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The group of self-distributivity is bi-orderable

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Abstract. We prove that the group of left self-distributivity, a cousin of Thompson's group F and of Artin's braid group B_∞ that describes the geometry of the identity $x(yz) = (xy)(xz)$, admits a bi-invariant linear ordering. To this end, we define a partial action of this group on finite binary trees that preserves a convenient linear ordering.

Keywords. Ordered groups, groups acting on trees, self-distributivity, Thompson's groups.

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There exists a close connection between Thompson's group F of [22], [17] and [1], and the associativity identity. Indeed, F acts on bracketed expressions by moving the brackets, *i.e.*, by applying associativity, and, conversely, every application of associativity comes from the action of an element of F. Thus, F can be called the geometry group of associativity, as it captures a number of specific geometrical properties of that identity, in particular those expressed in the well-known MacLane–Stasheff pentagon relation [6] [16].

When we replace the associativity identity $x(yz) = (xy)z$ with the left self-distributivity identity $x(yz) = (xy)(xz)$, Thompson's group F is no longer relevant, but there exists another group G_{LD} that similarly captures the geometrical aspects of the identity. The group G_{LD} happens to be an extension of Artin's braid group B_∞, of which it can be seen as a sort of tree version, a relation that explains the deep connection between braids and the self-distributive law. In the recent years, several new results about braids, in particular the existence of a linear ordering compatible with the product, have been discovered by projecting results initially established in G_{LD} [4], leading in turn to a number of further developments [7], [15], [10], [21]—see [13]. Thus the group G_{LD} (which will be defined by an explicit presentation below) may appear as an interesting object of study.

Order properties have been recently established for various groups connected with topology: besides the orderability of braid groups alluded to above, the orderability of the mapping class groups of surfaces with a nonempty boundary [20], the bi-orderability of the pure braid groups [14], the fact that Artin's braid groups are not bi-orderable in a strong sense [19]. Let us also mention work in progress by D. Rolfsen and B. Wiest about the orderability of knot groups. As for Thompson's group F, it can be realized as a group of diffeomorphisms of a real interval [12], and, as such, it acts on the reals, which easily implies that it is orderable, and even bi-orderable as shows the explicit form of the action [1].

As the group G_{LD} is closely connected both with Thompson's group F and with Artin's braid group B_∞, the question of whether it is bi-orderable, like F, or not bi-orderable, like B_∞, appears natural. It had been shown in [4] that G_{LD} is equipped with a linear left-invariant preordering (which projects on the canonical left-invariant linear ordering of the braids). However, this preordering is not an ordering, and it is not right invariant, so it does not answer the above question. In this paper, we shall prove that, as for orderability, G_{LD} is similar to F, and not to B_∞:

Proposition. *The group G_{LD} is bi-orderable, i.e., there exists a linear ordering on G_{LD} that is compatible with product on both sides.*

Our proof consists in defining an action of G_{LD} that is reminiscent of the action of F on the reals. However, due to an essential technical difference between associativity and self-distributivity, namely the fact that the variable x is repeated twice in the right-hand term of the identity $x(yz) = (xy)(xz)$, there is no natural way to let G_{LD} act on the reals via diffeomorphisms. Instead we shall let G_{LD} act on finite binary rooted trees and observe that this action preserves some linear ordering of such trees. A similar approach is also possible in the case of Thompson's group F, in which case one essentially re-obtains the action of F on $\mathbf{R}$, and, more generally, in the case of analog groups that can be associated with algebraic identities preserving the order of the variables [9].

The organization of the paper is as follows. In Section 1, we recall the definition of the group G_{LD} and introduce its partial action on finite binary trees and, more generally, on terms, which are finite binary trees with labeled leaves. In Section 2, we construct a linear ordering of terms connected with their coding by words using the left Polish form. In Section 3, we show that the action of G_{LD} on terms preserves the previous ordering, and we deduce a bi-invariant ordering on G_{LD}. Finally, in Section 4, we deduce from the action of G_{LD} on finite trees an action of the positive part of G_{LD}—a certain submonoid of G_{LD} of which G_{LD} is the groupe of fractions—on the Cantor line and on the reals.

1. The action of G_{LD} on terms

In this preliminary section, we recall the definition of the group G_{LD}, and its connections with the left self-distributivity identity, with Thompson's group F, and with Artin's braid group B_∞. We also define a partial action of G_{LD} on terms connected with the left self-distributivity identity.

The group G_{LD}

The group G_{LD} is a countable group that describes, in some sense explained below, the geometry of the left self-distributivity identity

$$x(yz) = (xy)(xz). \qquad (LD)$$

We shall define G_{LD} using an explicit presentation. The generators are in one-to-one correspondence with the vertices in a complete binary rooted tree: so we can specify a generator by using a finite sequence of 0's and 1's describing the path from the root to the considered vertex. Such finite sequences will be called *addresses*; we use A for the set of all addresses, and ϕ for the empty address, *i.e.*, the address of the root (Figure 1.1). For $\alpha, \beta \in A$, $\alpha\beta$ denotes the concatenation of α and β. We say that two addresses α, β are orthogonal, written $\alpha \perp \beta$, if there exists an adress γ such that $\gamma 0$ is a prefix of α and $\gamma 1$ is a prefix of β, or vice versa.

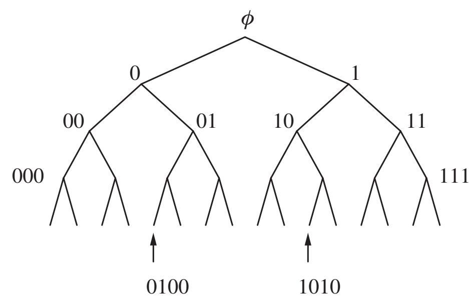


Figure 1.1. Binary addresses.

Definition. We denote by G_{LD} the group $\langle \{g_\alpha\,;\, \alpha \in A\}\,;\, R_{LD}\rangle$, where R_{LD} consists of the following five families of relations:

$$\begin{aligned}
g_\alpha \cdot g_\beta &= g_\beta \cdot g_\alpha \qquad \text{for } \alpha \perp \beta, && \text{(type } \perp)\\
g_{\alpha 0\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 10\beta} \cdot g_{\alpha 00\beta}, && \text{(type 0)}\\
g_{\alpha 10\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 01\beta}, && \text{(type 10)}\\
g_{\alpha 11\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 11\beta}, && \text{(type 11)}\\
g_{\alpha 1} \cdot g_\alpha \cdot g_{\alpha 1} \cdot g_{\alpha 0} &= g_\alpha \cdot g_{\alpha 1} \cdot g_\alpha. && \text{(type 1)}
\end{aligned}$$

Let us recall that Artin's braid group B_∞ can be defined as the group generated by an infinite sequence $\sigma_1, \sigma_2, \ldots$ subject to the relations

$$\begin{aligned}
\sigma_i \cdot \sigma_j &= \sigma_j \cdot \sigma_i \qquad \text{for } |i-j| \geq 2, && \text{type (i)}\\
\sigma_{i+1} \cdot \sigma_i \cdot \sigma_{i+1} &= \sigma_i \cdot \sigma_{i+1} \cdot \sigma_i. && \text{type (ii)}
\end{aligned}$$

Then, from the presentation, it is obvious that the mapping

$$\mathrm{pr} : g_\alpha \mapsto \begin{cases} \sigma_i & \text{for } \alpha = 1^{i-1} \ (\textit{i.e.}, 1 \text{ repeated } i-1 \text{ times}), \\ 1 & \text{if } \alpha \text{ contains at least one } 0 \end{cases}$$

defines a surjective homomorphism of G_{LD} onto B_∞: B_∞ is what remains from G_{LD} when we collapse every generator associated with a vertex of the complete binary tree not lying on the right branch. As B_∞ is not finitely generated, G_{LD} is not either finitely generated. The kernel of the projection of G_{LD} onto B_∞ is large (and complicated): if H_i denotes the parabolic subgroup of G_{LD} generated by all generators g_α with α beginning with 1^i0, then, by type $\perp$ relations, the elements of H_i and H_j commute for $i \neq j$, and Ker(pr) includes the direct product $H_0 \times H_1 \times \cdots$. It can then be shown that, for every i, the mapping $g_\alpha \mapsto g_{1^i0\alpha}$ induces an isomorphism of G_{LD} onto H_i. More generally, a parabolicity theorem asserts that, for every address γ, mapping g_α to $g_{\gamma\alpha}$ defines an isomorphism of G_{LD} onto the subgroup of G_{LD} generated by those generators g_β such that β begins with γ.

The syntactic form of the relations R_{LD} defining G_{LD} is reminiscent of the Coxeter relations that define Artin groups, though they do not preserve the length and are not symmetric. It is proved in [4] and [5] that most of the tools developed by Garside in his study of braid groups [11] can be extended to groups defined by such generalized Coxeter relations. The specific case of G_{LD} is made difficult by the fact that, in contradistinction to B_∞, G_{LD} is not the inductive limit of an increasing family of groups of finite type. However, by introducing local counterparts to Garside's fundamental braids Δ_n, one can extend some of the results, and, in particular, prove that G_{LD} is a group of fractions:

Proposition 1.1. [4], [8] *Let G_{LD}^+ be the submonoid of G_{LD} generated by the elements g_α with $\alpha \in A$. Then every element of G_{LD} can be written as ab^{-1} with $a, b \in G_{LD}^+$.*

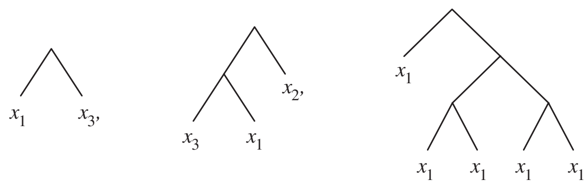
We claim nothing about the presentation of the monoid G_{LD}^+: whether G_{LD}^+ admits, as a monoid, the above presentation of G_{LD} is currently unknown.

Terms and trees

Terms will play a central role in the sequel. Several equivalent definitions are possible. For our current purpose, it will be convenient to consider terms as finite trees.

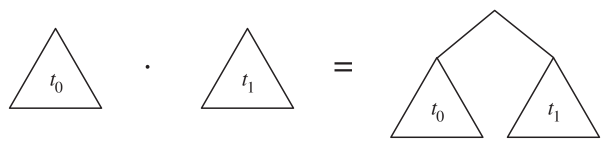
Definition. Let $x_1, x_2, \ldots$ be a fixed sequence of variables (= letters); a *term* is defined to be a finite binary rooted tree whose leaves (*i.e.*, vertices of degree 1) wear labels in $\{x_1, x_2, \ldots\}$. We write T_∞ for the set of all terms, and T_1 for the subset of T_∞ consisting of those terms where all leaves are labeled x_1.

Thus,



are typical terms in T_∞. The latter belongs to T_1. In the case of T_1, we can of course forget about the labels, and identify a term with an unlabeled tree.

Terms are equipped with a natural product, namely the operation that associates with two terms t_0, t_1 the term, denoted $t_0{\cdot}t_1$, consisting of a root with two successors, a left one which is t_0, and a right one which is t_1:



Then, provided we identify the variable x_i with the tree consisting of a single vertex labeled x_i, $(T_\infty, \cdot)$ is a free magma based on $\{x_1, x_2, \dots\}$, and $(T_1, \cdot)$ is a free magma based on $\{x_1\}$.

Each vertex in a finite binary rooted tree can be specified by an address in A describing the path from the root to that vertex. For t a term, we define the *outline* of t to be the collection of all addresses of leaves in (the tree associated with) t, and the *skeleton* of t to be the collection of the addresses of vertices in t: thus, for instance, the outline of the term $(x_3{\cdot}x_1){\cdot}x_2$ is the set $\{00, 01, 1\}$, while its skeleton is $\{00, 01, 0, 1, \phi\}$, as t comprises three leaves and two inner vertices.

For t a term, and α an address in the skeleton of t, we have the natural notion of the α-subterm of t, denoted $\mathrm{sub}(t, \alpha)$: this is the subtree of t whose root lies at address α. This amounts to defining inductively

$$\mathrm{sub}(t, \alpha) = \begin{cases} t & \text{if } t \text{ is a variable or } \alpha = \lambda \text{ holds,} \\ \mathrm{sub}(t_0, \beta) & \text{for } t = t_0{\cdot}t_1 \text{ and } \alpha = 0\beta, \\ \mathrm{sub}(t_1, \beta) & \text{for } t = t_0{\cdot}t_1 \text{ and } \alpha = 1\beta. \end{cases}$$

For instance, the 0-subterm of the term $(x_3{\cdot}x_1){\cdot}x_2$ is the term $x_3{\cdot}x_1$, its 01-subterm is the term x_1, while its 010-subterm is not defined. Observe that the outline of a term t is the set of those addresses α such that $\mathrm{sub}(t, \alpha)$ is a variable.

The action of G_{LD} on terms

We shall now describe the connection between the group G_{LD} and the left self-distributivity identity by means of a partial actions of G_{LD} on terms.

In the sequel, a set equipped with a left self-distributive operation will be called an *LD-system* (the names LD-magma and LD-groupoid have also been used occasionally). Let us say that two terms t, t' in T_∞ are *LD-equivalent*, denoted $t =_{LD} t'$, if we can transform t to t' by repeatedly applying Identity (LD). By standard arguments, the quotient structure $T_\infty/{=_{LD}}$ is a free LD-system based on $\{x_1, x_2, \dots\}$, and studying free LD-systems amounts to studying LD-equivalence of terms.

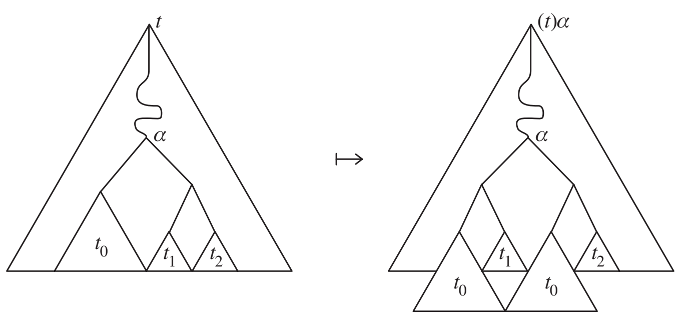
Applying the left self-distributivity identity to a term t consists in replacing some subterm of t which has the form $t_1{\cdot}(t_2{\cdot}t_3)$ with the corresponding term $(t_1{\cdot}t_2){\cdot}(t_1{\cdot}t_3)$, or *vice versa*. Having defined the α-subterm of a term precisely, we can take into account the position, *i.e.*, the address, of the subterm where the identity is applied. This leads to defining a partial action on T_∞ of the free monoid $(A \cup A^{-1})^*$ generated by A and a disjoint copy A^{-1} of A comprising a formal inverse α^{-1} for each address α.

Definition. (i) For t a term, and α an address such that the α-subterm of t exists and can be written as $t_1{\cdot}(t_2{\cdot}t_3)$, we define $(t)\alpha$ to be the term obtained from t by replacing the α-subterm with the corresponding term $(t_1{\cdot}t_2){\cdot}(t_1{\cdot}t_3)$.

(ii) For t a term, and α an address, we define $(t)\alpha^{-1}$ to be the unique term t' verifying $t = (t')\alpha$, when it exists.

(iii) For t a term, and w a word on $A \cup A^{-1}$, say $w = \alpha_1^{e_1}{\cdot}\ldots{\cdot}\alpha_p^{e_p}$, with $\alpha_i \in A$ and $e_i = \pm 1$, we define $(t)w$ to be $(\ldots((t)\alpha_1^{e_1})\alpha_2^{e_2}\ldots)\alpha_p^{e_p}$, when it exists.

Thus $(t)\alpha$ is the term obtained by expanding t at α using left self-distributivity:



Example 1.2. Let $t = x_1{\cdot}x_2{\cdot}x_3{\cdot}x_4$—here, and everywhere in the sequel, we take the convention that missing brackets are to be added on the right, so, for instance, the previous expression stands for $x_1{\cdot}(x_2{\cdot}(x_3{\cdot}x_4))$—then the only addresses α for which $(t)\alpha$ exists are λ and 1, and we have $(t)\phi = (x_1{\cdot}x_2){\cdot}(x_1{\cdot}x_3{\cdot}x_4)$, and $(t)1 = x_1{\cdot}(x_2{\cdot}x_3){\cdot}(x_2{\cdot}x_4)$.

By construction, the term $(t)\alpha$ is defined if and only if the address $\alpha 10$ belongs to the skeleton of t, and that $(t)\alpha^{-1}$ exists if and only if the addresses $\alpha 00$ and $\alpha 10$ belong to the skeleton of t, and, in addition, $\mathrm{sub}(t, \alpha 10) = \mathrm{sub}(t, \alpha 00)$ holds.

We thus have obtained a partial right action of the free monoid $(A \cup A^{-1})^*$ on the set T_∞. By construction, we have:

Lemma 1.3. *Two terms t, t' in T_∞ are LD-equivalent if and only if $t' = (t)w$ holds for some word w in $(A \cup A^{-1})^*$.*

The previous action is partial, *i.e.*, not everywhere defined, in essence. In particular, there exist words w such that $(t)w$ is defined for no term t: this happens for instance for $w = \phi{\cdot}1{\cdot}\phi^{-1}$, as, by construction, no term of the form $(t)\phi{\cdot}1$ may have equal subterms at 00 and 10, hence be eligible for the action of ϕ^{-1}. This situation is unpleasant, but—in contradistinction to easier cases like the case of associativity—there exists no way of avoiding it by using a convenient quotient or subset, or by replacing groups by groupoids (small categories with inverse).

By definition, the group G_{LD} is a quotient of the free group generated by the g_α's, $\alpha \in A$, hence of the free monoid $(A \cup A^{-1})^*$: for w a word on $A \cup A^{-1}$, we denote by $\overline{w}$ the image of w in G_{LD} under the homomorphism that maps α to g_α and α^{-1} to g_α^{-1}.

The connection between G_{LD} and left self-distributivity comes from the fact that the partial action of $(A \cup A^{-1})^*$ on terms described above factors through G_{LD} and the resulting action is faithful in the following sense:

Proposition 1.4. [4] *Assume that w, w' are words on $A \cup A^{-1}$ and there exists at least one term t such that both $(t)w$ and $(t)w'$ are defined Then the following are equivalent:*

(i) There exists at least one term t satisfying $(t)w = (t)w'$;

(ii) For every term t such that $(t)w$ and $(t)w'$ exist, we have $(t)w = (t)w'$;

(iii) The words w and w' represent the same element of G_{LD}.

In the particular case when w and w' are words on A, the condition that there exists at least one term t such that both $(t)w$ and $(t)w'$ are defined is always satisfied.

The previous statements may appear convoluted, but, because there exist words w such that $(t)w$ is defined for no t, there is no way to obtain a simpler statement: the action of α^{-1} is not an exact inverse of the action of α, as $(t)\alpha{\cdot}\alpha^{-1} = t$ holds only if $(t)\alpha$ is defined. The proof of Proposition 1.4 is delicate: as one can expect, it is not very difficult to check that (iii) implies (ii), *i.e.*, that the action factors through G_{LD}, but proving that (i) implies (iii), *i.e.*, that the factorized action is faithful, requires a nontrivial argument.

Owing to the previous result, we obtain a well-defined partial action of G_{LD} on T_∞: for t a term, and a in G_{LD}, we define $(t)a$ to be $(t)w$ where w is any word on $A \cup A^{-1}$ that represents a and is such that $(t)w$ exists, if such a word exists.

The action is partial, as there exist some elements a of G_{LD}, like $g_\phi g_1 g_\phi^{-1}$, such that $(t)w$ exist for no expression w of a, but it is well-defined in the sense that, if w and w' are distinct expressions of some element a such that both $(t)w$ and $(t)w'$ exist, then the latter terms are equal.

Lemma 1.3 and Proposition 1.4 immediately yield:

Proposition 1.5. *For every term t, the LD-equivalence class of t is the orbit of t under the (partial) action of G_{LD}, and this action is faithful: we have $t' =_{LD} t$ if and only if $t' = (t)a$ holds for some a in G_{LD}, and, in this case, the involved element a is unique.*

This statement should make it natural to call G_{LD} the geometry group of Identity (LD).

The connection between G_{LD} and Thompson's group F

A similar approach can be developed when left self-distributivity is replaced with associativity. This amounts to considering an alternative action, here denoted $\bullet$, of the free monoid $(A \cup A^{-1})^*$ on terms, namely the action obtained by replacing the basic instance

$$(t_1 \cdot (t_2 \cdot t_3))\,\phi = (t_1 \cdot t_2) \cdot (t_1 \cdot t_3)$$

with

$$(t_1 \cdot (t_2 \cdot t_3)){\bullet}\phi = (t_1 \cdot t_2) \cdot t_3.$$

Studying the $\bullet$-action leads to introducing new relations, and, therefore, to a new group.

Definition. We denote by G_A the group $\langle \{g_\alpha\,;\,\alpha \in A\}\,;\,R_A\rangle$, where R_A consists of

$$\begin{aligned}
g_\alpha \cdot g_\beta &= g_\beta \cdot g_\alpha \qquad \text{for } \alpha \perp \beta, && \text{(type } \perp\text{)}\\
g_{\alpha 0\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 00\beta}, && \text{(type 0)}\\
g_{\alpha 10\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 01\beta}, && \text{(type 10)}\\
g_{\alpha 11\beta} \cdot g_\alpha &= g_\alpha \cdot g_{\alpha 1\beta}, && \text{(type 11)}\\
g_{\alpha 1} \cdot g_\alpha \cdot g_{\alpha 0} &= g_\alpha \cdot g_\alpha. && \text{(type 1)}
\end{aligned}$$

It can now be proved that the (partial) action $\bullet$ of $(A \cup A^{-1})^*$ on T_∞ factors through G_A, and, if we say that two terms are A-equivalent if we can transform the first into the second using asociativity, we have the following counterpart to Proposition 1.5:

Proposition 1.6. [6] *For every term t, the A-equivalence class of t is the orbit of t under the (partial) action of G_A, and this action is faithful: the term t' is A-equivalent to t if and only if $t' = (t)\bullet a$ holds for some a in G_A, and, in this case, the involved element a is unique.*

Thus, the group G_A is an exact counterpart to the group G_{LD}. From a technical point of view, the results and the proofs are much easier in the case of associativity because the action is never empty in the latter case.

Proposition 1.7. *The group G_A is (isomorphic to) Thompson's group F.*

Proof. (sketch) One of the standard presentations of F is [1]

$$\langle X_0, X_1, X_2, \dots\, ;\, X_k^{-1} X_n X_k = X_{n+1} \text{ for } k < n\rangle.$$

Let us consider the elements g_{1^i} in G_A. An induction on the number of 0's in α shows that, for every address α, g_α belongs to the subgroup of G_A generated by the g_{1^i}'s, *i.e.*, the elements g_{1^i} generate G_A. Moreover, for $k < n$, we have ${g_{1^k}}^{-1} g_{1^n} g_{1^k} = g_{1^{n+1}}$ by type 11 relations. Hence the mapping $X_i \mapsto g_{1^i}$ induces a surjective morphism of F onto G_A. Conversely, for each address α, we define an element Y_α in F inductively on the number of 0's in α by $Y_\alpha = X_i$ for $\alpha = 1^i$, and

$$Y_\alpha = Y_\beta^{-1} Y_{\beta 1}^{-1} \cdots Y_{\beta 1^{k-1}} Y_{\beta 1^k}^{-1} Y_{\beta 1^{k+1}}^{-1} Y_{\beta 1^k}^2 Y_{\beta 1^{k-1}} \cdots Y_{\beta 1} Y_\beta$$

for $\alpha = \beta 0 1^k$. The elements Y_α satisfy the relations R_A, so $g_\alpha \mapsto Y_\alpha$ induces a surjective morphism of G_A onto F, which is the inverse of the above morphism of F onto G_A. $\square$

Let us mention that a similar approach can be developed for every family of algebraic identities, and refer to [9], where studying the associated group leads to a solution of the word problem of the identity $x(yz) = (xy)(yz)$.

2. A linear ordering on finite binary trees

Terms (*i.e.*, finite labeled binary trees) can be equipped with several orderings. Here we consider the linear ordering on T_∞ that uses the left height as a discriminant, the latter being defined as the length of the leftmost branch in the associated tree. To make the definition precise, we encode every term by a word and then use a lexicographical ordering.

Definition. For t a term, the *left Polish form* of t is the word $[\![t]\!]$ over the alphabet $\{x_1, x_2, \dots, \bullet\}$ defined by the following inductive clauses:

$$[\![t]\!] = \begin{cases} t & \text{if } t \text{ is a variable,} \\ \bullet[\![t_1]\!][\![t_2]\!] & \text{for } t = t_1{\cdot}t_2. \end{cases}$$

For instance, the left Polish form of the term $x_1{\cdot}(x_2{\cdot}x_3{\cdot}x_4){\cdot}x_5$ is the word $\bullet x_1\bullet\bullet x_2\bullet x_3x_4x_5$. When the term t is viewed as a tree, the word $[\![t]\!]$ is obtained by enumerating the variables of t from left to right and letting each occurrence of a variable be preceded by as many letters $\bullet$ as there are final 0's in the corresponding address. For w a word over the alphabet $\{x_1, x_2, \ldots, \bullet\}$, we denote by $\#_x(w)$ and $\#_\bullet(w)$ the number of letters x_i and of letters $\bullet$ in w respectively. By standard arguments, we have the following characterization:

Lemma 2.1. *Assume that w is a word over the alphabet $\{x_1, x_2, \ldots, \bullet\}$. Then w is the left Polish form of a well formed term if and only if we have $\#_x(w) = \#_\bullet(w)+1$, and $\#_x(u) \le \#_\bullet(u)$ for every proper prefix u of w.*

Definition. Assume that t_1, t_2 are terms in T_∞. We say that $t_1 <_L t_2$ holds if the word $[\![t_1]\!]$ precedes the word $[\![t_2]\!]$ in the lexicographical extension of the linear ordering $x_1 < x_2 < \cdots < \bullet$.

By construction, the relation $<_L$ is a linear ordering on T_∞, and x_1 is minimal for $<_L$. If $\mathrm{ht}_L(t)$ denotes the left height of the term t, the word $[\![t]\!]$ begins with $\mathrm{ht}_L(t)$ letters $\bullet$ followed by a variable. So, $\mathrm{ht}_L(t_1) < \mathrm{ht}_L(t_2)$ implies $t_1 <_L t_2$.

Lemma 2.2. *The inequality $t_1 <_L t_2$ implies $t_1{\cdot}t_3 <_L t_2{\cdot}t_4$ for all terms t_3, t_4.*

Proof. Lemma 2.1 implies that a proper prefix of the left Polish form of a term is never the left Polish form of a well formed term. Hence $t_1 <_L t_2$ holds if and only if the words $[\![t_1]\!]$ and $[\![t_2]\!]$ have a variable clash of the type "variable *vs.* $\bullet$". Then the words $[\![t_1{\cdot}t_3]\!]$ and $[\![t_2{\cdot}t_4]\!]$, *i.e.*, $\bullet[\![t_1]\!][\![t_3]\!]$ and $\bullet[\![t_2]\!][\![t_4]\!]$, have a similar clash. □

We deduce several equivalent characterizations of $<_L$.

Lemma 2.3. *Assume that t_1, t_2 are terms in T_∞. If t_1 is a variable, say x_i, then $t_1 <_L t_2$ holds unless t_2 is a variable x_j with $j < i$. If t_1 is not a variable, then $t_1 <_L t_2$ holds if and only if either $\mathrm{sub}(t_1, 0) <_L \mathrm{sub}(t_2, 0)$ holds, or $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$ and $\mathrm{sub}(t_1, 1) <_L \mathrm{sub}(t_2, 1)$ hold.*

Proof. Assume $t_1 <_L t_2$, and neither t_1 nor t_2 are variables. Three cases are possible. For $\mathrm{sub}(t_1, 0) <_L \mathrm{sub}(t_2, 0)$, Lemma 2.2 implies $t_1 <_L t_2$. For $\mathrm{sub}(t_1, 0) >_L \mathrm{sub}(t_2, 0)$, we obtain $t_1 >_L t_2$ symmetrically. Finally, for $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$, $t_1 <_L t_2$ is equivalent to $\mathrm{sub}(t_1, 1) <_L \mathrm{sub}(t_2, 1)$ by definition. □

In order to state the next result, we need the easy notion of the left edge of an address.

Definition. For α an address, the *left edge of* α is the finite sequence $(\alpha_1 0, \ldots, \alpha_p 0)$, where $\alpha_1, \ldots, \alpha_p$ are those prefixes of α such that $\alpha_1 1, \ldots, \alpha_p 1$ are prefixes of α, enumerated in increasing order.

For instance, the left edge of 010011 is the sequence $(00, 01000, 010010)$. As an induction shows, the length of the left edge of the address α is the number of 1's in α.

Lemma 2.4. *Assume that t_1, t_2 are terms in T_∞. Then the following are equivalent:*

(i) The relation $t_1 <_L t_2$ holds;

(ii) There exists an address α in the skeletons both of t_1 and of t_2 such that $\mathrm{sub}(t_1, \beta) = \mathrm{sub}(t_2, \beta)$ *holds for every β in the left edge of α, and* $\mathrm{sub}(t_1, \alpha) <_L \mathrm{sub}(t_2, \alpha)$ *holds.*

(iii) There exists an address α both in the outline of t_1 and in the skeleton of t_2 such that $\mathrm{sub}(t_1, \beta) = \mathrm{sub}(t_2, \beta)$ *holds for every β in the left edge of α, and either* $\mathrm{sub}(t_2, \alpha)$ *is a variable larger than* $\mathrm{var}(t_1, \alpha)$*, or it is not a variable.*

Proof. An induction on α shows that, if α belongs to the skeleton of the term t and $(\alpha_1, \dots, \alpha_p)$ is the left edge of α, then the word $[\![t]\!]$ begins with

$$\bullet^{k_1} [\![\mathrm{sub}(t, \alpha_1)]\!] \bullet^{k_2} [\![\mathrm{sub}(t, \alpha_2)]\!] \dots \bullet^{k_p} [\![\mathrm{sub}(t, \alpha_p)]\!] \bullet^{k} [\![\mathrm{sub}(t, \alpha)]\!], \tag{2.1}$$

where k_i is the number of final 0's in α_i and k is the number of final 0's in α. The result is obvious for $\alpha = \phi$, and, otherwise, it follows from an easy induction on t. Then, by definition of a lexicographical ordering, it follows from (2.1) and from the fact that a proper prefix of a left Polish form is never the left Polish form of a well formed term that (ii) implies (i).

By construction, (iii) implies (ii). Finally, assuming (i), and letting α be the address of the first position where the words $[\![t_1]\!]$ and $[\![t_2]\!]$ disagree, we obtain (iii) using the explicit expansion of (2.1). $\square$

For the next result, we introduce another preordering on terms.

Definition. Assume that t_1, t_2 are terms. We say that $t_1 \sqsubseteq t_2$ holds if and only if t_1 is an iterated left subterm of t_2, *i.e.*, $t_1 = \mathrm{sub}(t_2, 0^k)$ holds for some $k \geq 0$. We say that $t_1 \sqsubseteq_{LD} t_2$ holds if there exist two terms t'_1, t'_2 satisfying $t'_1 =_{LD} t_1$, $t'_2 =_{LD} t_2$, and $t'_1 \sqsubseteq t'_2$.

It is known [4] that the relation $\sqsubseteq_{LD}$ induces an ordering on $T_\infty/{=_{LD}}$, whose restriction to $T_1/{=_{LD}}$ is linear.

Lemma 2.5. *Assume that t_1, t_2 are $\sqsubseteq_{LD}$-comparable terms in T_∞. Then the following are equivalent:*

(i) The relation $t_1 <_L t_2$ holds;

(ii) There exists an address α in both in the outline of t_1 and in the skeleton of t_2 such that $\mathrm{sub}(t_1, \beta) = \mathrm{sub}(t_2, \beta)$ *holds for every β in the left edge of α, and* $\mathrm{sub}(t_2, \alpha)$ *is a term of left height at least* 1 *whose leftmost variable is* $\mathrm{var}(t_1, \alpha)$*.*

Proof. Assume (i). Then there exists an address α satisfying the conditions of Lemma 2.4(iii). We claim that $\mathrm{sub}(t_2, \alpha)$ cannot be a variable. Indeed, assume $\mathrm{sub}(t_1, \alpha) = x_i$ and $\mathrm{sub}(t_2, \alpha) = x_j$ with $j > i$. Let $(\alpha_1, \dots, \alpha_q)$ denote the left edge of α. Let us consider for a while the right Polish form of terms: for t a term, we denote by $[t]$ the word inductively defined by: $[t] = t$ if t is a variable, and $[t] = [t_1][t_2]\bullet$ for $t = t_1{\cdot}t_2$. Then, the word $[t_1]$ begins with $[\mathrm{sub}(t_1, \alpha_1)] \dots [\mathrm{sub}(t_1, \alpha_q)]x_i$, while the word $[t_2]$ begins with $[\mathrm{sub}(t_1, \alpha_1)] \dots [\mathrm{sub}(t_1, \alpha_q)]x_j$. By the results of [4], this is known to contradict the hypothesis that t_1 and t_2 are $\sqsubseteq_{LD}$-comparable. So the only possibility is that $\mathrm{sub}(t_2, \alpha)$ is not a variable, and that its leftmost variable is x_i. This gives (ii). That (ii) implies (i) follows from Lemma 2.4. $\square$

The left ordering of terms satisfies several invariance properties. Let us define a *substitution* to be a mapping of $\{x_1, x_2, \dots\}$ into T_∞. If h is a subtitution and t is a term in T_∞, we denote by t^h the term obtained from t by replacing each variable x_i occurring in t with the corresponding term $h(x_i)$. Note that the mapping $t \mapsto t^h$ is an endomorphism of the free magma $(T_\infty, \cdot)$, and that every endomorphism of $(T_\infty, \cdot)$ has this form.

Proposition 2.6. *Assume that t_1, t_2 are terms in T_∞ and h is a substitution of T_∞. Assume in addition that at least one of the following conditions holds:*

(i) We have $h(x_i) <_L h(x_{i+1})$ and $\mathrm{ht}_L(h(x_i)) = \mathrm{ht}_L(h(x_{i+1}))$ for every i;

(ii) The terms t_1 and t_2 are $\sqsubseteq_{LD}$-comparable.

Then $t_1 <_L t_2$ holds if and only if $t_1^h <_L t_2^h$ does.

Proof. As $<_L$ is a linear ordering, it suffices that we show that $t_1 <_L t_2$ implies $t_1^h <_L t_2^h$. So assume $t_1 <_L t_2$. By Lemma 2.4, there exists an address α such that $\mathrm{sub}(t_1, \beta) = \mathrm{sub}(t_2, \beta)$ holds for every β in the left edge of α, $\mathrm{sub}(t_1, \alpha)$ is a variable say x_i, and $\mathrm{sub}(t_2, \alpha)$ is either a variable x_j with $j > i$, or it is a term that is not a variable. When Condition (ii) holds, by Lemma 2.5, we can assume in addition that $\mathrm{sub}(t_2, \alpha)$ is a term with leftmost variable x_i and left height at least 1. Applying the substitution h, we obtain $\mathrm{sub}(t_1^h, \beta) = \mathrm{sub}(t_2^h, \beta)$ for every β in the left edge of α. Then we have $\mathrm{sub}(t_1^h, \alpha) = h(x_i)$. Three cases are to be considered.

If Condition (i) holds and we have $\mathrm{sub}(t_2, \alpha) = x_j$ with $j > i$, we obtain $\mathrm{sub}(t_2^h, \alpha) = h(x_j) >_L h(x_i) = \mathrm{sub}(t_1^h, \alpha)$. If Condition (i) holds and $\mathrm{sub}(t_2, \alpha)$ is not a variable, the hypothesis on h implies $\mathrm{ht}_L(\mathrm{sub}(t_2^h, \alpha)) > \mathrm{ht}_L(\mathrm{sub}(t_1^h, \alpha))$, hence $\mathrm{sub}(t_1^h, \alpha) <_L \mathrm{sub}(t_2^h, \alpha)$. Finally, if Condition (ii) holds and $\mathrm{sub}(t_2, \alpha)$ is a term with leftmost variable x_i and left height $k \geq 1$, we find $\mathrm{ht}_L(\mathrm{sub}(t_1^h, \alpha)) = \mathrm{ht}_L(h(x_i))$, and $\mathrm{ht}_L(\mathrm{sub}(t_2^h, \alpha)) = \mathrm{ht}_L(h(x_i)) + k$, hence $\mathrm{sub}(t_1^h, \alpha) <_L \mathrm{sub}(t_2^h, \alpha)$. So, $\mathrm{sub}(t_1^h, \alpha) <_L \mathrm{sub}(t_2^h, \alpha)$ holds in every case. By Lemma 2.4, this implies $t_1^h <_L t_2^h$. $\square$

Definition. For t a term in T_∞, we denote by $t^\dagger$ the projection of t in T_1, *i.e.*, the image of t under the substitution that maps every variable to x_1.

Corollary 2.7. *(i) Every substitution of T_1 preserves the ordering $<_L$.*
(ii) If t_1 and t_2 are $\sqsubseteq_{LD}$-comparable terms, $t_1 <_L t_2$ is equivalent to $t_1^\dagger <_L t_2^\dagger$.

Other characterizations of the linear ordering $<_L$ can be mentioned. For instance, if we assume that t, t_1, t_2 are terms and the outline of t is included in the skeleton of t_1 and t_2, then, letting $(\alpha_1, \ldots, \alpha_p)$ be the left–right enumeration of the outline of t, $t_1 <_L t_2$ holds if and only if the sequence $(\mathrm{sub}(t_1, \alpha_1), \ldots, \mathrm{sub}(t_1, \alpha_p))$ precedes the sequence $(\mathrm{sub}(t_2, \alpha_1), \ldots, \mathrm{sub}(t_2, \alpha_p))$ in the lexicographical extension of $<_L$ to T_∞^*.

In the special case of T_1, it can also be checked that $t_1 <_L t_2$ holds if and only if the left–right increasing enumeration of the outline of t_1 precedes the left–right increasing enumeration of the outline of t_2 with respect to the lexicographical extension of the prefix ordering of addresses to A^*.

3. The linear ordering on G_{LD}

We use now the partial action of the group G_{LD} on the linearly ordered set $(T_\infty, <_L)$ to define a linear ordering on G_{LD}. The ordering so defined has nice properties, in particular it is compatible with multiplication on both sides, so G_{LD} is a bi-orderable group.

The first step is to prove that the action of G_{LD} on T_∞ preserves the ordering $<_L$.

Proposition 3.1. *For all terms t_1, t_2 in T_∞, and every a in G_{LD} such that $(t_1)a$ and $(t_2)a$ exist, $t_1 <_L t_2$ holds if and only if $(t_1)a <_L (t_2)a$ does.*

Proof. As G_{LD} is generated by the elements g_α with $\alpha \in A$, it suffices to prove the result for the latter elements, *i.e.*, to prove that, if α is an address, and t_1, t_2 are terms then $t_1 <_L t_2$ is equivalent to $(t_1)\alpha <_L (t_2)\alpha$ when the latter terms are defined. As the action of α is injective, it suffices to prove that $t_1 <_L t_2$ implies $(t_1)\alpha <_L (t_2)\alpha$. We use induction on α. Assume first that α is the empty address. The hypothesis that $(t_1)\phi$ and $(t_2)\phi$ exist implies that $\mathrm{sub}(t_e, 0)$, $\mathrm{sub}(t_e, 10)$, and $\mathrm{sub}(t_e, 11)$ exist for $e = 1, 2$, and we have the explicit decompositions

$$[\![t_e]\!] = \bullet[\![\mathrm{sub}(t_e, 0)]\!]\bullet[\![\mathrm{sub}(t_e, 10)]\!][\![\mathrm{sub}(t_e, 11)]\!],$$

and

$$[\![(t_e)\lambda]\!] = \bullet\bullet[\![\mathrm{sub}(t_e, 0)]\!][\![\mathrm{sub}(t_e, 10)]\!]\bullet[\![\mathrm{sub}(t_e, 0)]\!][\![\mathrm{sub}(t_e, 11)]\!].$$

By Lemma 2.3, only three cases are possible, namely

- $\mathrm{sub}(t_1, 0) <_L \mathrm{sub}(t_2, 0)$, or
- $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$ and $\mathrm{sub}(t_1, 10) <_L \mathrm{sub}(t_2, 10)$, or

- $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$, $\mathrm{sub}(t_1, 10) = \mathrm{sub}(t_2, 10)$ and $\mathrm{sub}(t_1, 11) <_L \mathrm{sub}(t_2, 11)$,

and the result is clear in each case.

Assume now $\alpha = 0\beta$. Then we have $(t_e)\alpha = \mathrm{sub}(t_e, 0)\beta{\cdot}\mathrm{sub}(t_e, 1)$. Two cases are possible. For $\mathrm{sub}(t_1, 0) <_L \mathrm{sub}(t_2, 0)$, by induction hypothesis, we have $\mathrm{sub}(t_1, 0)\beta <_L \mathrm{sub}(t_2, 0)\beta$, and, therefore, $(t_1)\alpha <_L (t_2)\alpha$. For $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$ and $\mathrm{sub}(t_1, 1) <_L \mathrm{sub}(t_2, 1)$, we have $\mathrm{sub}(t_1, 0)\beta = \mathrm{sub}(t_2, 0)\beta$, and, again, $(t_1)\alpha <_L (t_2)\alpha$.

Assume finally $\alpha = 1\beta$. Then we have $(t_e)\alpha = \mathrm{sub}(t_e, 0){\cdot}\mathrm{sub}(t_e, 1)\beta$. Two cases are possible again. For $\mathrm{sub}(t_1, 0) <_L \mathrm{sub}(t_2, 0)$, we deduce $(t_1)\alpha <_L (t_2)\alpha$ directly. For $\mathrm{sub}(t_1, 0) = \mathrm{sub}(t_2, 0)$ and $\mathrm{sub}(t_1, 1) <_L \mathrm{sub}(t_2, 1)$, the latter inequality implies $\mathrm{sub}(t_1, 1)\beta <_L \mathrm{sub}(t_2, 1)\beta$ by induction hypothesis, and we deduce $(t_1)\alpha <_L (t_2)\alpha$ again. □

Lemma 3.2. *Assume that a belongs to $G^+_{LD} \setminus \{1\}$. Then $t <_L (t)a$ holds whenever $(t)a$ is defined.*

Proof. It suffices to consider the case of a single address α. If α is the empty address, the result follows from the equality $\mathrm{ht}_L((t)\phi) = \mathrm{ht}_L(t) + 1$. Otherwise, we use an induction on α, or simply resort to Lemma 2.4: by the previous argument, we have $\mathrm{sub}(t, \alpha) <_L \mathrm{sub}((t)\alpha, \alpha)$, and, by construction, $\mathrm{sub}(t, \beta) = \mathrm{sub}((t)\alpha, \beta)$ holds for every β in the left edge of α. □

The next step consists in using the action of the submonoid G^+_{LD} of G_{LD} on T_∞ to order G^+_{LD}. For each element a of G^+_{LD}, we shall need a characterization of those terms t for which $(t)a$ is defined. Let us say that a term t is *canonical* if the list of all variables that occur in t, enumerated from left to right ignoring repetitions, is an initial segment of $(x_1, x_2, \dots)$. The following result is proved in [3] (in a general framework).

Proposition 3.3. *Assume that $a_1, \dots, a_k$ are elements of G^+_{LD}. Then there exists a unique canonical term $t_L(a_1, \dots, a_k)$ such that, for every term t, the terms $(t)a_1, \dots, (t)a_k$ all are defined if and only if $t = t_L(a_1, \dots, a_k)^h$ holds for some substitution h.*

Lemma 3.4. *For $a, b \in G^+_{LD}$, the following are equivalent:*

(i) There exists a term t in T_∞ such that $(t)a <_L (t)b$ holds;

(ii) The inequality $(t_L(a, b))a <_L (t_L(a, b))b$ holds;

(iii) For every term t in T_∞ such that $(t)a$ and $(t)b$ exist, $(t)a <_L (t)b$ holds.

Proof. That (ii) implies (i) and (iii) implies (ii) is clear. So assume (i). By construction, there exists a substitution h satisfying $t = t_L(a, b)^h$, and our hypothesis is the inequality $(t_L(a, b)^h)a <_L (t_L(a, b)^h)b$, *i.e.*, $((t_L(a, b))a)^h <_L ((t_L(a, b))b)^h$. The terms $(t_L(a, b))a$ and $(t_L(a, b))b$ are LD-equivalent, hence, by Proposition 2.6, the previous inequality is equivalent to $(t_L(a,b))a <_L (t_L(a,b))b$, which gives (ii), and, then, to $((t_L(a,b))a)^g <_L ((t_L(a,b))b)^g$ for every substitution g, which gives (iii). $\square$

Definition. For $a, b \in G_{LD}^+$, we say that $a < b$ holds if the equivalent conditions of Lemma 3.4 are satisfied.

Proposition 3.5. *The relation $<$ is a linear ordering on the monoid G_{LD}^+ that is compatible with multiplication on both sides; it admits 1 as a minimal element.*

Proof. That the relation $<$ is irreflexive is clear as $<_L$ is an ordering on terms. Assume $a < b < c$, and let t be a term such that $(t)a$, $(t)b$, and $(t)c$ are defined, for instance $t = t_L(a,b,c)$. By Lemma 3.4, $a < b$ implies $(t)a <_L (t)b$, and $b < c$ implies $(t)b <_L (t)c$. We deduce $(t)a <_L (t)c$, which in turn gives $a < c$ by Lemma 3.4. So $<$ is an ordering on G_{LD}^+, and it is linear as $<_L$ is a linear ordering on T_∞.

Assume now $a < b$, and let c be an arbitrary element of G_{LD}^+. Let t be a term such that both $(t)ca$ and $(t)cb$ exist. By construction, we have $(t)ca = ((t)c)a$ and $(t)cb = ((t)c)b$, so the hypothesis $a < b$ implies $((t)c)a <_L ((t)c)b$, which in turn implies $ca < cb$ by definition. With the same hypotheses, assume that $(t)ac$ and $(t)bc$ are defined. Then $(t)a < (t)b$ holds by hypothesis, and, by Proposition 3.1, this implies $((t)a)c <_L ((t)b)c$, which in turn implies $ac < bc$ by definition. Finally, assume $a \neq 1$. By Lemma 3.2, $t <_L (t)a$ holds, so, by definition, we have $1 < a$. $\square$

It is now easy to extend the ordering of G_{LD}^+ to the whole of G_{LD}.

Lemma 3.6. *For $a, b, a', b' \in G_{LD}^+$ satisfying $ab^{-1} = a'b'^{-1}$, $a < b$ is equivalent to $a' < b'$.*

Proof. By Proposition 1.1, there exist c, c' in G_{LD}^+ satisfying $ac = a'c'$ and $bc = b'c'$. Assume $a < b$. Using the compatibility of the order with multiplication on the right, we deduce $ac < bc$, *i.e.*, $a'c' < b'c'$, hence $a' < b'$. $\square$

Definition. For $c, d \in G_{LD}$, we say that $c < d$ holds if $cd^{-1} = ab^{-1}$ holds for some a, b in G_{LD}^+ satisfying $a < b$.

Proposition 3.7. *The relation $<$ is a linear order on the group G_{LD} that extends the order $<$ on G_{LD}^+. This order is compatible with multiplication on both sides, and, therefore, it is compatible with conjugacy.*

Proof. For a, b in G_{LD}^+, $1 = ab^{-1}$ implies $a = b$, hence $a \not< b$, hence, for every c in G_{LD}, $c < c$ is impossible. Assume $c < d < e$ in G_{LD}. There exist a_1, b_1, a_2, b_2 in G_{LD}^+ satisfying $cd^{-1} = a_1 b_1^{-1}$, $de^{-1} = a_2 b_2^{-1}$, $a_1 < b_1$, and $a_2 < b_2$. Let a_3, b_3 be elements of G_{LD}^+ satisfying $a_2 b_3 = b_1 a_3$. We find

$$ce^{-1} = a_1 b_1^{-1} a_2 b_2^{-1} = (a_1 a_3)(b_2 b_3)^{-1}.$$

The hypothesis $a_1 < b_1$ implies $a_1a_3 < b_1a_3$, the hypothesis $a_2 < b_2$ implies $a_2b_3 < b_2b_3$. By hypothesis, we have $b_1a_3 = a_2b_3$, so we deduce $a_1a_3 < b_2b_3$, and, therefore, $c < e$. Hence the relation $<$ is an ordering on G_{LD}.

Assume $a, b \in G^+_{LD}$ and $a < b$ holds in the sense of G^+_{LD}. Then ab^{-1} is an expression of ab^{-1} with a, b in G^+_{LD} and $a < b$, *i.e.*, $a < b$ in the sense of G_{LD} holds. Thus the order $<$ on G_{LD} extends the previous order $<$ on G^+_{LD}.

Assume now $c, d, e \in G_{LD}$ and $c < d$. By definition, there exist a, b in G^+_{LD} satisfying $cd^{-1} = ab^{-1}$ and $a < b$. Then we have $(ce)(de)^{-1} = ab^{-1}$, so $ce < de$ holds as well. On the other hand, let us express e as $a_0b_0^{-1}$ with a_0, b_0 in G^+_{LD}. There exist a_1, b_1, a_2, b_2, a_3, b_3 in G^+_{LD} satisfying $b_0a_1 = aa_2$, $b_0b_1 = bb_2$, $a_2a_3 = b_2b_3$ (Figure 3.1). Then, we find

$$(ec)(ed)^{-1} = a_0b_0^{-1}ab^{-1}b_0a_0^{-1} = (a_0a_1a_3)(a_0b_1b_3)^{-1}. \tag{3.1}$$

The hypothesis $a < b$ implies $b_0a_1a_3 = aa_2a_3 < ba_2a_3 = bb_2b_3 = b_0b_1b_3$, whereas we deduce $a_1a_3 < b_1b_3$, and, therefore, $a_0a_1a_3 < a_0b_1b_3$ using compatibility with multiplication on the left twice. By (3.1), this gives $ec < ed$. $\square$

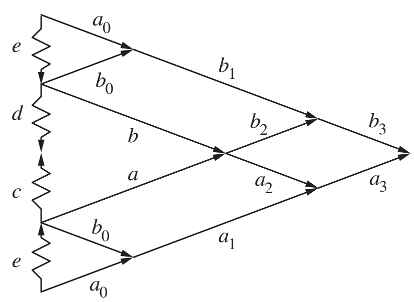


Figure 3.1. Compatibility of order with multiplication on the left.

We thus have proved our main result, namely that G_{LD} is a bi-orderable group. By general results [18], we deduce

Corollary 3.8. *The group G_{LD} is torsion free, the group algebra $\mathbf{C}G_{LD}$ admits no zero divisor, and it embeds in a skew field.*

The action of the group G_{LD} on terms is a partial action. In particular, some elements of G_{LD} do not act, *i.e.*, the domain of the associated operator is empty. Hence, we cannot compare all elements of G_{LD} using their action on terms directly. However, using the action gives a sufficient condition when it is defined.

Proposition 3.9. *(i) Assume $c, d \in G_{LD}$ and there exists a term t such that $(t)c$ and $(t)d$ are defined. Then $c < d$ holds in G_{LD} if and only if $(t)c <_L (t)d$ holds in T_∞.*

(ii) Assume $c \in G_{LD}$ and there exists at least one term t such that $(t)c$ exists. Then $c > 1$ holds in G_{LD} if and only if $(t)c >_L t$ holds for any term t such that $(t)c$ exists.

Proof. (i) Assume that t is a term and $(t)c$ and $(t)d$ are defined. By Proposition 1.1, there exist a and b in G_{LD}^+ satisfying $c^{-1}d = ab^{-1}$, hence $ca = db$. We cannot claim that $(t)ca$ is defined in general, but, as a belongs to G_{LD}^+, the only possible obstruction for $(t)ca$ to be defined is the skeleton of $(t)c$ being too small. Now, for every substitution h, the term t^h is eligible for the action of c as well as t is, and we can choose h such that the skeleton of $(t^h)c$ is arbitrary large. Then $(t^h)ca$ exists, so does $(t^h)db$, and we have $(t^h)ca = (t^h)db$. Assume now $c < d$, hence $a > b$. We deduce $(t)cb <_L (t)ca = (t)db$, and, therefore, $(t)c <_L (t)d$. The argument is symmetric for $c > d$.

Point (ii) follows by taking $d = 1$ in (i). □

4. Action of G_{LD}^+ on the Cantor space and the reals

We conclude the paper with the observation that the previous action of the monoid G_{LD}^+ on finite binary trees induces an action on the Cantor space, viewed as a line at infinity for the set A of all binary addresses.

We first introduce a partial action of G_{LD}^+ on addresses by using the *origin* function. The idea is that, if a is an element of G_{LD}^+ and t is a term large enough to make sure that $(t)a$ exists, then every address β in the skeleton of $(t)a$ has a well-defined origin in the skeleton of t. A direct definition can be posed easily.

Definition. Assume that α, β are addresses. The *origin* $\alpha(\beta)$ of β under α is defined by

$$\alpha(\beta) = \begin{cases} \beta & \text{if } \beta \perp \alpha \text{ holds or } \alpha 11 \text{ is a prefix of } \beta, \\ \alpha 0\gamma & \text{for } \beta = \alpha 00\gamma \text{ and } \beta = \alpha 10\gamma, \\ \alpha 10\gamma & \text{for } \beta = \alpha 01\gamma, \\ \text{undefined} & \text{if } \beta \text{ is a prefix of } \alpha 1. \end{cases}$$

Lemma 4.1. *(i) Defining $\alpha_1 \cdot \ldots \cdot \alpha_k(\beta)$ to be $\alpha_1(\ldots(\alpha_p(\beta)\ldots)$ induces a partial left action of G_{LD}^+ on A.*

(ii) For $a \in G_{LD}^+$, denote by $a(\beta)$ the image of β under the action of a, when it exists. Then $a(\beta)$ is defined if and only if some prefix β' of β lies in the outline of the term $t_L(a)$, and, in this case, we have $a(\beta) = a(\beta')\gamma$ where β is $\beta'\gamma$.

(iii) If t is a term with pairwise distinct variables, then, for every address β in the skeleton of $(t)a$, the address $a(\beta)$ is the unique address in the skeleton of t such that the variable occurring at β in $(t)a$ is the variable occurring at $a(\beta)$ in t.

The easy verifications are left to the reader.

Remark. We have switched from a right action to a left action here because the origin function actually goes backwards: we could have considered instead the inheriting function that associates with every address in the skeleton of a term t its heirs in the term $(t)a$. Inheriting corresponds to a right action, but, in contradistinction to the case of associativity, it does not define a function on addresses, as a given address may have several heirs: for instance, the heir of the address 0 under the action of ϕ consists of the two addresses 00 and 10, since the variable x in $x(yz)$ has two copies at 00 and 10 respectively in $(xy)(xz)$. However, inheriting is injective, and we obtain a function by considering its inverse, which is the current origin function.

The action of G^+_{LD} on A is partial: by Lemma 4.1(ii), for each a in G^+_{LD}, $a(\beta)$ is defined only when β is long enough, *i.e.*, it does not lie in some neighbourhood of ϕ for the topology $\mathcal{T}$ on A associated with the distance defined by $d(\alpha, \beta) = 2^{-n}$ if $\alpha \neq \beta$ holds and n is the length of the greatest common prefix of α and β. Now, by Lemma 4.1(ii) again, the action is $\mathcal{T}$-continuous on A, so we can extend it into an everywhere defined action on the $\mathcal{T}$-boundary of A, which is the Cantor line $\widehat{A}$ consisting of all $\mathbf{N}$-indexed sequences of 0's and 1's.

Definition. For $s \in \widehat{A}$ and $a \in G^+_{LD}$, the element $a(s)$ of $\widehat{A}$ is defined to be $a(\beta)s_0$, where β is the unique prefix of s lying in the outline of $t_L(a)$ and $s = \beta s_0$ holds.

By Lemma 4.1, the previous action is defined everywhere. The reader can easily check the equalities $g_\phi(000\cdots) = g_\phi(100\cdots) = 000\cdots$, $g_\phi(00111\cdots) = g_\phi(10111\cdots) = 0111\cdots$, $g_\phi(01000\cdots) = 1000\cdots$. More generally, the action of g_ϕ on $\widehat{A}$ is displayed on Figure 4.1.

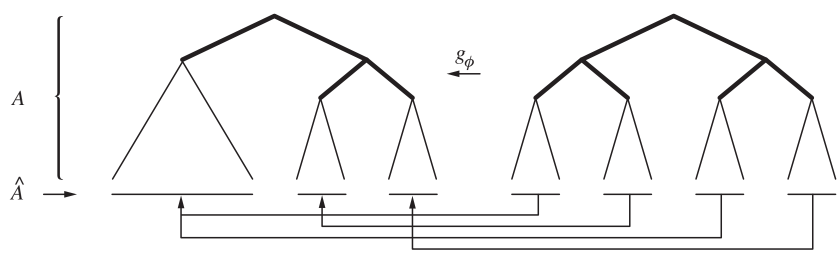


Figure 4.1. Action of G^+_{LD} on the Cantor set.

Let us equip $\widehat{A}$ with the lexicographical ordering, which corresponds to the usual ordering of dyadic numbers, and with the associated topology.

Proposition 4.2. *For every a in G^+_{LD}, the action of a on $\widehat{A}$ is surjective, it is continuous on the right, and it admits finitely many left discontinuities.*

Proof. The result is clear for every generator γ_α, and it is preserved under multiplication. $\square$

The linear order of G^+_{LD} can be defined in terms of the action of G^+_{LD} on the Cantor line $\widehat{A}$:

Proposition 4.3. *The action of G^+_{LD} on $\widehat{A}$ preserves the order in the sense that $a < b$ holds in G^+_{LD} if and only if there exists s_0 in $\widehat{A}$ such that $a(s) = b(s)$ holds for $s \leq s_0$, but $a(s) < b(s)$ holds for $s > s_0$, s close enough to s_0.*

Proof. Assume $a < b$ in G^+_{LD}. Let t be a term with pairwise distinct variables such that $(t)a$ and $(t)b$ exist. By Lemma 3.4, we have $(t)a < (t)b$, hence, as $(t)a$ and $(t)b$ are LD-equivalent, Lemma 2.5 tells us that there exists an address α such that α lies in the outline of t_1 and in the skeleton of t_2, $\mathrm{sub}((t)a, \beta) = \mathrm{sub}((t)b, \beta)$ holds for every β in the left edge of α (hence for every β on the left of α such that the considered subterms exist), $\mathrm{sub}((t)a, \alpha)$ is a variable say x_i, and $\mathrm{sub}((t)b, \alpha)$ is a term of size at least 2 whose leftmost variable is x_i. Let s_0 be $\alpha000\cdots$. By construction, $a(s) = b(s)$ holds for $s \leq s_0$. In particular, we have $a(s_0) = b(s_0) = \gamma000\cdots$, where γ is the address where x_i occurs in t. Let q be the left height of the term $\mathrm{sub}((t)b, \alpha)$. For $p \geq q$, we have

$$a(\alpha0^p1000\cdots) = \gamma0^p1000\cdots, \quad \text{and} \quad b(\alpha0^p1000\cdots) = \gamma0^{p-q}1000\cdots$$

Hence we have $a(s) < b(s)$ for points s arbitrarily close on the right of s_0. As the action is continuous on the right, this is enough to conclude. $\square$

Finally, we can copy the previous left action of G^+_{LD} on $\widehat{A}$ into an action on the real interval $[0, 1)$ using the dyadic expansion. This amounts to associating with every element a of G^+_{LD} a piecewise affine mapping f_a of $[0, 1)$ into itself. For instance, f_ϕ is defined by

$$f_\phi(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1/4, \\ x + 1/4 & \text{for } 1/4 \leq x < 1/2, \\ 2x - 1 & \text{for } 1/2 \leq x < 3/4, \\ x & \text{for } 3/4 \leq x < 1. \end{cases}$$

In Figure 4.2 we have displayed the function f_ϕ associated with the action of left self-distributivity at ϕ, *i.e.*, at the root of the tree, and its counterpart when associativity replace self-distributivity (when compared with the diagrams of [12], the current diagram is inversed because we consider the origin function). Similarly, we have represented in Figure 4.3 the rectangle diagrams associated with a few positive words both in the case of associativity, as in [1], and left self-distributivity.

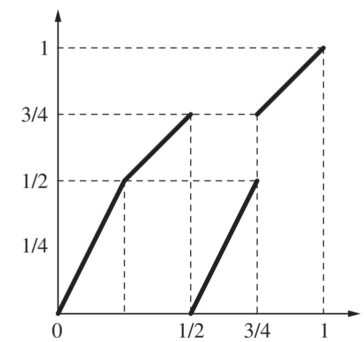


Figure 4.2. Left self-distributivity *vs.* associativity: action at ϕ on the reals.

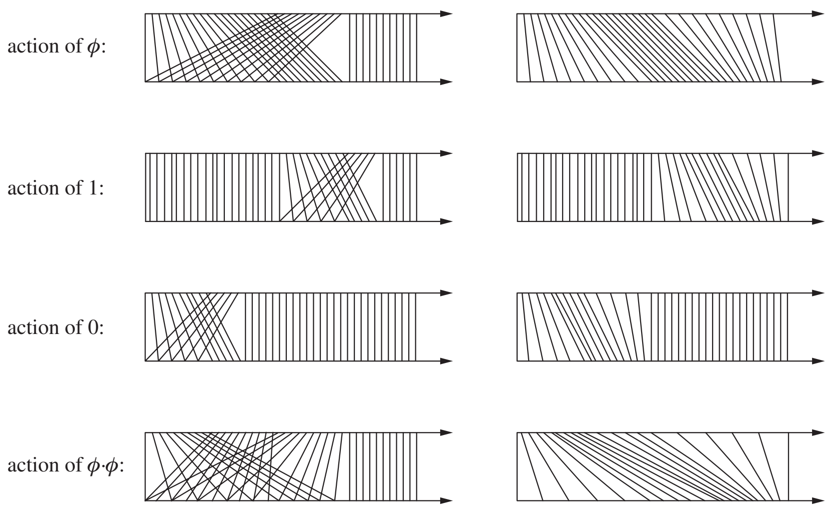


Figure 4.3. Left self-distributivity *vs.* associativity: rectangle diagrams.

Using Proposition 4.3, we deduce:

Proposition 4.4. *The relation $a < b$ holds in G^+_{LD} if and only if there exists a real x_0 satisfying $a(x) = b(x)$ for $x \leq x_0$ and $a(x_0 + \varepsilon) < b(x_0 + \varepsilon)$ for ε small enough.*

In the case of associativity, using the previous approach amounts to defining the action of Thompson's group F on the reals considered in [12]. Then the mappings f_a are bijections, the action is defined on the group, and not only on the monoid, the counterpart of Proposition 4.4 is straightforward, and we obtain a linear ordering on F trivially. In the case of self-distributivity, the result is *not* so easy, for some form of the nontrivial result expressed in Lemma 2.5 is required.

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