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Lusternik–Schnirelman theory for closed 1-forms

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Dedicated to S.P. Novikov on the occasion of his 60th birthday

Abstract. S. P. Novikov developed an analog of the Morse theory for closed 1-forms. In this paper we suggest an analog of the Lusternik - Schnirelman theory for closed 1-forms. For any cohomology class $\xi \in H^1(M, \mathbf{R})$ we define an integer $\mathrm{cl}(\xi)$ (*the cup-length associated with* ξ); we prove that any closed 1-form representing ξ has at least $\mathrm{cl}(\xi) - 1$ critical points. The number $\mathrm{cl}(\xi)$ is defined using cup-products in cohomology of some flat line bundles, such that their monodromy is described by complex numbers, which are not Dirichlet units.

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§1. The main result

1.1. Let M be a closed manifold and let $\xi \in H^1(M; \mathbf{R})$ be a nonzero cohomology class. The Novikov inequalities [N1], [N2], [N3] estimate the numbers of zeros $c_i(\omega)$ of different indices of any closed 1-form ω with Morse type singularities on M lying in the class ξ.

Novikov type inequalities were constructed in [BF1] for closed 1-forms with slightly more general singularities (non-degenerate in the sense of Bott [B]). In [BF2] an equivariant generalization of the Novikov inequalities was found.

In this paper we will consider the problem of estimating the number of critical points of closed 1-forms ω with no non-degeneracy assumption. We suggest here a version of the Lusternik - Schnirelman theory for closed 1-forms.

An announcement [F1] describes some results of this paper.

My recent preprint [F2] suggests a different approach to the Lusternik - Schnirelman theory of closed 1-forms; it uses untwisted cohomology and Massey products. Examples computed in [F2], show that the results of [F2] and of the present paper

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are independent.

1.2 Let $\xi \in H^1(M;\mathbf{Z})$ be an integral cohomology class. We will define below a nonnegative integer $\mathrm{cl}(\xi)$, which we will call *the cup-length associated with* ξ.

Recall, that a complex flat vector bundle E over M is determined by its monodromy, a linear representation of the fundamental group $\pi_1(M,x_0)$ in $\mathrm{GL}_{\mathbf{C}}(E_0)$, where E_0 is the fiber over the base point $x_0 \in M$; this representation is given by the parallel transport of vectors along loops. For example, a flat line bundle is determined by a homomorphism $H_1(M;\mathbf{Z}) \to \mathbf{C}^*$, where $\mathbf{C}^*$ is considered as a multiplicative abelian group.

Given class ξ as above and a nonzero complex number $a \in \mathbf{C}^*$, we have the complex flat line bundle over M with the following property: the monodromy along any loop $\gamma \in \pi_1(M)$ is the multiplication by $a^{\langle \xi,\gamma\rangle}$. We will denote this bundle by a^ξ. If $a, b \in \mathbf{C}^*$, we have the canonical isomorphism of flat line bundles

$$a^\xi \otimes b^\xi \simeq ab^\xi.$$

A lattice $\mathcal{L} \subset V$ in a finite dimensional vector space V is a finitely generated subgroup with $\operatorname{rank}\mathcal{L} = \dim_{\mathbf{C}} V$. We will say that a complex flat bundle $E \to M$ of rank m *admits an integral lattice* if its monodromy representation $\pi_1(M,x_0) \to \mathrm{GL}_{\mathbf{C}}(E_0)$ is conjugate to a homomorphism $\pi_1(M,x_0) \to \mathrm{GL}_{\mathbf{Z}}(\mathcal{L}_0)$, where $\mathcal{L}_0 \subset E_0$ is a lattice in the fiber. This condition is equivalent to the assumption that E is obtained from a local system $\tilde{E}$ of finitely generated free abelian groups over M by tensoring on $\mathbf{C}$.

1.3. Definition. *The cup-length* $\mathrm{cl}(\xi)$ *is the largest integer* k *such that there exists a nontrivial* k*-fold cup product*

$$H^{d_1}(M;E_1) \otimes H^{d_2}(M;E_2) \otimes \cdots \otimes H^{d_k}(M;E_k) \to H^d(M;E), \qquad (1\text{-}1)$$

where $d = d_1 + \cdots + d_k$, $E = E_1 \otimes E_2 \otimes \cdots \otimes E_k$, $d_1 > 0, \ldots, d_k > 0$, *and the first two flat bundles* E_1 *and* E_2 *have the following property: there exist nonzero complex numbers* $a_1, a_2 \in \mathbf{C}^*$, *and complex flat bundles* F_1 *and* F_2 *over* M, *admitting integral lattices, so that*

$$E_i \simeq {a_i}^\xi \otimes F_i, \quad \text{for} \quad i = 1,2, \qquad (1\text{-}2)$$

and both numbers a_1 *and* a_2 *are not Dirichlet units.*

Recall that *a Dirichlet unit* is defined as a complex number $b \neq 0$ such that b and its inverse b^{-1} are algebraic integers. In other words, Dirichlet units can be characterized as roots of polynomial equations

$$b^n + \gamma_1 b^{n-1} + \cdots + \gamma_{n-1} b + \gamma_n = 0,$$

where all γ_i are integers and $\gamma_n = \pm 1$.

Note that the cup-length $\mathrm{cl}(\xi)$, defined by 1.3, satisfies $0 \le \mathrm{cl}(\xi) \le \dim M$. We will see examples below showing that $\mathrm{cl}(\xi) = \dim M$ is possible.

The definition of the cup-length $\mathrm{cl}(\xi)$ above is slightly different from the one given in [F1]; following the present definition, we may have a larger cup-length $\mathrm{cl}(\xi)$.

Theorem 1. *Let ω be a closed 1-form on M lying in an integral cohomology class $\xi \in H^1(M; \mathbf{Z})$. Let $S(\omega)$ denote the set of zeros of ω, i.e. the set of points $p \in M$ such that $\omega_p = 0$. Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies*

$$\mathrm{cat}(S(\omega)) \ge \mathrm{cl}(\xi) - 1. \tag{1-3}$$

In particular, if the set of zeros $S(\omega)$ is finite, then for the total number $|S(\omega)|$ of zeros

$$|S(\omega)| \ge \mathrm{cl}(\xi) - 1. \tag{1-4}$$

Here $\mathrm{cat}(S)$ denotes the Lusternik - Schnirelman category of $S = S(\omega)$, i.e. the least number k, so that S can be covered by k closed subsets $A_1 \cup \cdots \cup A_k$ such that each inclusion $A_j \to S$ is null-homotopic.

Proof of Theorem 1 is given in §2.

1.4. Corollary ([F1])**.** *Suppose that there exist complex numbers $a_1, a_2, \dots, a_m \in \mathbf{C}^*$, not all Dirichlet units, such that a cup product*

$$H^{d_1}(M; {a_1}^{\xi}) \otimes H^{d_2}(M; {a_2}^{\xi}) \otimes \cdots \otimes H^{d_k}(M; {a_k}^{\xi}) \to H^d(M; a^{\xi}),$$

with $d_j > 0$, $j = 1, 2, \dots k$, is nontrivial. Then for any closed 1-form ω on manifold M, lying in class $\xi \in H^1(M; \mathbf{Z})$, holds $\mathrm{cat}(S(\omega)) \ge k - 1$.

Proof. We may assume that $\xi \ne 0$; otherwise the statement follows from the Lusternik - Schnirelman theory for functions.

Corollary 1.4 directly follows from Theorem 1, if there are at least two non Dirichlet units among $a_1, a_2, \dots, a_k$. Suppose that there is precisely one non Dirichlet unit. Denote $a = a_1 a_2 \cdots a_k$. Then a is not a Dirichlet unit, and, in particular, $a \ne 1$. Hence $H^n(M; a^{\xi}) = 0$. Therefore, the dimension of the nontrivial cup-product above $d = d_1 + d_2 + \cdots + d_k < n = \dim M$ is less than n. By the Poincaré duality, the cup-product pairing

$$H^d(M; a^{\xi}) \otimes H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M) \to H^n(M; \mathcal{L}_M)$$

is non-degenerate. Here $\mathcal{L}_M$ denotes the orientation flat line bundle of M. The monodromy of $\mathcal{L}_M$ along any loop γ equals ± 1 depending on whether the orientation of M is preserved or reversed by γ. Note that $\mathcal{L}_M$ admits an integral lattice.

Hence, we may find a nontrivial cup-product of length $k+1$ with an extra factor in $H^{n-d}(M; a^{-\xi} \otimes \mathcal{L}_M)$. Now, Theorem 1 applies and gives $\mathrm{cat}(S(\omega)) \geq k$. □

1.5. It is clear that Corollary 1.4 becomes false if we remove the requirement that one of the numbers a_i are not Dirichlet units. The simplest example is provided by the torus T^n; any cohomology class $\xi \in H^1(T^n; \mathbf{R})$ of the torus $M = T^n$ contains a closed 1-form without zeros, but the cup-length of T^n is n.

1.6. Remark. A crude estimate for the cup-length $\mathrm{cl}(\xi)$ can be obtained by taking the maximal length of a non-trivial product (1-1) with $E_j = a_j^{\xi}$ and $a_j \in \mathbf{C}^*$ being *transcendental*, $j = 1, 2, \dots, k$. We will give an example (cf. 1.10, example 3) showing that this estimate can be really worse than the one provided by Theorem 1.

1.7. Remark. *In the longest nontrivial product (1-1) the number d must be equal the dimension of the manifold $n = \dim M$.* Indeed, any nontrivial cup-product (1-1) with $d < n$ can be made longer by using the Poincaré duality.

1.8. Forms with non-integral periods. In general, the cohomology class determined by a closed 1-form ω belongs to $H^1(M, \mathbf{R})$, i.e. it has real coefficients. It is clear that multiplying ω by a non-zero constant $\lambda \neq 0$ does not change the set of critical points $S(\omega)$ and multiplies the cohomology class by λ. Hence Theorem 1 also gives estimates in the case of *cohomology classes $\xi \in H^1(M, \mathbf{R})$ of rank 1* (i.e. for classes, which are real multiples of integral classes) if we define the associated cup-length $\mathrm{cl}(\xi)$ as follows

$$\mathrm{cl}(\lambda\xi) = \mathrm{cl}(\xi), \quad \lambda \in \mathbf{R},\ \lambda \neq 0, \quad \xi \in H^1(M, \mathbf{Z}).$$

Recall, that given a cohomology class $\xi \in H^1(M, \mathbf{R})$, its *rank* is defined as the rank of the abelian group, which is the image of the homomorphism $H_1(M, \mathbf{Z}) \to \mathbf{R}$, determined by ξ. Note that the cohomology classes of rank 1 are dense in $H^1(M, \mathbf{R})$. Therefore the following definition makes sense.

Definition. Given a class $\xi \in H^1(M, \mathbf{R})$ of rank > 1, we define $\mathrm{cl}(\xi)$ as the largest number k, such that there exists a sequence of rank 1 classes $\xi_m \in H^1(M, \mathbf{R})$ with

$$\mathrm{cl}(\xi_m) \geq k, \qquad \lim_{m \to \infty} \xi_m = \xi, \tag{1-5}$$

and each ξ_m, considered as a homomorphism $H_1(M; \mathbf{Z}) \to \mathbf{R}$, vanishes on the kernel of the homomorphism $\xi : H_1(M; \mathbf{Z}) \to \mathbf{R}$.

Theorem 2. *Let ω be a closed 1-form on M lying in a cohomology class $\xi \in H^1(M; \mathbf{R})$. Let $S(\omega)$ denote the set of zeros of ω. Then the Lusternik - Schnirelman category of $S(\omega)$ satisfies*

$$\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi) - 1. \tag{1-6}$$

In particular, if the set of critical points $S(\omega)$ is finite then for the total number $|S(\omega)|$ of the critical points,

$$|S(\omega)| \geq \mathrm{cl}(\xi) - 1. \tag{1-7}$$

For the proof see §3.

1.9. Connected sums. Let M_1 and M_2 be two closed n-dimensional manifolds. Assume for simplicity, that $n > 2$. We will denote by $M_1 \# M_2$ the connected sum of M_1 and M_2. Given cohomology classes $\xi_\nu \in H^1(M_\nu; \mathbf{R})$, where $\nu = 1, 2$, the class $\xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$ is well defined, in an obvious way.

In the description of examples (cf. 1.10) we will use the following statement:

Proposition 1. *In the situation described above,*

$$\mathrm{cl}(\xi_1 \# \xi_2) = \max\{\mathrm{cl}(\xi_1), \mathrm{cl}(\xi_2)\}. \tag{1-8}$$

Proof is given in §3.

1.10. Examples. 1. In the notations of the previous subsection, let $\xi_1 = 0$ and suppose that $\xi_2 \neq 0$ can be realized by a closed 1-from with no critical points (for example, fibration over the circle). Then we obtain from Proposition 1 that $\mathrm{cl}(\xi_1 \# \xi_2) = \mathrm{cl}(\xi_1)$. Since $\xi_1 = 0$, the cup-length $\mathrm{cl}(\xi_1)$ can be estimated from below by the usual cup-length of the manifold M_1 with complex coefficients.

To have a specific example, let us take $M_1 = T^n$, $M_2 = S^1 \times S^{n-1}$, $\xi_1 = 0$ and $\xi_2 \in H^1(M_2; \mathbf{Z})$ being a generator, where $n > 2$. Then we have for $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$

$$\mathrm{cl}(\xi_1 \# \xi_2) = n. \tag{1-9}$$

Therefore, by Theorem 1, *any closed 1-form ω on $M_1 \# M_2$ lying in class ξ has a least $n - 1$ critical points.*

2. In a similar way one may construct examples of cohomology classes of higher rank with many critical points. Namely, suppose that $M_1 = T^n$, where $n > 2$ and $\xi_1 = 0$; take for M_2 arbitrary closed manifold of dimension n with a cohomology class $\xi_2 \in H^1(M_2; \mathbf{R})$ of rank q. Then for the class $\xi = \xi_1 \# \xi_2 \in H^1(M_1 \# M_2; \mathbf{R})$ (having rank q) we again obtain $\mathrm{cl}(\xi) = n$ (by Proposition 1).

One may take, for example, $M_2 = T^q \times S^{n-q}$ with ξ_2 induced from a maximally irrational class on the torus T^q.

3. Let M be a 3-dimensional manifold obtained by 0-framed surgery on the knot 5_2:

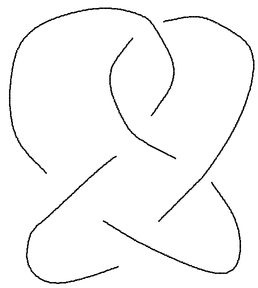

Figure 1.

This knot has Alexander polynomial $\Delta(\tau) = 2 - 3\tau + 2\tau^2$. Then $H^1(M;\mathbf{Z}) = \mathbf{Z}$ and taking $\xi \in H^1(M;\mathbf{Z})$ to be a generator we find that $H^1(M;a^\xi)$ is trivial for all $a \in \mathbf{C}^*$, which are not the roots of the Alexander polynomial. It is easy to check that if a is one of the roots of $2 - 3a + 2a^2 = 0$ then $H^1(M;a^\xi) \neq 0$. Note that the roots of $2 - 3a + 2a^2 = 0$ are not Dirichlet units. Hence we obtain that all Novikov Betti numbers are trivial (since, as it is known [N3], that the Novikov Betti numbers equal to $\dim H^*(M;a^\xi)$ for generic $a \in \mathbf{C}$). However by Corollary 1.4 we obtain that any closed 1-forms in class ξ has at least 1 critical point.

§2. Proof of Theorem 1

2.1. Since we assume that the cohomology class ξ of ω is integral, $\xi \in H^1(M,\mathbf{Z})$, there exists a smooth map $f : M \to S^1$, such that $\omega = f^*(d\theta)$, where $d\theta$ is the standard angular form on the circle $S^1 \subset \mathbf{C}$, $S^1 = \{z; |z| = 1\}$.

Denote $f^{-1}(b)$ by $V \subset M$, where $b \in S^1$ is a regular value; it is a codimension one submanifold. Let N denote the manifold obtained by cutting M along V. Note that N and V could be disconnected.

Each connected component of V yields two connected components of ∂N, the positive and the negative. In order to distinguish between the positive and the negative boundary components of ∂N, we use the orientation of the normal bundle to V in M, given by the form ω. The positive components are defined as those with the internal normal vector field to N being positive. The union of all positive (negative) boundary components of N will be denoted by $\partial_+ N$, or $\partial_- N$, correspondingly.

Let $p : N \to M$ denotes the natural projection. Then $p^*\omega = dg$, where $g : N \to \mathbf{R}$ is a smooth function, determined up to a constant on each connected component of N. It is clear that g is constant on each connected component of ∂N. The points of $\partial_+ N$ are points of local minimum of g; the points of $\partial_- N$ are points of local maximum of g. The map g sends the set $S(g)$ of critical points of g diffeomorphically onto the set $S(\omega)$.

2.2. Relative Lusternik - Schnirelman category. We will use the well-known notion of relative Lusternik - Schnirelman category, cf. [Fa], [Fo], [S]. Let's recall it.

For any subset $X \subset N$ containing $\partial_+ N$ we will denote by $\text{cat}_{(N,\partial_+ N)}(X)$ the minimal number k such that X can be covered by $k+1$ closed subsets

$$X \subset A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_k \subset N$$

with the following properties:

(1) A_0 contains $\partial_+ N$ and the inclusion $A_0 \to N$ is homotopic to a map $A_0 \to \partial_+ N$ keeping the points of $\partial_+ N \subset A$ fixed;

(2) for $j = 1, 2, \dots, k$, each inclusion $A_j \to N$ is null-homotopic.

We claim, that

$$\text{cat}\, S(\omega) = \text{cat}\, S(g) \geq \text{cat}_{(N,\partial_+ N)}(N). \tag{2-1}$$

This follows from known results, cf., for example, [Fo], Th. 4.2. We apply Theorem 4.2 of [Fo] to each of the connected components of N and to the restriction of function g on it; we use the additivity of the relative Lusternik - Schnirelman category with respect to disjoint union, cf. [Fo], Prop. 2.8.

Our next purpose will be to prove the inequality

$$\text{cat}_{(N,\partial_+ N)}(N) \geq \text{cl}(\xi) - 1. \tag{2-2}$$

Together with (2-1) this will complete the proof of the Theorem.

2.3. The deformation complex. The proof of (2-2) will consist of building a *polynomial deformation*, a finitely generated free cochain complex C^* over the ring $P = \mathbf{Z}[\tau]$ of polynomials with integral coefficients, having properties (a), (b) described below. With the help of the deformation complex we will prove the Lifting Property, cf. Corollary 2.6, playing a crucial role in the proof.

In [F3] we show how the deformation complex leads to inequalities, which are stronger than the Novikov inequalities.

The construction of the deformation complex is similar to [F2]; the difference is that in the present paper we will work over the integers, and in [F2] over a field.

Claim. *Let $E \to M$ be a flat vector bundle over M, admitting an integral lattice, and let $\tilde{E}$ be a local system of free abelian groups over M such that $\tilde{E} \otimes \mathbf{C} \simeq E$. Denote by $\tilde{E}_0 = p^*(\tilde{E})$; it is a local system over N. There exists a free finitely generated cochain complex C^* over the ring $P = \mathbf{Z}[\tau]$ having the following properties:*

(a) for any nonzero complex number $a \in \mathbf{C}^*$ there is a canonical isomorphism

$$H^q(C^* \otimes_P \mathbf{C}_a) \xrightarrow{\simeq} H^q(M; a^{-\xi} \otimes E). \tag{2-3}$$

Here $\mathbf{C}_a$ *is* $\mathbf{C}$*, which is viewed as a* P*-module with the following structure:* $\tau x = ax$ *for* $x \in \mathbf{C}$.

(b) for $a = 0$ *there is a canonical evaluation isomorphism*

$$H^q(C^* \otimes_P \mathbf{Z}_0) \to H^q(N, \partial_+ N; \tilde{E}_0), \tag{2-4}$$

where $\mathbf{Z}_0$ *is* $\mathbf{Z}$ *with the following* P*-module structure:* $\tau x = 0$ *for any* $x \in \mathbf{Z}$*.*

To construct C^*, we shall assume that N is triangulated and ∂N is a subcomplex. Let $i_\pm : V \to N$ be the inclusions, which identify V with $\partial_\pm N$ correspondingly. $\tilde{E}$ determines also an isomorphism of local systems $\sigma : i_+^* \tilde{E}_0 \to i_-^* \tilde{E}_0$ over V.

Denote by $C^q(N)$ and $C^q(V)$ the free abelian groups of $\tilde{E}_0$-valued cochains; $\delta_N : C^q(N) \to C^{q+1}(N)$ and $\delta_V : C^q(V) \to C^{q+1}(V)$ will denote the corresponding coboundary homomorphisms.

Let $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ denote the free P-modules formed by polynomials with coefficients in the corresponding abelian groups; for example, an element $c \in C^q(N)[\tau]$ is a formal sum $c = \sum_{i \ge 0} c_i \tau^i$ with $c_i \in C^q(N)$ and only finitely many c_i's are nonzero. The P-module structure is given as follows: $\tau \cdot c = \sum_{i \ge 0} c_i \tau^{i+1}$. It is clear that $C^q(N)[\tau]$ and $C^{q-1}(V)[\tau]$ are free finitely generated P-modules.

The natural P-module extensions

$$\delta_N : C^q(N)[\tau] \to C^{q+1}(N)[\tau], \quad \text{and} \quad \delta_V : C^q(V)[\tau] \to C^{q+1}(V)[\tau]. \tag{2-5}$$

of the boundary homomorphisms act coefficientwise, so that δ_N and δ_V are P-homomorphisms. If $\alpha = \sum_{i \ge 0} \alpha_i \tau^i \in C^q(N)[\tau]$, then $\delta_N(\alpha) = \sum_{i \ge 0} \delta_N(\alpha_i) \tau^i$.

Define a finitely generated free cochain complex C^* over $P = \mathbf{Z}[\tau]$ (*the deformation complex*) as follows: $C^* = \oplus C^q$, where

$$C^q = C^q(N)[\tau] \oplus C^{q-1}(V)[\tau].$$

Elements of chain complex C^q will be denoted as pairs (α, β), where $\alpha \in C^q(N)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. The differential $\delta : C^q \to C^{q+1}$ is given by the following formula

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-6}$$

where $\alpha \in C^q(N)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. Obviously, C^* is the cylinder of the chain map $\sigma \otimes i_+^* - \tau i_-^*$ with a shifted grading.

To show (a) we note that M is obtained from N by identifying all points $i_+(v)$ with $i_-(v)$, where $v \in V$; the flat bundle E over M is obtained from the flat bundle $\tilde{E}$ over N by identifying the vectors $e_+ \in \tilde{E}|_{\partial_+ N}$ and $e_- \in \tilde{E}|_{\partial_- N}$ with $\sigma i_+^*(e_+) = a i_-^*(e_-)$. Hence $H^q(M; a^{-\xi} \otimes E)$ can be identified with the cohomology of complex $C^*(M; a^{-\xi} \otimes E)$, consisting of cochains $\alpha \in C^q(N)$ satisfying the boundary conditions

$$a i_-^*(\alpha) = \sigma \otimes i_+^*(\alpha) \in C^q(V).$$

The complex $C^q \otimes_P \mathbf{C}_a = C^q(N) \oplus C^{q-1}(V)$ has the differential given by

$$\delta(\alpha, \beta) = (\delta_N(\alpha), (\sigma \otimes i_+^* - a i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-7}$$

where $\alpha \in C^q(N)$ and $\beta \in C^{q-1}(V)$. It is clear that there is a chain homomorphism $C^*(M; a^{-\xi} \otimes E) \to C^* \otimes_P \mathbf{C}_a$ (acting by $\alpha \mapsto (\alpha, 0)$). It is easy to see that it induces an isomorphism on the cohomology. Indeed, suppose that a cocycle $\alpha \in C^q(M; a^{-\xi} \otimes E)$ bounds in the complex $C^* \otimes_P \mathbf{C}_a$. Then there are $\alpha_1 \in C^{q-1}(N)$, $\beta_1 \in C^{q-2}(V)$ such that $\alpha = \delta_N(\alpha_1)$, $\sigma \otimes i_+^*(\alpha_1) - a i_-^*(\alpha_1) - \delta_V(\beta_1) = 0$. We may find a cochain $\beta_2 \in C^{q-2}(N)$ such that $\sigma i_+^*(\beta_2) = \beta_1$ and $i_-^*(\beta_2) = 0$ (by extending β_1 into a neighborhood of $\partial_+ N$). Then setting $\alpha_2 = \alpha_1 - \delta_N(\beta_2)$ we have

$$\alpha = \delta_N(\alpha_2), \qquad \sigma i_+^*(\alpha_2) - a i_-^*(\alpha_2) = 0, \tag{2-8}$$

which means that α also bounds in $C^q(M; a^{-\xi} \otimes E)$.

Similarly, suppose that (α, β) is a cocycle of complex $C^* \otimes_P \mathbf{C}_a$. As above we may find a cochain $\beta' \in C^{q-1}(N)$ with $i_+^*(\beta') = \beta$ and $i_-^*(\beta') = 0$. Then $(\alpha - \delta_N(\beta'), 0)$ it is a cocycle of $C^*(M; a^{-\xi} \otimes E)$ and it is cohomologous to the initial cocycle (α, β). This proves (a).

(b) follows similarly. □

2.4. Relative deformation complex. We will define now a relative version of the deformation complex C^*.

Let $A \subset N$ be a simplicial subcomplex. We will assume that A *is disjoint from* $\partial_+ N$. Let $C^q(N, A)$ denote the free abelian group of $\tilde{E}_0$-valued cochains on N which vanish on A. Let $C^q(N, A)[\tau]$ be constructed similarly to $C^q(N)[\tau]$, cf. above. We define the complex C_A^* as follows:

$$C_A^q = C^q(N, A)[\tau] \oplus C^{q-1}(V)[\tau]. \tag{2-9}$$

The differential $\delta : C_A^q \to C_A^{q+1}$ is defined by the following formula:

$$\delta(\alpha, \beta) = (\delta_{N,A}(\alpha), (\sigma i_+^* - \tau i_-^*)(\alpha) - \delta_V(\beta)), \tag{2-10}$$

where $\alpha \in C^q(N, A)[\tau]$ and $\beta \in C^{q-1}(V)[\tau]$. Here $\delta_{N,A} : C^q(N, A) \to C^{q+1}(N, A)$ and $\delta_V : C^q(V) \to C^{q+1}(V)$ denote the coboundary homomorphisms and also their P-module extension. $i_\pm^* : C^q(N, A) \to C^q(V)$ denote the restriction maps of cochains, and the same symbols denote also their polynomial extensions $i_\pm^* : C^q(N, A)[\tau] \to C^q(V)[\tau]$.

Similarly to (a) and (b) in 2.3 we have:

(a') for any $a \in \mathbf{C}^*$ there is a natural isomorphism

$$H^i(C_A^* \otimes_P \mathbf{C}_a) \simeq H^i(M, p(A); a^{-\xi} \otimes E), \tag{2-11}$$

where $p : N \to M$ is the identification map, cf. 2.1;

(b') also,

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0). \tag{2-12}$$

2.5. Algebraic integers and lifting. In this section it will become clear why our definition of the cup-length $\mathrm{cl}(\xi)$ involves the condition of not being a Dirichlet unit.

Proposition 2. *Suppose that $A \subset N$ is a subcomplex, disjoint from $\partial_+ N$, such that the inclusion $A \to N$ is homotopic to a map $A \to \partial_+ N$. Let $a \in \mathbf{C}^*$ be a complex number, such that a^{-1} is not an algebraic integer. Then the homomorphism $C^*_A \to C^*$ induces an epimorphism on the cohomology*

$$H^i(C^*_A \otimes_P \mathbf{C}_a) \to H^i(C^* \otimes_P \mathbf{C}_a), \qquad i = 0, 1, 2, \dots. \tag{2-13}$$

Proof. Let $\mathbf{Z}_0$ denote the group $\mathbf{Z}$ considered as a P-module with the trivial τ action, i.e. $\mathbf{Z}_0 = P/\tau P$. We will show first that

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \to H^i(C^* \otimes_P \mathbf{Z}_0) \tag{2-14}$$

is an epimorphism. We know from (2-4) and (2-12) that

$$H^i(C^*_A \otimes_P \mathbf{Z}_0) \simeq H^i(N, A \cup \partial_+ N; \tilde{E}_0) \quad \text{and} \quad H^i(C^* \otimes_P \mathbf{Z}_0) \simeq H^i(N, \partial_+ N; \tilde{E}_0).$$

In the exact sequence

$$\dots \to H^i(N, A \cup \partial_+ N; \tilde{E}_0) \to H^i(N, \partial_+ N; \tilde{E}_0) \xrightarrow{j^*} H^i(A \cup \partial_+ N, \partial_+ N; \tilde{E}_0) \to \dots$$

j^* acts trivially (since the inclusion $(A \cup \partial_+ N, \partial_+ N) \to (N, \partial_+ N)$ is null-homotopic) and hence $H^i(N, A \cup \partial_+ N; \tilde{E}_0) \to H^i(N, \partial_+ N; \tilde{E}_0)$ is an epimorphism. This proves that (2-14) is an epimorphism. Now, Proposition 2 follows from Proposition 3 below. □

Proposition 3. *Let C and D be chain complexes of free finitely generated $P = \mathbf{Z}[\tau]$-modules and let $f : C \to D$ be a chain map. Suppose that for some q the induced map $f_* : H_q(C \otimes_P \mathbf{Z}_0) \to H_q(D \otimes_P \mathbf{Z}_0)$ is an epimorphism; here $\mathbf{Z}_0$ is $\mathbf{Z}$ considered with the trivial P-action: $\mathbf{Z}_0 = P/\tau P$. Then for any complex number $a \in \mathbf{C}^*$, such that a^{-1} is not an algebraic integer, the homomorphism*

$$f_* : H_q(C \otimes_P \mathbf{C}_a) \to H_q(D \otimes_P \mathbf{C}_a) \tag{2-15}$$

is an epimorphism; here $\mathbf{C}_a$ denotes $\mathbf{C}$ with τ acting as the multiplication by a.

Proof. Denote by $Z_q(C), Z_q(D)$ the sets of cycles of C and D and by $B_q(C)$ and $B_q(D)$ the sets of their boundaries. Recall that the homological dimension of P is 2. We have the exact sequence

$$0 \to Z_q(C) \to C_q \to B_{q-1}(C) \to 0$$

and hence $Z_q(C)$ is a free P-module (since $B_{q-1}(C)$ is a submodule of a free module and so has a homological dimension ≤ 1). Similarly $Z_q(D)$ is free.

Choose free bases for $Z_q(C), Z_q(D)$ and D_{q+1}, and express in terms of these bases the map

$$f \oplus d : Z_q(C) \oplus D_{q+1} \to Z_q(D). \tag{2-16}$$

The resulting matrix $\mathcal{G}$ is rectangular, with entries in P.

We claim: *there exist integers $b_j \in \mathbf{Z}$ and minors $A_j(\tau) \in P$ of the matrix $\mathcal{G}$ of size* $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$, *such that the polynomial with integer coefficients*

$$p(\tau) = \sum_j b_j A_j(\tau) \tag{2-17}$$

satisfies

$$p(0) = 1. \tag{2-18}$$

In fact, we will show that our claim is *equivalent* to the requirement that $f_* : H_q(C \otimes_P \mathbf{Z}_0) \to H_q(D \otimes_P \mathbf{Z}_0)$ is an epimorphism. Namely, using the resolvent $0 \to P \xrightarrow{\tau} P \to \mathbf{Z}_0 \to 0$ it is easy to see that $\operatorname{Tor}_1^P(B_{q-1}(C), \mathbf{Z}_0) = 0$ (since $B_{q-1}(C)$ is a submodule of a free module). Hence we have the exact sequence

$$0 \to Z_q(C) \otimes_P \mathbf{Z}_0 \to C_q \otimes_P \mathbf{Z}_0 \to B_{q-1}(C) \otimes \mathbf{Z}_0 \to 0.$$

This means that $Z_q(C) \otimes_P \mathbf{Z}_0 = Z_q(C \otimes_P \mathbf{Z}_0)$, and $B_{q-1}(C) \otimes_P \mathbf{Z}_0 = B_{q-1}(C \otimes_P \mathbf{Z}_0)$. Hence, the hypothesis of the Proposition, the homomorphism

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{Z}_0) \oplus (D_{q+1} \otimes_P \mathbf{Z}_0) \to Z_q(D) \otimes_P \mathbf{Z}_0$$

is an epimorphism. This epimorphism is described by the matrix $\mathcal{G}(0)$, where we substitute $\tau = 0$ into $\mathcal{G}$. Therefore, there are minors $A_j(\tau)$ of $\mathcal{G}$ of size $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$, so that the ideal in $\mathbf{Z}$, generated by the integers $A_j(0)$ contains 1. This proves (2-18).

Since $p(\tau)$ is an integral polynomial with $p(0) = 1$ and a^{-1} is not an algebraic integer it follows that

$$p(a) \neq 0. \tag{2-19}$$

Let us show that (2-19) is equivalent to the statement that (2-15) is an epimorphism. We have the exact sequence

$$0 \to Z_q(C) \otimes_P \mathbf{C}_a \to C_q \otimes_P \mathbf{C}_a \to B_{q-1} \otimes \mathbf{C}_a \to 0$$

(here we may work over $\mathbf{C}[\tau]$ which is a PID). Hence, similarly to the arguments above, we obtain that the map

$$f \oplus d : (Z_q(C) \otimes_P \mathbf{C}_a) \oplus (D_{q+1} \otimes_P \mathbf{C}_a) \to Z_q(D) \otimes_P \mathbf{C}_a \tag{2-20}$$

is described by the matrix $\mathcal{G}$ with substitution $\tau = a$. We conclude that at least one of the $\operatorname{rk} Z_q(D) \times \operatorname{rk} Z_q(D)$ minors $A_j(a)$ is nonzero because of (2-19), and hence (2-20) and (2-15) are epimorphisms. □

2.6. Corollary (Lifting Property). *Let $E \to M$ be a flat vector bundle admitting an integral lattice. Let $a \in \mathbf{C}^*$ be a complex number, not an algebraic integer. Let $A \subset M$ be a closed subset such that $A = p(A')$, where $A' \subset N - \partial_+ N$ is a closed polyhedral subset such that the inclusion $A' \to N$ is homotopic to a map with values in $\partial_+ N$. Then the restriction map*

$$H^q(M, A; a^\xi \otimes E) \to H^q(M; a^\xi \otimes E) \tag{2-21}$$

is an epimorphism.

Proof. We just combine the isomorphisms (2-3) and (2-11) and Proposition 2. □

2.7. End of proof of Theorem 1. We need to establish inequality (2-2). In other words, we want to prove the triviality of any cup-product

$$v_0 \cup v_1 \cup v_2 \cup \cdots \cup v_{m+1} = 0, \quad \text{where} \quad v_j \in H^{d_j}(M; E_j), \tag{2-22}$$

(where m denotes $m = \operatorname{cat}_{(N, \partial_+ N)}(N)$) assuming that $d_j > 0$ for $j = 0, 1, 2, \dots, m+1$, and the bundles E_0 and E_1 are of the form $a_i^\xi \otimes F_i$, where $i = 0, 1$, with the numbers $a_0, a_1 \in \mathbf{C}$ not Dirichlet units, and the bundles F_0 and F_1 admitting integral lattices.

Moreover, we will assume that one of the numbers a_0 and a_1 is not an algebraic integer. In the case when both a_0 and a_1 are algebraic integers, the inverse numbers a_0^{-1} and a_1^{-1} are not algebraic integers, and we shall apply the arguments following below to the form $-\omega$ (representing the cohomology class $-\xi$), which obviously has the same set of critical points.)

Since we may always rename the numbers a_0 and a_1, we will assume below that a_0 is not an algebraic integer.

Suppose that N can be covered by closed subsets $A_0, A_1 \cup \cdots \cup A_m = N$ so that A_0 contains $\partial_+ N$ and the inclusion $A_0 \to N$ is homotopic to a map into $\partial_+ N$ keeping the points of $\partial_+ N$ fixed, (cf. 2.2), and for $j = 1, 2, \dots, m$ the subset A_j is null-homotopic in N. Without loss of generality we may assume that all A_j are polyhedral.

Let $U_\pm$ be a small cylindrical neighborhood of $\partial_\pm N$ in N. We observe that for $j = 2, 3, \dots, m+1$ we may lift the class v_j to a relative cohomology class lying in

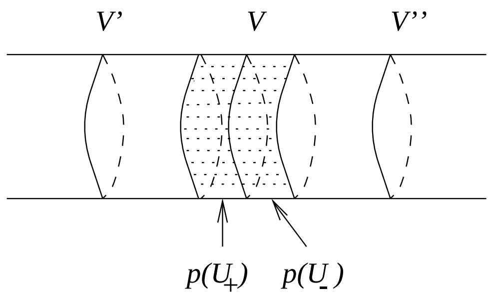

Figure 2.

$\tilde{v}_j \in H^{d_j}(M, B_j; E_j)$, where $B_j = p(A_{j-1} - U_+)$, since B_j is null-homotopic in M and $d_j > 0$. Recall that $p : N \to M$ denotes the natural identification map.

Applying Corollary 2.6, class v_0 can be lifted to a class $\tilde{v}_0 \in H^{d_0}(M, B_0; E_0)$, where $B_0 = p(A_0 - U_+)$.

Let B_1 be a closed cylindrical neighborhood of V in M containing $\overline{p(U_-)} \cup \overline{p(U_+)}$. We claim that we may lift the class $v_1 \in H^{d_1}(M; E_1)$ to a class $\tilde{v}_1 \in H^{d_1}(M, B_1; E_1)$. We will use Corollary 2.6. First, find two shifts of V into $M - B_1$, one (denoted V') in the positive normal direction and the other (denoted V'') in the negative normal direction (cf. Figure 2). If the number a_1 is not an algebraic integer we may apply Corollary 2.6 to the cut V''. If the number a_1^{-1} is not an algebraic integer we may apply Corollary 2.6 to the cut V'.

Now, it is clear that the product $v_0 \cup \cdots \cup v_{m+1}$ must be trivial since it is obtained from the product $\tilde{v}_0 \cup \cdots \cup \tilde{v}_{m+1}$ (lying in $H^d(M, \cup_{j=0}^{m+1} B_j; E)$, where $E = \otimes_{j=0}^{m+1} E_j$) by restricting onto M, and the group $H^d(M, \cup_{j=0}^{m+1} B_j; E)$ vanishes, since $M = \cup_{j=0}^{m+1} B_j$. □

§3. Proofs of Theorem 2 and Proposition 1

3.1. Proof of Theorem 2. Let ω be a closed 1-form lying in a cohomology class $\xi \in H^1(M; \mathbf{R})$ of rank $= r > 1$. Let $S = S(\omega)$ denote the set of zeros of ω. It is clear that $\xi|_S = 0$.

Let r be the rank of ξ and let $\xi_1, \dots, \xi_r \in H^1(M; \mathbf{Z})$ be a basis of the free abelian group $\mathrm{Hom}(H_1(M)/\ker(\xi); \mathbb{Z})$. We may write $\xi = \sum_{i=1}^r \alpha_i \xi_i$, and the coefficients are real $\alpha_i \in \mathbf{R}$.

Suppose that ξ_m is a sequence of rank 1 classes with $\mathrm{cl}(\xi_m) \geq \mathrm{cl}(\xi)$, which converges to ξ as $m \to \infty$, and each of the classes ξ_m vanishes on $\ker(\xi)$. Then we have $\xi_m = \sum_i \alpha_{i,m}\xi_i$, where $\alpha_{i,m} = \lambda_m \cdot n_{i,m}$, $\lambda_m \in \mathbf{R}$, and $n_{i,m} \in \mathbf{Z}$ for $i = 1, 2, \dots, r$. Each sequence $\alpha_{i,m}$ converges to α_i as m tends to ∞.

Choose a closed 1-form ω_i in the class ξ_i for $i = 1, \dots, r$; since $\xi_i|_S = 0$ we may choose it so that it vanishes identically on a neighborhood of S. Define the following sequence of closed 1-forms

$$\omega_m = \omega - \sum_{i=1}^{r}(\alpha_i - \alpha_{i,m})\omega_i.$$

It is clear that ω_m has rank 1 and for m large enough $S(\omega_m) = S(\omega)$. The cohomology class of ω_m is ξ_m. By Theorem 1 we have $\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi_m) - 1$. Hence we obtain $\mathrm{cat}(S(\omega)) \geq \mathrm{cl}(\xi) - 1$. □

3.2. Proof of Proposition 1. It is clear that it is enough to prove (1-8) assuming that the classes ξ_1 and ξ_2 are integral $\xi_\nu \in H^1(M_\nu; \mathbf{Z})$ for $\nu = 1, 2$. The general statement then follows automatically due to the nature of our definition of $\mathrm{cl}(\xi)$ for general ξ, cf. 1.8. One may use here an equivalent definition of the cup-length $\mathrm{cl}(\xi)$ for $\mathrm{rk}(\xi) > 1$, which can be obtained from the definition given in 1.8 if in (1-5) we will additionally require that the approximating rank 1 classes ξ_m belong to $H^1(M; \mathbf{Q})$.

Position M_1 and M_2 so that their intersection is a small n-dimensional disk D^n, where $n = \dim M_1 = \dim M_2$, and then the connected sum $M_1\#M_2$ is obtained from the union $M_1 \cup M_2$ by removing the interior of D^n. Let E be a flat bundle over the connected sum $M_1\#M_2$ and let E_ν be a flat bundle over M_ν so that

$$E|_{M_\nu - \overset{\circ}{D}{}^n} \simeq E_\nu|_{M_\nu - \overset{\circ}{D}{}^n}, \tag{3-1}$$

for $\nu = 1, 2$. Here we use the assumption that $n > 2$ and so the sphere S^{n-1} is simply connected.

As follows from the Mayer - Vietoris sequence, there is a canonical isomorphism

$$\psi : H^q(M_1; E_1) \oplus H^q(M_2; E_2) \to H^q(M_1\#M_2; E)$$

for $0 < q < n$. It will be clear from the rest of the proof that we do not need to worry about the case $q = n$. ψ is multiplicative in the following sense. Suppose that we have another flat bundle F over the connected sum $M_1\#M_2$ and let F_ν be flat bundles over M_ν, $\nu = 1, 2$, satisfying condition (3-1). Then for any $v \in H^i(M_1; E_1)$ and $w \in H^j(M_1; F_1)$ with $0 < i$, $0 < j$, and $i + j < d$, holds $\psi(v \cup w, 0) = \psi(v, 0) \cup \psi(w, 0)$. Similar property holds with respect to the other variable.

Suppose now that $k = \mathrm{cl}(\xi_1)$ and we have cohomology classes $v_j \in H^{d_j}(M_1; E_j)$, where $j = 1, 2, \ldots, k$, satisfying all the properties of Definition 1.3; in particular, their product $v_1 \cup \cdots \cup v_k$ is non-trivial. Then $\sum d_j = n$ (cf. 1.7). Extend each flat bundle E_j to a flat bundle $\tilde{E}_j$ over M; for $j = 1, 2$ we will make this extension so, that $\tilde{E}_1$ and $\tilde{E}_2$ will still satisfy condition (1-2).

We will first assume that $k > 2$. Then the classes

$$u_j = \psi(v_j, 0) \in H^{d_j}(M; \tilde{E}_j), \quad j = 1, 2, \ldots, k - 1,$$

have non-trivial cup product $u_1 \cup \cdots \cup u_{k-1}$ and satisfy all the properties of Definition 1.3. Using the Poincaré duality (as in the proof of Corollary 1.4), we may find a non-trivial cup product $u_1 \cup \cdots \cup u_{k-1} \cup u$, where $u \in H^{d_k}(M; E^* \otimes \mathcal{L}_M)$, $E = \otimes_{j=1}^{k-1} \tilde{E}_j$, and $\mathcal{L}_M$ is the orientation flat line bundle of M.

In case, when $k = 2$ by the same reasons we will have a non-trivial cup-product $u_1 \cup u$, where $u \in H^{d_2}(M; \tilde{E}_1^* \otimes \mathcal{L}_M)$ and the bundle $\tilde{E}_1^* \otimes \mathcal{L}_M$ satisfies (1-2) assuming that E_1 does.

This proves inequality $\mathrm{cl}(\xi) \geq \mathrm{cl}(\xi_1)$. Therefore $\mathrm{cl}(\xi) \geq \max\{\mathrm{cl}(\xi_1), \mathrm{cl}(\xi_2)\}$.

The inverse inequality follows similarly, using the properties of the map ψ mentioned above. □

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