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On holomorphic forms on compact complex threefolds

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Abstract. We study the structure of holomorphic 1. forms on compact complex threefolds of positive algebraic dimension. We obtain a rather detailed description of integrable 1. forms. We use this result to extend Castelnuovo - De Franchis lemma (as well as Catanese's generalization) to non-Kähler threefolds.

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A very useful tool in the study of Kähler manifolds is the classical Castelnuovo - De Franchis lemma: it says that if ω_1 and ω_2 are two linearly independent holomorphic 1-forms on a connected compact Kähler manifold M, which satisfy the collinearity relation $\omega_1 \wedge \omega_2 \equiv 0$, then there exists a holomorphic map $\pi : M \to C$ onto an algebraic curve C of genus greater or equal than 2 such that ω_1, ω_2 are pull-back by π of two holomorphic 1-forms on C. This simple lemma has several nontrivial consequences on the topology of Kähler manifolds [Cat], their Chern numbers [Bog] [Miy], and many others things. A nice generalization has recently been found by Catanese [Cat] (and, independently, other mathematicians; see [Cat] for the references). We state only a particular case, sufficient for our purposes: if ω_1, ω_2 and ω_3 are holomorphic 1-forms on a connected compact Kähler manifold M, such that $\omega_1 \wedge \omega_2 \wedge \omega_3 \equiv 0$ and $\omega_1 \wedge \omega_2$, $\omega_2 \wedge \omega_3$, $\omega_3 \wedge \omega_1$ are linearly independent, then there exists a holomorphic map $\pi : M \to S$ onto a normal algebraic surface S of Albanese general type such that ω_1, ω_2, ω_3 are pull-back by π of three holomorphic 1-forms on S. We shall recall in section 1 the definition of "variety of Albanese general type", for the moment we only say that it is one of the possible higherdimensional generalizations of "curve of genus greater or equal than 2".

These results are based on the closedness of holomorphic forms on compact Kähler manifolds; in fact, the Kähler assumption is exploited only to ensure that closedness. There are, however, many examples of compact complex *non-Kähler* manifolds which support *non-closed* holomorphic 1-forms: the most classical ones are compact quotients of certain Lie groups [Ue1,§17]. It is not clear to us if Castelnuovo - De Franchis - Catanese statement is still true outside the Kähler world. For instance, it is false on algebraic varieties in positive characteristic, and it frequently happens that positive characteristic algebraic geometry presents the same pathologies of non-Kähler complex geometry.

In this paper we shall study the three-dimensional situation (holomorphic 1-forms on compact surfaces are always closed, by Stokes theorem). In order to obtain a complete result, we shall need a (unnecessary?) integrability hypothesis: recall that a 1-form ω is said to be *integrable* if $\omega \wedge d\omega$ is identically zero.

Theorem. *Let M be a connected compact complex threefold.*
1) If $\omega_1, \omega_2 \in \Omega^1(M)$ are two holomorphic 1-forms such that:
1.i) $\omega_1 \wedge d\omega_1 \equiv 0$;
1.ii) $\omega_1 \wedge \omega_2 \equiv 0$;
1.iii) ω_1 and ω_2 are linearly independent;
then ω_1 and ω_2 are closed. Hence there exists a holomorphic map $\pi : M \to C$ onto an algebraic curve C of genus greater or equal than 2 such that $\omega_i = \pi^*(\eta_i)$ for suitable $\eta_i \in \Omega^1(C)$, $i = 1, 2$.
2) If $\omega_1, \omega_2, \omega_3 \in \Omega^1(M)$ are three holomorphic 1-forms such that:
2.i) $\omega_1 \wedge d\omega_1 \equiv \omega_2 \wedge d\omega_2 \equiv 0$;
2.ii) $\omega_1 \wedge \omega_2 \wedge \omega_3 \equiv 0$;
2.iii) $\omega_1 \wedge \omega_2$, $\omega_2 \wedge \omega_3$ and $\omega_3 \wedge \omega_1$ are linearly independent;
then ω_1, ω_2 and ω_3 are closed. Hence there exists a holomorphic map $\pi : M \to S$ onto a normal algebraic surface S of Albanese general type such that $\omega_i = \pi^*(\eta_i)$ for suitable $\eta_i \in \Omega^1(S)$, $i = 1, 2, 3$.

Let us spend some words about the proof. In both cases the threefold M has some nonconstant meromorphic function, given by a "quotient" of holomorphic forms. Hence its algebraic dimension [Ue1] $a(M)$ is at least 1. If $a(M) = 3$ then (Moishezon) M is bimeromorphic to a projective threefold and therefore every 1-form on M is closed. If $a(M) = 1$ or 2 we can efficiently use an algebraic reduction of M, which is a fibration over a curve or a surface [Ue1]. If $a(M) = 2$ then M may have non-closed 1-forms, but we shall see that they are quite special, and in particular never integrable. However if $a(M) = 2$ we shall prove the theorem even without the integrability hypothesis and also in a higherdimensional context. The difficult case, where integrability will play an important role, is the case $a(M) = 1$, and in fact our paper is mostly devoted to a rather detailed description of integrable 1-forms on threefolds whose algebraic dimension is equal to 1 (description which may eventually be useful for other purposes). We also note some point of contact with [C-P], where the authors study 2-forms on Kähler threefolds with the help of canonical fibrations; however in our case the difficulties arise from the non-Kähler setting [Ue2], whereas in [C-P] they have a different nature.

As an application of the theorem we shall prove the

Corollary. *Let M be a connected compact complex threefold with* $\dim \Omega^3(M) \leq 1$ *and* $\dim \Omega^1(M) - \dim \Omega^3(M) \geq 3$. *Then M fibers over a curve of genus greater or equal than 2 or a normal surface of Albanese general type.*

1. Preliminaries and some general results

We shall use in the following the basic properties of algebraic dimension and algebraic reduction of a compact complex space, which can be found in [Ue1,§3,§12]. We only recall that given a connected compact complex manifold M of algebraic dimension $a(M)$ (by definition, this is the transcendence degree of the field of meromorphic functions on M, $\mathcal{M}(M)$) we can construct a modification $\tilde{M} \xrightarrow{r} M$ and a holomorphic map $\pi : \tilde{M} \to V$ with connected fibres onto an algebraic manifold V of dimension $a(M)$, such that $\mathcal{M}(M) \overset{r}{\simeq} \mathcal{M}(\tilde{M}) = \pi^*\mathcal{M}(V)$. A similar construction can be done starting with any algebraically closed subfield of $\mathcal{M}(M)$, instead of the full $\mathcal{M}(M)$.

A compact complex manifold (or space) M is called *Moishezon* if $a(M) = \dim(M)$. Any Moishezon space is bimeromorphic to a complex projective manifold, and therefore every holomorphic form on it is closed. This fact can be elementarily proved by remarking that any k-form ω on a compact complex space of dimension $(k+1)$ is closed (by Stokes theorem applied to the exact and non negative form $d\omega \wedge \overline{d\omega}$) and by observing that a Moishezon space M contains a lot of compact complex subspaces, of any dimension (more precisely, given a generic $p \in M$ and a l-subspace $E \subset T_pM$ we can find a l-dimensional compact complex subspace $N \subset M$ with $p \in N$ and $T_pN = E$).

Given a Moishezon space M, we can consider its Albanese map $a_M : M \to A_M$, where A_M is the Albanese torus of M. Then any holomorphic 1-form on M is the pull-back by a_M of a unique holomorphic (linear) 1-form on A_M. We shall say that M is of *Albanese general type* [Cat] if $\dim(a_M(M)) = \dim(M)$ and $a_M(M)$ is a general type variety (equivalently, by results of Ueno and Kawamata, $a_M(M) \subset A_M$ is not invariant by translations along a nontrivial subtorus of A_M). Remark that this definition is more restrictive than the one given by Catanese (he requires only $\dim a_M(M) = \dim(M) < \dim(A_M)$).

As in [Cat], we shall say that a collection of holomorphic 1-forms $\omega_1, ..., \omega_{k+1}$ on M (smooth, connected, compact, n-dimensional) generate a *strict k-wedge* if:
i) $\omega_1 \wedge ... \wedge \omega_{k+1} \equiv 0$;
ii) the k-forms $\Omega_j = \omega_1 \wedge ... \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge ... \wedge \omega_{k+1}$, $j = 1, ..., k+1$, are linearly independent.
More explicitely, this means that there are meromorphic functions $f_1, ..., f_k \in \mathcal{M}(M)$ such that

$$\omega_{k+1} = \sum_{i=1}^{k} f_i \omega_i$$

and $\{1, f_1, ..., f_k\}$ are linearly independent (over $\mathbf{C}$). In particular, each f_i is not a constant and so the algebraic dimension of M is at least 1. We necessarily have $1 \le k \le n$. If $k = 1$ then we are in the setting of Castelnuovo - De Franchis lemma: two linearly independent 1-forms whose wedge product is identically zero. At the opposite side, one can show that a Moishezon n-space is of Albanese general type if and only if it admits a strict n-wedge [Kaw].

We can now recall Castelnuovo - De Franchis - Catanese lemma. We shall give a proof slightly different from that of [Cat] in order to see how and where the closedness hypothesis is really exploited.

Lemma 1 [Cat]. *Let M be a connected compact complex manifold and let $\omega_1, \ldots, \omega_{k+1}$ be closed 1-forms on M generating a strict k-wedge. Then there exists a holomorphic map $\pi : M \to V$ onto a k-dimensional normal Moishezon space of Albanese general type such that $\omega_i \in \pi^*(\Omega^1(V))$ for every $i = 1, \ldots, k+1$.*

Proof. The closed 1-forms $\{\omega_i\}_{i=1}^{k+1}$ generate a singular holomorphic foliation $\mathcal{F}$ on M, given at a generic point by the intersection of the kernels of the 1-forms. Clearly the codimension of the leaves of $\mathcal{F}$ is equal to k. Let $\mathcal{M}(\mathcal{F})$ be the field of meromorphic functions on M which are constant on the leaves of $\mathcal{F}$. As usual, we can construct a modification $\tilde{M} \xrightarrow{r} M$ and a holomorphic map $\tilde{M} \xrightarrow{\pi} \tilde{V}$ with connected fibres onto a smooth algebraic variety $\tilde{V}$ such that

$$\mathcal{M}(\mathcal{F}) \overset{r}{\simeq} \mathcal{M}(\tilde{\mathcal{F}}) = \pi^* \mathcal{M}(\tilde{V})$$

where $\tilde{\mathcal{F}}$ is the foliation lifted to $\tilde{M}$.

We claim that $l = \dim(\tilde{V})$ is equal to k. Assume by contradiction that $l \neq k$, i.e. $l < k$. Then the foliation $\tilde{\mathcal{F}}$, whose leaves are contained in the fibres of π, restricts on a generic fibre to a foliation of codimension $(k-l) > 0$. We can find among the 1-forms $\tilde{\omega}_i = r^*\omega_i$ a collection of $(k-l)$ 1-forms (say, $\tilde{\omega}_1, \ldots, \tilde{\omega}_{k-l}$) whose restrictions to a generic fibre F generate the foliation $\tilde{\mathcal{F}}|_F$ (that is, $\tilde{\omega}_1 \wedge \ldots \wedge \tilde{\omega}_{k-l}|_F \not\equiv 0$). Hence for every $i = k-l+1, \ldots, k+1$ the 1-form $\tilde{\omega}_i|_F$, which vanishes on the leaves of $\tilde{\mathcal{F}}|_F$, can be expressed as a linear combination of $\tilde{\omega}_1|_F, \ldots, \tilde{\omega}_{k-l}|_F$ with meromorphic coefficients. Varying F we find for every $i = k-l+1, \ldots, k+1$ a collection of meromorphic functions $f_{ij} \in \mathcal{M}(\tilde{M})$, $j = 1, \ldots, k-l$, such that

$$\tilde{\omega}_i|_{\text{fibres}} = \sum_{j=1}^{k-l} f_{ij} (\tilde{\omega}_j|_{\text{fibres}}).$$

The closedness of $\tilde{\omega}_i$ on fibres, $i = 1, \ldots, k+1$, implies that

$$\sum_{j=1}^{k-l} (df_{ij}|_{\text{fibres}}) \wedge (\tilde{\omega}_j|_{\text{fibres}}) \equiv 0$$

and therefore $f_{ij}|_{\text{fibres}}$ is constant on the leaves of $\tilde{\mathcal{F}}|_{\text{fibres}}$, that is f_{ij} is constant on the leaves of $\tilde{\mathcal{F}}$: $f_{ij} \in \mathcal{M}(\tilde{\mathcal{F}})$. Hence every f_{ij} is also constant on the fibres of π.

We now look at the cohomology classes $[\tilde{\omega}_i|_F] \in H^1(F, \mathbf{C})$, for F a generic fibre. Varying F, the class of $[\tilde{\omega}_i|_F]$ is locally constant (with respect to a local trivialization which preserves integral cohomology) because $\tilde{\omega}_i$ is closed on $\tilde{M}$. On the other hand $[\tilde{\omega}_1|_F], ..., [\tilde{\omega}_{k-l}|_F]$ are linearly independent because $\tilde{\omega}_1|_F, ..., \tilde{\omega}_{k-l}|_F$ are (and because a closed nontrivial holomorphic 1-form on a compact manifold is never exact, so that the space of closed holomorphic 1-forms injects into the first cohomology group). From

$$[\tilde{\omega}_i|_F] = \sum_{j=1}^{k-l} f_{ij}[\tilde{\omega}_j|_F]$$

we deduce that f_{ij} are constant on all of $\tilde{M}$ and not only on the fibres of π.

Finally, by definition of strict k-wedge we have

$$\tilde{\omega}_1 = \sum_{j=2}^{k+1} f_j \tilde{\omega}_j$$

where $f_j \in \mathcal{M}(\tilde{M})$ and $\{1, f_2, ..., f_{k+1}\}$ are linearly independent. Taking the product with $\tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{k-l}$, restricting to fibres and dividing by $\tilde{\omega}_1 \wedge ... \wedge \tilde{\omega}_{k-l}|_{\text{fibres}}$ we finally obtain

$$1 = \sum_{j=k-l+1}^{k+1} (f_j \sum_{i=1}^{k-l} f_{ji})$$

contradicting the linear independence of $\{1, f_{k-l+1}, ..., f_{k+1}\}$ because every f_{ij} is constant. This proves, as desired, that the dimension of $\tilde{V}$ is equal to k.

As a consequence of this, the foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$ coincides with the fibration $\pi : \tilde{M} \to \tilde{V}$. Because $\mathcal{F}$ on M is defined by closed holomorphic 1-forms and therefore it is locally defined by holomorphic maps to $\mathbf{C}^k$, we see that π descends to a holomorphic map $\hat{\pi} : M \to V$ which defines a fibration which coincides with $\mathcal{F}$. The space V is a normal Moishezon space of dimension k and $\tilde{V} \to V$ is a modification. The 1-forms ω_i vanish on the fibres of $\hat{\pi}$ and therefore they are projectable on V: $\omega_i = \hat{\pi}^*(\eta_i)$, $\eta_i \in \Omega^1(V)$ (singular fibres give no problem, see for instance and more generally [Eno, lemma 3.3]). These η_i generate a strict k-wedge on V, so that V is of Albanese general type. □

In the previous proof we tried to use as less as possible the closedness of the holomorphic 1-forms. Remark that the crucial point was to prove that the functions f_{ij} are constant, and this was done in two steps. As a by-product we can easily prove the next two lemmata.

Lemma 2. *Let M be a connected compact complex manifold of dimension n and let $\omega_1, ..., \omega_{n+1} \in \Omega^1(M)$ be generators of a strict n-wedge. Then M is a Moishezon manifold (and therefore of Albanese general type).*

Proof. Suppose by contradiction that $l = a(M) < n$ and take an algebraic reduction $\pi : \tilde{M} \to V$, $\dim(V) = l$. As in the proof of lemma 1 we choose $(n-l)$ 1-forms among the $\tilde{\omega}_i$ (say, $\tilde{\omega}_1, ..., \tilde{\omega}_{n-l}$) whose $(n-l)$-fold exterior product do not vanish identically on the fibres of π. Then for every $i = n-l+1, ..., n+1$ we obtain

$$\tilde{\omega}_i \wedge \tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{n-l}|_{\text{fibres}} = g_i(\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{n-l}|_{\text{fibres}})$$

where $g_i \in \mathcal{M}(\tilde{M})$ is now constant on the fibres simply because $\mathcal{M}(\tilde{M}) = \pi^*\mathcal{M}(V)$.

Even if $\tilde{\omega}_i \wedge \tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{n-l}$ is possibly non-closed, we still have that its cohomology class $[\tilde{\omega}_i \wedge \tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{n-l}|_F] \in H^{n-l}(F, \mathbf{C})$ (F a generic fibre) is locally constant: to see this, take a generic algebraic curve $C \subset V$ and observe that $\tilde{\omega}_i \wedge \tilde{\omega}_2 \wedge ... \wedge \tilde{\omega}_{n-l}|_{\pi^{-1}(C)}$ is closed by Stokes theorem and $\dim(\pi^{-1}(C)) = n - l + 1$. Hence every g_i is constant on all of $\tilde{M}$, and as in the proof of lemma 1 we rapidly arrive to a contradiction. □

Lemma 3. *Let M be a connected compact complex manifold with $a(M) = \dim(M) - 1$ and let $\omega_1, ..., \omega_{k+1} \in \Omega^1(M)$ be generators of a strict k-wedge. Then $d\omega_i \equiv 0$ for every $i = 1, ..., k+1$ (and therefore we can apply lemma 1).*

Proof.

The generic fibres of an algebraic reduction $\tilde{M} \xrightarrow{\pi} V$ are elliptic curves. If one of the $\tilde{\omega}_i$ does not vanish on a generic fibre then we can work as in the proof of lemma 2 and we arrive to a contradiction. Hence $\tilde{\omega}_i|_{\text{fibres}} \equiv 0$ for every i, so that $\tilde{\omega}_i$ is projectable on V and therefore closed (again, by [Eno, lemma 3.3], singular fibres give no problem). □

If $\dim(M) - a(M) = 2$ (or more) then the situation is more complicated: it may happen that every $\tilde{\omega}_i \wedge \tilde{\omega}_j$ vanishes on the fibres of the algebraic reduction, so that the arguments of lemma 2 do not work, and at the same time some of the $\tilde{\omega}_i$ do not vanish on the fibres, so that we cannot project on V. Moreover, it seems difficult to analyse the variation of the cohomology class of $\tilde{\omega}_i$ restricted to fibres.

Returning to the case $\dim(M) - a(M) = 1$, we also note the following property of integrable 1-forms.

Proposition 1. *Let M be a connected compact complex manifold with $a(M) = \dim(M) - 1$. Then any integrable 1-form on M is closed.*

Proof. Take an algebraic reduction $\tilde{M} \xrightarrow{\pi} V$, whose generic fibres are elliptic curves. Every non-closed 1-form $\tilde{\omega} \in \Omega^1(\tilde{M})$ becomes closed when restricted to any surface $\pi^{-1}(C)$, where C is a generic curve on V. It follows that for generic $p \in \tilde{M}$ the kernel of $d\tilde{\omega}$ at p contains the vertical direction $T_p(\pi^{-1}(\pi(p)))$. On the other hand, for generic $p \in \tilde{M}$ the kernel of $\tilde{\omega}$ at p do not contain the same vertical direction, otherwise $\tilde{\omega}$ would be projectable on V and therefore closed. Hence $(\tilde{\omega} \wedge d\tilde{\omega})(p) \neq 0$. □

Let us also observe that a manifold M with $a(M) = \dim(M) - 1 \geq 2$ may possess non-closed (hence non-integrable) 1-forms. For instance, take an algebraic surface S with a holomorphic 2-form $\Omega \not\equiv 0$ whose periods belong to a lattice $\mathbf{Z} \oplus \tau\mathbf{Z} \subset \mathbf{C}$. Let E be the elliptic curve $\mathbf{C}/\mathbf{Z} \oplus \tau\mathbf{Z}$. We can choose an open covering $\{U_j\}$ of S such that $\Omega|_{U_j} = d\omega_j$, $\omega_j \in \Omega^1(U_j)$, and $\omega_i - \omega_j = dF_{ij}$, $F_{ij} \in \mathcal{O}(U_i \cap U_j)$. Then $\{F_{ij} + F_{jk} + F_{ki}\}$ is a locally constant cocycle which represents the class of Ω in $H^2(S, \mathbf{C})$, hence $F_{ij} + F_{jk} + F_{ki} \in \mathbf{Z} \oplus \tau\mathbf{Z}$ for every i, j, k. We can consider each F_{ij} as a translation on E and so we can construct an E-bundle M over S by glueing the pieces $\{U_j \times E\}$ via the translations $\{F_{ij}\}$. The 1-forms $\omega_j + dt \in \Omega^1(U_j \times E)$ glue to a global 1-form $\omega \in \Omega^1(M)$, whose differential $d\omega$ projects on S to Ω. The 3-form $\omega \wedge d\omega$ vanishes exactly on the preimage on M of the zero set of Ω.

Looking at the proof of proposition 1 we see that this example is not far from the general case. On the other hand, a particular case of this construction (S a complex torus) gives the "solvmanifolds of type 2" [Ue1,p.214].

From now on we shall restrict to the three-dimensional situation. If M is a threefold and $a(M) \geq 2$ or $k = 3$ then by the previous results there is nothing more to do concerning k-wedges. In the next two sections we shall analyse the structure of holomorphic 1-forms on threefolds of algebraic dimension 1.

2. Non-closed 1-forms on threefolds with $a(M) = 1$

Let M be a connected compact complex threefold of algebraic dimension equal to 1. There exists a modification $r : \tilde{M} \to M$ and a surjective holomorphic map $\pi : \tilde{M} \to C$ onto an algebraic curve C, which induces an isomorphism between $\mathcal{M}(C)$ and $\mathcal{M}(\tilde{M})$ (and hence $\mathcal{M}(M)$). The fibres of π are connected, and a generic fibre of π has non-positive Kodaira dimension [Ue1,§12]. There are several possibilities for such a generic fibre, but in this section we shall prove that the existence of a non-closed 1-form on M strongly restricts the choice.

Proposition 2. *If $\omega \in \Omega^1(M)$ is not closed then a generic fibre of π is a surface bimeromorphic to a complex torus.*

In order to prove this proposition we shall bound the first Betti number b_1 of the generic fibre of π.

Lemma 4. $b_1(\text{generic fibre}) \geq 2$.

Proof. If $b_1(\text{generic fibre}) \leq 1$ then the generic fibre of π has no holomorphic 1-forms (see for instance the appendix of [Ue1] for the rudiments of Kodaira's classification of surfaces). Hence the restriction of $\tilde{\omega} = r^*\omega$ to a generic fibre is identically zero, that is $\tilde{\omega}$ is projectable on C and therefore closed, contradiction. □

Lemma 5. b_1(generic fibre) ≥ 4.

Proof. If b_1(generic fibre) $= 2$ or 3 then the generic fibre F_t of π has a one-dimensional space of holomorphic 1-forms, or equivalently its Albanese torus A_t is an elliptic curve over which F_t fibers (with connected fibres) via the Albanese map $\alpha_t : F_t \to A_t$. Over a Zariski - open subset $C_0 \subset C$ we can glue together these Albanese tori $\{A_t\}_{t\in C_0}$ and Albanese maps $\{\alpha_t\}_{t\in C_0}$ to obtain an elliptic surface $A \xrightarrow{p} C_0$ and a holomorphic map $\alpha : \tilde{M}_0 = \pi^{-1}(C_0) \to A$ such that $A_t = p^{-1}(t)$ and the restriction of α to $F_t = \pi^{-1}(t)$ coincides with α_t, for every $t \in C_0$ (see for instance [Cam, lemme 2]; the important fact is that the Albanese map is unique modulo automorphisms of the Albanese torus).

The 1-form $\tilde{\omega} = r^*\omega$ restricts on F_t to a 1-form which is induced by the Albanese map α_t and therefore vanishes on the fibres of α_t. That is, $\tilde{\omega}|_{\tilde{M}_0}$ vanishes on the fibres of α and so it is projectable on A: there exists $\eta \in \Omega^1(A)$ such that $\alpha^*\eta = \tilde{\omega}|_{\tilde{M}_0}$. We want to prove that η is closed: this fact would be obvious if A were compact (Stokes) or at least if A were a Zariski - open subset of a compact surface B such that $\alpha : \tilde{M}_0 \to A$ extends to a holomorphic map $\beta : \tilde{M} \to B$ [Eno, lemma 3.3]. However, we note that the arguments of [Eno] can be applied also to our noncompact (and perhaps noncompactifiable) situation, in the following way.

For any $\epsilon > 0$ (small) let $\gamma_\epsilon \subset C_0$ be the boundary of a ϵ-neighbourhood of $C \setminus C_0$ in C, with respect to any smooth metric on C. We have to prove that the non-negative function

$$F(\epsilon) = \int_{p^{-1}(\gamma_\epsilon)} |\eta \wedge \overline{d\eta}| \quad , \qquad \epsilon > 0,$$

tends to zero as $\epsilon \to 0$, so that by Stokes theorem $\int_A d\eta \wedge \overline{d\eta} = 0$ and hence $d\eta = 0$. Take a hermitian metric on $\tilde{M}$ (not only $\tilde{M}_0$) and denote by Θ its hermitian (1,1)-form. Then the non-negative function

$$G(\epsilon) = \int_{\pi^{-1}(\gamma_\epsilon)} |\tilde{\omega} \wedge \overline{d\tilde{\omega}} \wedge \Theta| \quad , \qquad \epsilon > 0,$$

tends to zero as $\epsilon \to 0$, because the volume of $\pi^{-1}(\gamma_\epsilon)$ tends to zero and $\tilde{\omega}$, Θ are defined on all of $\tilde{M}$. The map $\pi^{-1}(\gamma_\epsilon) \xrightarrow{\alpha} p^{-1}(\gamma_\epsilon)$ is a regular fibration, along which $\tilde{\omega}$ is projectable to η; we therefore obtain, by Fubini's theorem,

$$G(\epsilon) \geq F(\epsilon) \cdot \inf_{q \in p^{-1}(\gamma_\epsilon)} \int_{\alpha^{-1}(q)} \Theta = F(\epsilon) \cdot \inf_{q \in p^{-1}(\gamma_\epsilon)} \{Area(\alpha^{-1}(q))\}.$$

But on a compact hermitian manifold the areas of compact complex curves are uniformly bounded from below by a strictly positive constant, therefore we finally obtain that $F(\epsilon)$ tends to zero, as desired. Hence $d\eta \equiv 0$, $d\tilde{\omega} \equiv 0$, contradiction.
□

Proof of proposition 2. From the classification of surfaces [Ue1, appendix] the only surfaces whose Kodaira dimension is non-positive and whose first Betti number is at least 4 are are the surfaces whose a minimal model is a complex torus or a ruled surface of genus at least 2. But this latter case never appears as a fibre of an algebraic reduction [Kuh]. □

Remark. There exists another way to see that the generic fibre of π is not a ruled surface, of any genus. In that case, $\ker d\tilde{\omega}$ would define on a Zariski open subset of $\tilde{M}$ a rational fibration. If Ψ is a closed (1,1)-form representing the first Chern class of $\tilde{M}$ then the integral of Ψ over a rational fibre is strictly positive (=2) and hence $\int_{\tilde{M}} d\tilde{\omega} \wedge \overline{d\tilde{\omega}} \wedge \Psi$ is also strictly positive. But the same integral must be zero by Stokes theorem. This argument, however, cannot be exploited in the situation of lemma 5, where $\ker d\tilde{\omega}$ defines on $\tilde{M}_0$ an elliptic fibration.

We conclude this section by recalling that there are several examples of compact complex threefolds of algebraic dimension 1 which possess non-closed 1-forms, integrable or not. The simplest ones are suspensions of torus automorphisms [G-V]. Take a 2×2 complex matrix A which preserves a lattice $\Gamma \subset \mathbf{C}^2$ and has an eigenvalue λ of modulus smaller than 1. Let T be the complex torus $\mathbf{C}^2/\Gamma$, over which A acts as a holomorphic diffeomorphism; there exists on T a 1-form $\eta \not\equiv 0$ such that $A^*\eta = \lambda\eta$. Remark that the leaves of the foliation defined by $\ker \eta$ are directed along the eigenspace of A corresponding to the eigenvalue of modulus bigger than 1, and arithmetical or dynamical considerations immediately show that each leaf is dense in T. Let M be the complex threefold $\mathbf{C}^* \times T/(z,p) \sim (\lambda^{-1}z, A(p))$. It is a torus bundle over $E = \mathbf{C}^*/z \sim \lambda^{-1}z$, its algebraic dimension is equal to 1, and the bundle projection $\pi : M \to E$ coincides with the algebraic reduction. The 1-form $z\eta \in \Omega^1(\mathbf{C}^* \times T)$ quotients to a 1-form $\omega \in \Omega^1(M)$, which is non-closed, integrable, and moreover satisfies the relation $d\omega = \beta \wedge \omega$, where $\beta = \pi^*(\frac{dz}{z}) \in \pi^*\Omega^1(E)$.

The determinant of A has necessarily modulus equal to 1, but it can be different from 1 and even different from any root of 1 [G-V,appendix]. Hence we distinguish two cases:

1) $\det A \neq 1$: then $\Omega^1(M)$ is bidimensional, spanned by β and ω;

2) $\det A = 1$: then the second eigenvalue of A is λ^{-1} and we can construct a 1-form $\omega' \in \Omega^1(M)$ with the same procedure as before but starting with $\eta' \in \Omega^1(T)$ satisfying $A^*\eta' = \lambda^{-1}\eta'$ and quotienting $\frac{1}{z}\eta' \in \Omega^1(\mathbf{C}^* \times T)$. We obtain $d\omega' = -\beta \wedge \omega'$, and $\omega + \omega'$ is not integrable. The space $\Omega^1(M)$ is threedimensional and spanned by β, ω and ω'. M is a so-called "solvmanifold of type 3" [Ue1,p.214].

One can take ramified coverings in order to obtain examples of threefolds fibered over a curve of higher genus. All these examples are torus bundles, i.e. the holomorphic type of the fibre is constant, but it should be possible to construct examples where that holomorphic type is variable.

In the next section we shall see that many features of these examples survive in the general case.

3. Integrable 1-forms on threefolds with $a(M) = 1$

We continue with the same assumptions and notations of the previous section, and moreover we shall assume that ω is integrable: $\omega \wedge d\omega \equiv 0$. Our main result is the following.

Proposition 3. *If $\omega \in \Omega^1(M)$ is non-closed and integrable then:*
i) there exists $\beta \in \Omega^1(C)$ such that

$$d\tilde{\omega} = \pi^*(\beta) \wedge \tilde{\omega}$$

ii) every fibre F of π contains an irreducible component F_0 such that

$$\tilde{\omega}|_{F_0} \not\equiv 0.$$

Remark. By i) the genus of C is strictly positive, hence the meromorphic map $\pi \circ r^{-1} : M - - - \to C$ is in fact holomorphic and so we can choose $\tilde{M} = M$ [Ue1,remark 12.7].

Let us firstly fix the notation. The integrable 1-form $\tilde{\omega}$ defines a codimension one holomorphic foliation $\mathcal{F}$ whose singular set $Sing(\mathcal{F})$ has codimension at least 2 (locally we can write $\tilde{\omega} = f\omega_0$ with f holomorphic and $Zero(\omega_0)$ of codimension at least 2, then $\mathcal{F}$ is the foliation generated by ω_0 and $Sing(\mathcal{F}) = Zero(\omega_0)$). Similarly, the 2-form $d\tilde{\omega}$ defines a one dimensional foliation $\mathcal{L}$, whose singular set $Sing(\mathcal{L})$ has codimension at least 2. We have, outside the singular sets, $\mathcal{L} \subset \mathcal{F}$. On the other hand, $d\tilde{\omega}$ is identically zero on every fibre of π and therefore $\mathcal{L}$ is tangent to the fibres of π. Neglecting singular sets, this means that the leaves of $\mathcal{L}$ are the "intersections" of the leaves of $\mathcal{F}$ and the fibres of π (at least generically: certain fibres of π can be leaves of $\mathcal{F}$, but a generic fibre is not a leaf of $\mathcal{F}$, otherwise ω would be closed).

We know, from proposition 2, that a generic fibre of π is bimeromorphic to a complex torus, but of course it can contain exceptional curves. However, as it is shown in [Ue2,cor.1.11], these exceptional curves belong to an hypersurface of $\tilde{M}$ which can be contracted, perhaps after some blow-ups. This operation does not affect our problem (the new threefold we obtain is still an algebraic reduction of M, and if the statement of proposition 2 is true for *some* algebraic reduction then it is true for *every* algebraic reduction), and so we may and shall suppose that the generic fibres of π are minimal surfaces.

Let us consider now the restriction of $\tilde{\omega}$ on a generic fibre: it is a holomorphic 1-form which is not identically zero, hence it has no zero at all since the generic fibre is a torus. This means that $\mathcal{F}$ is transverse to the generic fibre, and therefore the differentiable type (but perhaps not the holomorphic type) of the foliation induced by $\mathcal{F}$ on the generic fibre (that is the foliation $\mathcal{L}$ restricted to the generic fibre) is constant. For a foliation on a torus given by a holomorphic 1-form there are two possibilities: either every leaf is compact (an elliptic curve) or no leaf is compact. These two possibilities are obviously differentiably distinct.

Lemma 6. *On the generic fibre of π the foliation induced by $\mathcal{F}$ has no compact leaf.*

Proof. It is a straightforward modification of lemma 5 of the previous section. The only difference is that instead of taking the Albanese reduction of every generic fibre we take only the "component" of the Albanese reduction which is obtained by integrating $\tilde{\omega}$ (if, by contradiction, the leaves of the foliation on a generic fibre were compact then the periods of $\tilde{\omega}$ on a generic fibre would be rational). □

In order to prove proposition 3 we will firstly construct a *meromorphic* 1-form as in i), and then we shall verify that it is actually holomorphic.

Lemma 7. *There exists a meromorphic 1-form β on C such that $d\tilde{\omega} = \pi^*(\beta) \wedge \tilde{\omega}$.*

Proof. Take any non-constant meromorphic function f on $\tilde{M}$, that is $f = f_0 \circ \pi$, $f_0 \in \mathcal{M}(C)$, f_0 not a constant. Outside the polar set of f, which is a union of fibres of π, the 1-form $f\tilde{\omega}$ is still holomorphic, integrable, and defines the same foliation as $\tilde{\omega}$. Its differential $d(f\tilde{\omega})$ is still identically zero on fibres, and hence it defines the same foliation as $d\tilde{\omega}$. Therefore $d(f\tilde{\omega}) = gd\tilde{\omega}$ for a suitable $g \in \mathcal{M}(\tilde{M})$, that is

$$gd\tilde{\omega} = df \wedge \tilde{\omega} + fd\tilde{\omega}$$

$$d\tilde{\omega} = \frac{df}{g-f} \wedge \tilde{\omega}.$$

But $g = g_0 \circ \pi$ for some $g_0 \in \mathcal{M}(C)$, and so we can set

$$\beta = \frac{df_0}{g_0 - f_0}.$$

q.e.d.

We shall compute the residue of β at every point of C. Take $t \in C$ and set $F_t = \pi^{-1}(t)$. We shall say [C-C] that F_t is $\mathcal{F}$*-dicritical* if there exists a modification (composition of blow-ups with smooth centres) $\hat{M} \xrightarrow{m} \tilde{M}$ such that $\hat{F}_t = m^{-1}(F_t) = (\pi \circ m)^{-1}(t)$ is *not* invariant by the foliation $\hat{\mathcal{F}} = m^*(\mathcal{F})$ (more precisely, there is an irreducible component of $\hat{F}_t$ which is not invariant by $\hat{\mathcal{F}}$). Here $\hat{\mathcal{F}}$ is the codimension one foliation, with codimension two singular set, generated by $\hat{\omega} = m^*\tilde{\omega} \in \Omega^1(\hat{M})$. Remark that $d\hat{\omega} = (\pi \circ m)^*(\beta) \wedge \hat{\omega}$. The proof of the next lemma will distinguish dicritical and non-dicritical case.

Lemma 8. *Let $t \in C$.*

i) β *has at* t *at most a first order pole;*
ii) $\mathrm{Res}_t\beta \geq 0$*;*
iii) if $\mathrm{Res}_t\beta = 0$ *(i.e.* β *is holomorphic at* t*) then there is an irreducible component* $(F_t)_0 \subset F_t$ *such that* $\tilde{\omega}|_{(F_t)_0} \not\equiv 0$.

Proof. *First case:* F_t *is* $\mathcal{F}$*-dicritical.*

Take a modification $\hat{M} \stackrel{m}{\to} \tilde{M}$ such that $\hat{F}_t = m^{-1}(F_t)$ contains an irreducible component $(\hat{F}_t)_0$ which is not $\hat{\mathcal{F}}$-invariant. Take a generic point of $(\hat{F}_t)_0$, where $\hat{\mathcal{F}}$ is transverse to $(\hat{F}_t)_0$, and choose local coordinates (x, y, z) centered at that point and a local coordinate w centered at $t \in C$ such that:

1) the projection $\pi \circ m$ is expressed by $w = z^n$, where n is the multiplicity of $(\hat{F}_t)_0$;

2) $\hat{\mathcal{F}}$ is given by the kernel of dx.

Hence

$$\hat{\omega} = h(x, y, z)dx$$

for a suitable holomorphic function h, and

$$(\pi \circ m)^*(\beta) = b(z^n)d(z^n)$$

for a suitable meromorphic function b. From $d\hat{\omega} = (\pi \circ m)^*(\beta) \wedge \hat{\omega}$ it easily follows that h factorizes as $h(x, y, z) = h_0(z)h_1(x)$, where

$$\frac{h_0'(z)}{h_0(z)} = nz^{n-1}b(z^n).$$

Clearly this implies that b has at 0 at most a first order pole. Moreover, if h_0 vanishes at 0 at order $k \geq 0$ (i.e. $h_0(z) = cz^k + ...$, $c \neq 0$) then

$$\mathrm{Res}_t\beta = \frac{k}{n} \geq 0.$$

If $\mathrm{Res}_t\beta = 0$ then $k = 0$, i.e. h_0 do not vanish on $(\hat{F}_t)_0$; also h_1 do not vanish identically on $(\hat{F}_t)_0$, otherwise, being independent on z, it would vanish everywhere. Therefore $\mathrm{Res}_t\beta = 0$ is equivalent to $\hat{\omega}|_{(\hat{F}_t)_0} \not\equiv 0$, and the proof is completed by observing that the existence of such a component implies that also F_t contains an irreducible component $(F_t)_0$ such that $\tilde{\omega}|_{(F_t)_0} \not\equiv 0$.

Second case: F_t *is not* $\mathcal{F}$*-dicritical.*

Up to a base change $C' \to C$ ramified at t we may assume that F_t contains an irreducible component $(F_t)^0$ whose multiplicity is equal to 1 (see for instance [F-M,p.3]). We choose local coordinates (x, y, z) near a point of that component such that π is given by $(x, y, z) \mapsto z$. Then

$$\tilde{\omega} = A(x, y, z)dx + B(x, y, z)dy + (xC_1(x, y, z) + yC_2(x, y, z))dz + C_0(z)dz$$

where A, B, C_i are local holomorphic functions. Observe that $C_0(z)dz$ is the pull-back by π of a holomorphic 1-form γ on C (defined on a neighbourhood of t), and hence $C_0(z)dz$ is in fact defined on a full neighbourhood of F_t. Hence the difference $\tilde{\omega} - C_0(z)dz$ is also defined on a full neighbourhood of F_t, and so we can decompose on that neighbourhood

$$\tilde{\omega} = \tilde{\omega}_0 + \pi^*(\gamma).$$

The holomorphic 1-form $\tilde{\omega}_0$ is still integrable, because $d\tilde{\omega} \wedge \pi^*(\gamma) \equiv 0$, and defines (near F_t) a codimension one foliation $\mathcal{F}_0$. Moreover $\tilde{\omega}_0$ and $\tilde{\omega}$ coincide when restricted to fibres, therefore $\mathcal{F}_0$ induces on a generic fibre a foliation without compact leaves.

We claim that F_t is $\mathcal{F}_0$-dicritical. To see this, observe that the curve $\{x = y = 0\}$ is tangent to $\mathcal{F}_0$. If F_t were $\mathcal{F}_0$-nondicritical then by [C-C] (see especially part IV) that curve could be "continued" to a surface Σ analytic on a neighbourhood of F_t and tangent to $\mathcal{F}_0$. The intersection of Σ with a generic fibre would be a *compact* analytic curve invariant by $\mathcal{F}_0$, and we said that this cannot happen.

Now we can apply the first part of the proof to $\mathcal{F}_0$ (the fact that this foliation is defined only on a neighbourhood of the fibre is clearly inessential). From

$$d\tilde{\omega}_0 = d\tilde{\omega} = \pi^*(\beta) \wedge \tilde{\omega} = \pi^*(\beta) \wedge \tilde{\omega}_0$$

we obtain i) and ii). If $\mathrm{Res}_t\beta = 0$ then there exists $(F_t)_0 \subset F_t$ such that $\tilde{\omega}_0|_{(F_t)_0} \not\equiv 0$, but $\pi^*(\gamma)$ vanishes on the fibres and so $\tilde{\omega}|_{(F_t)_0} \not\equiv 0$. □

Remark. One can try to prove the nondicritical case of lemma 8 by a purely local argument, as it is done in the dicritical case, avoiding any reference to the deep theorem of Cano and Cerveau. Near a generic point of $(F_t)^0$ we can choose coordinates (x, y, z) such that $\tilde{\omega}$ is expressed by $A(x, z)dx + C(x, z)dz$ and the projection is still $(x, y, z) \mapsto z$, and then we will find that $\pi^*(\beta)$ is something like $\frac{A_z - C_x}{A} dz$, but we don't know how to control the term C_x (e.g. why isn't possible $A(x, z) = z$, $C(x, z) = 1 + 2x$, which would give a negative residue?). A global argument seems here unavoidable.

Proof of proposition 3.

By lemma 8 the residue of β at each point of C is real and non negative, but by the residue theorem $\sum_{t \in C} \mathrm{Res}_t\beta = 0$, therefore the only possibility is that $\mathrm{Res}_t\beta = 0$ for every $t \in C$. Hence, by the same lemma, β is holomorphic and every fibre of π contains an irreducible component over which the 1-form do not vanish identically. □

We stop here our analysis of integrable 1-forms, even if it is perhaps possible to obtain further informations concerning the structure of the singular fibres of π. The interested reader may look at [F-M] for a comprehensive study of singular fibres of torus fibrations, and [Ue2] for some patologies specific to the non-Kähler context. Probably, the presence of an integrable 1-form on the threefold excludes some of these pathologies; for instance our proposition 2 implies that the direct image (on C) of the sheaf of 1-forms which vanish on $\mathcal{F}$ is trivial, and this fact should say something about the direct image of the sheaf of relative 1-forms.

4. Proof of the theorem and its corollary

Concerning the *proof of the theorem*, by the results of section 1 it remains only to consider the case where $a(M) = 1$.
Case 1. We have $\omega_2 = f\omega_1$ for a suitable non-constant $f \in \mathcal{M}(M)$, therefore ω_2 is also integrable. If ω_1 or ω_2 were not closed (say, $d\omega_1 \not\equiv 0$) then from proposition 3.ii we deduce that f (which is constant on the fibres of the algebraic reduction) has no poles and therefore it is a constant, contradiction.
Case 2. We shall prove that $a(M) = 1$ leads to a contradiction. We have $\omega_3 = f_1\omega_1 + f_2\omega_2$ for suitable $f_1, f_2 \in \mathcal{M}(M)$, with $\{1, f_1, f_2\}$ linearly independent. The one-dimensional foliation defined by the common kernel of the 2-forms $\omega_i \wedge \omega_j$ is tangent to the fibres of the algebraic reduction, because the closedness of 2-forms implies $df_i \wedge \omega_1 \wedge \omega_2 \equiv 0$, $i = 1, 2$. At least one of the 1-forms ω_1 and ω_2 do not vanish identically when restricted to fibres (say, $\omega_1|_{\text{fibres}} \not\equiv 0$), therefore we can write

$$\omega_2|_{\text{fibres}} = h_2(\omega_1|_{\text{fibres}}), \qquad \omega_3|_{\text{fibres}} = h_3(\omega_1|_{\text{fibres}})$$

for suitable meromorphic functions h_j, and then we obtain (cf. lemma 1)

$$h_3 = f_1 + f_2 h_2.$$

If $d\omega_1 \not\equiv 0$ then by proposition 3.ii we deduce that h_3 and h_2 have no poles and so they are constant, contradicting the linear independence of $\{1, f_1, f_2\}$. If $d\omega_1 \equiv 0$ we still have the same non-vanishing conclusion of proposition 3.ii, for cohomological reasons, and therefore the same contradiction. □

Concerning the *proof of the corollary*, we have either $\dim \Omega^3(M) = 0$, $\dim \Omega^1(M) \geq 3$ or $\dim \Omega^3(M) = 1$, $\dim \Omega^1(M) \geq 4$. In both cases we can find a 3-dimensional subspace $E \subset \Omega^1(M)$ mapped to zero by the natural linear map $\bigwedge^3 E \to \Omega^3(M)$. Because $\dim \Omega^3(M) \leq 1$, the map $E \ni \omega \mapsto \omega \wedge d\omega \in \Omega^3(M)$ vanishes on a (homogeneous) surface $S \subset E$, hence we can find a basis $\omega_1, \omega_2, \omega_3$ of E with ω_1 and ω_2 integrable. If $\omega_i \wedge \omega_j$ are linearly independent then, by the theorem, M fibers over a normal algebraic surface of Albanese general type. Otherwise we can find a bidimensional subspace $F \subset E$ generated by two 1-forms whose wedge product vanishes. These 1-forms are necessarily integrable (again by $\dim \Omega^3(M) \leq 1$) and, by the theorem, M fibers over an algebraic curve of genus greater or equal than 2. □

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