

Invariant currents on limit sets

John Lott

Abstract. We relate the L^2 -cohomology of a complete hyperbolic manifold to the invariant currents on its limit set.

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1. Introduction

Let M be a complete oriented connected n -dimensional hyperbolic manifold. We can write $M = H^n/\Gamma$, where Γ is a torsion-free discrete subgroup of $\text{Isom}^+(H^n)$, the group of orientation-preserving isometries of the hyperbolic space H^n . The action of Γ on H^n extends to a conformal action on S_∞^{n-1} , the sphere at infinity. For basic notions of hyperbolic geometry, we refer to [2]. Unless otherwise indicated, we assume that Γ is nonelementary, i.e. does not have an abelian subgroup of finite index.

A major theme in the study of hyperbolic manifolds is the relationship between the properties of M and the action of Γ on S_∞^{n-1} . For example, let $\lambda_0(M) \in [0, \infty)$ be the infimum of the spectrum $\sigma(\Delta)$ of the Laplacian on M . Let $\Lambda \subset S_\infty^{n-1}$ be the limit set of Γ and let $D(\Gamma)$ be its Hausdorff dimension. Sullivan [15] showed that if M is geometrically finite then

$$\lambda_0(M) = \begin{cases} (n-1)^2/4 & \text{if } D(\Gamma) \leq \frac{n-1}{2}, \\ D(\Gamma)(n-1-D(\Gamma)) & \text{if } D(\Gamma) \geq \frac{n-1}{2}. \end{cases} \quad (1.1)$$

Thus there is a strong relationship between the spectrum of the Laplacian, acting on functions on M , and the geometry of the limit set. There is also a Laplacian Δ_p on p -forms on M (see, for example, [9]). The motivating question of this paper is: What, if any, is the relationship between the spectrum of Δ_p and the geometry of the limit set?

If $p > 0$, it is clear that the infimum of the spectrum of Δ_p depends on more than just the limit set as a set. For example, let M be a closed hyperbolic 3-manifold. From Hodge theory, $0 \in \sigma(\Delta_1)$ if and only if the first Betti number $b_1(M)$ of M is nonzero. There are examples with $b_1(M) = 0$ and examples with $b_1(M) \neq 0$. However, in either case, $\Lambda = S_\infty^2$.

In this paper, we address the question of whether $\text{Ker}(\Delta_p) \neq 0$ for a hyperbolic manifold M . We show how the answer to the question is related to the existence of Γ -invariant p -currents on S_∞^{n-1} , of a certain regularity. In some sense, these currents probe the finer geometry of the limit set.

In order to state our results, let us recall the notion of harmonic extension of p -forms. We use the hyperbolic ball model for H^n , with boundary S^{n-1} . The space of p -hyperforms on S^{n-1} is the dual space to the space of real-analytic $(n-1-p)$ -forms on S^{n-1} . We think of a p -hyperform on S^{n-1} as a p -form whose coefficient functions are hyperfunctions. A p -current on S^{n-1} is a p -hyperform whose coefficient functions are distributions.

There is a Poisson transform Φ_p from p -hyperforms on S^{n-1} to coclosed harmonic p -forms on H^n [6]. To describe Φ_p in terms of visual extension, let ω be a p -hyperform on S^{n-1} . Given $x \in H^n$, let S_x be the unit sphere in $T_x H^n$ and let $A_x : S_x \rightarrow S^{n-1}$ be the visual map. Given $v \in T_x H^n \cong T_0(T_x H^n)$, define a vector field V on S_x by saying that at $y \in S_x$, V is the translation of v in $T_x H^n$ from 0 to y , followed by orthogonal projection onto $T_y S_x$. Then for $v_1, \dots, v_p \in T_x H^n$,

$$\langle \Phi_p(\omega), v_1 \wedge \dots \wedge v_p \rangle = \frac{1}{\text{vol}(S^{n-1})} \int_{S_x} \langle A_x^* \omega, V_1 \wedge \dots \wedge V_p \rangle d\text{vol}. \quad (1.2)$$

Equivalently, given $x \in H^n$ and $v \in T_x H^n$, take an upper-half-space model

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\} \quad (1.3)$$

for H^n in which $x = (0, \dots, 0, 1)$ and $v = c \frac{\partial}{\partial x_n}$ for some $c \in \mathbb{R}$. Consider the Killing vector field $c \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$. It restricts to a conformal vector field W on $\partial H^n = S^{n-1}$. Then for $v_1, \dots, v_p \in T_x H^n$,

$$\langle \Phi_p(\omega), v_1 \wedge \dots \wedge v_p \rangle = \frac{1}{\text{vol}(S^{n-1})} \int_{S^{n-1}} \langle \omega, W_1 \wedge \dots \wedge W_p \rangle d\text{vol}. \quad (1.4)$$

By a result of Gaillard, for $p > 0$, Φ_p is an isomorphism from *exact* p -hyperforms on S^{n-1} to *closed and coclosed* p -forms on H^n [6, Théorème 2]. Following [6], we say that a p -form α on H^n has slow growth if there are constants $a, b > 0$ such that for some (or any) $m_0 \in H^n$,

$$|\alpha(m)| \leq a e^{bd(m_0, m)} \quad (1.5)$$

for all $m \in H^n$. Then for $p > 0$, Φ_p is also an isomorphism from *exact* p -currents on S^{n-1} to *closed and coclosed* p -forms on H^n of slow growth [6, Théorème 3].

Let $\pi : H^n \rightarrow H^n/\Gamma$ be the quotient map. Let $\Omega = S^{n-1} - \Lambda$ be the domain of discontinuity.

By Gaillard's theorem, if $p > 0$ then $\Phi_p^{-1} \circ \pi^*$ induces an isomorphism between closed and coclosed p -forms on H^n/Γ , and Γ -invariant exact p -hyperforms on S^{n-1} . Let α be an L^2 -harmonic p -form on H^n/Γ . By Hodge theory, α is closed and coclosed. Thus we can use results about the L^2 -cohomology of H^n/Γ to construct Γ -invariant exact p -hyperforms on S^{n-1} , and vice versa. The questions that we address are :

1. What can we say about the regularity of these hyperforms?
2. Are they supported on the limit set?

Under Hodge duality, the space of L^2 -harmonic p -forms on H^n/Γ is isomorphic to the space of L^2 -harmonic $(n-p)$ -forms. Without loss of generality, hereafter we assume that $p \in [1, \frac{n}{2}]$.

Theorem 1. *If n is even then up to a constant, $\Phi_{\frac{n}{2}}$ is an isometric isomorphism between exact $\frac{n}{2}$ -forms on S^{n-1} which are Sobolev $H^{-\frac{1}{2}}$ -regular, and L^2 -harmonic $\frac{n}{2}$ -forms on H^n .*

From Theorem 1, we obtain that the $\frac{n}{2}$ -hyperforms that we construct on S^{n-1} cannot be too regular.

Corollary 1. *Suppose that α is a nonzero L^2 -harmonic $\frac{n}{2}$ -form on H^n/Γ . If Γ is infinite then $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$ is not Sobolev $H^{-\frac{1}{2}}$ -regular.*

We now give some positive regularity results. Let us recall that Γ is said to be cocompact if H^n/Γ is compact. It is said to be convex-cocompact if there is a compact subset K of H^n/Γ such that all nontrivial closed geodesics in H^n/Γ lie in K . If Γ is convex-cocompact then H^n/Γ consists of K along with a finite number of flaring ends attached to K .

Theorem 2. *A. If Γ is cocompact then for any $p \in [1, \frac{n}{2}]$, there are isomorphisms between the following vector spaces :*

$V_1 = \{\text{Harmonic } p\text{-forms on } H^n/\Gamma\}$.

$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1}\}$.

$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}$.

$V_4 = H^p(H^n/\Gamma, \mathbb{R})$, the p -dimensional real cohomology group of H^n/Γ .

B. If Γ is convex-cocompact then for any $p \in [1, \frac{n-1}{2})$, there are isomorphisms between the following vector spaces :

$V_1 = \{L^2\text{-harmonic } p\text{-forms on } H^n/\Gamma\}$.

$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1} \text{ which are supported on the limit set}\}$.

$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are supported on the limit set and which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$
 $V_4 = H_c^p(H^n/\Gamma, \mathbb{R}),$ the p -dimensional real compactly-supported cohomology group of H^n/Γ .

In Theorem 2, we show that the injection $V_3 \rightarrow V_2$ is surjective and that Φ_p induces an isomorphism from V_2 to V_1 . In case A, there is an isomorphism between V_4 and V_1 from standard Hodge theory. By [12], this is also true in case B.

There are extensions of Theorem 2 to hyperbolic manifolds with vanishing injectivity radius. We state one such extension here.

Theorem 3. *If $n = 3$, suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on H^3/Γ . Let α be an L^2 -harmonic 1-form on H^3/Γ . Then for all $\epsilon > 0$, the hyperform $\Phi_1^{-1}(\pi^*\alpha)$ is Sobolev $H^{-1-\epsilon}$ -regular.*

We show that the regularity estimate in Theorem 2 is sharp in some cases. We find an interesting distinction between cocompact groups, and convex-cocompact groups which are not cocompact.

Theorem 4. *A. Suppose that Γ is cocompact. Let α be a nonzero harmonic 1-form on H^n/Γ . Then $\Phi_1^{-1}(\pi^*\alpha)$ is not Sobolev H^{-1} -regular.
 B. Let Γ be a convex-cocompact group which is not cocompact. Let α be an L^2 -harmonic 1-form on H^n/Γ . Then $\Phi_1^{-1}(\pi^*\alpha)$ is Sobolev H^{-1} -regular.*

We look at what our general results become in the case of surfaces and 3-manifolds. In the case of surfaces, we obtain results about the actions of Fuchsian groups on certain function spaces on S^1 . Let $\mathcal{A}'(S^1)$ denote the hyperfunctions on S^1 and let $\mathcal{A}'_0(S^1)$ denote those which vanish on constant functions. Let $\mathcal{D}'(S^1)$ denote the distributions on S^1 and let $\mathcal{D}'_0(S^1)$ denote those which vanish on constant functions. Recall that a Zygmund function on S^1 is a function $f : S^1 \rightarrow \mathbb{C}$ such that

$$\sup_{x \in S^1, h \in \mathbb{R}^+} \frac{|f(x+h) + f(x-h) - 2f(x)|}{h} < \infty. \tag{1.6}$$

A Zygmund function is continuous and lies in the Sobolev space $H^{1-\epsilon}(S^1)$ for all $\epsilon > 0$. Let $\mathcal{DZ}(S^1)$ denote the generalized functions on S^1 which are derivatives of Zygmund functions, plus constants. If Γ is a subgroup of $\text{PSL}(2, \mathbb{R})$, let $(\mathcal{A}'_0(S^1))^\Gamma$ denote the Γ -invariant subspace of $\mathcal{A}'_0(S^1)$, and similarly for $(\mathcal{D}'_0(S^1))^\Gamma$ and $(\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$.

Theorem 5. *A. Let Γ be a torsion-free uniform lattice in $\text{Isom}^+(H^2)$, with H^2/Γ a closed surface of genus g . Then*

1. $\dim (\mathcal{A}'_0(S^1))^\Gamma = 2g.$

- 2. $\dim (\mathcal{D}'_0(S^1))^\Gamma = 2g.$
 - 3. $\dim (\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g.$
 - 4. $\dim (L^2(S^1)/\mathbb{C})^\Gamma = 0.$
- B. Let Γ be a torsion-free nonuniform lattice in $\text{Isom}^+(H^2)$, with H^2/Γ the complement of k points in a closed surface S of genus g . Then*
- 1. $\dim (\mathcal{A}'_0(S^1))^\Gamma = \infty.$
 - 2. $\dim (\mathcal{D}'_0(S^1))^\Gamma = \max(2g, 2g + 2k - 2).$
 - 3. $\dim (H^{-\frac{1}{2}}(S^1)/\mathbb{C})^\Gamma = 2g.$
 - 4. $\dim (\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g.$
 - 5. $\dim (L^2(S^1)/\mathbb{C})^\Gamma = 0.$

Parts A.2 and B.2 of Theorem 5 are due to Haefliger and Banghe [8].

Next, we look at the case of quasi-Fuchsian 3-manifolds. We follow the philosophy of Connes and Sullivan [5, Section IV.3.γ]. Let S be a closed oriented surface of genus $g > 1$. Let Γ be a quasi-Fuchsian subgroup of $\text{Isom}^+(H^3)$ which is isomorphic to $\pi_1(S)$. Then H^3/Γ is diffeomorphic to $\mathbb{R} \times S$ and $H^1_c(H^3/\Gamma; \mathbb{C}) = \mathbb{C}$. Thus there is a nonzero L^2 -harmonic 1-form α on H^3/Γ .

We show that $\Phi_1^{-1}(\pi^*\alpha)$ is a Γ -invariant exact 1-current supported on the limit set $\Lambda \subset S^2$. The domain of discontinuity $\Omega \subset S^2$ is the union of two 2-disks D_+ and D_- , with D_+/Γ and D_-/Γ homeomorphic to S . Let $\chi_{D_+} \in L^2(S^2)$ be the characteristic function of D_+ . We show that $\Phi_1^{-1}(\pi^*\alpha)$ is proportionate to the exact 1-current $d\chi_{D_+}$ on S^2 .

Let $Z : D^2 \rightarrow D_+$ be a uniformization of D_+ . By Carathéodory's theorem, Z extends to a continuous homeomorphism $\bar{Z} : \bar{D}^2 \rightarrow \bar{D}_+$. The restriction of \bar{Z} to $\partial\bar{D}^2$ gives a homeomorphism $\partial\bar{Z} : S^1 \rightarrow \Lambda$.

The 1-current $d\chi_{D_+}$ defines a cyclic 1-cocycle τ on the algebra $C^1(S^2)$ by

$$\tau(F^0, F^1) = \int_{S^2} d\chi_{D_+} \wedge F^0 dF^1. \tag{1.7}$$

Lemma 1. *The function space $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ is a Banach algebra with the norm*

$$\|f\| = \left(\int_{\mathbb{R}^+} \int_{S^1} \frac{|f(\theta + h) - f(\theta)|^2}{h^2} d\theta dh \right)^{\frac{1}{2}} + \|f\|_\infty. \tag{1.8}$$

Given $f^0, f^1 \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$, let

$$f^i(\theta) = \sum_{j \in \mathbb{Z}} c_j^i e^{\sqrt{-1}j\theta} \tag{1.9}$$

be the Fourier expansion. Define a bilinear function

$$\bar{\tau} : \left(\mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \times \left(\mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \rightarrow \mathbb{C} \tag{1.10}$$

by

$$\bar{\tau}(f^0, f^1) = -2\pi i \sum_{j \in \mathbb{Z}} j c_j^0 c_{-j}^1. \tag{1.11}$$

Then $\bar{\tau}$ is a continuous cyclic 1-cocycle on $\mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$.

We relate the function-theoretic 1-cocycle $\bar{\tau}$ to the 1-cocycle τ .

Theorem 6. *Given $F^0, F^1 \in C^1(S^2)$, put $f^i = (\partial\bar{Z})^* F^i$, $i \in \{1, 2\}$. Then $f^i \in \mathbb{H}^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ and*

$$\tau(F^0, F^1) = -\bar{\tau}(f^0, f^1). \tag{1.12}$$

In Subsection 5.2 we give examples of discrete subgroups Γ of $\text{Isom}^+(H^3)$ with limit set S^2 such that for all $\epsilon > 0$, the Γ -invariant subspace of $H^{-\epsilon}(S^2)/\mathbb{C}$ is infinite-dimensional. This contrasts with the fact that from ergodicity, the Γ -invariant subspace of $L^2(S^2)/\mathbb{C}$ vanishes.

Let us remark that our results could be extended to eigenfunctions of Δ_p with nonzero eigenvalue. In this paper we only deal with L^2 -harmonic forms since the dimension of the space of such forms can often be computed in terms of topological data, such as when M is a geometrically-finite hyperbolic manifold [12].

2. Regularity

Let p be an integer in $[1, \frac{n}{2}]$. Take coordinates $(r, \theta) \in (0, 1) \times S^{n-1}$ for $H^n - \{0\}$, with metric

$$ds^2 = \frac{4(dr^2 + r^2 d\theta^2)}{(1 - r^2)^2}. \tag{2.1}$$

For $k \geq 0$, consider the hypergeometric function

$$F_{p,k}(z) = F\left(1 + p - \frac{n}{2}, 1 + p + k; 1 + \frac{n}{2} + k; z\right). \tag{2.2}$$

Put

$$c_{p,k} = \frac{2^{p+1} \Gamma(n - p + k) \Gamma(\frac{n}{2} + 1)}{n \Gamma(n - p) \Gamma(\frac{n}{2} + k + 1)} = \frac{2^{p+1} (n - p)(n - p + 1) \dots (n - p + k - 1)}{n (\frac{n}{2} + 1)(\frac{n}{2} + 2) \dots (\frac{n}{2} + k)}. \tag{2.3}$$

Let $\{\alpha_i\}_{i=1}^\infty$ be a sequence of coclosed $(p - 1)$ -forms on S^{n-1} such that

1. α_i is an eigenvector for the Laplacian with eigenvalue $(k_i + p)(k_i + n - p)$, $k_i \in \mathbb{Z} \cap [0, \infty)$.
2. $\{d\alpha_i\}_{i=1}^\infty$ is an orthonormal basis of the exact p -forms on S^{n-1} .

Then

$$\|\alpha_i\|_{L^2}^2 = \frac{1}{(k_i + p)(k_i + n - p)}. \tag{2.4}$$

Given an exact p -hyperform ω on S^{n-1} , let

$$\omega = \sum_{i=1}^\infty c_i d\alpha_i \tag{2.5}$$

be its Fourier expansion. Gaillard [6, p. 599] showed that the Poisson transform of ω is

$$\begin{aligned} \Phi_p(\omega) = \sum_{i=1}^\infty c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p,k_i} r^{p-1+k_i} \\ \left[\frac{r}{k_i + p} F_{p-1,k_i}(r^2) d\alpha_i + (1 - r^2) F_{p,k_i}(r^2) dr \wedge \alpha_i \right]. \end{aligned} \tag{2.6}$$

Put $S^{n-1}(r) = \{(r, \theta) : \theta \in S^{n-1}\} \subset H^n$. Given $\eta \in \Omega^{p-1}(S^{n-1})$, we can think of $d\eta$ and $dr \wedge \eta$ as p -forms on $H^n - \{0\}$. Their pointwise norms on $S^{n-1}(r)$ are

$$|d\eta|_{S^{n-1}(r)} = \left(\frac{1 - r^2}{2r}\right)^p |d\eta|_{S^{n-1}} \tag{2.7}$$

and

$$|dr \wedge \eta|_{S^{n-1}(r)} = \frac{1 - r^2}{2} \left(\frac{1 - r^2}{2r}\right)^{p-1} |\eta|_{S^{n-1}}. \tag{2.8}$$

Theorem 1. *If n is even then up to a constant, $\Phi_{\frac{n}{2}}$ is an isometric isomorphism between exact $\frac{n}{2}$ -forms on S^{n-1} which are Sobolev $H^{-\frac{1}{2}}$ -regular, and L^2 -harmonic $\frac{n}{2}$ -forms on H^n .*

Proof. We have

$$F_{\frac{n}{2},k}(z) = F\left(1, 1 + \frac{n}{2} + k; 1 + \frac{n}{2} + k; z\right) = (1 - z)^{-1}, \tag{2.9}$$

$$F_{\frac{n}{2}-1,k}(z) = F\left(0, \frac{n}{2} + k; 1 + \frac{n}{2} + k; z\right) = 1 \tag{2.10}$$

and

$$c_{\frac{n}{2},k} = \frac{2^{\frac{n}{2}}}{k + \frac{n}{2}}. \tag{2.11}$$

Then

$$\Phi_{\frac{n}{2}}(\omega) = \sum_{i=1}^{\infty} c_i 2^{\frac{n}{2}-1} \left[r^{\frac{n}{2}+k_i} d\alpha_i + \left(k_i + \frac{n}{2}\right) r^{\frac{n}{2}+k_i-1} dr \wedge \alpha_i \right]. \tag{2.12}$$

Thus

$$\begin{aligned} \int_{H^n} |\Phi_{\frac{n}{2}}(\omega)|^2 d\text{vol} &= \sum_{i=1}^{\infty} |c_i|^2 2^{n-2} \text{vol}(S^{n-1}) \int_0^1 \left[r^{n+2k_i} \left(\frac{1-r^2}{2r}\right)^n + \right. \\ &\quad \left. r^{n+2k_i-2} \left(\frac{1-r^2}{2}\right)^2 \left(\frac{1-r^2}{2r}\right)^{n-2} \right] \left(\frac{2r}{1-r^2}\right)^{n-1} \frac{2}{1-r^2} dr \\ &= \sum_{i=1}^{\infty} |c_i|^2 2^{n-1} \text{vol}(S^{n-1}) \int_0^1 r^{2k_i+n-1} dr \\ &= 2^{n-2} \text{vol}(S^{n-1}) \sum_{i=1}^{\infty} \frac{1}{k_i + \frac{n}{2}} |c_i|^2. \end{aligned} \tag{2.13}$$

The theorem follows. □

Corollary 1. *Suppose that α is a nonzero L^2 -harmonic $\frac{n}{2}$ -form on H^n/Γ . If Γ is infinite then $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$ is not Sobolev $H^{-\frac{1}{2}}$ -regular.*

Proof. If $\Phi_{\frac{n}{2}}^{-1}(\pi^*\alpha)$ were Sobolev $H^{-\frac{1}{2}}$ -regular then Theorem 1 would imply that $\pi^*\alpha$ is L^2 , contradicting the assumption that Γ is infinite. □

The following is the main technical result of the paper.

Theorem 7. *If ω is an exact p -hyperform on S^{n-1} and if $\Phi_p(\omega)$ is L^∞ -bounded on H^n then ω is Sobolev $H^{-p-\epsilon}$ -regular for all $\epsilon > 0$.*

Proof. By the assumptions, $\frac{1}{\text{vol}(S^{n-1}(r))} \int_{S^{n-1}(r)} |\Phi_p(\omega)|^2 d\text{vol}$ is uniformly bounded in $r \in (0, 1)$. Thus for $\epsilon > 0$,

$$\int_0^1 r(1-r^2)^{-1+2\epsilon} \frac{1}{\text{vol}(S^{n-1}(r))} \int_{S^{n-1}(r)} |\Phi_p(\omega)|^2 d\text{vol} dr < \infty. \tag{2.15}$$

In particular, just looking at the $dr \wedge \alpha$ component of $\Phi_p(\omega)$ in (2.6) gives

$$\sum_{i=1}^{\infty} (k_i + p)^2 (k_i + n - p)^2 c_{p,k_i}^2 |c_i|^2 \tag{2.16}$$

$$\int_0^1 r(1-r^2)^{-1+2\epsilon} r^{2p-2+2k_i} (1-r^2)^2 F_{p,k_i}^2(r^2) \left(\frac{1-r^2}{2}\right)^2 \left(\frac{1-r^2}{2r}\right)^{2p-2} \frac{1}{(k_i+p)(k_i+n-p)} dr < \infty, \quad (2.17)$$

or

$$\sum_{i=1}^{\infty} (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \int_0^1 z^{k_i} (1-z)^{2p+1+2\epsilon} F_{p,k_i}^2(z) dz < \infty. \quad (2.18)$$

For the regularity question, it is the regime of large k_i and z near 1 which is relevant. Thus our main problem is to derive uniform estimates for $F_{p,k_i}^2(z)$, for large k_i and z near 1.

Substituting $z = \frac{w-1}{w+1}$ gives

$$\sum_{i=1}^{\infty} (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \int_1^{\infty} (w-1)^{k_i} (w+1)^{-2p-k_i-3-2\epsilon} F_{p,k_i}^2\left(\frac{w-1}{w+1}\right) dw < \infty. \quad (2.19)$$

Restricting the summation to $k_i > 0$, the further substitution $w = k_i x$ gives

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 k_i^{-2p-2-2\epsilon} \int_{k_i^{-1}}^{\infty} x^{-2p-3-2\epsilon} \left(1 - \frac{1}{k_i x}\right)^{k_i} \left(1 + \frac{1}{k_i x}\right)^{-2p-k_i-3-2\epsilon} F_{p,k_i}^2\left(\frac{k_i x - 1}{k_i x + 1}\right) dx < \infty. \quad (2.20)$$

In order to estimate F_{p,k_i} , we use the transformation [1, 15.3.4]

$$\begin{aligned} F_{p,k}(z) &= F\left(1+p - \frac{n}{2}, 1+p+k; 1+\frac{n}{2}+k; z\right) \\ &= (1-z)^{\frac{n}{2}-p-1} F\left(1+p - \frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{z}{z-1}\right). \end{aligned} \quad (2.21)$$

Then

$$F_{p,k}\left(\frac{w-1}{w+1}\right) = \left(\frac{2}{w+1}\right)^{\frac{n}{2}-p-1} F\left(1+p - \frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{1}{2} - \frac{w}{2}\right). \quad (2.22)$$

From [11, (4) p. 246 and (15) p. 248],

$$P_{\frac{n}{2}-p-1}^{-\frac{n}{2}-k}(w) = \frac{1}{\Gamma\left(1+\frac{n}{2}+k\right)} \left(\frac{w+1}{w-1}\right)^{-\frac{n}{2}-\frac{k}{2}} F\left(1+p - \frac{n}{2}, \frac{n}{2}-p; 1+\frac{n}{2}+k; \frac{1}{2} - \frac{w}{2}\right) \quad (2.23)$$

and

$$\int_0^\infty e^{-wt} t^{\frac{n}{2}+k-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(k+p+1)\Gamma(n+k-p)}{(w^2-1)^{\frac{n+k}{4}+\frac{k}{2}}} P_{\frac{n}{2}-p-1}^{-\frac{n}{2}-k}(w). \tag{2.24}$$

We obtain

$$F_{p,k}\left(\frac{w-1}{w+1}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^{\frac{n}{2}-p-1} \frac{\Gamma(1+\frac{n}{2}+k)}{\Gamma(k+p+1)\Gamma(n+k-p)} (w+1)^{k+p+1} \int_0^\infty e^{-wt} t^{\frac{n}{2}+k-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt, \tag{2.25}$$

so

$$F_{p,k_i}\left(\frac{k_i x - 1}{k_i x + 1}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} 2^{\frac{n}{2}-p-1} \frac{k_i^{k_i+p+1} \Gamma(1+\frac{n}{2}+k_i)}{\Gamma(k_i+p+1)\Gamma(n+k_i-p)} x^{k_i+p+1} \left(1+\frac{1}{k_i x}\right)^{k_i+p+1} \int_0^\infty e^{-k_i x t} t^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) dt. \tag{2.26}$$

(Recall that for large t [1, 9.7.2 and 10.2.17],

$$K_{\frac{n}{2}-p-\frac{1}{2}}(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t}. \tag{2.27}$$

Then from (2.20),

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i-2\epsilon} \Gamma^2(1+\frac{n}{2}+k_i)}{\Gamma^2(k_i+p+1)\Gamma^2(n+k_i-p)} \int_{k_i^{-1}}^\infty x^{2k_i-1-2\epsilon} \int_0^\infty \int_0^\infty \left(1-\frac{1}{k_i x}\right)^{k_i} \left(1+\frac{1}{k_i x}\right)^{k_i-1-2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) K_{\frac{n}{2}-p-\frac{1}{2}}(t') dt dt' dx < \infty, \tag{2.28}$$

or

$$\sum_i (k_i+p)(k_i+n-p) c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i-2\epsilon} \Gamma^2(1+\frac{n}{2}+k_i)}{\Gamma^2(k_i+p+1)\Gamma^2(n+k_i-p)} \int_{k_i^{-1}}^\infty x^{2k_i-1-2\epsilon} \int_0^\infty \int_0^\infty \left(1-\frac{1}{k_i^2 x^2}\right)^{k_i} \left(1+\frac{1}{k_i x}\right)^{-1-2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2}+k_i-\frac{1}{2}} K_{\frac{n}{2}-p-\frac{1}{2}}(t) K_{\frac{n}{2}-p-\frac{1}{2}}(t') dt dt' dx < \infty. \tag{2.29}$$

Formally taking k_i large, we obtain

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{k_i^{2k_i - 2\epsilon} \Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} \int_0^\infty \int_0^\infty \int_0^\infty x^{2k_i - 1 - 2\epsilon} e^{-k_i x(t+t')} (tt')^{\frac{n}{2} + k_i - \frac{1}{2}} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dx dt dt' < \infty, \tag{2.30}$$

or

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} \int_0^\infty \int_0^\infty (t + t')^{-2k_i + 2\epsilon} (tt')^{\frac{n}{2} + k_i - \frac{1}{2}} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dt dt' < \infty. \tag{2.31}$$

That is,

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \int_0^\infty \int_0^\infty \left(\frac{t + t'}{2\sqrt{tt'}}\right)^{-2k_i} (tt')^{\frac{n}{2} - \frac{1}{2}} (t + t')^{2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(t) K_{\frac{n}{2} - p - \frac{1}{2}}(t') dt dt' < \infty. \tag{2.32}$$

Making the change of variables $t = e^u v$ and $t' = e^{-u} v$, we have

$$\sum_i (k_i + p)(k_i + n - p)c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \int_{-\infty}^\infty (\cosh u)^{-2k_i} (\cosh u)^{2\epsilon} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv du < \infty. \tag{2.33}$$

From [11, (8) p. 325],

$$\int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv = \frac{e^{2n+2\epsilon+1} e^{-(2n-2p+2\epsilon)u} \Gamma(n - p + \epsilon) \Gamma^2(\frac{1+n+2\epsilon}{2}) \Gamma(1 + p + \epsilon)}{8\Gamma(1 + n + 2\epsilon)} F(n - p + \epsilon, \frac{1 + n + 2\epsilon}{2}; 1 + n + 2\epsilon; 1 - e^{-4u}). \tag{2.34}$$

Using the asymptotics of the hypergeometric function from [1, 15.3.6], one finds that for large u ,

$$(\cosh u)^{2\epsilon} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2} - p - \frac{1}{2}}(e^u v) K_{\frac{n}{2} - p - \frac{1}{2}}(e^{-u} v) dv = O(e^{-2|u|}). \tag{2.35}$$

Thus we can apply steepest descent methods to (2.33) to obtain

$$\sum_i (k_i + p)(k_i + n - p) c_{p,k_i}^2 |c_i|^2 \frac{\Gamma(2k_i - 2\epsilon)\Gamma^2(1 + \frac{n}{2} + k_i)}{\Gamma^2(k_i + p + 1)\Gamma^2(n + k_i - p)} 4^{-k_i} \tag{2.36}$$

$$k_i^{-\frac{1}{2}} \int_0^\infty v^{n+2\epsilon} K_{\frac{n}{2}-p-\frac{1}{2}}^2(v) dv < \infty.$$

Using the asymptotics of the gamma function [1, 6.1.39], we find

$$\sum_i k_i^{-2p-2\epsilon} |c_i|^2 < \infty. \tag{2.37}$$

Recalling (2.5), this is equivalent to saying that ω is Sobolev $H^{-p-\epsilon}$ -regular.

To justify passing from (2.29) to (2.30), it is enough to note that if $x \geq k_i^{-1}$ then

$$\left(1 - \frac{1}{k_i^2 x^2}\right)^{k_i} \left(1 + \frac{1}{k_i x}\right)^{-1-2\epsilon} < 1. \tag{2.38}$$

Thus we have uniform bounds in the preceding arguments. □

Corollary 2. *Suppose that H^n/Γ has positive injectivity radius. Suppose that α is an L^2 -harmonic p -form on H^n/Γ , $p \in [1, \frac{n}{2}]$. Then $\Phi_p^{-1}(\pi^*\alpha)$ is Sobolev $H^{-p-\epsilon}$ -regular for all $\epsilon > 0$.*

Proof. By elliptic theory [4, Prop. 1.3], there is a constant $r > 0$ such that for all $m \in H^n/\Gamma$, $|\alpha(m)|$ is bounded in terms of the L^2 -norm of α on the ball $B_r(m) \subset H^n/\Gamma$. Then $\pi^*\alpha$ is uniformly bounded on H^n . The corollary follows from Theorem 7. □

Theorem 2.A. *If Γ is cocompact then for any $p \in [1, \frac{n}{2}]$, there are isomorphisms between the following vector spaces :*

$$V_1 = \{\text{Harmonic } p\text{-forms on } H^n/\Gamma\}.$$

$$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1}\}.$$

$$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$$

$$V_4 = H^p(H^n/\Gamma, \mathbb{R}), \text{ the } p\text{-dimensional real cohomology group of } H^n/\Gamma.$$

Proof. By standard Hodge theory, $V_1 \cong V_4$. In particular, V_1 is finite-dimensional. By Corollary 2, there is an injection $V_1 \rightarrow V_3$. There is an evident injection $V_3 \rightarrow V_2$. By Gaillard’s theorem [6, Théorème 2], if $\omega \in V_2$ then $\Phi_p(\omega)$ is a Γ -invariant closed and coclosed p -form on H^n . Hence $\Phi_p(\omega) = \pi^*\alpha$ for some closed and coclosed p -form α on H^n/Γ . Hence there is an injection $V_2 \rightarrow V_1$. The theorem follows. □

Corollary 3. *Suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on H^n/Γ . Suppose that all of the cusps of H^n/Γ have rank $n - 1$. If α is an L^2 -harmonic p -form on H^n/Γ , $p \in \{\frac{n-1}{2}, \frac{n}{2}\}$, then for all $\epsilon > 0$, the hyperform $\Phi_p^{-1}(\pi^*\alpha)$ is Sobolev $H^{-p-\epsilon}$ -regular.*

Proof. For some $\mu > 0$ less than the Margulis constant of H^n , the μ -thin part of H^n/Γ has a finite number of compact components. By the proof of Corollary 2, α is bounded on the μ -thick part of H^n/Γ . It follows from [12, Theorem 4.12] that α is bounded on the cusps of H^n/Γ . The corollary follows from Theorem 7. \square

Theorem 3. *In the case $n = 3$, suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on H^3/Γ . Let α be an L^2 -harmonic 1-form on H^3/Γ . Then for all $\epsilon > 0$, the hyperform $\Phi_1^{-1}(\pi^*\alpha)$ is Sobolev $H^{-1-\epsilon}$ -regular.*

Proof. Following the line of proof of Corollary 3, it suffices to analyze the asymptotics of an L^2 -harmonic 1-form ω on a rank-1 cusp. We can take a neighborhood of such a cusp to be the quotient of

$$\{(x, y, z) : y^2 + z^2 \geq R, z \geq 0\} \tag{2.39}$$

by the group generated by $x \rightarrow x + 2\pi$, for some $R > 0$. We follow the analysis of [12, Section 4], with care for constants. Make a change of coordinates to $y = r \cos \theta$, $z = r \sin \theta$, with $r \in [R, \infty)$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The Riemannian metric in these coordinates is

$$ds^2 = \frac{dx^2}{r^2 \cos^2 \theta} + \frac{dr^2}{r^2 \cos^2 \theta} + \frac{d\theta^2}{\cos^2 \theta}, \tag{2.40}$$

with volume form $d\text{vol} = \frac{dxdrd\theta}{r^2 \cos^3 \theta}$.

Let

$$\omega = \alpha_0 d\theta + \alpha_1 dx + \beta_0 dr \tag{2.41}$$

be an L^2 -harmonic 1-form on the cusp. Then

$$\int \left(r^{-2} |\alpha_0|^2 + |\alpha_1|^2 + |\beta_0|^2 \right) \frac{dxdrd\theta}{\cos \theta} < \infty. \tag{2.42}$$

The equations $d\omega = d^*\omega = 0$ become

$$\begin{aligned} 0 &= \partial_x \alpha_0 - \partial_\theta \alpha_1 = \partial_r \alpha_0 - \partial_\theta \beta_0 = \partial_r \alpha_1 - \partial_x \beta_0 \\ &= \cos \theta \partial_\theta \left(\frac{\alpha_0}{\cos \theta} \right) + r^2 \partial_x \alpha_1 + r^2 \partial_r \beta_0. \end{aligned} \tag{2.43}$$

From these equations, one obtains the Laplacian-type equations

$$\begin{aligned} -\partial_r^2 \alpha_0 - \partial_x^2 \alpha_0 - \frac{1}{r^2} \partial_\theta \left(\cos \theta \partial_\theta \left(\frac{\alpha_0}{\cos \theta} \right) \right) &= 0, \\ -\partial_r^2 \alpha_1 - \partial_x^2 \alpha_1 - \frac{1}{r^2} \cos \theta \partial_\theta \left(\frac{1}{\cos \theta} \partial_\theta \alpha_1 \right) &= 0, \\ -\partial_r^2 \beta_0 - \partial_x^2 \beta_0 - \frac{1}{r^2} \cos \theta \partial_\theta \left(\frac{1}{\cos \theta} \partial_\theta \beta_0 \right) &= -\frac{2}{r^3} \cos \theta \partial_\theta \left(\frac{\alpha_0}{\cos \theta} \right). \end{aligned} \quad (2.44)$$

We first analyze the second equation in (2.44). Given a function $f \in C^\infty(-\frac{\pi}{2}, \frac{\pi}{2})$, put

$$Lf = -\cos \theta \partial_\theta \left(\frac{1}{\cos \theta} \partial_\theta f \right) \quad (2.45)$$

Then L is the self-adjoint operator coming from the Dirichlet form on $L^2((-\frac{\pi}{2}, \frac{\pi}{2}), \frac{1}{\cos \theta} d\theta)$. Making the change of variable $u = \sin \theta$, the eigenfunction equation $Lf = \lambda f$ becomes

$$-(1-u^2)f''(u) = \lambda f. \quad (2.46)$$

The square-integrable solutions to this have $\lambda = (q+1)(q+2)$ with $q \in \mathbb{Z} \cap [0, \infty)$. The corresponding eigenfunction is given in terms of ultraspherical polynomials [1, 22.6.6] by

$$f_q(u) = (1-u^2)C_q^{3/2}(u). \quad (2.47)$$

Explicitly, $f_q(u)$ is proportionate to $\frac{d^q}{du^q}((1-u^2)^{q+1})$.

Performing separation of variables on the second equation in (2.44), suppose that

$$\alpha_1(x, r, \theta) = e^{imx} g(r) f_q(\theta), \quad (2.48)$$

with $m \in \mathbb{Z}$. Then

$$-g'' + m^2 g + \frac{(q+1)(q+2)}{r^2} g = 0. \quad (2.49)$$

If $m \neq 0$ then g decreases exponentially fast in r . Suppose that $m = 0$. One finds that for large r , $g(r) \sim r^{q+2}$ or $g(r) \sim r^{-q-1}$. For ω to be square-integrable, one must have $g(r) \sim r^{-q-1}$. If $q > 0$ then $|\alpha_1 dx| = r \cos \theta |g(r)| |f_q(\theta)|$ decays polynomially fast in r . In the critical case $q = 0$, $|\alpha_1 dx|$ remains bounded in r .

Next, put

$$L'f = -\partial_\theta \left(\cos \theta \partial_\theta \left(\frac{f}{\cos \theta} \right) \right). \quad (2.50)$$

and $\widehat{L} = \frac{1}{\cos \theta} \circ L' \circ \cos \theta$. Then \widehat{L} is the self-adjoint operator coming from the Dirichlet form on $L^2((-\frac{\pi}{2}, \frac{\pi}{2}), \cos \theta d\theta)$. It has a nonnegative discrete spectrum starting at 0, and hence so does L' . Suppose that $f(\theta)$ is an eigenfunction of L'

with eigenvalue $\lambda \geq 0$. Performing separation of variables on the first equation in (2.44), suppose that

$$\alpha_0(x, r, \theta) = e^{imx}g(r)f(\theta), \tag{2.51}$$

with $m \in \mathbb{Z}$. Then

$$-g'' + m^2g + \frac{\lambda}{r^2}g = 0. \tag{2.52}$$

If $m \neq 0$ then g decreases exponentially fast in r . Suppose that $m = 0$. One finds that for large r , $g(r) \sim r^{\frac{1+\sqrt{1+4\lambda}}{2}}$. For ω to be square-integrable, one must have $g(r) \sim r^{\frac{1-\sqrt{1+4\lambda}}{2}}$. If $\lambda > 0$ then $|\alpha_0 d\theta| = \cos \theta |g(r)| |f(\theta)|$ decays like a power in r . In the critical case $\lambda = 0$, $|\alpha_0 d\theta|$ remains bounded in r .

Finally, one can analyze the third equation in (2.44), an inhomogeneous equation, by similar methods. The upshot is that $|\omega|$ is bounded on the rank-1 cusp. \square

Proposition 1. *Suppose that there is a positive lower bound to the lengths of the nontrivial closed geodesics on H^n/Γ . Let α be an L^2 -harmonic p -form on H^n/Γ , $p \in [1, \frac{n}{2}]$. Then $\Phi_p^{-1}(\pi^*\alpha)$ is a current.*

Proof. For some $\mu > 0$ less than the Margulis constant of H^n , the μ -thin part of H^n/Γ has a finite number of compact components. As in the proof of Corollary 2, there is a uniform upper bound for $|\alpha|$ on the μ -thick part of H^n/Γ . On each cuspidal component of the μ -thin part, $|\alpha|$ has at most exponential growth, with a uniform exponential constant [12, Section 4]. The result follows from [6, Théorème 3]. \square

Proposition 2. *For $r \in (0, 1)$, let $i_r : S^{n-1} \rightarrow S^{n-1}(r)$ be the embedding of S^{n-1} as the r -sphere around 0 in the ball model of H^n . As in [6, p. 586], put*

$$C_p = \frac{2^p \Gamma(n - 2p + 1) \Gamma(\frac{n}{2} + 1)}{n \Gamma(n - p) \Gamma(\frac{n}{2} - p + 1)}. \tag{2.53}$$

Let ω be an exact p -current on S^{n-1} . Then as $r \rightarrow 1$, the forms $i_r^ \Phi_p(\omega)$ converge to $C_p \omega$ in the sense of convergence of currents.*

Proof. From (2.6),

$$i_r^* \Phi_p(\omega) = \sum_{i=1}^{\infty} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} r^{p-1+k_i} \frac{r}{k_i + p} F_{p-1, k_i}(r^2) d\alpha_i. \tag{2.54}$$

Given a smooth form $\eta \in \Omega^p(S^{n-1})$, let $\Pi(\eta)$ be the projection of η onto the square-integrable exact p -forms on S^{n-1} . Then $\Pi(\eta)$ is also smooth and has a Fourier expansion

$$\Pi(\eta) = \sum_{i=1}^{\infty} a_i d\alpha_i, \tag{2.55}$$

with $\sum_{i=1}^\infty k_i^N |a_i|^2 < \infty$ for all $N \in \mathbb{Z}^+$. The pairing

$$\langle i_r^* \Phi_p(\omega), \eta \rangle = \int_{S^{n-1}} i_r^* \Phi_p(\omega) \wedge \overline{* \eta} \tag{2.56}$$

is given by

$$\langle i_r^* \Phi_p(\omega), \eta \rangle = \sum_{i=1}^\infty \overline{a_i} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} r^{p-1+k_i} \frac{r}{k_i + p} F_{p-1, k_i}(r^2). \tag{2.57}$$

Then

$$\begin{aligned} \langle i_1^* \Phi_p(\omega), \eta \rangle &= \sum_{i=1}^\infty \overline{a_i} c_i \frac{(k_i + p)(k_i + n - p)}{2} c_{p, k_i} \frac{1}{k_i + p} \\ &\quad \frac{\Gamma(1 + \frac{n}{2} + k_i) \Gamma(1 - 2p + n)}{\Gamma(1 - p + n + k_i) \Gamma(1 - p + \frac{n}{2})} \\ &= C_p \sum_{i=1}^\infty \overline{a_i} c_i \\ &= C_p \langle \omega, \eta \rangle. \end{aligned} \tag{2.58}$$

As ω is a current, $\sum_{i=1}^\infty k_i^N |a_i| |c_i| < \infty$ for all $N \in \mathbb{Z}^+$.

Lemma 2. *As r increases from 0 to 1, the expression $r^{p-1+k_i} \frac{r}{k_i+p} F_{p-1, k_i}(r^2)$ increases monotonically from 0 to $\frac{1}{k_i+p} \frac{\Gamma(1+\frac{n}{2}+k_i)\Gamma(1-2p+n)}{\Gamma(1-p+n+k_i)\Gamma(1-p+\frac{n}{2})}$.*

Proof. The fact that the right-hand-side of (2.6) is closed implies that

$$\frac{d}{dr} \left(r^{p-1+k_i} \frac{r}{k_i + p} F_{p-1, k_i}(r^2) \right) = r^{p-1+k_i} (1 - r^2) F_{p, k_i}(r^2). \tag{2.59}$$

(Of course, this can be checked directly.) From [1, 15.3.3],

$$\begin{aligned} F_{p, k_i}(r^2) &= F\left(1 + p - \frac{n}{2}, 1 + p + k_i; 1 + \frac{n}{2} + k_i; r^2\right) \\ &= (1 - r^2)^{n-1-2p} F\left(n + k_i - p, \frac{n}{2} - p; 1 + \frac{n}{2} + k_i; r^2\right). \end{aligned} \tag{2.20}$$

As the arguments of $F(n + k_i - p, \frac{n}{2} - p; 1 + \frac{n}{2} + k_i; r^2)$ are all nonnegative, the lemma follows. □

Proposition 2 now follows from dominated convergence. □

Proposition 3. *Suppose that α is an L^2 -harmonic p -form on H^n/Γ , $p \in [1, \frac{n}{2})$. Suppose that $\Phi_p^{-1}(\pi^*\alpha)$ is a current. Then $\Phi_p^{-1}(\pi^*\alpha)$ is supported on the limit set Λ of Γ .*

Proof. Given a smooth form $\phi \in \Omega^p(S^{n-1})$ with relatively compact support in $S^{n-1} - \Lambda$, Proposition 2 implies that

$$\lim_{r \rightarrow 1} \langle i_r^* \pi^* \alpha, \phi \rangle = C_p \langle \Phi_p^{-1}(\pi^* \alpha), \phi \rangle. \tag{2.61}$$

If $\Lambda = \emptyset$, we assume that $\text{supp}(\phi) \neq S^{n-1}$; this is sufficient for the argument. Then we can use an upper-half-space model for H^n , with $\text{supp}(\phi) \subset \mathbb{R}^{n-1}$. Put $V = \text{supp}(\phi) \times (0, \infty) \subset H^n$. Using the coordinates (x_1, \dots, x_{n-1}, y) for H^n , let us write $\tilde{\alpha} = a(x, y) + dy \wedge b(x, y)$. Then [12, Theorem 4.3] states that on V , as $y \rightarrow 0$,

$$a = \begin{cases} a_{00}(x)y^{n-2p-1} + O(y^{n-2p} \log(y)) & \text{if } p < \frac{n-1}{2}, \\ a_{01}(x)y^2 \log(y) + O(y^2) & \text{if } p = \frac{n-1}{2} \end{cases} \tag{2.62}$$

and

$$b = \begin{cases} b_{01}(x)y^{n-2p} \log(y) + O(y^{n-2p}) & \text{if } p < \frac{n-1}{2}, \\ b_{00}(x)y + O(y^2 \log(y)) & \text{if } p = \frac{n-1}{2}. \end{cases} \tag{2.63}$$

(The statement of [12, Theorem 4.3] should read “ $y \rightarrow 0$ ”.) As $r \rightarrow 1$, the intersections $S^{n-1}(r) \cap V$ asymptotically approach the horosphere pieces

$$\{(x_1, \dots, x_{n-1}, y) \in H^n : y = \frac{1-r}{1+r}\} \cap V. \tag{2.64}$$

It follows that $\langle \Phi_p^{-1}(\pi^*\alpha), \phi \rangle = 0$ for all such ϕ , from which the proposition follows. \square

Remark. The analog of Proposition 3 is false if $p = \frac{n}{2}$. This can be seen in the case $\Gamma = \{e\}$ using Theorem 1.

We give a partial converse to Proposition 3, in the case of convex-cocompact groups.

Theorem 2.B. *If Γ is convex-cocompact then for any $p \in [1, \frac{n-1}{2})$, there are isomorphisms between the following vector spaces :*

$$V_1 = \{L^2\text{-harmonic } p\text{-forms on } H^n/\Gamma\}.$$

$$V_2 = \{\Gamma\text{-invariant exact } p\text{-hyperforms on } S^{n-1} \text{ which are supported on the limit set}\}.$$

$$V_3 = \{\Gamma\text{-invariant exact } p\text{-currents on } S^{n-1} \text{ which are supported on the limit set and which are Sobolev } H^{-p-\epsilon}\text{-regular for all } \epsilon > 0\}.$$

$$V_4 = H_c^p(H^n/\Gamma, \mathbb{R}), \text{ the } p\text{-dimensional real compactly-supported cohomology group of } H^n/\Gamma.$$

Proof. By [12], $V_1 \cong V_4$. In particular, V_1 is finite-dimensional. By Gaillard’s theorem [6, Théorème 2], Corollary 2 and Proposition 3, there are injections $V_1 \rightarrow V_3 \rightarrow V_2$. It remains to show that there is an injection $V_2 \rightarrow V_1$. In view of Gaillard’s theorem, it suffices to show that if $\omega \in V_2$ then $\Phi_p(\omega)$ descends to a form which is square-integrable on H^n/Γ . If Γ is cocompact then this is automatic, so assume that Γ is not cocompact. As Ω/Γ is compact, we can find a fundamental domain F for the action of Γ on H^n such that $\bar{F} \cap S^{n-1}$ is disjoint from Λ . Take an upper-half-space model for H^n with $\infty \in \Omega$. In terms of the upper-half-space coordinates (x_1, \dots, x_{n-1}, y) , [6, Lemme 3] implies that near $y = 0$,

$$\Phi_p(\omega)|_F = y^{n-2p-1}\phi(x, y), \tag{2.65}$$

where the p -form $\phi(x, y)$ is continuous up to $y = 0$. It follows that $\int_F |\Phi_p(\omega)|^2 d\text{vol} < \infty$. \square

3. 1-Forms

In this section we look in more detail at the case of L^2 -harmonic 1-forms on convex-cocompact hyperbolic manifolds. If the hyperbolic manifold is compact, we show that the Sobolev regularity estimate of Theorem 2.A is sharp. If the hyperbolic manifold is convex-cocompact but not compact, we show how to construct its L^2 -harmonic 1-forms explicitly in terms of the harmonic extension of functions. In this case, we show that the Sobolev regularity estimate of Corollary 2 can be slightly improved.

Proposition 4. *Suppose that Γ is cocompact. For $\epsilon > 0$, let V_ϵ^Γ be the Γ -invariant subspace of the function space $H^{-\epsilon}(S^{n-1})/\mathbb{C}$. Then V_ϵ^Γ is isomorphic to $H^1(\Gamma; \mathbb{C})$.*

Proof. We first define linear maps $I : H^1(\Gamma; \mathbb{C}) \rightarrow V_\epsilon^\Gamma$ and $J : V_\epsilon^\Gamma \rightarrow H^1(\Gamma; \mathbb{C})$. To define I , given $x \in H^1(\Gamma; \mathbb{C}) = H^1(H^n/\Gamma; \mathbb{C})$, let $\alpha \in \Omega^1(H^n/\Gamma)$ be the harmonic 1-form which represents x . Put $\tilde{\alpha} = \pi^*\alpha$. By Theorem 7, $\Phi_1^{-1}(\tilde{\alpha})$ is an exact $H^{-1-\epsilon}$ -regular Γ -invariant 1-form on S^{n-1} . Choose $f \in H^{-\epsilon}(S^{n-1})$ so that $\Phi_1^{-1}(\tilde{\alpha}) = df$. Then for all $\gamma \in \Gamma$,

$$d(f - \gamma \cdot f) = df - \gamma \cdot df = 0. \tag{3.1}$$

Thus

$$f - \gamma \cdot f = c(\gamma) \tag{3.2}$$

for some $c(\gamma) \in \mathbb{C}$. Put $I(x) = f \pmod{\mathbb{C}}$.

To define J , given $\bar{f} \in V_\epsilon^\Gamma$, let $f \in H^{-\epsilon}(S^{n-1})$ be a representative of \bar{f} , not necessarily Γ -invariant. As \bar{f} is Γ -invariant, for each $\gamma \in \Gamma$ there is a $c(\gamma) \in \mathbb{C}$ such that $f - \gamma \cdot f = c(\gamma)$. As

$$c(\gamma_1\gamma_2) = f - (\gamma_1\gamma_2) \cdot f = (f - \gamma_1 \cdot f) + \gamma_1 \cdot (f - \gamma_2 \cdot f) = c(\gamma_1) + \gamma_1 \cdot c(\gamma_2) = c(\gamma_1) + c(\gamma_2), \tag{3.3}$$

we have a cocycle $c : \Gamma \rightarrow \mathbb{C}$. Put $J(\bar{f}) = [c]$.

We show that $J \circ I$ is the identity. It suffices to show that the cocycle c of (3.2) represents $x \in H^1(\Gamma; \mathbb{C})$. For this, it suffices to show that for all $\gamma \in \Gamma$,

$$c(\gamma) = \int_{C_\gamma} \alpha, \tag{3.4}$$

where C_γ is a closed curve on H^n/Γ in the homotopy class of $\gamma \in \pi_1(H^n/\Gamma)$ and $\alpha \in \Omega^1(H^n/\Gamma)$ is the harmonic representative of x . Let \tilde{C}_γ be a lift of C_γ to H^n , ending at a point $m \in H^n$ and starting at $\gamma^{-1} \cdot m$. Then

$$\begin{aligned} \int_{C_\gamma} \alpha &= \int_{\tilde{C}_\gamma} \tilde{\alpha} = \int_{\tilde{C}_\gamma} \Phi_1(df) = \int_{\tilde{C}_\gamma} d\Phi_0(f) = (\Phi_0(f))(m) - (\Phi_0(f))(\gamma^{-1} \cdot m) \\ &= (\Phi_0(f) - \gamma \cdot \Phi_0(f))(m) = (\Phi_0(f - \gamma \cdot f))(m) = (\Phi_0(c(\gamma)))(m) = c(\gamma). \end{aligned} \tag{3.5}$$

This shows that $J \circ I$ is the identity. To see that $I \circ J$ is the identity, given $\bar{f} \in V_\epsilon^\Gamma$, let $f \in H^{-\epsilon}(S^{n-1})$ be a representative of \bar{f} , not necessarily Γ -invariant. Define $\tilde{\alpha} = \Phi_1(df)$. Then $\tilde{\alpha}$ is a smooth Γ -invariant harmonic 1-form on H^n and projects to a harmonic 1-form $\alpha \in \Omega^1(H^n/\Gamma)$. By the same sort of calculation as in (3.5), one finds that $J(\bar{f}) = [\alpha]$ in $H^1(\Gamma; \mathbb{C})$. By construction, $I([\alpha]) = \bar{f}$. Thus $I \circ J$ is the identity. \square

Theorem 4.A. *Suppose that Γ is cocompact. Let α be a nonzero harmonic 1-form on H^n/Γ . Then $\Phi_1^{-1}(\pi^*\alpha)$ is not Sobolev H^{-1} -regular.*

Proof. Suppose that $\Phi_1^{-1}(\pi^*\alpha)$ is Sobolev H^{-1} -regular. Then $\Phi_1^{-1}(\pi^*\alpha) = df$ for some $f \in L^2(S^{n-1})$. Extending the proof of Proposition 4 to the case $\epsilon = 0$, the equivalence class \bar{f} of f in $L^2(S^{n-1})/\mathbb{C}$ is Γ -invariant and satisfies $J(\bar{f}) = [\alpha]$. As Γ acts ergodically on S^{n-1} , we must have $\bar{f} = 0$ and hence $[\alpha]$ vanishes in $H^1(H^n/\Gamma; \mathbb{C})$, which is a contradiction. \square

We now consider groups Γ which are convex-cocompact but not compact. First, we prove some generalities about the relationship between compactly-supported cohomology and L^2 -cohomology.

Let M be a complete connected oriented Riemannian manifold. Let $H_{(2)}^p(M)$ be the p -th (reduced) L^2 -cohomology group of M . It is isomorphic to $\text{Ker}(\Delta_p)$. There is a map $i : H_c^p(M; \mathbb{C}) \rightarrow H_{(2)}^p(M)$. In general, i is not injective; think of $M = \mathbb{R}^n$. However, it is true, and well-known, that i always induces an injection of $\text{Im}(H_c^p(M; \mathbb{C}) \rightarrow H^p(M; \mathbb{C}))$ into $H_{(2)}^p(M)$ [9, Prop. 4]. The next result gives a sufficient condition for i to be injective on all of $H_c^1(M; \mathbb{C})$. Recall that there is a notion of the space of ends of M , and of an end being contained in an open set $U \subset M$; see, for example, [3, §1.2].

Proposition 5. *Suppose that for every end e of M , every open set U containing e has infinite volume. Suppose that M has a Green's operator $G : C_0^\infty(M) \rightarrow L^2(M)$ such that $\Delta \circ G = \text{Id}$. Then $i : H_c^1(M; \mathbb{C}) \rightarrow H_{(2)}^1(M)$ is injective.*

Proof. We have the decomposition

$$H_c^1(M; \mathbb{C}) = \left(\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \oplus \left(\text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right). \tag{3.6}$$

We first show that i is injective on $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$. A representative of $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$ is a closed compactly-supported 1-form α such that $\alpha = df$ for some function f . By construction, f is locally constant outside of a compact subset of M and so gives a function on the space of ends of M . Now $d(f - G\Delta f)$ is a harmonic 1-form on M . As

$$\langle dG\Delta f, dG\Delta f \rangle = \langle G\Delta f, \Delta f \rangle, \tag{3.7}$$

we have that $d(f - G\Delta f)$ is square-integrable. The map $\alpha \rightarrow d(f - G\Delta f)$ describes i on $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$. To see that it is injective, suppose that $d(f - G\Delta f) = 0$. Then $f - G\Delta f$ is constant. As $G\Delta f \in L^2(M)$, the volume assumption implies that f , as a function on the space of ends of M , is a constant c . Then $f - c$ is compactly-supported on M , with $d(f - c) = \alpha$, so $[\alpha] = 0$ in $H_c^1(M; \mathbb{C})$. In summary, we have realized an injection of $\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$ into $H_{(2)}^1(M)$.

It remains to show that

$$i \left(\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \cap i \left(\text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) = 0. \tag{3.8}$$

Suppose that $d(f - G\Delta f)$ is nonzero and lies in the image, under i , of $\text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C}))$. Then $d(f - G\Delta f) = \omega \pmod{\overline{\text{Im}(d)}}$ for some closed compactly-supported 1-form ω . Furthermore, by assumption, there is a closed compactly-supported $(\dim(M) - 1)$ -form η such that $\int_M \omega \wedge \eta = 1$. However, $\int_M d(f - G\Delta f) \wedge \eta = 0$. It follows that

$$i \left(\text{Ker}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) \cap i \left(\text{Im}(H_c^1(M; \mathbb{C}) \rightarrow H^1(M; \mathbb{C})) \right) = 0. \tag{3.9}$$

This proves the proposition. □

Suppose that Γ is convex-cocompact but not cocompact. Then H^n/Γ satisfies the hypotheses of Proposition 5 and so $i : H_c^1(H^n/\Gamma; \mathbb{C}) \rightarrow H_{(2)}^1(H^n/\Gamma)$ is injective. For the rest of this section, we assume that $n > 2$. It follows from [12, Theorem 3.13] that i is an isomorphism. This essentially comes from the fact that given an L^2 -harmonic 1-form ω on H^n/Γ , one can apply the Poincaré Lemma from infinity

to homotop ω to something with compact support. We show how to construct the L^2 -harmonic 1-forms on H^n/Γ explicitly.

Lemma 3. *There is an isomorphism between $H_c^1(H^n/\Gamma; \mathbb{C})$ and the quotient space*

$$W = \{(f, c) \in C^\infty(\Omega) \times H^1(\Gamma; \mathbb{C}) : f \text{ is locally-constant and for all } \gamma \in \Gamma, \quad (3.10)$$

$$f - \gamma \cdot f = c(\gamma)\} / \mathbb{C}.$$

(Here \mathbb{C} acts by addition on $C^\infty(\Omega)$ and fixes $H^1(\Gamma; \mathbb{C})$.)

Proof. Given $x \in H_c^1(H^n/\Gamma; \mathbb{C})$, represent it by a smooth closed compact-supported 1-form $\alpha \in \Omega^1(H^n/\Gamma)$. Put $\tilde{\alpha} = \pi^*\alpha$. As α is compactly-supported, we can extend $\tilde{\alpha}$ continuously by zero to become a closed 1-form on $H^n \cup \Omega$. Fix a point $s \in \Omega$. Define $f : \Omega \rightarrow \mathbb{C}$ by

$$f(z) = \int_{\tilde{C}} \tilde{\alpha}, \quad (3.11)$$

where \tilde{C} is a curve in $H^n \cup \Omega$ from s to z . Then

$$(f - \gamma \cdot f)(z) = \int_{\tilde{C}'} \tilde{\alpha}, \quad (3.12)$$

where \tilde{C}' is a curve in $H^n \cup \Omega$ from $\gamma^{-1} \cdot z$ to z . Now \tilde{C}' projects to a closed curve C' on the compact manifold-with-boundary $(H^n \cup \Omega)/\Gamma$. Then

$$(f - \gamma \cdot f)(z) = \int_{C'} \alpha. \quad (3.13)$$

It follows that $f - \gamma \cdot f = c(\gamma)$, where c is the image of x in $H^1((H^n \cup \Omega)/\Gamma; \mathbb{C}) \cong H^1(\Gamma; \mathbb{C})$. A different choice of s changes f by a constant.

Conversely, given $(f, c) \in W$, fix a point $m_0 \in H^n/\Gamma$. Let R be large enough that the convex core of H^n/Γ lies within $B_R(m_0)$. Let $\phi : [0, \infty) \rightarrow \mathbb{R}$ be a smooth function which is monotonically nonincreasing, identically one on $[0, R]$ and identically zero on $[R + 1, \infty)$. Let $\eta \in C^\infty(H^n)$ be the lift to H^n of $\phi(d(m_0, \cdot)) \in C^\infty(H^n/\Gamma)$. Extend f inward to a locally-constant smooth function $F : (H^n - \pi^{-1}(B_R(m_0))) \rightarrow \mathbb{C}$. Put $\tilde{\alpha} = d((1 - \eta)F)$ on $H^n - \pi^{-1}(B_R(m_0))$ and extend it by zero to H^n . Then $\tilde{\alpha}$ is a closed Γ -invariant 1-form on H^n which descends to a closed 1-form $\alpha \in \Omega^1(H^n/\Gamma)$ with support in $B_{R+1}(m_0)$, and hence an element $[\alpha] \in H_c^1(H^n/\Gamma; \mathbb{C})$.

One can check that these two maps are inverses. We omit the details. □

The map $W \rightarrow H^1(H^n/\Gamma; \mathbb{C})$ induced from $(f, c) \rightarrow c$ is the same as the map $H_c^1(H^n/\Gamma; \mathbb{C}) \rightarrow H^1(H^n/\Gamma; \mathbb{C})$. Its kernel can be identified with the Γ -invariant

locally-constant functions on Ω , modulo \mathbb{C} . This has dimension equal to the number of ends of H^n/Γ minus one, as it should.

Choose $x \in \mathbb{H}_c^1(H^n/\Gamma; \mathbb{C})$. Define the locally-constant function $f : \Omega \rightarrow \mathbb{C}$ as in the proof of Lemma 3. As Λ has measure zero, we can think of f as a measurable function on S^{n-1} .

Proposition 6. *f lies in $L^p(S^{n-1})$ for all $p \in [1, \infty)$.*

Proof. Let K be the convex core of H^n/Γ and let ∂K be its boundary. Put $\tilde{K} = \pi^{-1}(K)$, the convex hull of Λ , and put $\tilde{\partial K} = \pi^{-1}(\partial K)$. As \tilde{K} is convex and K is compact, it follows that $\tilde{\partial K}$ is quasi-convex, meaning that there is an $R > 0$ such that if $y_1, y_2 \in \tilde{\partial K}$ then the geodesic from y_1 to y_2 , in H^n , lies in an R -neighborhood of $\tilde{\partial K}$. We take a ball model B^n for H^n such that $x_0 = \pi(0)$ lies in K .

If $\Omega \subset S^{n-1}$ is connected then the result is trivial, so we assume that Ω has more than one connected component. Let D be a connected component of Ω . We first estimate the spherical volume of D . There is an end e of H^n/Γ such that if a curve c in H^n goes to D then $\pi \circ c$ exits e . Let $\partial_e K$ be the connected component of ∂K corresponding to e . Then there is a component $\tilde{\partial}_D K$ of $\pi^{-1}(\partial_e K)$ such that D retracts onto $\tilde{\partial}_D K$ under the nearest-point retraction. Furthermore, the closure of $\tilde{\partial}_D K$ in \tilde{B}^n separates D from $K - \tilde{\partial}_D K$. Let r_D be the hyperbolic distance from 0 to $\tilde{\partial}_D K$. Then $\tilde{\partial}_D K \subset H^n - B_{r_D}(0)$. We are interested in what happens when r_D is large. If $z_1, z_2 \in \tilde{\partial}_D K$ then the geodesic from z_1 to z_2 cannot enter $B_{r_D-R}(0)$, as this would violate the quasi-convexity of $\tilde{\partial}_D K$. Quantitatively, this implies that the spherical distance from z_1 to z_2 cannot exceed $2 \sin^{-1} \left(\frac{1}{\cosh(r_D-R)} \right)$. Thus D lies within a spherical ball of radius $r_0 = 4 \sin^{-1} \left(\frac{1}{\cosh(r_D-R)} \right)$. As the volume of this spherical ball is bounded above by a constant times r_0^{n-1} , we conclude that there is a constant $C > 0$ such that $\text{vol}(D) \leq C e^{-(n-1)r_D}$, uniformly in the choice of D .

The connected components of Ω are in one-to-one correspondence with the set $\pi_1(K, \partial K)$. Fix an end e of M , with associated connected component $\partial_e K$ of ∂K . Take the ball model so that $x_0 \in \partial_e K$. The connected components D of Ω corresponding to e form the preimage of $\partial_e K$ under the map $\pi_1(K, \partial K) \rightarrow \pi_0(\partial K)$. Given D , let $c(s), 0 \leq s \leq r_D$, be a normalized minimal geodesic from 0 to $\tilde{\partial}_D K$. Consider a loop L_D in H^n/Γ which starts at x_0 , follows $\pi \circ c$ to $\pi(c(r_D)) \in \partial_e K$ and then returns to x_0 by a length-minimizing path in $\partial_e K$. The length of L_D will be bounded above by $r_D + \text{diam}(\partial_e K)$. On the other hand, L_D describes a class $[L_D] \in \pi_1(K, x_0)$. It follows that $d(0, [L_D] \cdot 0) \leq \text{length}(L_D)$. Also, as c is

minimal from 0 to $c(r_D)$, we have $r_D \leq d(0, [L_D] \cdot 0) + \text{diam}(\partial_e K)$. Thus

$$d(0, [L_D] \cdot 0) \leq \text{length}(L_D) \leq r_D + \text{diam}(\partial_e K) \leq d(0, [L_D] \cdot 0) + 2\text{diam}(\partial_e K). \tag{3.14}$$

In terms of the homotopy sequence

$$\pi_1(K, x_0) \xrightarrow{\alpha} \pi_1(K, \partial K) \xrightarrow{\beta} \pi_0(\partial K), \tag{3.15}$$

we have defined a map $s : \beta^{-1}(\partial_e K) \rightarrow \pi_1(K, x_0)$ which sends D to $[L_D]$, with $\alpha \circ s = \text{Id}$ on $\beta^{-1}(\partial_e K)$. Thus s is injective. By the construction of f , there is a bound

$$|f(D)| \leq A \text{length}(L_D) + B \leq A d(0, [L_D] \cdot 0) + B' \tag{3.16}$$

for $D \in \beta^{-1}(\partial_e K)$. Then

$$\sum_{D \in \beta^{-1}(\partial_e K)} |f(D)|^p \text{vol}(D) \leq \sum_{D \in \beta^{-1}(\partial_e K)} (A d(0, [L_D] \cdot 0) + B')^p. \tag{3.17}$$

$$C e^{-(n-1)(d(0, [L_D] \cdot 0) - \text{diam}(\partial_e K))}.$$

By [14], there is an $\epsilon > 0$ such that

$$\sum_{\gamma \in \Gamma} e^{-(n-1-\epsilon)d(0, \gamma \cdot 0)} < \infty. \tag{3.18}$$

It follows that f is L^p on $\bigcup\{D \in \beta^{-1}(\partial_e K)\}$. Considering together the finite number of ends of H^n/Γ , the proposition follows. \square

Lemma 4. For $f \in L^2(S^{n-1})$, let $\Phi_0 f \in C^\infty(H^n)$ be its harmonic extension. For $1 \leq j \leq n$, let x_j be the restriction to S^{n-1} of the j -th coordinate function on \mathbb{R}^n . Then

$$|\nabla(\Phi_0 f)|^2(0) = (n-1)^2 \sum_{j=1}^n \left| \frac{\int_{S^{n-1}} x_j f \, d\text{vol}}{\text{vol}(S^{n-1})} \right|^2. \tag{3.19}$$

Proof. Let $\{\beta_i\}_{i=1}^\infty$ be an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenvectors of $\Delta_{S^{n-1}}$ with eigenvalue $(k_i + 1)(k_i + n - 1)$, $k_i \in \mathbb{Z} \cap [-1, \infty)$. Let $f = \sum_{i=1}^\infty a_i \beta_i$ be the Fourier expansion of f . Then from [6, p. 599],

$$(\Phi_0 f)(r, \theta) = \frac{\Gamma(\frac{n}{2})}{\Gamma(n-1)} \sum_{i=1}^\infty a_i \frac{\Gamma(n+k_i)}{\Gamma(\frac{n}{2} + k_i + 1)} r^{1+k_i} F(1 - \frac{n}{2}, 1+k_i; 1 + \frac{n}{2} + k_i; r^2) \beta_i(\theta). \tag{3.20}$$

It follows that

$$|\nabla(\Phi_0 f)|^2(0) = \frac{(n-1)^2}{n^2} \sum_{k_i=0} |a_i|^2 (|\beta_i|^2 + |\nabla_{S^{n-1}} \beta_i|^2). \tag{3.21}$$

We can take the β_i 's with $k_i = 0$ to be the functions $\left\{ \left(\frac{n}{\text{vol}(S^{n-1})} \right)^{\frac{1}{2}} x_j \right\}_{j=1}^n$. In this case, one can verify that $|\beta_i|^2 + |\nabla_{S^{n-1}} \beta_i|^2$ is constant on S^{n-1} . Its integral is

$$\int_{S^{n-1}} \left(|\beta_i|^2 + |\nabla_{S^{n-1}} \beta_i|^2 \right) d\text{vol} = \langle \beta_i, \beta_i \rangle + \langle \beta_i, \Delta_{S^{n-1}} \beta_i \rangle = 1 + (n-1) = n. \quad (3.22)$$

Hence

$$|\beta_i|^2 + |\nabla_{S^{n-1}} \beta_i|^2 = \frac{n}{\text{vol}(S^{n-1})} \quad (3.23)$$

and so

$$\begin{aligned} |\nabla(\Phi_0 f)|^2(0) &= \frac{(n-1)^2}{n \text{vol}(S^{n-1})} \sum_{k_i=0} |a_i|^2 \quad (3.24) \\ &= \frac{(n-1)^2}{n \text{vol}(S^{n-1})} \sum_{j=1}^n \left| \int_{S^{n-1}} \left(\frac{n}{\text{vol}(S^{n-1})} \right)^{\frac{1}{2}} x_j f d\text{vol} \right|^2 \\ &= (n-1)^2 \sum_{j=1}^n \left| \frac{\int_{S^{n-1}} x_j f d\text{vol}}{\text{vol}(S^{n-1})} \right|^2. \end{aligned}$$

The lemma follows. \square

Proposition 7. $d(\Phi_0 f)$ is a Γ -invariant harmonic 1-form on H^n . It descends to an L^2 -harmonic 1-form on H^n/Γ .

Proof. As f is L^2 , $\Phi_0 f$ is well-defined. As $\Phi_0 f$ is harmonic, $\Delta_1 d(\Phi_0 f) = d(\Delta_0 \Phi_0 f) = 0$. Thus $d(\Phi_0 f)$ is harmonic. Furthermore, for all $\gamma \in \Gamma$,

$$d(\Phi_0 f) - \gamma \cdot d(\Phi_0 f) = d(\Phi_0(f - \gamma \cdot f)) = d(\Phi_0 c_\gamma) = dc_\gamma = 0. \quad (3.25)$$

Thus $d(\Phi_0 f)$ is Γ -invariant. It remains to show that the descent of $d(\Phi_0 f)$ to H^n/Γ is L^2 .

Let m be a point in the connected component of $H^n/\Gamma - K$ corresponding to an end e . Take a ball model B^n of H^n with $\pi(0) = m$. Let D be the connected component of Ω adjacent, in $\overline{B^n}$, to the connected component of $H^n - \tilde{K}$ containing 0. Changing f by a constant, we may assume that f vanishes on D . The method of proof of Proposition 6 implies that the L^1 -norm of f , as seen in the visual sphere at m , is $O(e^{-(n-1)d(m,K)})$ with respect to m . Then by Lemma 4,

$$|\nabla(\Phi_0 f)|^2(0) = O(e^{-2(n-1)d(m,K)}). \quad (3.26)$$

On the other hand, the volume of $\{m \in H^n/\Gamma : d(m, K) \in [j, j+1]\}$ is $O(e^{(n-1)j})$. The proposition follows. \square

Thus we have constructed $\dim(\mathbb{H}_c^1(H^n/\Gamma; \mathbb{C}))$ linearly-independent L^2 -harmonic 1-forms on H^n/Γ .

Theorem 4.B. *Let Γ be a convex-cocompact group which is not cocompact. Let α be a nonzero L^2 -harmonic 1-form on H^n/Γ . Then $\Phi_1^{-1}(\pi^*\alpha)$ is Sobolev H^{-1} -regular.*

Proof. We know that $\pi^*\alpha = d(\Phi_0 f)$ for some $f \in L^2(S^{n-1})$ constructed as in Lemma 3. Then $\pi^*\alpha = \Phi_1(df)$, with df being Sobolev H^{-1} -regular. \square

4. Surfaces

Theorem 5.A. *Let Γ be a torsion-free uniform lattice in $\text{Isom}^+(H^2)$, with H^2/Γ a closed surface of genus g . Then*

1. $\dim(\mathcal{A}'_0(S^1))^\Gamma = 2g$.
2. $\dim(\mathcal{D}'_0(S^1))^\Gamma = 2g$.
3. $\dim(\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g$.
4. $\dim(L^2(S^1)/\mathbb{C})^\Gamma = 0$.

Proof. The proof is similar to the proof of Theorem 2.A. If $F \in (\mathcal{A}'_0(S^1))^\Gamma$ then dF is a Γ -invariant exact hyperform on S^1 and $\Phi_1(dF)$ is a Γ -invariant closed and coclosed 1-form on H^2 . Thus $\Phi_1(dF) = \pi^*\alpha$ for a harmonic 1-form on H^2/Γ . In terms of the complex coordinate z on D^2 , we can write $\Phi_1(dF) = h_1(z)dz + h_2(\bar{z})d\bar{z}$ where $h_1(z)$ and $h_2(z)$ are holomorphic functions. Let $k_1(z)$ and $k_2(z)$ satisfy $h_i(z) = k_i''(z)$ for $i \in \{1, 2\}$. Then

$$d(\Phi_0 F) = \Phi_1(dF) = d(k'_1(z) + k'_2(\bar{z})), \tag{4.1}$$

so $\Phi_0 F = k'_1(z) + k'_2(\bar{z}) + \text{const}$. As α is bounded, $\Phi_1(dF)$ is uniformly bounded on H^2 and so

$$\sup_{z \in D^2} (1 - |z|^2) |k_i''(z)| < \infty. \tag{4.2}$$

That is, k'_i is an element of the Bloch space and so k_i has a boundary value in the Zygmund functions \mathcal{Z} [7, p. 282,442]. Thus $F(\theta) = k'_1(e^{i\theta}) + k'_2(e^{-i\theta}) + \text{const.}$, showing that F has the required regularity.

Part (4) follows from the fact that Γ acts ergodically on S^1 . \square

Theorem 5.B. *Let Γ be a torsion-free nonuniform lattice in $\text{Isom}^+(H^2)$, with H^2/Γ the complement of k points in a closed surface S of genus g . Then*

1. $\dim(\mathcal{A}'_0(S^1))^\Gamma = \infty$.

- 2. $\dim (\mathcal{D}'_0(S^1))^\Gamma = \max(2g, 2g + 2k - 2).$
- 3. $\dim \left(H^{-\frac{1}{2}}(S^1)/\mathbb{C} \right)^\Gamma = 2g.$
- 4. $\dim (\mathcal{D}Z(S^1)/\mathbb{C})^\Gamma = 2g.$
- 5. $\dim (L^2(S^1)/\mathbb{C})^\Gamma = 0.$ □

Proof. Sending $f \in (\mathcal{A}'_0(S^1))^\Gamma$ to $\Phi_1(df)$, we see that $(\mathcal{A}'_0(S^1))^\Gamma$ is isomorphic to the space of closed and coclosed 1-forms on H^2/Γ . Let p be a puncture point in S and let \mathbb{Z} be the subgroup of Γ generated by a loop around p . Then the cusp of H^2/Γ corresponding to p embeds in H^2/\mathbb{Z} . We model the latter by the upper-half-plane quotiented by $z \rightarrow z + 1$. Consider the pullback of $\Phi_1(df)$ under the quotient map $H^2/\mathbb{Z} \rightarrow H^2/\Gamma$. As in [8], such a 1-form on H^2/\mathbb{Z} can be written as $h_1(z)dz + h_2(\bar{z})d\bar{z}$, where $h_i(z) = h_i(z + 1)$. Each h_i has a Fourier expansion

$$h_i(z) = \sum_{j \in \mathbb{Z}} c_{i,j} e^{2\pi\sqrt{-1}jz}. \tag{4.3}$$

If $c_{1,j} = 0$ for $j < -J$ then a change of variable $w = e^{2\pi\sqrt{-1}z}$ gives

$$h_1(z)dz = \sum_{j \geq -J} c_{1,j} w^{j-1} \frac{dw}{2\pi\sqrt{-1}}, \tag{4.4}$$

and similarly for $h_2(\bar{z})d\bar{z}$.

To each puncture point $p_l \in S$, $1 \leq l \leq k$, assign an integer J_l and let $i\left(-\sum_{l=1}^k (J_l + 1)p_l\right)$ denote the space of holomorphic differentials on S whose Laurent expansion around each p_l has the form of the right-hand-side of (4.4) with $J = J_l$. By the Riemann-Roch theorem, $i(D) \geq g - 1 + \sum_{l=1}^k (J_l + 1)$. Taking the numbers $\{J_l\}_{l=1}^k$ large, part (1) follows.

Part (2) was proven in [8]. For completeness, we repeat the argument. On the upper-half-plane, $|h_1(z)dz| = |h_1(x + iy)|y$. As $d(i, iy) = |\ln(y)|$, if $h_1(z)dz$ has slow growth as $y \rightarrow \infty$ then we must have $c_{1,j} = 0$ for $j < 0$. The space of such holomorphic differentials on S has dimension $i\left(-\sum_{l=1}^k p_l\right)$. The Riemann-Roch theorem implies that $i\left(-\sum_{l=1}^k p_l\right) = \max(g + k, g + k - 1)$. Part (2) follows.

Suppose that $f \in \left(H^{-\frac{1}{2}}(S^1)/\mathbb{C}\right)^\Gamma$. Then df is $H^{-\frac{3}{2}}$ -regular. Considering $\Phi_1(df)$, we know that on a cusp, $h_1(z)$ has an expansion (4.3) with $c_{1,j} = 0$ for $j < 0$. If $c_{1,0} \neq 0$ then as $y \rightarrow \infty$, $h_1(z)dz \sim c_{1,0}dz$. To analyze the singularity at a cusp point on S^1 , we consider the 1-form $c_{1,0}dz$ on the upper-half-plane and perform the reflection $z \rightarrow \frac{z}{|z|^2}$. On the boundary of the upper-half-plane, this restricts to $x \rightarrow \frac{1}{x}$ and so $c_{1,0}dx \rightarrow -c_{1,0}\frac{dx}{x^2}$. The point $i\infty$ gets mapped

to 0 and so it is enough to look at the singularity of $-c_{1,0} \frac{dx}{x^2}$ near $x = 0$. The Fourier transform of $\frac{1}{x^2}$ is proportionate to $|k|$. Hence $\frac{1}{x^2}$ lies in H^s if and only if $\int_{\mathbb{R}} (1+k^2)^s |k|^2 dk < \infty$, i.e. if $s < -\frac{3}{2}$. This contradicts the assumption that df is $H^{-\frac{3}{2}}$ -regular. Thus $c_{1,0} = 0$. Then $\Phi_1(df)$ is bounded and as in the proof of Theorem 5.A, $f \in (\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$. Furthermore, $h_1(z)dz$ extends smoothly over the puncture points to give a holomorphic differential on S . We conclude that both $(H^{-\frac{1}{2}}(S^1)/\mathbb{C})^\Gamma$ and $(\mathcal{DZ}(S^1)/\mathbb{C})^\Gamma$ are isomorphic to two copies of the space of holomorphic differentials on S , the dimension of which is g . Parts (3) and (4) follow.

Finally, part (5) follows from the ergodicity of the Γ -action on S^1 . □

5. 3-Manifolds

5.1. Quasi-Fuchsian groups

Let S be a closed oriented surface of genus $g > 1$. Let Γ be a quasi-Fuchsian subgroup of $\text{Isom}^+(H^3)$ which is isomorphic to $\pi_1(S)$. Then H^3/Γ is diffeomorphic to $\mathbb{R} \times S$ and $H_c^1(H^3/\Gamma; \mathbb{C}) = \mathbb{C}$. (In terms of the projection $p : \mathbb{R} \times S \rightarrow \mathbb{R}$, a proper map, one has $H_c^1(H^3/\Gamma; \mathbb{C}) = p^*(H_c^1(\mathbb{R}; \mathbb{C}))$). Thus there is a nonzero L^2 -harmonic 1-form α on H^3/Γ .

By Corollary 2 and Proposition 3, $\Phi_1^{-1}(\pi^*\alpha)$ is a Γ -invariant exact 1-current supported on the limit set $\Lambda \subset S^2$. The domain of discontinuity $\Omega \subset S^2$ is the union of two 2-disks D_+ and D_- , with D_+/Γ and D_-/Γ homeomorphic to S . Let $\chi_{D_+} \in L^2(S^2)$ be the characteristic function of D_+ . By Proposition 7, $\Phi_1^{-1}(\pi^*\alpha)$ is proportionate to the exact 1-current $d\chi_{D_+}$ on S^2 .

In order to write $d\chi_{D_+}$ more directly on Λ , we follow the general scheme of [5, Section IV.3.γ]. Let $Z : D^2 \rightarrow D_+$ be a uniformization of D_+ . By Carathéodory's theorem, Z extends to a continuous homeomorphism $\bar{Z} : \bar{D}^2 \rightarrow \bar{D}_+$. The restriction of \bar{Z} to $\partial\bar{D}^2$ gives a homeomorphism $\partial\bar{Z} : S^1 \rightarrow \Lambda$.

From a general construction [5, Theorem 2, p. 208], the 1-current $d\chi_{D_+}$ defines a cyclic 1-cocycle τ on the algebra $C^1(S^2)$ by

$$\tau(F^0, F^1) = \int_{S^2} d\chi_{D_+} \wedge F^0 dF^1. \tag{5.1}$$

Lemma 1. *The function space $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ is a Banach algebra with the norm*

$$\|f\| = \left(\int_{\mathbb{R}^+} \int_{S^1} \frac{|f(\theta+h) - f(\theta)|^2}{h^2} d\theta dh \right)^{\frac{1}{2}} + \|f\|_\infty. \tag{5.2}$$

Given $f^0, f^1 \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$, let

$$f^i(\theta) = \sum_{j \in \mathbb{Z}} c_j^i e^{\sqrt{-1}j\theta} \tag{5.3}$$

be the Fourier expansion. Define a bilinear function

$$\bar{\tau} : \left(H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \times \left(H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1) \right) \rightarrow \mathbb{C} \tag{5.4}$$

by

$$\bar{\tau}(f^0, f^1) = -2\pi i \sum_{j \in \mathbb{Z}} j c_j^0 c_{-j}^1. \tag{5.5}$$

Then $\bar{\tau}$ is a continuous cyclic 1-cocycle on $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$.

Proof. It is straightforward to check that $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ is a Banach algebra with the given norm. It is also easy to check that $\bar{\tau}$ is continuous. If $f^0, f^1 \in C^\infty(S^1)$ then

$$\bar{\tau}(f^0, f^1) = \int_{S^1} f^0 df^1. \tag{5.6}$$

As in [5, p. 182], put

$$(b\bar{\tau})(f^0, f^1, f^2) = \bar{\tau}(f^0 f^1, f^2) - \bar{\tau}(f^0, f^1 f^2) + \bar{\tau}(f^2 f^0, f^1). \tag{5.7}$$

If $f^0, f^1, f^2 \in C^\infty(S^1)$ then $(b\bar{\tau})(f^0, f^1, f^2) = 0$. As $C^\infty(S^1)$ is dense in $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ and $b\bar{\tau}$ is continuous in its arguments, it follows that $b\bar{\tau} = 0$. \square

Theorem 6. Given $F^0, F^1 \in C^1(S^2)$, put $f^i = (\partial\bar{Z})^* F^i$, $i \in \{1, 2\}$. Then $f^i \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$ and

$$\tau(F^0, F^1) = -\bar{\tau}(f^0, f^1). \tag{5.8}$$

Proof. Consider S^2 as $\mathbb{C} \cup \infty$ with $\infty \in D_-$. For $r \in (0, 1)$, let $i_r : S^1 \rightarrow D^2$ be the embedding of S^1 as the circle of radius r around $0 \in D^2$. Thinking of Z as a map from D^2 to \mathbb{C} , let

$$Z(z) = \sum_{k=0}^{\infty} c_k z^k \tag{5.9}$$

be its Taylor's series. Then

$$\frac{i}{2} \int_{B_r(0)} dZ \wedge dZ^* = \frac{i}{2} \int_{S^1} i_r^* Z d(i_r^* Z^*) = \pi \sum_{k=0}^{\infty} k r^{2k} |c_k|^2. \tag{5.10}$$

As Z is univalent,

$$\frac{i}{2} \int_{D^2} dZ \wedge dZ^* = \text{area}(Z(D^2)) < \infty. \tag{5.11}$$

It follows that

$$\lim_{r \rightarrow 1} i_r^* Z = \partial \bar{Z} \tag{5.12}$$

in $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$. Then $f^i \in H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$.

We have

$$\begin{aligned} \tau(F^0, F^1) &= \int_{S^2} d\chi_{D^+} \wedge F^0 dF^1 \\ &= - \int_{S^2} \chi_{D^+} dF^0 \wedge dF^1 \\ &= - \int_{D^+} dF^0 \wedge dF^1 \\ &= - \int_{D^2} d(Z^* F^0) \wedge d(Z^* F^1). \end{aligned} \tag{5.13}$$

Then

$$\begin{aligned} \tau(F^0, F^1) &= \lim_{r \rightarrow 1} - \int_{B_r(0)} d(Z^* F^0) \wedge d(Z^* F^1) \\ &= \lim_{r \rightarrow 1} - \int_{S^1} i_r^* Z^* F^0 \wedge d(i_r^* Z^* F^1) \\ &= \lim_{r \rightarrow 1} -\tau(i_r^* Z^* F^0, i_r^* Z^* F^1). \end{aligned} \tag{5.14}$$

From (5.12),

$$\lim_{r \rightarrow 1} i_r^* Z^* F^i = f^i \tag{5.15}$$

in $H^{\frac{1}{2}}(S^1) \cap L^\infty(S^1)$. The theorem follows. \square

Example. Let Σ be a closed oriented surface of genus $g > 2$, let $\phi \in \text{Diff}(\Sigma)$ be an orientation-preserving pseudo-Anosov diffeomorphism and let M be the mapping torus of ϕ . Then M is a 3-manifold which fibers over the circle and admits a hyperbolic structure [16, 13]. Let $\widehat{M} = H^3/\Gamma$ be the corresponding cyclic cover of M , with the pullback hyperbolic metric. The group Γ is isomorphic to $\pi_1(\Sigma)$. From [10, Proposition 9], \widehat{M} has no nonzero L^2 -harmonic 1-forms. This contrasts with the quasi-Fuchsian case.

5.2. Covering spaces

If M is a closed 3-manifold then M has nontrivial L^2 -harmonic 1-forms if and only if $b_1(M) > 0$. There are many examples of hyperbolic manifolds 3-manifolds M with $b_1(M) > 0$, such as those which fiber over a circle. It is less obvious that there are infinite normal covers $\widehat{M} = H^3/\Gamma$ of closed hyperbolic 3-manifolds such that \widehat{M} has nonzero L^2 -harmonic 1-forms. We give some examples. The limit sets will be all of S^2 .

Let M be a closed oriented hyperbolic 3-manifold with a surjective homomorphism $\alpha : \pi_1(M) \rightarrow F_r$ onto a free group with $r > 1$ generators. Let $\widehat{M} = H^3/\Gamma$ be the corresponding cover with $\Gamma \cong \text{Ker}(\alpha)$. The space of ends of \widehat{M} is a Cantor set. As F_r is nonamenable, Proposition 5 applies to show that \widehat{M} has an infinite-dimensional space of L^2 -harmonic 1-forms. Thus for all $\epsilon > 0$, $(H^{-\epsilon}(S^2)/\mathbb{C})^\Gamma$ is infinite-dimensional.

For another example, let Σ be a closed oriented surface of genus $g > 2$. Let ρ be a nonzero element of $H^1(\Sigma; \mathbb{Z}) = \mathbb{Z}^{2g}$. Let $\widehat{\Sigma}$ be the cyclic cover of Σ coming from the homomorphism $\pi_1(\Sigma) \rightarrow H_1(\Sigma; \mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}$. It is an infinite-genus surface.

Let ϕ be an orientation-preserving pseudo-Anosov diffeomorphism of Σ which acts trivially on $H^1(\Sigma; \mathbb{Z})$; it is a surprising fact that such diffeomorphisms exist [17]. It lifts to a diffeomorphism $\widehat{\phi}$ of $\widehat{\Sigma}$. Let M be the mapping torus of ϕ , with its hyperbolic metric. It follows from the Wang sequence that $H^1(M; \mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z}$. Let $\widehat{M} = H^3/\Gamma$ be the cyclic covering of M coming from $\rho \oplus 0 \in H^1(M; \mathbb{Z})$. Equivalently, \widehat{M} is the mapping torus of $\widehat{\phi}$.

Given $e^{i\theta} \in U(1)$, let $\rho_\theta : \mathbb{Z} \rightarrow U(1)$ be the representation $\rho_\theta(n) = e^{in\theta}$. Let E_θ be the flat unitary line bundle on Σ coming from the representation $\pi_1(\Sigma) \rightarrow H_1(\Sigma; \mathbb{Z}) \xrightarrow{\rho} \mathbb{Z} \xrightarrow{\rho_\theta} U(1)$. Let F_θ be the flat unitary line bundle on M coming from the representation $\pi_1(M) \rightarrow H_1(M; \mathbb{Z}) \xrightarrow{\rho \oplus 0} \mathbb{Z} \xrightarrow{\rho_\theta} U(1)$; it is the mapping torus for the action of ϕ on E_θ . As in [10, Section 4], it follows from Fourier analysis that \widehat{M} has a nonzero L^2 -harmonic 1-form if and only if $H^1(M; F_\theta) \neq 0$ for all θ . Furthermore, because of the \mathbb{Z} -action on \widehat{M} , if there is one nonzero L^2 -harmonic 1-form then there is an infinite-dimensional space.

From the Euler characteristic identity and Poincaré duality,

$$2 - 2g = 2 \dim H^0(\Sigma; E_\theta) - \dim H^1(\Sigma; E_\theta). \tag{5.16}$$

As $\dim H^0(\Sigma; E_\theta) \leq 1$, it follows that

$$\dim H^1(\Sigma; E_\theta) = 2g - 2 \left(1 - \dim H^0(\Sigma; E_\theta) \right) > 0. \tag{5.17}$$

From the Wang sequence,

$$H^1(M; F_\theta) \cong H^0(\Sigma; E_\theta) \oplus H^1(\Sigma; E_\theta) \neq 0. \tag{5.18}$$

Thus \widehat{M} has nonzero L^2 -harmonic 1-forms and for all $\epsilon > 0$, $(H^{-\epsilon}(S^2)/\mathbb{C})^\Gamma$ is infinite-dimensional. The L^2 -harmonic 1-forms on \widehat{M} arise from the fact that $\text{Im} \left(H_c^1(\widehat{M}; \mathbb{C}) \rightarrow H^1(\widehat{M}; \mathbb{C}) \right)$ is nonzero.

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John Lott
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1109
USA
e-mail: lott@math.lsa.umich.edu

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