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Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups

Gilbert Levitt and Martin Lustig

Abstract. Let F_n be the free group of rank n, and ∂F_n its boundary (or space of ends).

For any $\alpha \in \operatorname{Aut} F_n$, the homeomorphism $\partial\alpha$ induced by α on ∂F_n has at least two periodic points of period $\leq 2n$. Periods of periodic points of $\partial\alpha$ are bounded above by a number M_n depending only on n, with $\log M_n \sim \sqrt{n \log n}$ as $n \to +\infty$.

Using the canonical Hölder structure on ∂F_n, we associate an algebraic number $\lambda \geq 1$ to any attracting fixed point X of $\partial\alpha$; if $\lambda > 1$, then for any Y close to X the sequence $\partial\alpha^p(Y)$ approaches X at about the same speed as $e^{-\lambda^p}$. This leads to a set of Hölder exponents $\Lambda_h(\Phi) \subset (1, +\infty)$ associated to any $\Phi \in \operatorname{Out} F_n$. This set coincides with the set of nontrivial exponential growth rates of conjugacy classes of F_n under iteration of Φ.

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Introduction and statement of results

Let φ be a homeomorphism of a closed surface Σ, with $\chi(\Sigma) < 0$. In [14], Nielsen studied φ by lifting it to the universal covering D of Σ and considering the induced homeomorphism f on the circle at infinity S. In more algebraic terms, the mapping class of φ corresponds to an outer automorphism Φ of $\pi_1\Sigma$, various lifts of φ to D correspond to various automorphisms α of $\pi_1\Sigma$ representing Φ, and $f : S \to S$ corresponds to the homeomorphism $\partial\alpha$ induced by α on the boundary of the group $\pi_1\Sigma$.

Let F_n be the free group of rank n. We will study automorphisms α of F_n, and outer automorphisms $\Phi \in \operatorname{Out} F_n$, through the homeomorphisms $\partial\alpha$ induced on the boundary ∂F_n. The space ∂F_n, homeomorphic to a Cantor set if $n \geq 2$, may be viewed as the (Gromov) boundary of F_n, or its space of ends, or the set of right-infinite reduced words in the generators and their inverses.

In the case of a surface group, Nielsen proved among many other things that $f = \partial\alpha : S \to S$ always has at least two periodic points. Furthermore, the period of these points may be bounded in terms of $|\chi(\Sigma)|$.

Our first main result has a similar flavor.

Theorem 1. *Let* $\alpha \in \operatorname{Aut} F_n$.

(1) The homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$ *has at least two periodic points of period* $\leq 2n$. *If it has only one orbit of periodic points, then this orbit has order two.*

(2) Suppose $X \in \partial F_n$ *is periodic of period* q *under* $\partial\alpha$. *Then* $q \leq M_n$, *where* M_n *depends only on* n *and* $\log M_n \sim \sqrt{n \log n}$ *as* $n \to \infty$.

The bound $2n$ and the bound on q are sharp. The quantity $\sqrt{n \log n}$ is asymptotic to the logarithm of the maximum order of torsion elements in $\operatorname{Aut} F_n$, see [11]. As a special case of assertion 2, there is a bound depending only on n for periods of elements $g \in F_n$ under the action of α. One may also establish a uniform bound for periods of conjugacy classes under the action of $\Phi \in \operatorname{Out} F_n$. It is proved in [9] that, for "most" $\alpha \in \operatorname{Aut} F_n$, the homeomorphism $\partial\alpha$ has exactly two fixed points, and no other periodic point.

Like many results of the present paper, the proof of Theorem 1 uses **R**-trees and techniques introduced in [5]. The proof of assertion 2 uses the main result of [5], and Bestvina-Handel's bound [1] for the rank of the fixed subgroup (the "Scott conjecture").

Let us now consider local properties of fixed points of $\partial\alpha$, using the canonical Hölder structure on ∂F_n (see [3, 7]). Let X be a fixed point of $\partial\alpha$ not belonging to the limit set of the fixed subgroup $\operatorname{Fix}\alpha \subset F_n$. It is either attracting or repelling [5]. In the attracting case, we show that, for $Y \in \partial F_n$ close enough to X, the sequence $\partial\alpha^p(Y)$ converges to X *super-exponentially* in the sense that

$$\lim_{p\to+\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty,$$

where d is any distance on ∂F_n defining the Hölder structure. We say that X is *superattracting* (see the beginning of Section 4 for a detailed discussion).

Theorem 2. *Let* $\alpha \in \operatorname{Aut} F_n$. *Let* $X \in \partial F_n$ *be a superattracting fixed point of* $\partial\alpha$. *There exists an algebraic number* $\lambda = \lambda(\alpha, X) \geq 1$ *such that*

$$\lim_{p\to+\infty} \frac{1}{p} \log\Big(-\log d(\partial\alpha^p(Y), X)\Big) = \log\lambda$$

for $Y \in \partial F_n$ *close to* X *(and* d *a distance on* ∂F_n *as above).*

Thus, when $\lambda > 1$, the sequence $\partial\alpha^p(Y)$ converges to X at about the same speed as $f^p(x)$ approaches 0, where f is the map $x \mapsto x^\lambda : [0,1) \to [0,1)$.

Example. Consider $\alpha : F_2 \to F_2$ given by $\alpha(a) = aba$, $\alpha(b) = ab$. The number associated to $X = \lim_{p\to+\infty} \alpha^p(a) = ababaaba\ldots$ is the Perron-Frobenius eigenvalue of the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. On the other hand, for the polynomially growing

$\alpha : F_3 \to F_3$ given by $\alpha(a) = a$, $\alpha(b) = ba$, $\alpha(c) = cba$, the number associated to the superattracting point $X = \lim_{p\to+\infty} \alpha^p(c) = cbaba^2ba^3 \dots$ equals 1.

We now associate a canonical set of *Hölder exponents* $\Lambda_h(\Phi) \subset (1, +\infty)$ to any $\Phi \in \operatorname{Out} F_n$. View Φ as a collection of automorphisms $\alpha \in \operatorname{Aut} F_n$. We say that $\mu > 1$ belongs to $\Lambda_h(\Phi)$ if there exist $\beta \in \Phi^q$, with $q \geq 1$, and a superattracting fixed point X of $\partial\beta$ with $\lambda(\beta, X) = \mu^q$. The set $\Lambda_h(\Phi)$ is a conjugacy invariant of Φ.

Example. If Φ is induced by a homeomorphism φ of a compact surface Σ with $\pi_1\Sigma \simeq F_n$, then $\Lambda_h(\Phi)$ consists of (roots of) the expansion factors of the pseudo-Anosov pieces of φ. They are algebraic units.

If $\alpha \in \operatorname{Aut} F_3$ is given by $\alpha(a) = ab^{-1}$, $\alpha(b) = bac^{-1}$, $\alpha(c) = ca^{-3}$ (see [6, Example II.7]), then $\Lambda_h(\Phi)$ consists of the real root λ of $x^3 - 3x^2 + 2x - 3$. Note that λ is not an algebraic unit, and therefore cannot be read off the graph of groups constructed by Sela in Theorem 4.1 of [15].

Theorem 3. *Given $\Phi \in \operatorname{Out} F_n$, the set of Hölder exponents $\Lambda_h(\Phi)$ equals the set $\Lambda(\Phi)$ of nontrivial exponential growth rates of conjugacy classes of F_n under iteration of Φ.*

The *(exponential) growth rate* of a conjugacy class γ is $\lambda(\gamma) = \lim_{p\to+\infty} |\Phi^p(\gamma)|^{1/p}$ (see Proposition 3.3). It is nontrivial if $\lambda(\gamma) > 1$. It will be shown in [10] that $\Lambda(\Phi)$ has at most $[\frac{3n-2}{4}]$ elements and consists of certain Perron-Frobenius eigenvalues of the transition matrix associated to a relative train track representative of Φ.

This paper is organized as follows. In Section 1 we prove the existence of periodic points for $\partial\alpha$. The proof of Theorem 1 is completed in Section 2 (Theorems 2.1 and 2.3). In Section 3 we briefly discuss growth rates. We start Section 4 by a general discussion of superattractivity, valid for an arbitrary hyperbolic group. We then prove Theorem 2.

1. Existence of periodic points

Let F_n be a free group. We consider its boundary ∂F_n, equipped with the natural action of F_n by left-translations. It is a Cantor set if $n \geq 2$ (it consists of two points if $n = 1$). In section 4, we will view ∂F_n as a set of right-infinite reduced words. A finitely generated subgroup $J \subset F_n$ is quasiconvex [16]. In particular, we can identify the boundary (or limit set) ∂J with a subset of ∂F_n.

An automorphism $\alpha \in \operatorname{Aut} F_n$ is a quasi-isometry of F_n. It induces a homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$, and a homeomorphism $\overline{\alpha} = \alpha \cup \partial\alpha$ on the compact space $\overline{F}_n = F_n \cup \partial F_n$.

The fixed subgroup $\operatorname{Fix} \alpha = \{g \in F_n \mid \alpha(g) = g\}$ has finite rank (Gersten, see e.g. [2]). Its boundary $\partial(\operatorname{Fix}\alpha)$ is a subspace of ∂F_n upon which $\partial\alpha$ acts as the identity. Note that for any integer q the subgroup $\operatorname{Fix}\alpha^q$ is α-invariant (i.e. it is mapped to itself by α).

Following Nielsen [14], we say that a fixed point X of $\partial\alpha$ is *singular* if $X \in \partial(\operatorname{Fix}\alpha)$, *regular* otherwise.

It is shown in [5, Proposition 1.1] that a regular fixed point X of $\partial\alpha$ is either *attracting* or *repelling*. Attracting means that $\overline{\alpha}^p(Y)$ converges to X for every Y in a neighborhood of X in $F_n \cup \partial F_n$ (as $p \to +\infty$), repelling means attracting for α^{-1} (see a detailed discussion in Section 4).

We say that $X \in \partial F_n$ is *periodic* if there exists $q \geq 1$ with $\partial\alpha^q(X) = X$. The smallest such q is the *period* of X and the set $\{X, \partial\alpha(X), \dots, \partial\alpha^{q-1}(X)\}$ is a *periodic orbit of order* q. We define X to be regular, attracting... if it is as a fixed point of $\partial\alpha^q$. We give a similar definition for a periodic orbit, noting that all its elements have the same type.

Theorem 1.1. *Let $\alpha \in \operatorname{Aut} F_n$. The homeomorphism $\partial\alpha : \partial F_n \to \partial F_n$ has at least two periodic points. More precisely, either $\partial\alpha$ has at least two periodic orbits, or the unique periodic orbit has order 2 and is the boundary of an α-invariant infinite cyclic subgroup.*

Example 1.2. We construct a $\partial\alpha$ with only one periodic orbit. First define $\beta : F_2 \to F_2$ by $a \mapsto a$, $b \mapsto aba$. Then $\partial\beta$ has two singular fixed points $a^{\pm\infty} = \lim_{p\to+\infty} a^{\pm p}$. It is easily checked that these are the only periodic points of $\partial\beta$. The automorphism β is the square of $\alpha : a \mapsto a^{-1}, b \mapsto a^{-1}b^{-1}$. The map $\partial\alpha$ permutes a^{∞} and $a^{-\infty}$.

The proof of Theorem 1.1 (to be found below) uses an α-invariant $\mathbf{R}$-tree T. The main properties of T are summarized as follows.

Theorem 1.3. **([5])** *For every automorphism α of F_n there exists an $\mathbf{R}$-tree T and a number $\lambda \geq 1$ such that:*

(1) F_n acts on T non-trivially, minimally, with trivial arc stabilizers.

(2) There exists a homothety H: $T \to T$ with stretching factor λ such that

$$\alpha(g)H = Hg$$

for all $g \in F_n$ (viewing elements of F_n as isometries of T). If $\lambda = 1$, then T is simplicial.

(3) There exists an F_n-equivariant injection $j : \partial T \to \partial F_n$ satisfying $\partial\alpha \circ j = j \circ H$. □

Furthermore:

Theorem 1.4. **([6])** *Given $Q \in T$, its stabilizer* $\operatorname{Stab} Q$ *has rank $\leq n-1$, and the* *action of* $\operatorname{Stab} Q$ *on* $\pi_0(T \setminus \{Q\})$ *has at most* $2n$ *orbits. The number of* F_n*-orbits of branch points is at most* $2n-2$. □

A *homothety* is a map H such that $d(Hx, Hy) = \lambda d(x, y)$ for some $\lambda > 0$ (the *stretching factor*). We denote ∂T the set of equivalence classes of infinite rays $\rho : (0, +\infty) \to T$, and again $H : \partial T \to \partial T$ the induced map. See [5, Sections 2 and 3] for other definitions and a proof of Theorem 1.3. Theorem 1.4 follows from Theorem III.2 of [6]. Given α and T, the number λ and the homothety H satisfying $\alpha(g)H = Hg$ are unique.

A homothety H with $\lambda > 1$ has a unique fixed point Q, which may be in T or only in its metric completion $\overline{T}$. We define an *eigenray* of H as in [5], as an isometric map $\rho : (0, \infty) \to T$ such that $\rho(\lambda x) = H\rho(x)$. We note:

Proposition 1.5. *If* $HR = R$*, the stabilizer* $\operatorname{Stab} R$ *is* α*-invariant. If* ρ *is an eigenray, then* $j(\rho)$ *is a fixed point of* $\partial\alpha$*. Now suppose* $\lambda > 1$*, and let* Q *be the fixed point of* H*. If* $Q \in \overline{T} \setminus T$*, then there exists a unique eigenray. If* $Q \in T$*, then any component of* $T \setminus \{Q\}$ *that is fixed by* H *contains a unique eigenray.* □

Proof of Theorem 1.1. First assume that the fixed subgroup $\operatorname{Fix} \alpha^q$ is nontrivial for some $q \geq 1$. If it is cyclic, its two boundary points are either fixed points of $\partial\alpha$ or a periodic orbit of order 2. If $\operatorname{Fix} \alpha^q$ has rank ≥ 2, we get uncountably many periodic orbits. From now on we assume that $\operatorname{Fix} \alpha^q$ is trivial for every q, and we construct an attracting periodic orbit of $\partial\alpha$. The same argument, applied to α^{-1}, will yield a second orbit.

Let T be as in Theorem 1.3. If H fixes some $Q \in T$ with $\operatorname{Stab} Q$ nontrivial, recall that $\operatorname{Stab} Q$ is α-invariant. Since it has rank less that n and $\partial\operatorname{Stab} Q$ embeds into ∂F_n, we will be able to use induction on n (of course $n = 1$ is trivial). Also note that, if ρ is an eigenray of H (with $\lambda > 1$), then the fixed point $j(\rho)$ of $\partial\alpha$ is attracting (see the proof of Assertion 2 of Proposition 4.4 in [5]).

Recall that we want to find an attracting periodic orbit of $\partial\alpha$. First assume $\lambda > 1$. Let $Q \in \overline{T}$ be the fixed point of H. If $Q \in \overline{T} \setminus T$, there is an eigenray ρ and $j(\rho)$ is an attracting fixed point of $\partial\alpha$. Suppose $Q \in T$. If $\operatorname{Stab} Q$ is nontrivial, we use induction on n. Otherwise $T \setminus \{Q\}$ has at most $2n$ components by Theorem 1.4, and some power of H has an eigenray. This gives an attracting periodic orbit as before.

Now we assume $\lambda = 1$. In this case T is simplicial and H is an isometry.

First suppose H fixes some Q. We may assume $\operatorname{Stab} Q$ is trivial (otherwise, we do induction). Then some H^k fixes an edge e. Replacing α by α^k, we assume $k = 1$. Collapse to a point every edge not in the orbit of e (under the action of F_n). We get a new tree T' with an isometry H' satisfying the conclusions of Theorem 1.3. The map H' fixes some point with nontrivial stabilizer (since all vertices now have nontrivial stabilizer) and we use induction.

The last possibility if that H is a hyperbolic isometry of T. In this case H has a translation axis A and fixes two ends of T. Orienting A by the action of H, we consider the positive end A^+ of A and the associated fixed point $X^+ = j(A^+)$ of $\partial\alpha$. We complete the proof by showing that X^+ is not repelling (and therefore is attracting since we assume $\operatorname{Fix}\alpha^q$ trivial for all q). Choose any point $Q \in A$, and $g \in F_n$ acting on T as a hyperbolic isometry whose axis has compact intersection with A. Writing $\alpha^p(g)Q = H^p g H^{-p} Q$ we see that the projection of $\alpha^p(g)Q$ onto A goes to A^+ as $p \to \infty$. By Section 3 of [5] this implies $\lim_{p\to\infty} \alpha^p(g) = X^+$. Thus X^+ cannot be repelling. □

2. Bounding periods

Theorem 2.1. *Let $\alpha \in \operatorname{Aut} F_n$. Suppose $X \in \partial F_n$ is periodic of period q under $\partial\alpha$. Then $q \le M_n$, where M_n depends only on n and $\log M_n \sim \sqrt{n \log n}$ as $n \to \infty$.*

The quantity $\sqrt{n \log n}$ is Landau's asymptotic estimate for $\log g(n)$, where $g(n)$ is the maximum order of elements in the symmetric group S_n [8]. It is shown in [11] that the same estimate holds for the maximum order of torsion elements in $GL(n, \mathbf{Z})$ and $\operatorname{Aut} F_n$.

We first prove the following special case of Theorem 2.1:

Lemma 2.2. *If $g \in F_n$ is periodic of period q under $\alpha \in \operatorname{Aut} F_n$, then $q \le A_n$, where A_n is the maximum order of torsion elements in $\operatorname{Aut} F_n$.*

Proof. The subgroup $\operatorname{Fix}\alpha^q$ is α-invariant, and the restriction of α has order exactly q in $\operatorname{Aut}(\operatorname{Fix}\alpha^q)$. Since the rank of $\operatorname{Fix}\alpha^q$ is $\le n$ by [1], and $\operatorname{Aut} F_k$ naturally embeds into $\operatorname{Aut} F_\ell$ for $k < \ell$, the group $\operatorname{Aut} F_n$ contains an element of order q. □

Remark. Before the Scott conjecture was proved, Stallings showed [17] that, for a given α, there is a bound for periods of elements $g \in F_n$.

Proof of Theorem 2.1. Lemma 2.2 shows that singular periodic points of $\partial\alpha$ have period $\le A_n$. Now suppose X is regular, say attracting.

The points $X, \partial\alpha(X), \dots, \partial\alpha^{q-1}(X)$ are attracting fixed points of $\partial\alpha^q$. By Theorem 1 of [5], the action of $\operatorname{Fix}\alpha^q$ on the set of attracting fixed points of $\partial\alpha^q$ has at most $2n$ orbits. Thus there exist $r \le 2n$ and $u \in \operatorname{Fix}\alpha^q$ such that

$$\partial\alpha^r(X) = uX.$$

By Lemma 2.2 we have

$$\alpha^s(u) = u$$

for some $s \le A_n$.

The above equations yield $\partial\alpha^{rs}(X) = aX$ with

$$a = \alpha^{(s-1)r}(u) \dots \alpha^r(u)u.$$

If $a = 1$ we get $q \leq rs \leq 2nA_n$. Otherwise we note that $a \in \mathrm{Fix}\,\alpha^s$, and from $X = \partial\alpha^{qrs}(X) = a^q X$ we conclude that X is singular, a contradiction.

We have thus shown $q \leq M_n = 2nA_n$. Since $\log A_n \sim \sqrt{n \log n}$ by [11], we have $\log M_n \sim \sqrt{n \log n}$. □

Remark. The bound $q \leq 2nA_n$ is not quite sharp. But if $\alpha \in \mathrm{Aut}\,F_n$ has order A_n then generic points of ∂F_n have period A_n under $\partial\alpha$. Therefore the estimate $\log M_n \sim \sqrt{n \log n}$ cannot be improved.

Theorem 2.3. *For any $\alpha \in \mathrm{Aut}\,F_n$, the map $\partial\alpha : \partial F_n \to \partial F_n$ has at least two periodic points of period $\leq 2n$.*

For the automorphism defined by $a_i \mapsto a_{i+1}$ $(1 \leq i \leq n-1)$, $a_n \mapsto a_1^{-1}$, every point of ∂F_n has period $2n$.

Proof. There are two cases. If α has no periodic element $g \neq 1$, then $\partial\alpha$ has at most $2n$ periodic points of a given type (attracting or repelling) by Theorem 1 of [5]. The other case is taken care of by the following result. □

Proposition 2.4. *Let $\alpha \in \mathrm{Aut}\,F_n$. If there is a nontrivial α-periodic element $g \in F_n$, then there is one of period $\leq 2n$.*

Proof. Let q be the smallest period of nontrivial periodic elements. Arguing as in the proof of Lemma 2.2, we may assume that α has order q. Such an α may be realized as an automorphism of a graph ([4], [18]): there exist a finite graph Λ, an automorphism f of Λ fixing a vertex v, and an isomorphism $F_n \to \pi_1(\Lambda, v)$ such that the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{\alpha} & F_n \\ \downarrow & & \downarrow \\ \pi_1(\Lambda, v) & \xrightarrow{f_*} & \pi_1(\Lambda, v). \end{array}$$

We choose Λ with minimal number of vertices. We claim that the action of $\mathbf{Z}/q\mathbf{Z} = < f >$ on the set of germs of edges at v is free. This will show $q \leq 2n$ since v has valence at most $2n$.

Assume the action is not free. Then some f^r $(1 \leq r \leq q-1)$ fixes an edge containing v. Let Λ_0 be the component of the fixed point set of f^r containing v. It is a tree since otherwise α would have a nontrivial periodic element of period $\leq r$. We may therefore collapse Λ_0 to a point, contradicting the choice of Λ. □

3. Growth rates

In this section we fix $\Phi \in \operatorname{Out} F_n$, and sometimes also an automorphism $\alpha \in \Phi$. We write $|g|$ for the word length of $g \in F_n$, and $|\gamma|$ for the length of a conjugacy class γ (equal to the length of a cyclically reduced word representing γ).

Let M be the transition matrix of a relative train track map representing Φ (see [1]). The largest positive eigenvalue (spectral radius) of the matrix M is denoted $\lambda(\Phi)$, or $\lambda(\alpha)$. It is an algebraic integer of degree bounded by $3n-3$.

For $g \in F_n$, the length of $\alpha^p(g)$ is bounded from above by a constant times $\|M\|^p\,|g|$. If $\lambda(\Phi)=1$, the growth of $\alpha^p(g)$ is polynomial and Φ is called *polynomially growing*. For future reference we note:

Remark 3.1. Given $\nu > \lambda(\alpha)$, there exists $C>0$ such that $|\alpha^p(g)| \le C\nu^p|g|$ for all $g \in F_n$ and $p \ge 1$.

Now let $\ell : F_n \to \mathbf{R}^+$ be the length function of an action of F_n on an $\mathbf{R}$-tree T. It is bounded from above by a constant times word length. In particular, if T is an α-invariant $\mathbf{R}$-tree as in Theorem 1.3, we have (up to multiplicative constants) $\lambda^p\ell(g) = \ell(\alpha^p(g)) \le |\alpha^p(g)| \le \|M\|^p|g|$ and therefore $\lambda \le \lambda(\alpha)$. Conversely:

Proposition 3.2. *There exists an α-invariant $\mathbf{R}$-tree T as in Theorem 1.3 with $\lambda = \lambda(\alpha)$.*

Proof. This is proved by the same arguments as in [5, section 2], but instead of using only the top stratum of the train track (which may lead to $\lambda < \lambda(\alpha)$) we use the whole relative train track and an eigenvector v of M associated to $\lambda(\alpha)$. One shows that the resulting action on an $\mathbf{R}$-tree T is nontrivial and has trivial arc stabilizers as in [5]. Minimality of the action may be achieved by restricting to the minimal invariant subtree. It is often more convenient, though, to work with the metric completion $\overline{T}$ of T so as to ensure that H has a fixed point Q when $\lambda(\alpha) > 1$. □

Now let J be a finitely generated malnormal subgroup of F_n (recall that J is *malnormal* if $gJg^{-1} \cap J \neq \{1\} \implies g \in J$). We say that J is Φ-*periodic* if there exist $q \ge 1$ and $\beta \in \Phi^q$ with $\beta(J) = J$. Note that, by malnormality, the class of β in $\operatorname{Out} J$ is uniquely determined.

For example, suppose that T is an $\mathbf{R}$-tree as in Theorem 1.3 and $J = \operatorname{Stab} Q$ for some branch point Q. Then J is malnormal (because arc stabilizers are trivial). By Theorem 1.4, it has rank $< n$. We claim that it is Φ-periodic. Indeed, by Theorem 1.4 there exist $m \in F_n$ and $q \ge 1$ such that mH^q fixes Q. Denoting $i_m(g) = mgm^{-1}$, the automorphism $\beta = i_m \circ \alpha^q \in \Phi^q$ maps J to itself.

If J is finitely generated, malnormal, Φ-periodic, we define $\lambda_J = \lambda(\beta_{|J})^{\frac{1}{q}}$.

Proposition 3.3. *Let $\Phi \in \operatorname{Out} F_n$.*

(1) Each conjugacy class γ in F_n has a growth rate $\lambda(\gamma) = \lim_{p\to+\infty} |\Phi^p(\gamma)|^{1/p}$.

(2) Given $\lambda \geq 1$, the following are equivalent:

- *$\lambda = \lambda(\gamma)$ for some conjugacy class γ.*
- *$\lambda = \lambda_J$ for some malnormal Φ-periodic subgroup $J \subset F_n$ of rank $\leq n$.*

The existence of the limit in assertion 1 is folklore (compare [1, Remark 1.8]). Simple examples show that one cannot restrict to free factors in assertion 2.

Proof. The proof is by induction on n. Let T be an α-invariant $\mathbf{R}$-tree with $\lambda = \lambda(\Phi)$ (see Proposition 3.2). We distinguish two cases, by evaluating the length function on γ.

If $\ell(\gamma) > 0$, we write $|\Phi^p(\gamma)| \geq \ell(\Phi^p(\gamma)) = \lambda^p \ell(\gamma)$ (up to a constant) and we conclude that γ has growth rate $\lambda(\gamma) = \lambda = \lambda(\Phi)$ (recall that the exponential growth of $\Phi^p(\gamma)$ is bounded from above by $\lambda(\Phi)$). Note that there exist classes with $\ell(\gamma) > 0$, hence there exist classes with growth rate $\lambda(\Phi)$.

If $\ell(\gamma) = 0$, an element $g \in F_n$ representing γ fixes some branch point $Q \in T$, and we argue by induction by considering γ as a conjugacy class in $J = \operatorname{Stab} Q$. We have pointed out earlier that J is malnormal, Φ-periodic, of rank $< n$. If $\beta = i_m \circ \alpha^q$ leaves J invariant, note that, by quasiconvexity of J, the growth rate of γ under $\beta_{|J}$ is the same as the growth rate of γ, viewed as a conjugacy class in F_n, under Φ^q.

These arguments show that every γ has a growth rate, which is of the form λ_J with J as in the proposition. Conversely, given J, let ℓ_J be the length function of a $\beta_{|J}$-invariant tree with $\lambda = \lambda(\beta_{|J})$. Conjugacy classes with $\ell_J(\gamma) > 0$ have growth rate λ_J under Φ. □

Definition. We call $\lambda(\Phi)$ the *top growth rate* of Φ. The *set of growth rates* $\Lambda(\Phi) \subset (1, \infty)$ consists of the growth rates $\lambda(\gamma)$ which are bigger than 1.

Note that $\Lambda(\Phi)$ consists of algebraic integers of degree $\leq 3n-3$, and that $\lambda(\Phi)$ is the largest element of $\Lambda(\Phi) \cup \{1\}$. See [10] for more results about $\Lambda(\Phi)$.

4. Hölder dynamics

Superattractivity

The discussion in this subsection (including Proposition 4.1) is valid for automorphisms of arbitrary (word) hyperbolic groups, but for simplicity we restrict to the case of F_n (the generalization is almost immediate using [13]).

Fixing a free basis of F_n, we may view ∂F_n as the set of right-infinite reduced words. Let $X \in \partial F_n$ be a fixed point of the homeomorphism $\partial\alpha$ induced by

$\alpha \in \operatorname{Aut} F_n$ on ∂F_n. We say that X is *singular* if it belongs to the limit set of the fixed subgroup $\operatorname{Fix}\alpha$, *regular* otherwise (recall that $\operatorname{Fix}\alpha$ has finite rank).

As explained in [5], there is a basic trichotomy: *either X is singular, or X is attracting, or X is repelling (i.e. attracting for α^{-1}).* Attractivity has a strong meaning here (see section 1 of [5]): given A, there exists m such that for $Y \in F_n \cup \partial F_n$

$$c_X Y \geq m \implies c_X(\partial\alpha(Y)) - c_X Y > A, \tag{1}$$

where $c_X Y$ is the length of the maximal common initial segment between the reduced words X and Y (i.e. the Gromov scalar product $< X, Y >$ with basepoint at the identity in the Cayley graph).

In particular, we have $\lim_{p\to+\infty} \overline{\alpha}^p(Y) = X$ uniformly on a neighborhood of X in $F_n \cup \partial F_n$ if X is attracting (whereas if X is singular there are fixed points of α in F_n arbitrarily close to X). For the automorphism β studied in Example 1.2, the (singular) fixed points $a^{\pm\infty}$ of $\partial\beta$ are partly repelling and partly attracting: for any $k \in \mathbf{Z}$ we have $\lim_{p\to+\infty} \partial\beta^p(a^k bY) = a^\infty$ if Y is a right-infinite reduced word not starting with b^{-1}, but $\lim_{p\to+\infty} \partial\beta^p(a^k b^{-1} Y) = a^{-\infty}$ if Y does not start with b.

Also note that an isolated fixed point of $\partial\alpha$ is singular if and only if it belongs to the limit set of an α-invariant cyclic subgroup (for the "only if" direction, simply observe that α leaves invariant the stabilizer of X for the action of F_n on ∂F_n). In particular, the natural action of $\operatorname{Fix}\alpha$ on the set of regular fixed points of $\partial\alpha$ is free. This action has finitely many orbits [2], indeed it follows from [5] that the number of orbits is at most $4n$. It is not clear to us whether there is a bound depending only on G when G is an arbitrary hyperbolic group.

Now recall that the boundary of F_n (of any hyperbolic group, in fact) has a canonical *Hölder structure* (see [3], [7]). It may be viewed as a collection $\mathcal{D}$ of distance functions on ∂F_n that are pairwise bi-Hölder equivalent: Given $d, d' \in \mathcal{D}$, there exist $C > 0$ and $\beta \in (0, 1]$ such that $\frac{1}{C} d^{\frac{1}{\beta}} \leq d' \leq C d^\beta$. This Hölder structure is preserved by $\partial\alpha$ for every $\alpha \in \operatorname{Aut} F_n$. If $J \subset F_n$ has finite rank, the inclusion $\partial J \hookrightarrow \partial F_n$ is bi-Hölder.

We represent the Hölder structure by the visual metrics $d_\varepsilon(X, Y) = \exp(-\varepsilon c_X Y)$.

Let $X \in \partial F_n$ be a fixed point of $\partial\alpha$, and $d = d_\varepsilon$ a visual metric. If X is regular, attracting, it follows from (1) that

$$\lim_{Y\to X} \frac{d(\partial\alpha(Y), X)}{d(Y, X)} = 0. \tag{2}$$

If X is repelling or singular, however, the above quotient is bounded away from 0 on a neighborhood of X (if X is singular, $c_X(\partial\alpha(Y)) - c_X Y$ is bounded near X because $\operatorname{Fix}\alpha$ is quasiconvex and α is a quasi-isometry).

Thus (2) is a metric characterization of attracting regular fixed points, similar to the characterization of a superattracting fixed point c of a holomorphic map

$f : \mathbf{C} \to \mathbf{C}$ by $f'(c) = 0$. For this reason, we call an attracting regular point *superattracting* (and a repelling regular point *superrepelling*).

Of course the map $\partial\alpha$ is a homeomorphism, and superattracting fixed points may exist only because $\partial\alpha$ is bi-Hölder but in general not bi-Lipschitz. For instance, if t is any lift to the Poincaré disc of a pseudo-Anosov diffeomorphism of a closed hyperbolic surface, then the homeomorphism induced by t on the circle at infinity is never bi-Lipschitz (see Remark (22.14) in [12]).

Characterization (2) above does not depend on the chosen visual metric d, but it is not valid for arbitrary metrics in $\mathcal{D}$. The following characterization will apply to every $d \in \mathcal{D}$.

Proposition 4.1. *Let $\alpha \in \operatorname{Aut} F_n$. A fixed point X of $\partial\alpha$ is superattracting if and only if*

$$\lim_{p\to+\infty} \frac{1}{p} \log d(\partial\alpha^p(Y), X) = -\infty$$

for $Y \in \partial F_n$ close to X, where d is any metric on ∂F_n defining the Hölder structure.

This equation means that $\partial\alpha^p(Y)$ converges to X super-exponentially as $p \to \infty$. Unlike (2), it is true for every metric in $\mathcal{D}$ if it is true for one.

Proof. We may assume that d is a visual metric. Suppose X is superattracting. We have to prove $\lim_{p\to\infty} \frac{1}{p} c_X(\partial\alpha^p(Y)) = +\infty$ for Y close to X. Given $A > 0$, let m be as in (1). If $\lim_{p\to\infty} \partial\alpha^p(Y) = X$, there exists n_0 such that $c_X(\partial\alpha^p(Y)) \geq m$ for $p \geq n_0$. For p large, we then have

$$c_X(\partial\alpha^p(Y)) \geq A(p - n_0) + m,$$

and the result follows.

Conversely, if X is singular, then $c_X(\partial\alpha(Z)) - c_X Z$ is bounded in a neighborhood of X, and therefore $\frac{1}{p} \log d(\partial\alpha^p(Y), X)$ is bounded from below as $p \to \infty$. □

Speed of convergence

We consider $\alpha \in \operatorname{Aut} F_n$, and the associated $\Phi \in \operatorname{Out} F_n$. Recall that $\Lambda(\Phi) \subset (1, \infty)$ is the set of nontrivial growth rates. It may also be viewed as a set of λ_J (see Proposition 3.3).

Theorem 4.2. *Let $\alpha \in \operatorname{Aut} F_n$. Let $X \in \partial F_n$ be a superattracting fixed point of $\partial\alpha$. There exists $\lambda = \lambda(\alpha, X) \in \Lambda(\Phi) \cup \{1\}$ such that*

$$\lim_{p\to+\infty} \frac{1}{p} \log\Big(-\log d(\partial\alpha^p(Y), X)\Big) = \log\lambda \tag{3}$$

for $Y \in \partial F_n$ close to X (and any metric d on ∂F_n defining the Hölder structure).

Conversely, given $\mu \in \Lambda(\Phi)$, there exist $q \geq 1$, an automorphism $\beta \in \Phi^q$, and a superattracting fixed point X of $\partial\beta$ with $\lambda(\beta, X) = \mu^q$.

It follows that the set $\Lambda_h(\Phi)$ of Hölder exponents defined in the introduction equals $\Lambda(\Phi)$. Note that replacing d by a metric bi-Hölder equivalent to d does not affect the validity of (3).

Proof of Theorem 4.2. We fix a basis of F_n and we consider the corresponding Cayley tree Γ.

Let X be a superattracting fixed point of $\partial\alpha$. We need to prove

$$\lim_{p \to +\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) = \log \lambda.$$

We will bound the left-hand side, first from above and then from below.

Lemma 4.3. *Suppose $X \in \partial J$, with $J \subset F_n$ a finitely generated α-invariant malnormal subgroup. Then*

$$\limsup_{p \to +\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) \leq \log \lambda_J$$

for all $Y \in \partial F_n$.

Recall that λ_J is the top growth rate of $\alpha_{|J}$.

Proof. Let x_{t_p} be the projection of $\partial\alpha^p(Y)$ onto the geodesic from $1 \in F_n$ to X in Γ. By quasiconvexity of J, we can find $j_p \in J$ within a fixed distance of x_{t_p}. We need to prove $\limsup_{p \to +\infty} \frac{1}{p} \log |j_p| \leq \log \lambda_J$. We will work with word length $|j_p|_J$ in J, which is comparable to $|j_p|$.

Define $w_p \in J$ by $j_p = \alpha(j_{p-1})w_p$. Since α is a quasi-isometry, there is a uniform bound for $|w_p|$, hence also for $|w_p|_J$ because J is quasiconvex. Now write

$$j_p = \alpha^p(j_0)\alpha^{p-1}(w_1) \cdots \alpha(w_{p-1})w_p.$$

For $\nu > \lambda_J$ we have

$$|j_p|_J \leq C\nu^p|j_0|_J + C\nu^{p-1}|w_1|_J + \cdots + C\nu|w_{p-1}|_J + |w_p|_J,$$

with C given by Remark 3.1. Thus $|j_p|_J = O(\nu^p)$ for all $\nu > \lambda_J$, and the lemma is proved. □

Corollary 4.4. *Theorem 4.2 holds if α is polynomially growing (i.e. $\lambda(\alpha) = 1$), with $\lambda(\alpha, X) = 1$.* □

Fix a subgroup J as in Lemma 4.3, and consider an $\mathbf{R}$-tree T with an action of J satisfying the conditions of Theorem 1.3 with respect to $\alpha_{|J}$. Using Proposition 3.2, we assume that the stretching factor of the homothety H equals λ_J. Suppose furthermore $\lambda_J > 1$.

Lemma 4.5. *Suppose $X = j(\rho)$, where ρ is an eigenray of $H : T \to T$ (in particular, $X \in \partial J$). Then*

$$\liminf_{p \to +\infty} \frac{1}{p} \log c_X(\partial\alpha^p(Y)) \geq \log \lambda_J$$

for $Y \in \partial F_n$ close enough to X.

Proof. With the notations of Section 1, let $Q \in \overline{T}$ be the fixed point of H (i.e. the origin of ρ). Choose j_p as in the proof of Lemma 4.3 and define d_p as $\overline{d}(Q, j_pQ)$ (where $\overline{d}$ denotes the distance in $\overline{T}$). Note that

$$\overline{d}(Q, \alpha(j_p)Q) = \overline{d}(Q, \alpha(j_p)HQ) = \overline{d}(Q, Hj_pQ) = \lambda_J \overline{d}(Q, j_pQ).$$

On the other hand, recall that the distance in J from $\alpha(j_p)$ to j_{p+1} is bounded independently of p (and of Y). Thus we obtain an inequality of the form

$$d_{p+1} \geq \lambda_J d_p - A,$$

with A independent of p and Y.

If Y is close enough to X in ∂F_n, then j_0 is close to X in $J \cup \partial J$, and therefore d_0 is large (by bounded backtracking, see section 3 of [5]). This implies

$$\liminf_{p \to +\infty} \frac{1}{p} \log d_p \geq \log \lambda_J.$$

Finally, we observe that $d_p = \overline{d}(Q, j_pQ)$ is bounded above by a constant times $|j_p|_J$, hence by a constant times $|j_p|$. □

Now we complete the proof of Theorem 4.2. If $\lambda(\alpha) = 1$, then we are done by Corollary 4.4. Assume $\lambda(\alpha) > 1$, and consider a tree T as in Proposition 3.2, with stretching factor $\lambda(\alpha)$. If $X = j(\rho)$ as in Lemma 4.5, we are done, with $\lambda = \lambda(\alpha)$. If not, then by Proposition 4.3 of [5] we have $X \in \partial \mathrm{Stab}\, Q$, where $Q \in T$ is the fixed point of H (recall that points of $\overline{T} \setminus T$ have trivial stabilizer).

The subgroup $\mathrm{Stab}\, Q$ is α-invariant, malnormal, and has rank $< n$ (see section 3). Repeat the argument, working with $\alpha_{|\mathrm{Stab}\, Q}$. After a finite number of steps we find that $X \in \partial J$ (with J invariant, malnormal, of rank $< n$), and either $\lambda_J = 1$ or $X = j(\rho)$. It follows from Lemmas 4.3 and 4.5 that Theorem 4.2 holds, with $\lambda(\alpha, X) = \lambda_J$.

Conversely, consider $\mu \in \Lambda(\Phi)$. First suppose $\mu = \lambda(\alpha)$. Consider an $\mathbf{R}$-tree T as in Theorem 1.3, with $\lambda = \lambda(\alpha)$. By Theorem 1.4 and Proposition 1.5, there exist $m \in F_n$ and $q \geq 1$ such that mH^q has an eigenray ρ. Let $\beta = i_m \circ \alpha^q$, with $i_m(g) = mgm^{-1}$. Then $X = j(\rho)$ is a fixed point of $\partial\beta$, and $\lambda(\beta, X) = \lambda(\beta) = \mu^q$.

For arbitrary $\mu = \lambda_J \in \Lambda(\Phi)$, let $\alpha' \in \Phi^r$ leave J invariant. The previous argument yields $\beta \in \Phi^{rq}$ and a fixed point X of $\partial\beta$ in ∂J such that $\lambda(\beta_{|J}, X) = \mu^{rq}$. Since the inclusion $\partial J \hookrightarrow \partial F_n$ is bi-Hölder, $\lambda(\beta, X) = \lambda(\beta_{|J}, X)$ has the required form. □

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Gilbert Levitt
Laboratoire Émile Picard
UMR CNRS 5580
Université Paul Sabatier
F-31062 Toulouse Cedex 4
France
e-mail: levitt@picard.ups-tlse.fr

Martin Lustig
Laboratoire de mathématiques
Université d'Aix-Marseille III
F-13397 Marseille Cedex 20
France
e-mail: Martin.Lustig@math.u-3mrs.fr

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