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On the Aronszajn Property for an Implicit Differential Equation of Fractional Order

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Abstract. In this paper we investigate some topological properties of solution sets of an implicit differential equation of fractional order in Banach spaces.

Keywords. Differential equation, derivative of fractional order, measure of noncompactness, solution sets

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1. Introduction

Assume that $I = [0, a]$, E is a Banach space, $B = \{x \in E : \|x - x_0\| \le b\}$ and $g : I \times B \times E \mapsto E$ is a continuous function which satisfies the following conditions:

1^o $\|g(t, x, z) - g(t, x, y)\| \le \phi(\|z - y\|)$, where ϕ is a continuous nondecreasing function such that $\phi(0) = 0$, $\phi(u) < u$ for $u > 0$;

2^o there exists a constant M_0 such that $\|g(t, x, z)\| \le M_0 + k\|z\|$, where $k < 1$.

In this paper we prove an existence theorem for the nonlinear implicit differential equation of fractional order:

$$D^\beta x = g(t, x, D^\beta x), \quad x(0) = x_0, \tag{1}$$

where $0 < \beta < 1$ and D^β denotes the fractional derivative of order β in the Caputo sense (cf.[5]). More precisely, we prove that the set of solutions of (1) is a compact R_δ, i.e., it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Obviously, any R_δ set is nonempty and connected.

Let us mention that differential equations of fractional order create an interesting and important branch of the theory of differential equations. The

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theory of such differential equations is developed intensively in recent years. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. as valuable tools to the modeling of many different phenomena. For details, see [10, 12–14] and references therein.

In principle, we may reduce such an equation to an integral equation with weak singularity and apply to it basic techniques of nonlinear analysis.

2. Results

We define a mapping $\tilde{g}$ by the following:

$$\tilde{g}(z)(t,x) = g(t,x,z(t,x)),$$

where $z : I \times B \mapsto E$ is a continuous function from $I \times B$ into E, so $z \in C(I \times B, E)$ with the norm $\|z\|_C = \sup_{(t,x)\in I\times B} \|z(t,x)\|$.

Lemma. *There exists a unique point $v \in C$ such that $v = \tilde{g}(v)$ and $v = \lim_{n\mapsto\infty} v_n$, where (v_n) is the sequence of successive approximations, i.e., $v_0 = 0$, $v_{n+1} = \tilde{g}(v_n)$.*

Proof. Choose any $z, \tilde{z}$ in $C(I \times B, E)$. Then by 1^o we obtain

$$\begin{aligned}\|\tilde{g}(z) - \tilde{g}(\tilde{z})\|_C &= \sup_{(t,x)\in I\times B} \|g(t,x,z(t,x)) - g(t,x,\tilde{z}(t,x))\| \\ &\le \sup_{(t,x)\in I\times B} \phi(\|z(t,x) - \tilde{z}(t,x)\|) \\ &\le \phi\Big(\sup_{(t,x)\in I\times B} \|z(t,x) - \tilde{z}(t,x)\|\Big) \\ &= \phi(\|z - \tilde{z}\|_C).\end{aligned}$$

Hence, by applying the well known Browder fixed point principle for nonlinear contractions [4, Theorem 1], we deduce that $\tilde{g}$ has the unique fixed point v, where v is the limit of successive approximations (v_n). □

Let α denote the Kuratowski measure of noncompactness in E (cf. [3, 9]). Assume that

3^o $\alpha(g(t, X \times Y)) \le \max(\omega(\alpha(X)), \alpha(Y))$ for $t \in I$, $X \subset B$ and bounded $Y \subset E$, where $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous nondecreasing function such that $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and

$$\int_0^\delta \frac{1}{s}\left[\frac{s}{\omega(s)}\right]^{\frac{1}{\beta}} ds = \infty \quad (\delta > 0,\ 0 < \beta < 1). \tag{2}$$

The main result of this paper is the following:

Theorem. *If g satisfies the assumptions $1^o - 3^o$, then the set of solutions of* (1) *defined on J is a compact R_δ.*

Proof. According to the above lemma, there exists a function $f(t,x)$ such that $f(t,x) = g(t,x,f(t,x))$, i.e., $f = \tilde{g}(f)$, and

$$f(t,x) = \lim_{n\to\infty} f_n(t,x) \quad \text{uniformly in } (t,x) \in I \times B, \tag{3}$$

where the sequence of functions $f_n :\ I \times B \mapsto E$ is defined by

$$f_0(t,x) = 0 \ \text{ and } \ f_{n+1}(t,x) = g(t,x,f_n(t,x)) \quad (t \in I,\ x \in B,\ n \in N).$$

From 2^o it follows that $\|f(t,x)\| \le M$, where $M = \frac{M_0}{1-k}$. Moreover, by 3^o we obtain

$$\alpha(f_n(t,X)) \le \omega(\alpha(X)) \quad \text{for } X \subset B \text{ and } t \in I. \tag{4}$$

Next, in view of (3) we have

$$f(t,X) \subset f_n(t,X) + K(0,\varepsilon) \quad \text{for } X \subset B,\ t \in I,$$

and for sufficiently large $n \in N$, where $K(0,\varepsilon)$ is the ball with center 0 and radius ε. Hence by (4)

$$\alpha(f(t,X)) \le \alpha(f_n(t,X)) + 2\varepsilon \le \omega(\alpha(X)) + 2\varepsilon \quad \text{for each } \ X \subset B \text{ and } t \in I.$$

Since the above inequality holds for any $\varepsilon > 0$, we get

$$\alpha(f(t,X)) \le \omega(\alpha(X)) \quad \text{for } X \subset B \text{ and } t \in I. \tag{5}$$

Next, we choose a positive number d such that $d \le a$ and $\frac{Md^{1-r}}{(1-r)\Gamma(1-r)} \le b$. We introduce the following notation: $J = [0,d]$; $C = C(J,E)$ the Banach space of continuous functions $J \mapsto E$ with the supremum norm $\|\cdot\|_C$; $\tilde{B} = \{x \in C : \|x(t) - x_0\| \le b,\ t \in J\}$.

Let us remark that a continuous function $u : J \mapsto B$ is a solution of the Cauchy problem

$$D^\beta x = f(t,x) \quad (0 < \beta < 1), \quad x(0) = x_0, \tag{6}$$

if and only if u is a solution of (1), where $f(t,x)$ is given at the beginning of this proof.

Notice that the problem (6) is equivalent to the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds, \quad (0 < \beta < 1). \tag{7}$$

Putting in (7) $r = 1 - \beta$ we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(1-r)} \int_0^t \frac{f(s, x(s))}{(t-s)^r} ds.$$

We introduce an operator F defined by

$$F(x)(t) = x_0 + \frac{1}{\Gamma(1-r)} \int_0^t \frac{f(s, x(s))}{(t-s)^r} ds \quad \text{for } x \in \tilde{B},\ t \in J.$$

Observe that F is an operator acting from $\tilde{B}$ into itself. Indeed, for $x \in \tilde{B}$ we have

$$\|F(x)(t) - x_0\| = \left\| \frac{1}{\Gamma(1-r)} \int_0^t \frac{f(s, x(s))}{(t-s)^r} ds \right\| \leq \frac{1}{\Gamma(1-r)} \cdot \frac{Md^{1-r}}{(1-r)} \leq b.$$

Put $\bar{f}(t, x) = f(t, r(x))$, where

$$r(x) = \begin{cases} x & \text{for } x \in B \\ x_0 + \frac{(x-x_0)b}{\|x-x_0\|} & \text{for } x \notin B, \end{cases}$$

and define a mapping $\tilde{F}$ by

$$\tilde{F}(x)(t) = x_0 + \frac{1}{\Gamma(1-r)} \int_0^t \frac{\bar{f}(s, x(s))}{(t-s)^r} ds \quad (x \in C,\ t \in J).$$

Let us recall that $\|\bar{f}(t, x)\| \leq M$ for $t \in J$, $x \in E$. In the same way as in [8] (cf. also [1, 16]) we can prove that the set $\tilde{F}(C)$ is equiuniformly continuous and $\tilde{F}$ is a continuous mapping from C into itself.

Now we shall show that $I - \tilde{F}$ is a proper mapping , i.e., $(I - \tilde{F})^{-1}(Z)$ is relatively compact for each relatively compact subset Z of C (I denotes the identity map).

Let Z be a given relatively compact subset of C and let (u_n) be a sequence in $(I - \tilde{F})^{-1}(Z)$. Put $V = \{u_n : n \in N\}$. Then the set $(I - \tilde{F})(V) \subset Z$ is relatively compact. As $V \subset (I - \tilde{F})(V) + \tilde{F}(V)$, the set V is equicontinuous and the function $t \mapsto v(t) = \alpha(V(t))$ is continuous on J. From the inclusion $r(X) \subset x_0 + \bigcup_{0 \leq \lambda \leq 1} \lambda(X - x_0)$ it follows that

$$\alpha(r(X)) \leq \alpha\left(\bigcup_{0 \leq \lambda \leq 1} \lambda(X - x_0) \right) \leq \alpha(X - x_0) = \alpha(X).$$

Since $V(t) \subset (I-\tilde{F})(V)(t) + \tilde{F}(V)(t)$, by (5), Heinz's lemma [7] and the corresponding properties of α we obtain

$$\begin{aligned}\alpha(V(t)) &\le \alpha((I-\tilde{F})(V)(t)) + \alpha(\tilde{F}(V)(t))\\ &= \alpha(\tilde{F}(V)(t))\\ &= \alpha\left(\frac{1}{\Gamma(1-r)}\int_0^t \frac{\bar{f}(s,x(s))}{(t-s)^r}ds : \; x \in V\right)\\ &\le \frac{2}{\Gamma(1-r)}\int_0^t \frac{1}{(t-s)^r}\alpha\big(\bar{f}(s,V(s))\big)ds\\ &\le \frac{2}{\Gamma(1-r)}\int_0^t \frac{1}{(t-s)^r}\omega\big(\alpha(V(s))\big)ds,\end{aligned}$$

i.e.,

$$v(t) \le \frac{2}{\Gamma(1-r)}\int_0^t \frac{1}{(t-s)^r}\omega(v(s))ds \quad \text{for } t \in J.$$

Moreover, ω satisfies (2). Applying the Mydlarczyk–Gripenberg theorem [11] with $\alpha = 1-r$ and theorem on integral inequalities [2, Theorem 2], from this we deduce that $v(t) = 0$ for $t \in J$. Thus $\alpha(V(t)) = 0$ for $t \in J$. Therefore for each $t \in J$ the set $V(t)$ is relatively compact in E, and by Ascoli's theorem the set V is relatively compact in C. Hence we can find a subsequence (u_{n_k}) of (u_n) which converges in C. Consequently, the set $(I-\tilde{F})^{-1}(Z)$ is relatively compact.

Notice that if $x = \tilde{F}(x)$, then $x \in \tilde{B}$. Indeed,

$$\begin{aligned}\|x(t)-x_0\| &= \|\tilde{F}(x)(t)-x_0\|\\ &= \left\|\frac{1}{\Gamma(1-r)}\int_0^t \frac{\bar{f}(s,x(s))}{(t-s)^r}ds\right\|\\ &\le \frac{1}{\Gamma(1-r)}\int_0^t \frac{\|\bar{f}(s,x(s))\|}{(t-s)^r}ds\\ &\le \frac{1}{\Gamma(1-r)}M\frac{d^{1-r}}{1-r}\\ &\le b.\end{aligned}$$

Thus $x(t) \in B$, so that $x \in \tilde{B}$. Applying now Vidossich's theorem [17, Theorem 1.1] (see also [15, Theorem 5]) we conclude that the set of all solutions of (1) on J is a compact R_δ. □

Example. As an important example, which illustrate the assumptions related to (6), we consider the function $\omega(\xi) = \xi|\ln\xi|^\beta$ for $0 < \xi \le e^{-\beta}$, $0 < \beta < 1$, and $\omega(0) = 0$. It can be easily verified that ω is continuous, nondecreasing and

$$|\omega(\xi)-\omega(\eta)| \le \omega(|\xi-\eta|) \quad \text{for } 0 \le \xi,\, \eta \le e^{-\beta}. \tag{8}$$

Moreover,

$$\int_{0+} \frac{1}{s}\left[\frac{s}{\omega(s)}\right]^{\frac{1}{\beta}} = \int_{0+} \frac{ds}{s|\ln s|} = \infty.$$

Let $E = C(0,1)$ and $B = \{x \in E : \| x \| \le \frac{1}{2}e^{-\beta}\}$. We define a function $f_1 : B \mapsto E$ by

$$f_1(x)(\tau) = \omega(|x(\tau)|) \quad \text{for } \tau \in [0,1] \text{ and } x \in B.$$

By (8) we get $\| f_1(x) - f_1(y) \| \le \omega(\| x - y \|)$ for $x, y \in B$. From this we deduce that for a given completely continuous function $f_2 : B \mapsto E$ the function $f = f_1 + f_2$ satisfies the inequality $\alpha(f(X)) \le \omega(\alpha(X))$ for $X \subset B$. Therefore, our equation has the form

$$D^{\beta}x = f(x), \quad 0 < \beta < 1.$$

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