mathbf{H}dt = \int_0^1 \big[y_t\, \dot{x}_t - \mathbf{H}_t(x_t, y_t)\big]\, dt \tag{23}$$

where $(x_t, y_t) = \Phi_t(z)$ for $t \in [0,1]$.

To introduce generating functions, we will use the symplectic isomorphism

$$\begin{array}{rcl} I : \overline{\mathbb{R}}^{2n} \times \mathbb{R}^{2n} & \to & T^*\mathbb{R}^{2n} \cong \mathbb{R}^{2n} \times \mathbb{R}^{2n} \\ (z, z') & \mapsto & (w, w') = \Big(\dfrac{z+z'}{2}, i(z-z')\Big) \end{array} \tag{24}$$

where $\overline{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$ denotes the vector space $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ endowed with the symplectic form $(-\Omega_{\mathbb{R}^{2n}}) \oplus \Omega_{\mathbb{R}^{2n}}$. For $t \in [0,1]$, let $\Gamma_t \subset \overline{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$ be the graph of Φ_t, and $\widetilde{\Gamma}_t \subset T^*\mathbb{R}^{2n}$ be its image under I.

Definition A.2. (see [14]). Let k be an arbitrary integer. A smooth function $S = S(w,\xi) : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ is a *generating function* if $0 \in (\mathbb{R}^k)^*$ is a regular value of $\partial_\xi S = \partial S/\partial\xi$. In that case, $\partial_\xi S^{-1}(0)$ is a smooth $2n$-manifold, and we have a smooth Lagrangian immersion $i_S : \partial_\xi S^{-1}(0) \to T^*\mathbb{R}^{2n}$ defined by $i_S(w,\xi) = \big(w, \partial_w S(w,\xi)\big)$. If i_S is an embedding, we say that S generates the embedded Lagrangian submanifold $L \subset T^*\mathbb{R}^{2n}$.

Notice that the critical points of S correspond to the intersection points of L with the zero section of $T^*\mathbb{R}^{2n}$.

Now the $\widetilde{\Gamma}_t$'s are Lagrangian submanifolds of $T^*\mathbb{R}^{2n}$, $\widetilde{\Gamma}_0$ is the zero section and obviously there is a compactly supported Hamiltonian isotopy $(\Psi_t)_{t\in[0,1]}$ of $T^*\mathbb{R}^{2n}$ such that $\widetilde{\Gamma}_t = \Psi_t(\widetilde{\Gamma}_0)$.

The next existence result was proved by Marc Chaperon [4], although not in this formulation, which comes from Jean-Claude Sikorav [14].

Theorem A.3. ([4]) *There exists a (a priori non-unique) smooth family of generating functions $S_t : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$, $t \in [0,1]$ such that*

(i) *S_t generates $\widetilde{\Gamma}_t$ for each $t \in [0,1]$*

(ii) *the whole family is quadratic at infinity: we have $S_t(w,\xi) = Q_\infty(\xi)$ outside a compact subset of $[0,1] \times \mathbb{R}^{2n} \times \mathbb{R}^k$, where $Q_\infty : \mathbb{R}^k \to \mathbb{R}$ is a non-degenerate quadratic form .*

Because of the choice of the identification (24), this implies that the fixed points of Φ_1 are in 1-1 correspondence with the critical points of S_1. Furthermore, if z is a fixed point of Φ_1, then the corresponding critical point is of the form (z,ξ), and an easy computation shows that

$$A_{\mathbf{H}}(z) = S_1(z,\xi) \tag{25}$$

By a so-called minimax method using the behaviour at infinity, it is possible to select two critical values of S_1. First, remark that we can extend the S_t's to $S^{2n} \times \mathbb{R}^k$, where $S^{2n} \cong \mathbb{R}^{2n} \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^{2n}$, by $S_t(\infty,\xi) = Q_\infty(\xi)$. Then, for $\alpha \in \mathbb{R}$, let $S_1^\alpha = \{S_1 \leq \alpha\}$. For $\alpha > 0$ large enough, the homotopy type of the pair $(S_1^\alpha, S_1^{-\alpha})$ is constant, and we denote it by $(S_1^{+\infty}, S_1^{-\infty})$. If i denotes the index of the quadratic form Q_∞, then it follows from the Künneth isomorphism that

$$H^*(S_1^{+\infty}, S_1^{-\infty}) \cong H^*(S^{2n}) \otimes H^*(D^i, S^{i-1}) \cong H^{*-i}(S^{2n})$$

where D^i (resp. S^{i-1}) is the unit disk (resp. the unit sphere) in $\mathbb{R}^i$. Hence

$$\begin{aligned} H^k(S_1^{+\infty}, S_1^{-\infty}) &= 0 \qquad \text{if } k \neq i, i+2n \\ H^i(S_1^{+\infty}, S_1^{-\infty}) &\cong H^{2n+i}(S_1^{+\infty}, S_1^{-\infty}) \cong \mathbb{R} \end{aligned}$$

Let u_- (resp. u_+) be a generator of $H^i(S_1^{+\infty}, S_1^{-\infty})$ (resp. of $H^{2n+i}(S_1^{+\infty}, S_1^{-\infty})$). Then define

$$c_\pm = \inf\{\alpha \in \mathbb{R}\,;\, u_\pm \text{ does not vanish in } H^*(S_1^\alpha, S_1^{-\infty})\}$$

It is easy to show that $H^i(S_1^{c_-+\eta}, S_1^{c_--\eta}) \neq 0$ and $H^{2n+i}(S_1^{c_++\eta}, S_1^{c_+-\eta}) \neq 0$ if $\eta > 0$ is small enough. This implies that $c_\pm$ are critical values of S_1. Furthermore, it can be proved that they do not depend on the particular family $(S_t)_{t\in[0,1]}$ chosen but only on the Hamiltonian $\mathbf{H}$, so we may call them $c_\pm(\mathbf{H})$. We list some of their properties in the next statement (some inequalities differ from those of [19], because some sign conventions differ).

Theorem A.4. ([19]). *To any $\mathbf{H} \in \mathcal{H}_V$ generating the isotopy $(\Phi_t)_{t\in[0,1]}$, we can associate two real numbers $c_\pm(\mathbf{H})$ with the following properties.*

1. $c_-(\mathbf{H}) \leq 0 \leq c_+(\mathbf{H})$.
2. $c_-(\mathbf{H}) = c_+(\mathbf{H})$ *if and only if* $\Phi_1 = \mathrm{Id}_{\mathbb{R}^{2n}}$.
3. *There are points* $z_\pm \in \mathbb{R}^{2n}$ *such that* $\Phi_1(z_\pm) = z_\pm$ *and* $A_{\mathbf{H}}(z_\pm) = c_\pm(\mathbf{H})$.
4. *If* $\mathbf{H} \leq 0$ *then* $c_-(\mathbf{H}) = 0$.
5. *If* $\mathbf{H} \leq \mathbf{K}$ *then* $c_\pm(\mathbf{H}) \geq c_\pm(\mathbf{K})$.
6. *The maps* $H \mapsto c_\pm(\mathbf{H})$ *are continuous for the* C^0*-topology on* $\mathcal{H}_V$. *More precisely, if* $\mathbf{H}$, $\mathbf{K}$ *are in* $\mathcal{H}_V$ *and satisfy* $\|\mathbf{H}-\mathbf{K}\|_{C^0} \leq \varepsilon$, *then* $|c_\pm(\mathbf{H})-c_\pm(\mathbf{K})| \leq \varepsilon$

Definition A.5. ([19]). The *gf-capacity* $c_{\mathrm{gf}}(V)$ of the open set $V \subset \mathbb{R}^{2n}$ is now defined as

$$c_{\mathrm{gf}}(V) = \sup\{c_+(\mathbf{H})\,;\mathbf{H} \in \mathcal{H}_V\} \tag{26}$$

Theorem A.6. ([19]). *The map* $V \mapsto c_{\mathrm{gf}}(V)$ *satisfies the following properties.*
1. *If* $V_1 \subset V_2$ *then* $c_{\mathrm{gf}}(V_1) \leq c_{\mathrm{gf}}(V_2)$
2. *If* $(\Phi_t)_{t\in[0,1]}$ *is a compactly supported Hamiltonian isotopy of* $\mathbb{R}^{2n}$, *then* $c_{\mathrm{gf}}\big(\Phi_t(V)\big)$ *is constant.*
3. $c_{\mathrm{gf}}\big(B^{2n}(0,r)\big) = c_{\mathrm{gf}}\big(B^2(0,r) \times \mathbb{R}^{2n-2}\big) = \pi r^2$.
4. *The Symplectic Camel Theorem stated at the beginning of this paper.*

Appendix B. The Maslov-Duistermaat index

In this appendix, we recall Duistermaat's generalisation of the Maslov index [7], and relate it to another index obtained with quadratic generating forms.

B.1. On a symplectic vector space

Let (F, σ) be a symplectic vector space of dimension $2m$, and $\Lambda(F) = \Lambda(F, \sigma)$ be the set of its Lagrangian subspaces. If $\alpha \in \Lambda(F)$ and $k = 0, \dots, m$, we consider

$$\Lambda^k(\alpha) = \{\beta \in \Lambda(F); \dim(\alpha \cap \beta) = k\}$$

and then $\Sigma(\alpha) = \Lambda(F) - \Lambda^0(\alpha)$, which is an algebraic hypersurface of $\Lambda(F)$ whose principal part is $\Lambda^1(\alpha)$.

Generically, a smooth loop $L : S^1 \to \Lambda(F)$ intersects $\Sigma(\alpha)$ in $\Lambda^1(\alpha)$ only; $\Lambda^1(\alpha)$ being coorientable, the algebraic intersection number of L with $\Sigma(\alpha)$ can be defined; and because $\Lambda(F)$ is connected, this number does not depend on the choice of $\alpha \in \Lambda(F)$. It is the *Maslov index of the loop* L, denoted by $\mathrm{ind}(L)$, see [1]. A loop is contractible if and only if its Maslov index vanishes.

The sign convention we use (following Duistermaat) is that, in $\mathbb{R}^2$ with the standard structure for instance, the loop $L = (L_t)_{t\in[0,1]}$ defined by $L_0 = \mathbb{R} \times 0$ and $L_t = e^{i\pi t}(L_0)$ has index -1 (ie. turning positively with respect to the natural orientation gives negative Maslov index).

In [7] (see also [6]), Duistermaat generalizes this index to non-closed curves of Lagrangian subspaces, as follows. Let $L : [0,1] \to \Lambda(F)$ be such a (continuous) path. We choose $\alpha \in \Lambda(F)$ transversal to L_0 and L_1. As $\Lambda^0(\alpha)$ is simply-connected (it has the structure of an affine space), there is a path L' in $\Lambda^0(\alpha)$ joining L_1 to L_0, and all such paths are homotopic. The *intersection index of* L *with* α, denoted by $[L : \alpha]$, will be the Maslov index of the loop $\widetilde{L} = L * L'$:

$$[L : \alpha] = \operatorname{ind}(\widetilde{L}) \tag{27}$$

Duistermaat then adds a boundary term to obtain an integer *independent of* α. Because of the transversality assumption, there is a linear map $C : L_1 \to \alpha$ such that L_0 is the graph of C, ie. $L_0 = \{u + Cu; u \in L_1\}$. Then a quadratic form denoted by $Q\big(L_1, \alpha; L_0\big)$ can be defined on L_1:

$$\begin{aligned} Q\big(L_1, \alpha; L_0\big) : L_1 &\to \mathbb{R} \\ u &\mapsto \sigma(Cu, u) \end{aligned} \tag{28}$$

The *Maslov-Duistermaat index* $\operatorname{ind}(L)$ *of the path* L is now

$$\operatorname{ind}(L) = [L : \alpha] + \operatorname{ind} Q\big(L_1, \alpha; L_0\big) \tag{29}$$

As notation suggests, it does not depend on the choice of $\alpha \in \Lambda^0(L_0) \cap \Lambda^0(L_1)$, and it obviously gives the same index as before when L is a loop.

Proposition B.1. *Let* $L : [0,1] \to \Lambda(F)$ *be a path.*

1. *The integer* $\operatorname{ind}(L)$ *depends only on the homotopy class (with endpoints fixed) of* L.
2. *If* $A \in \operatorname{Sp}(F, \sigma)$ *and* AL *denotes the path* $(AL)_t := A(L_t)$ *in* $\Lambda(F)$*, then* $\operatorname{ind}(AL) = \operatorname{ind}(L)$
3. *If* L' *is a loop in* $\Lambda(F)$ *based at* L_1*, then* $\operatorname{ind}(L * L') = \operatorname{ind}(L) + \operatorname{ind}(L')$ *(note that the Maslov-Duistermaat index is not additive for the concatenation of all paths).*

Proof. These properties come directly from the definition and from the analogous (standard) properties of the ordinary Maslov index for loops. □

To extend the Maslov-Duistermaat index to a symplectic vector bundle over the circle, we will need the following result.

Corollary B.2. *Let* L*,* L' *be two paths in* $\Lambda(F)$*, and* $A = (A_t)_{t \in [0,1]}$ *be a loop in* $\operatorname{Sp}(F)$*. Let* AL *denote the path in* $\Lambda(F)$ *defined by* $(AL)_t := A_t(L_t)$*, and similarly for* AL'*. Then we have*

$$\operatorname{ind}(AL) - \operatorname{ind}(AL') = \operatorname{ind}(L) - \operatorname{ind}(L')$$

Proof. The path AL is homotopic (with endpoints fixed) to the path A_0L followed by the loop AL_1, hence $\operatorname{ind}(AL) = \operatorname{ind}(L) + \operatorname{ind}(AL_1)$ by Proposition B.1. Similarly, $\operatorname{ind}(AL') = \operatorname{ind}(L') + \operatorname{ind}(AL'_1)$. But, since $\Lambda(F)$ is connected, the two loops AL_1 and AL'_1 are homotopic, hence they have the same ordinary Maslov index. □

B.2. On a symplectic vector bundle over the circle

Consider next a symplectic vector bundle $E \to S^1$ with fiber (F, σ). We see S^1 as the interval $[0, 1]$ with endpoints identified, and denote by t its generic point; the fiber of E over t will be called E_t.

We consider $V = \cup_{t \in S^1} V_t$ a Lagrangian subbundle of E, and $R : [0, 1] \to \Lambda(E)$ a path of Lagrangian subspaces $R_t \subset E_t$ (without imposing $R_0 = R_1$) .

Because $\operatorname{Sp}(F)$ is connected, the symplectic bundle E is trivial, ie. there is a symplectic isomorphism $\tau : E \cong S^1 \times (F, \sigma)$. Then $\tau(V)$ can be identified to a loop in $\Lambda(F)$, and $\tau(R)$ to a path. According to Corollary B.2 the difference

$$\operatorname{ind}_V(R) := \operatorname{ind}\big(\tau(R)\big) - \operatorname{ind}\big(\tau(V)\big) \tag{30}$$

does not depend on the trivialization τ chosen. It is called the *Maslov index of* R *with respect to* V.

Remark B.3. Suppose that R_t and V_t are transverse for all $t \in [0, 1]$. Then

$$\operatorname{ind}_V(R) = \operatorname{ind} Q(R_1, V_0; R_0) \tag{31}$$

where the definition of $Q(R_1, V_0; R_0)$ is a straightforward generalization of (28). Indeed, by the very definition of $\operatorname{ind}_V(R)$, we may suppose that V (resp. R) is a loop (resp. a path) in $\Lambda(F)$. Since $V_0 = V_1$ is transverse to R_0 and R_1 by assumption, we may take $\alpha = V_0$ to compute $\operatorname{ind}(R)$. Let R' be a path in $\Lambda^0(V_0)$, joining R_1 to R_0. Then $\operatorname{ind}(R) = \operatorname{ind}(R' \cdot R) + \operatorname{ind} Q(R_1, V_0; R_0)$ by definition, and we just need to prove that $\operatorname{ind}(R' \cdot R) = \operatorname{ind}(V)$. But it is clear that $R' \cdot R$ is homotopic to a loop S in $\Lambda(F)$ such that $S_t \cap V_t = 0$ for all t, and this implies that S and V have the same (ordinary) Maslov index.

If Γ_1 and Γ_2 be two Lagrangian subbundles of E, then the *Maslov class* $\mu(\Gamma_1, \Gamma_2)$ *of the pair* (Γ_1, Γ_2) is defined as $\mu(\Gamma_1, \Gamma_2) = \operatorname{ind}_{\Gamma_2}(\Gamma_1)$. It vanishes if and only Γ_1 and Γ_2 are homotopic through Lagrangian subbundles of E. In that case, we have $\operatorname{ind}_{\Gamma_1}(R) = \operatorname{ind}_{\Gamma_2}(R)$; more generally, the following relation holds:

$$\operatorname{ind}_{\Gamma_1}(R) - \operatorname{ind}_{\Gamma_2}(R) = \mu(\Gamma_2, \Gamma_1) = -\mu(\Gamma_1, \Gamma_2) \tag{32}$$

B.3. Using generating functions

We consider the space $\mathbb{R}^{2m}$ endowed with the symplectic form $\Omega_{\mathbb{R}^{2m}}$.

Let k be an arbitrary integer, and $Q = Q(w,\xi) : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ be a quadratic form. Using matrix representation with respect to the canonical bases of $\mathbb{R}^m$ and $\mathbb{R}^k$, we write $Q(X) = \frac{1}{2}\,{}^tXBX$, with $X = \begin{pmatrix} w \\ \xi \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ {}^tb & c \end{pmatrix}$ a symmetric $(n+k)$-matrix.

We say that Q is a *generating form* if it is a generating function in the sense of Definition A.2, ie. if the $k \times (n+k)$-matrix $({}^tb, c)$ is of maximal rank k. Then $\Sigma_Q = \left\{(w,\xi); {}^tbw + c\xi = 0\right\}$ is a m-dimensional vector subspace of $\mathbb{R}^m \times \mathbb{R}^k$, and the map $i_Q : \Sigma_Q \to \mathbb{R}^{2m} \cong T^*\mathbb{R}^m$ defined by $i_Q(w,\xi) = (w, aw + b\xi)$ is a Lagrangian linear embedding. The Lagrangian subspace $L = \operatorname{Im}(i_Q)$ is said to be generated by Q. The spaces $\operatorname{Ker} Q$ and $(\mathbb{R}^m \times 0) \cap L$ are obviously isomorphic.

Example B.4. Let W be a Lagrangian submanifold of $\mathbb{R}^{2m}$ admitting a generating function $S : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$. If (w,w') is a point on W and (w,ξ) is the corresponding element of Σ_S, then $d^2S(w,\xi)$ is a generating form for the Lagrangian subspace $T_{(w,w')}W \in \Lambda(\mathbb{R}^{2m})$.

As in the non-linear case of Section A, there are existence and uniqueness results for forms generating a continuous path of Lagrangian subspaces. The proofs are much simpler, however, in the linear case: see [16].

Theorem B.5. and Definition). *Let $L : [0,1] \to \Lambda(\mathbb{R}^{2m})$ be a path of Lagrangian subspaces. Then there is a path $(Q_t)_{t\in[0,1]}$ of generating forms, such that Q_t generates L_t for all $t \in [0,1]$. Furthermore, if $(Q_t)_{t\in[0,1]}$ is any such path, then the integer* $\operatorname{ind} Q_1 - \operatorname{ind} Q_0$ *depends only on $L = (L_t)_{t\in[0,1]}$. It is called the gf-index of L, denoted by* $\operatorname{ind}_{\mathrm{gf}}(L)$. *If L is a loop, then* $\operatorname{ind}_{\mathrm{gf}}(L)$ *coincides with the standard Maslov index of L.*

Now, if $\mathrm{Sp}(\mathbb{R}^{2n})$ is the manifold of linear symplectomorphisms of $(\mathbb{R}^{2n}, \Omega_{\mathbb{R}^{2n}})$ and $A : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n})$ is a continuous path, we use the identification (24) to define a path L in $\Lambda(\mathbb{R}^{2m}, \Omega_{\mathbb{R}^{2m}})$, with $2m = n$: for $t \in [0,1]$, the graph of A_t is a Lagrangian subspace of $\overline{\mathbb{R}}^{2n} \times \mathbb{R}^{2n}$, and we set $L_t = I(\operatorname{graph} A_t)$.

Definition B.6. The *gf-index* of the path A is $\operatorname{ind}_{\mathrm{gf}}(A) := \operatorname{ind}_{\mathrm{gf}}(L)$.

Proposition B.7. *Let $\Phi = (\Phi_t)_{t\in[0,1]}$ be a Hamiltonian isotopy of $\mathbb{R}^{2n}$ with compact support, and $(S_t)_{t\in[0,1]}$ be a family of generating functions as in Theorem A.3. Let $z \in \mathbb{R}^{2n}$ be a fixed point of Φ_1 and (z,ξ) be the corresponding critical point of S_1. If A denotes the path of symplectomorphisms $A_t = d\Phi_t(z) \in \mathrm{Sp}(\mathbb{R}^{2n})$, then*

$$\operatorname{ind}_{\mathrm{gf}}(A) = \operatorname{ind} d^2 S_1(z,\xi) - \operatorname{ind} Q_\infty$$

Proof. We follow the notations of Appendix A. In particular, $(\Psi_t)_{t\in[0,1]}$ is the Hamiltonian isotopy of $T^*\mathbb{R}^{2n}$ given by $\Psi_t = I \circ (\mathrm{id} \times \Phi_t) \circ I^{-1}$. There is a continuous path $(w_t, \xi_t) \in \mathbb{R}^{2n} \times \mathbb{R}^k$ ending at (z,ξ), such that $(w_t,\xi_t) \in \Sigma_{S_t}$ and $i_{S_t}(w_t,\xi_t) = \Psi_t(z,0)$ for all t. Then $Q_t = d^2 S_t(w_t,\xi_t)$ is a quadratic generating form of $T_{\Psi_t(0,z)}\widetilde{\Gamma}_t$, and that vector subspace is precisely L_t. Hence $\operatorname{ind}_{\mathrm{gf}}(A) = \operatorname{ind}_{\mathrm{gf}}\big(I(\operatorname{graph} A)\big) = \operatorname{ind} Q_1 - \operatorname{ind} Q_0$ by definition.

Since S_0 generates the zero section and $S_0 = Q_\infty$ outside a compact set, it is easy to see that $\operatorname{ind} d^2 S_0(w_0,\xi_0) = \operatorname{ind} Q_\infty$ (consider a path γ_t on Σ_{S_0}, joining (w_0,ξ_0) to a point at infinity; it is immediate that $\operatorname{Ker} d^2 S_0(\gamma_t)$ has constant dimension, so the index of $d^2 S_0(\gamma_t)$ is also constant).

Hence $\operatorname{ind}_{\mathrm{gf}}(A) = \operatorname{ind} d^2 S_1(z,\xi) - \operatorname{ind} Q_\infty$ as claimed. □

On the other hand, the path $(\operatorname{graph} A_t)_{t\in[0,1]}$ also has a well-defined Maslov-Duistermaat index, from Appendix B. We show that the two indices are equal if the path starts at the identity map.

Proposition B.8. *Let $A = (A_t)_{t\in[0,1]}$ be a path in $\mathrm{Sp}(\mathbb{R}^{2n})$. If $A_0 = \mathrm{Id}$, then*

$$\operatorname{ind}_{\mathrm{gf}}(A) = \operatorname{ind}(\operatorname{graph} A)$$

Proof. Let us begin with a simple but important remark: to prove that ind and $\operatorname{ind}_{\mathrm{gf}}$ coincide for all paths joining two fixed Lagrangians L_0 and L_1, it is enough to show that they coincide for one of them. This follows easily from the additive property of $\inf_{\mathrm{gf}}$ under concatenation of paths, from the weaker corresponding statement for the Maslov-Duistermaat index (see Proposition B.1), and the fact that the indices do coincide on *loops* of Lagrangian subspaces.

Since

- (i) A_1 gives a decomposition $\mathbb{R}^{2n} = F' \oplus F''$ as the direct sum of symplectic A_1-invariant subspaces such that the restriction of A_1 to F' does not have the eigenvalue -1, and the restriction of A_1 to F'' has only the eigenvalue -1,
- (ii) the symplectic group of a symplectic vector space is always connected,
- (iii) the indices are additive with respect to symplectic direct sums,

we may suppose that A_1 does not have the eigenvalue -1 or that it has only this eigenvalue.

1. Let us first assume that -1 is not an eigenvalue of A_1. Then, in view of (24), the hypotheses mean that $L_0 = \mathbb{R}^{2n} \times 0$ and that L_1 is transversal to $0 \times \mathbb{R}^{2n}$. Consequently, we may take $\alpha = 0 \times \mathbb{R}^{2n}$ in (27)–(28)–(29).

   First, let L' be path in $\Lambda^0(\alpha)$, joining L_1 to L_0. Then $[L : \alpha] = \operatorname{ind}(L * L')$

by definition, see (27). But ind and $\mathrm{ind}_{\mathrm{gf}}$ coincide on loops of Lagrangian subspaces, so $[L : \alpha] = \mathrm{ind}_{\mathrm{gf}}(L * L')$. Since $\mathrm{ind}_{\mathrm{gf}}$ is additive (this is obvious), we have

$$[L : \alpha] = \mathrm{ind}_{\mathrm{gf}}(L) + \mathrm{ind}_{\mathrm{gf}}(L')$$

Now L' is a path of Lagrangian subspaces that never meets the vertical $0 \times \mathbb{R}^{2n}$. This implies that the L'_t's are graphs of (symmetric) linear maps $\ell'_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Then $Q'_t(w) := \frac{1}{2} < \ell'_t w, w >$ defines a quadratic form generating L'_t, for $t \in [0,1]$. Since $L'_1 = L_0 = \mathbb{R}^{2n} \times 0$, we have $\ell'_1 = 0$, hence $Q'_1 = 0$. Therefore,

$$\mathrm{ind}_{\mathrm{gf}}(L') = \mathrm{ind}\, Q'_1 - \mathrm{ind}\, Q'_0 = -\,\mathrm{ind}\, Q'_0$$

Finally, we relate Q'_0 and $Q(L_1, \alpha; L_0)$. Consider the linear map $C : L_1 \to \alpha$ such that $u + Cu \in L_0 = \mathbb{R}^{2n} \times 0$ for all $u \in L_1$. Since $L_1 = L'_0$ is the graph of ℓ'_0, we write $u = (w, \ell'_0 w) = (w, 0) + (0, \ell'_0 w) \in (\mathbb{R}^{2n} \times 0) \oplus (0 \times \mathbb{R}^{2n})$. Hence $Cu = (0, -\ell'_0 w)$, and then

$$Q(L_1, \alpha; L_0)(u) = \Omega(Cu, u) = \Omega(0, -\ell'_0 w; w, \ell'_0 w) = \; < \ell'_0 w, w > \; = 2Q'_0(w)$$

This proves in particular that

$$\mathrm{ind}\, Q'_0 = \mathrm{ind}\, Q(L_1, \alpha; L_0)$$

whence

$$\begin{aligned} \mathrm{ind}(L) &= [L : \alpha] + \mathrm{ind}\, Q(L_1, \alpha; L_0) \\ &= \mathrm{ind}_{\mathrm{gf}}(L) + \mathrm{ind}_{\mathrm{gf}}(L') + \mathrm{ind}\, Q(L_1, \alpha; L_0) \\ &= \mathrm{ind}_{\mathrm{gf}}(L) - \mathrm{ind}\, Q'_0 + \mathrm{ind}\, Q'_0 \\ &= \mathrm{ind}_{\mathrm{gf}}(L) \end{aligned}$$

2. Let us now assume that -1 is the only eigenvalue of A_1. We choose α to be $I\big(\mathrm{graph}(B)\big)$, where $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then α is transversal to L_0 and L_1, and furthermore it is possible to join L_1 to $-\,\mathrm{Id}$ through symplectomorphisms that have only -1 as eigenvalue. It is then easy to see that both indices do not change if we compose our path (A_t) with this path from L_1 to $-\,\mathrm{Id}$. Hence we may assume that $A_1 = -\,\mathrm{Id}$. But then we only need to check equality of the indices to one given path from Id to $-\,\mathrm{Id}$, and again we may assume that $\mathbb{R}^{2n} = \mathbb{R}^2$ and A_t is rotation of angle $2\pi t$. A direct application of the definitions shows that in this case both indices are equal to 0. □

References

[1] V. I. Arnold, Characteristic class entering in quantization conditions, *Functional Analysis and its Applications* **1** (1967), 1–14.

[2] M. Audin, Quelques remarques sur les surfaces lagrangiennes de Givental, *Journ. of Geom. and Physics* **7** (1990), 583–598.

[3] A. Banyaga, Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comm. Math. Helv.* **53** (1978), 174–227.

[4] M. Chaperon, Une idée du type géodésiques brisées pour les systèmes hamiltoniens, *C.R.A.S Paris* **298** (1984), 293–296.

[5] Y. Chekanov, Lagrangian tori in a symplectic vector space and global symplectomorphisms, *Math.Z.* **223** (1996), 547–559.

[6] P. Dazord, Une interprétation géométrique de la classe de Maslov-Arnold, *Journ. Math. Pures et Appl.* **56** (1977), 231–250.

[7] J. J. Duistermaat, On the Morse index in variational calculus, *Advances in math.* **21** (1976), 173–195.

[8] Y. Eliashberg and M. Gromov, Convex symplectic manifolds, *Several Complex Variables Proceedings*, Summer Research Insitute, Santa-Cruz 1989, Part 2 (E. Bedford et al., ed.), vol. **52**, 1991, pp. 135–162.

[9] A. B. Givental, Lagrange embeddings of surfaces and unfolded Whitney umbrellas, *Funct. Anal. and Appl.* **20** (1986), 35–41.

[10] W. Klingenberg, *Riemannian geometry*, de Gruyter, 1982.

[11] D. McDuff, Blow ups and symplectic embeddings in dimension 4, *Topology* **30** (1991), 409–421.

[12] D. McDuff and L. Traynor, The 4-dimensional camel problem and related results, *Symplectic Geometry* (D. Salamon, ed.), vol. **192**, London Math. Soc. Lecture Notes Ser., 1993, pp. 162–182.

[13] L. Polterovich, Symplectic displacement energy for Lagrangian submanifolds, *Erg. Theory and Dyn. Systems* **13** (1993), 357–367.

[14] J.-C. Sikorav, Problèmes d'intersection et de points fixes en géométrie hamiltonienne, *Comment. Math. Helvetici* **62** (1987), 62–73.

[15] D. Théret, A complete proof of Viterbo's uniqueness theorem on generating functions, To appear in *Topology and its Applications*, 1996.

[16] D. Théret, Utilisation des fonctions génératrices en géométrie symplectique globale, Thèse de Doctorat, Université Paris 7 (France), 1996.

[17] C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.* **100** (1990), 301–320.

[18] C. Viterbo, Plongements lagrangiens et capacités symplectiques de tores dans $\mathbb{R}^{2n}$, *C.R.A.S. Paris*, série I **311** (1990), 487–490.

[19] C. Viterbo, Symplectic topology as the geometry of generating functions, *Math. Ann.* **292** (1992), 685–710.

[20] A. Weinstein, Lectures on symplectic manifolds, *CBMS Reg. Conf. Series in Math.*, vol. **29**, AMS, Providence, R.I. 1977.

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