

On orbit closures of spherical subgroups in flag varieties

Michel Brion

Abstract. Let \mathcal{F} be the flag variety of a complex semi-simple group G , let H be an algebraic subgroup of G acting on \mathcal{F} with finitely many orbits, and let V be an H -orbit closure in \mathcal{F} . Expanding the cohomology class of V in the basis of Schubert classes defines a union V_0 of Schubert varieties in \mathcal{F} with positive multiplicities. If G is simply-laced, we show that these multiplicities are equal to the same power of 2. For arbitrary G , we show that V_0 is connected in codimension 1. If moreover all multiplicities are 1, we show that the singularities of V are rational and we construct a flat degeneration of V to V_0 in \mathcal{F} . Thus, for any effective line bundle L on \mathcal{F} , the restriction map $H^0(\mathcal{F}, L) \rightarrow H^0(V, L)$ is surjective, and $H^n(V, L) = 0$ for all $n \geq 1$.

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Introduction

Let G be a connected complex reductive group, B a Borel subgroup and H a spherical subgroup, that is, H is an algebraic subgroup having only finitely many orbits in the flag variety G/B . The classification of these orbits and the geometry of their closures are of importance in representation theory. Equivalent questions concern the B -orbits in G/H and their closures in that space. More generally, one may consider B -orbit closures in an equivariant embedding X of G/H ; these play an essential role in the geometry and topology of X .

The set $\mathcal{B}(G/H)$ of B -orbit closures in G/H is partially ordered by inclusion. A weaker order \preceq on $\mathcal{B}(G/H)$ is defined by: $Y \preceq Y'$ if there exists a sequence (P_1, \dots, P_n) of parabolic subgroups containing B such that $Y' = P_1 \cdots P_n Y$. In this paper, we investigate this weak order and its associated graph, with applications to B -orbit closures in “regular” completions of G/H , and in turn to the geometry of H -orbit closures in G/B .

The case where H equals B is well known: then $\mathcal{B}(G/H)$ identifies to the Weyl group W , and the inclusion (resp. weak) order is the Bruhat-Chevalley (resp. left weak) order, see e.g. [13] 5.9. The H -orbit closures in G/B are the Schubert vari-

eties; using an explicit desingularization, reduction to positive characteristics and Frobenius splitting, one shows that their singularities are rational, in particular, they are normal and Cohen-Macaulay ([19]).

Another important class of spherical subgroups consists of symmetric subgroups, that is, fixed points of involutions of G . Orbit closures of symmetric subgroups in flag varieties admit explicit resolutions of singularities as well, and their inclusion and weak orders have been studied in detail by Richardson and Springer ([20], [21], [23]). But their singularities are far from being fully understood; non-normal examples were constructed by Barbasch and Evens ([1]).

Returning to a spherical subgroup H , classical projective geometry yields many examples of H -orbit closures in flag varieties that are neither normal nor Cohen-Macaulay, e.g., the variety of all secant lines to a given conic in projective n -space where $n \geq 3$ (see Example 6 for details). A criterion for having rational singularities will be formulated below, in terms of the oriented graph $\Gamma(G/H)$ associated with the weak order.

For this, we endow that graph with additional data, as in [20]: each edge from Y to Y' is labeled by a simple root of G corresponding to a minimal parabolic subgroup P such that $PY = Y'$. The degree of the associated morphism $P \times {}^B Y \rightarrow Y'$ being 1 or 2, this defines simple and double edges. There may be several edges with the same endpoints, but they are simultaneously simple or double (Proposition 1).

In the symmetric case, a key result is that all minimal elements of $\Gamma(G/H)$ are closed orbits ([20]); thus, one may study orbit closures by going up along oriented paths in the graph. This does not extend to the general case (see e.g. Example 2 below), but it is easy to see that the unique maximal element is G/H . This enables an inductive study of orbit closures by going down along oriented paths; most of our results are obtained in this way.

For instance, $\Gamma(G/H)$ determines the cohomology class of any H -orbit closure V in G/B , that is, its intersection numbers with Schubert varieties, $\int_{G/B} [V] \cdot [\overline{BwB/B}] = d(V, w)$, for all $w \in W$. Consider indeed the associated B -orbit closure Y in G/H . Join it to G/H by an oriented path γ ; denote by $D(\gamma)$ its number of double edges, and by $w(\gamma)$ the product in W of the simple reflections associated with its labels. Then $d(V, w(\gamma)) = 2^{D(\gamma)}$ (in particular, $D(\gamma)$ depends only on V and $w(\gamma)$) and, moreover, all non-zero intersection numbers $d(V, w)$ are obtained in this way ([6]). Equivalently, we have in the cohomology ring of G/B :

$$[V] = \sum_{w=w(\gamma)} 2^{D(\gamma)} [\overline{Bw_0wB/B}],$$

where w_0 denotes the longest element of W .

The oriented paths in $\Gamma(G/H)$ and their associated Weyl group elements are studied in Section 1. The main tool is a notion of neighbor paths that reduces several questions to the case where G has rank two. Using this, we show that the

union of Schubert varieties

$$V_0 = \bigcup_{w=w(\gamma)} \overline{Bw_0wB}/B$$

is connected in codimension 1 (Corollary 7). If moreover G is simply-laced, then the number of double edges in any oriented path depends only on its endpoints (Proposition 5). As a consequence, all non-zero coefficients of $[V]$ in the basis of Schubert cycles are equal. For symmetric spaces, this can also be deduced from a combinatorial result of Richardson and Springer [24]. It does not extend to multiply-laced groups, see Example 4 in Section 1.

In Section 2, we analyze the intersections of B -orbit closures with G -orbit closures in an equivariant completion X of G/H that is *regular* in the sense of Bifet, De Concini and Procesi [2]; these intersections are proper, by [6]. The main ingredient is a “slice” $S_{Y,\gamma}$, a toric variety associated with a B -orbit closure Y meeting G/H and with an oriented path γ from Y to G/H . Given a G -orbit closure X' and an irreducible component Y' of $Y \cap X'$, there exists a “slice” meeting Y' , and the corresponding intersection multiplicity divides $2^{D(\gamma)}$, with equality if X' is a closed G -orbit (Theorem 1).

This distinguishes the B -orbit closures Y such that all oriented paths in $\Gamma(G/H)$ with source Y contain only simple edges; we call them *multiplicity-free*. In a regular completion of G/H , any irreducible component of the intersection of multiplicity-free Y with a G -orbit closure is multiplicity-free as well, and the corresponding intersection multiplicity is 1 (Corollary 3).

In Section 3, we relate our approach to work of Knop ([17], [18]). He defined an action of W on the set $\mathcal{B}(G/H)$ such that the W -conjugates of G/H are the orbit closures of maximal rank (in the sense of [18].) Moreover, the isotropy group $W_{(G/H)}$ is closely related to the “Weyl group of G/H ” studied in [17], a generalization of the little Weyl group of symmetric spaces.

It is easy to see that all orbit closures of maximal rank are multiplicity-free. We describe their intersections with G -orbit closures in a regular completion, in terms of W and $W_{(G/H)}$ (Proposition 9). As applications, we show that all elements of W of minimal length in a given $W_{(G/H)}$ -coset have the same length, and that $W_{(G/H)}$ is generated by reflections or products of two commuting reflections of W . This in turn yields another proof of the fact that the Weyl group of G/H is generated by reflections, a result of Knop [17].

Section 4 contains our main result, Theorem 5: *the singularities of any multiplicity-free B -orbit closure Y in a spherical variety X of G/H are rational. If moreover X is regular, then the scheme-theoretical intersection of Y with any G -orbit closure is reduced.* Since no explicit desingularization of Y is known in this generality, the proof is indirect; it goes by decreasing induction on dimension like Seshadri’s proof of normality of Schubert varieties [22].

For a H -orbit closure V in G/B , the corresponding B -orbit closure Y is multiplicity-free if and only if $[V] = [V_0]$. In that case, we construct a flat degeneration of V to V_0 where the latter is viewed as a reduced subscheme of

G/B (Corollary 6). Thus, the equality $[V] = [V_0]$ holds in the Grothendieck group of G/B as well. Moreover, for any effective line bundle L on G/B , the cohomology groups $H^n(V, L)$ vanish for $n \geq 1$, and the restriction map $H^0(G/B, L) \rightarrow H^0(V, L)$ is surjective (Corollary 8).

This yields a more concrete realization of the H -module $H^0(V, L)$: choose the Borel subgroup B such that V is the closure of HB/B and let λ be the dominant weight of B associated with L . Then $H^0(G/B, L)$ is the dual of the simple G -module $V(\lambda)$ with highest weight λ , and $H^0(V, L)$ is the dual of the H -submodule of $V(\lambda)$ generated by the highest weight line ℓ_λ . As V degenerates to V_0 , this H -module degenerates to a B -module of the same dimension, generated by the extremal weight lines $\ell_{w_0 w \lambda}$ for all w as above. This construction will be developed elsewhere.

When applied to symmetric spaces and combined with Theorem B of [1], this surjectivity result implies a version of the PRV conjecture, see [1] §6. It holds for another class of (smooth) H -orbit closures as well, but not to all of them, see [4]. In fact, surjectivity of the restriction map for all effective L 's and all H -orbit closures is equivalent to their multiplicity-freeness (Proposition 10.)

A remarkable example of a spherical homogeneous space where all orbit closures of a Borel subgroup have maximal rank (and hence are multiplicity-free) is the group G viewed as a homogeneous space under $G \times G$. If moreover G is adjoint, then it has a canonical regular completion X . It is proved in [8] that the $B \times B$ -orbit closures in X are normal and that their intersections are reduced. This follows from the fact that X is Frobenius split compatibly with all of these closures. It would be tempting to generalize this to any spherical variety X . By [5], X is Frobenius split compatibly with all G -orbit closures; but this does not extend to B -orbit closures, since their intersections may be not reduced. This happens, e.g., for the space of all symmetric $n \times n$ matrices of determinant 1, that is, the symmetric space $SL(n)/SO(n)$: consider the subvarieties $(a_{11} = 0)$ and $(a_{11}a_{22} - a_{12}^2 = 0)$.

So the present paper generalizes results of [8] to all regular completions of spherical homogeneous spaces, by other methods. It raises many further questions, e.g., is it true that the normalization of any B -orbit closure in a spherical variety has rational singularities? And do our results extend to positive characteristics?

Notation

Let G be a complex connected reductive algebraic group. Let B, B^- be opposite Borel subgroups of G , with unipotent radicals U, U^- and common torus T , a maximal torus of G . Let X be the character group of B ; we identify X with the character group of T and we choose a scalar product on X , invariant under the Weyl group W . Let Φ be the root system of (G, T) , with the subset Φ^+ of positive roots, i.e., of roots of (B, T) , and its subset Δ of simple roots.

For $\alpha \in \Delta$, let $s_\alpha \in W$ be the corresponding simple reflection, and let $P_\alpha = B \cup Bs_\alpha B$ be the corresponding minimal parabolic subgroup. For any subset I of Δ , let P_I be the subgroup of G generated by the P_α , $\alpha \in I$. The map $I \rightarrow P_I$ is a bijection from subsets of Δ to subgroups of G containing B , that is, to standard parabolic subgroups of G .

Let L_I be the Levi subgroup of P_I that contains T ; let Φ_I be the root system of (L_I, T) , with Weyl group W_I . We denote by ℓ the length function on W and by W^I the set of all $w \in W$ such that $\ell(ws_\alpha) = \ell(w) + 1$ for all $\alpha \in I$ (this amounts to: $w(I) \subseteq \Phi^+$). Then W^I is a system of representatives of the set of right cosets W/W_I .

1. The weak order and its graph

In what follows, we denote by G/H a spherical homogeneous space and by $\mathcal{B}(G/H)$ the set of B -orbit closures in G/H . One associates to a given $Y \in \mathcal{B}(G/H)$ several combinatorial invariants, see [18]: The *character group* $X(Y)$ is the set of all characters of B that arise as weights of eigenvectors of B in the function field $\mathbb{C}(Y)$. Then $X(Y)$ is a free abelian group of finite rank $r(Y)$, the *rank* of Y .

Let Y^0 be the open B -orbit in Y and let $P(Y)$ be the set of all $g \in G$ such that $gY^0 = Y^0$; then $P(Y)$ is a standard parabolic subgroup of G . Let $L(Y)$ be its Levi subgroup that contains T and let $\Delta(Y)$ be the corresponding subset of Δ : the set of *simple roots* of Y .

We note some easy properties of these invariants.

Lemma 1. (i) $X(Y)$ is isomorphic to the quotient of the group of invertible regular functions on Y^0 , by the subgroup of constant non-zero functions.

(ii) The derived subgroup $[L(Y), L(Y)]$ fixes a point of Y^0 .

(iii) The group $W_{\Delta(Y)}$ fixes pointwise $X(Y)$. Equivalently, any simple root of Y is orthogonal to $X(Y)$.

Proof. (i) Let f be an eigenvector of B in $\mathbb{C}(Y)$ with weight $\chi(f)$. Then f restricts to an invertible regular function on Y^0 , and is uniquely determined by $\chi(f)$ up to a constant. Conversely, let f be an invertible regular function on the B -orbit Y^0 . Then f pulls back to an invertible regular function on B , that is, to a scalar multiple of a character of B . Thus, f is an eigenvector of B in $\mathbb{C}(Y)$.

(ii) Choose $y \in Y^0$. Let B_y (resp. $P(Y)_y$) be the isotropy group of y in B (resp. $P(Y)$). Since $Y^0 = B_y = P(Y)_y$, we have $P(Y) = BP(Y)_y$. Thus, $P(Y)_y$ acts transitively on $P(Y)/B$, the flag variety of $P(Y)$. Using e.g. [9], it follows that $P(Y)_y$ contains a maximal connected semi-simple subgroup of $P(Y)$, that is, a conjugate of $[L(Y), L(Y)]$.

(iii) follows from [18] Lemma 3.2; it can be deduced from (ii) as well. \square .

Let $Y, Y' \in \mathcal{B}(G/H)$ and let $\alpha \in \Delta$. We say that α (or P_α) *raises* Y to Y' if $Y' = P_\alpha Y \neq Y$. Let then

$$f_{Y,\alpha} : P_\alpha \times^B Y \rightarrow P_\alpha/B$$

be the homogeneous bundle with fiber the B -variety Y and basis P_α/B (isomorphic to projective line.) The map $P_\alpha \times Y \rightarrow G/H$, $(p, y) \mapsto py$ factors through a proper morphism

$$\pi_{Y,\alpha} : P_\alpha \times^B Y \rightarrow Y' = P_\alpha Y$$

that restricts to a finite morphism $P_\alpha \times^B Y^0 \rightarrow P_\alpha Y^0$. In particular, $\dim(Y') = \dim(Y) + 1$.

By [20] or [18] Lemma 3.2, one of the following three cases occurs.

- Type U: $P_\alpha Y^0 = Y'^0 \cup Y^0$ and $\pi_{Y,\alpha}$ is birational. Then $X(Y') = s_\alpha X(Y)$; thus, $r(Y') = r(Y)$.
- Type T: $P_\alpha Y^0 = Y'^0 \cup Y^0 \cup Y_-^0$ for some $Y_- \in \mathcal{B}(G/H)$ of the same dimension as Y , and $\pi_{Y,\alpha}$ is birational. Then $r(Y) = r(Y_-) = r(Y') - 1$.
- Type N: $P_\alpha Y^0 = Y'^0 \cup Y^0$ and $\pi_{Y,\alpha}$ has degree 2. Then $r(Y) = r(Y') - 1$.
In particular, $r(Y) \leq r(P_\alpha Y)$ with equality if and only if α has type U.

Our notation for types differs from that in [20] and [18]; it can be explained as follows. Choose $y \in Y^0$ with isotropy group $(P_\alpha)_y$ in P_α . Then $(P_\alpha)_y$ acts on $P_\alpha/B \cong \mathbb{P}^1$ with finitely many orbits, since B acts on $P_\alpha Y^0 \cong P_\alpha/(P_\alpha)_y$ with finitely many orbits. By [20] or [18], the image of $(P_\alpha)_y$ in $\text{Aut}(P_\alpha/B) \cong \text{PGL}(2)$ is a torus (resp. the normalizer of a torus) in type T (resp. N); in type U, this image contains a non-trivial unipotent subgroup.

Definition. Let $\Gamma(G/H)$ be the oriented graph with vertices the elements of $\mathcal{B}(G/H)$ and edges labeled by Δ , where Y is joined to Y' by an edge of label α if that simple root raises Y to Y' . This edge is simple (resp. double) if $\pi_{Y,\alpha}$ has degree 1 (resp. 2.) The partial order \preceq on $\mathcal{B}(G/H)$ with oriented graph $\Gamma(G/H)$ will be called the *weak order*.

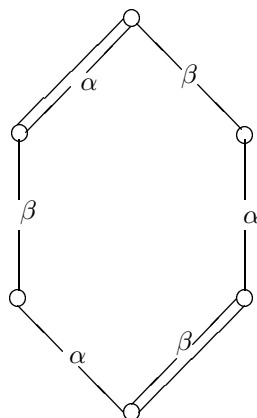
Note that the dimension and rank functions are compatible with \preceq . We shall see that $Y, Y' \in \mathcal{B}(G/H)$ satisfy $Y \preceq Y'$ if and only if $Y' = \overline{BwY}$ for some $w \in W$ (Corollary 1.)

In the case where H is a parabolic subgroup of G , the rank function is zero. Thus, all edges are of type U; in particular, they are simple. Here is another example, where double edges occur.

Example 1. Let $G = \text{GL}(3)$ with simple roots α and β . Let H be the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}$$

It is easy to see that G/H is spherical of rank one and that $\Gamma(G/H)$ is as follows:



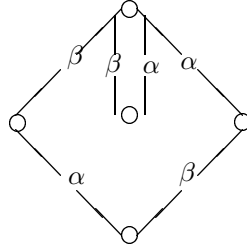
Observe that $\Gamma(G/H)$ is the same as $\Gamma(G/B)$, except for double edges. But the geometry of B -orbit closures is very different in both cases: all of them are smooth in G/B (the flag variety of \mathbb{P}^2), whereas G/H contains a B -stable divisor that is singular in codimension 1. This can be checked directly (see Example 6 for a generalization), or read off the graph of G/H as follows.

Let Z be the closed B -orbit in G/H . We claim that $Y = P_\beta P_\alpha Z$ is singular along $P_\alpha Z$. Indeed, the morphism $\pi : P_\beta \times^B P_\alpha Z \rightarrow Y$ is birational, and $\pi^{-1}(P_\beta Z)$ equals $P_\beta \times^B Z$. But the restriction $P_\beta \times^B Z \rightarrow P_\beta Z$ has degree two. Now our claim follows from Zariski's main theorem.

One checks that $r(P_\beta Z) = 1$, whereas $r(Y) = 0$. Thus, the rank function is not compatible with the inclusion order.

Obviously, all closed B -orbits in G/H are minimal elements for the weak order. In fact, these closed B -orbits are isomorphic and their codimension is the maximal length of all oriented paths in $\Gamma(G/H)$, see e.g. [7] 2.2. If moreover G/H is symmetric, then all minimal elements of $\Gamma(G/H)$ are closed orbits, see [20] Theorem 4.6; equivalently, all maximal oriented paths in $\Gamma(G/H)$ have the same length. But this does not extend to all spherical homogeneous spaces, as shown by

Example 2. Let $G = \mathrm{GL}(2) \times \mathrm{GL}(2)$ with simple roots α and β . Let H be the subgroup of G consisting of all matrices of the form (A, A) where A is upper triangular. One checks easily that G/H is spherical with graph



Returning to the general situation, observe that the simple roots of Y are determined by $\Gamma(G/H)$: indeed, $\alpha \in \Delta$ is not in $\Delta(Y)$ if and only if α is the label of an edge with endpoint Y . Similarly, if α raises Y then its type is determined by $\Gamma(G/H)$: it is U (resp. N) if there is a unique edge of label α and target $P_\alpha Y$ and this edge is simple (resp. double); and it is T if there are two such edges. It follows that the ranks of B -orbit closures are determined by $\Gamma(G/H)$ together with $r(G/H)$.

Example 2 shows that there may be two edges in $\Gamma(G/H)$ with the same endpoints. In fact, there is no restriction on the number of edges with prescribed endpoints, as shown by the example below suggested by D. Luna. But we shall see that all such edges have the same type.

Example 3. Let n be a positive integer. Let $G = \mathrm{SL}(2) \times \cdots \times \mathrm{SL}(2)$ (n terms) and let H be the subgroup of G consisting of those n -tuples $\begin{pmatrix} t & u_1 \\ 0 & t^{-1} \end{pmatrix}, \dots, \begin{pmatrix} t & u_n \\ 0 & t^{-1} \end{pmatrix}$ where $t \in \mathbb{C}^*$, $u_1, \dots, u_n \in \mathbb{C}$ and $u_1 + \cdots + u_n = 0$. One checks that G/H is spherical; the open H -orbit in $G/B \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n terms) consists of those (z_1, \dots, z_n) such that $z_i \neq \infty$ for all i , and that $z_1 + \cdots + z_n \neq 0$. Let Y be the B -stable hypersurface in G/H corresponding to the H -stable hypersurface $(z_1 + \cdots + z_n = 0)$ in G/B . One checks that Y is irreducible and raised to G/H by all simple roots of G (there are n of them). Thus, Y is joined to G/H by n edges of type T .

Proposition 1. *Let $Y, Y' \in \mathcal{B}(G/H)$ and let α, β be distinct simple roots raising Y to Y' . Then either α, β are orthogonal and of type U , or they are of type T .*

Proof. We begin with two lemmas that reduce the “local” study of $\Gamma(G/H)$ to simpler situations.

Let $Y \in \mathcal{B}(G/H)$ and let $P = P_I$ be a standard parabolic subgroup of G , with radical $R(P)$. Let $\mathcal{B}(PY^0)$ be the set of all closures in G/H of B -orbits in PY^0 ; in other words, $\mathcal{B}(PY^0)$ is the set of all $Z \in \mathcal{B}(G/H)$ such that $PZ = PY$. Let $\Gamma(PY^0)$ be the oriented graph with set of vertices $\mathcal{B}(PY^0)$, and with edges those edges of $\Gamma(G/H)$ that have both endpoints in $\mathcal{B}(PY^0)$ and labels in I .

Lemma 2. *The quotient $PY^0/R(P)$ exists and is a spherical homogeneous space under $P/R(P)$, with graph $\Gamma(PY^0)$.*

Proof. Since PY^0 is a unique P -orbit and $R(P)$ is a normal subgroup of P contained in B , the quotient $PY^0/R(P)$ exists and is homogeneous under $P/R(P)$; moreover, any $B/R(P)$ -orbit in $PY^0/R(P)$ pulls back to a unique B -orbit in PY^0 . Let \mathcal{O} be a B -orbit in PY^0 and let $\alpha \in I$. Then $R(P_\alpha)$ contains $R(P)$, the square

$$\begin{array}{ccc} P_\alpha \times^B \mathcal{O} & \rightarrow & P_\alpha \mathcal{O} \\ \downarrow & & \downarrow \\ P_\alpha \times^B \mathcal{O}/R(P) & \rightarrow & P_\alpha \mathcal{O}/R(P) \end{array}$$

is cartesian, and the map $P_\alpha \times^B \mathcal{O}/R(P) \rightarrow P_\alpha/R(P) \times^{B/R(P)} \mathcal{O}/R(P)$ is an isomorphism. Thus, the type is preserved under pull back. \square

Let H' be a closed subgroup of the normalizer $N_G(H)$ such that H' contains H , and that the quotient H'/H is connected. Let $Z(G)$ be the center of G , then $G/H'Z(G)$ is a spherical homogeneous space under the adjoint group $G/Z(G)$. The natural G -equivariant map $p : G/H \rightarrow G/H'Z(G)$ is the quotient by the right action of $H'Z(G)$ on G/H .

Lemma 3. *The pull-back under p of any B -orbit in $G/H'Z(G)$ is a unique B -orbit in G/H . This defines an isomorphism of $\Gamma(G/H'Z(G))$ onto $\Gamma(G/H)$.*

Proof. The first assertion follows from [7] Proposition 2.2 (iii). The second assertion is checked as in the proof of Lemma 2. \square

Lemma 4. *Let $Y \in \mathcal{B}(G/H)$, $Y \neq G/H$, and let $\alpha \in \Delta$. If $P_\alpha Y^0 = G/H$ then α is orthogonal to $\Delta - \{\alpha\}$, and the derived subgroup of $L_{\Delta - \{\alpha\}}$ fixes pointwise G/H .*

Proof. We may assume that G is semi-simple adjoint and acts faithfully on G/H ; then it suffices to check that $G \cong \text{PGL}(2)$. Let H be the isotropy group in G of a point of Y^0 . Since $P_\alpha Y^0 = G/H$, we have $P_\alpha H = G$. Equivalently, the map $H/P_\alpha \cap H \rightarrow G/P_\alpha$ is an isomorphism. But since $Y \neq G/H$, we have $Y^0 \neq P_\alpha Y^0$, so that the image of $P_\alpha \cap H$ in $P_\alpha/R(P_\alpha) \cong \text{PGL}(2)$ is a proper subgroup. It follows that $(P_\alpha \cap H)^0$ is solvable. Thus, $H/P_\alpha \cap H$ is the flag variety of H^0 . Let S be a maximal semi-simple subgroup of H^0 . Then the rank of S is the rank of the Picard group of its flag variety $H/P_\alpha \cap H \cong G/P_\alpha$, so that $r(S) = r(G) - 1$. Moreover, the connected automorphism group of this flag variety is $S/Z(S)$ (see e.g. [9]). Thus, we have a map $G \rightarrow S/Z(S)$ that “splits” the inclusion of S into G . It follows that S is a direct factor of G , whence S is trivial since G acts faithfully on G/H . Thus, $G = P_\alpha \cong \text{PGL}(2)$. \square

We now prove Proposition 1. Applying Lemma 2 to Y' and $P_{\alpha,\beta}$, we may assume that $Y' = G/H$ and that $\Delta = \{\alpha, \beta\}$.

If α has type U, then $r(Y) = r(G/H)$ whence β has type U as well. We claim that $\mathcal{B}(G/H)$ consists of Y and G/H . Indeed, if $Z \in \mathcal{B}(G/H)$ and $Z \neq G/H$, then Z is connected to G/H by an oriented path in $\Gamma(G/H)$. Consider the top edge of that path; let Z' be its source and (say) α its label. Since α has type U, it follows that $Z' = Y$. Hence $Z = Y$ (for otherwise, Y would be stable by P_α or P_β). Thus, $P_\alpha Y^0 = G/H$; then α and β are orthogonal by Lemma 4.

If α has type N, then $r(Y) = r(G/H) - 1$, whence β has type N or T. We consider two subcases.

If α and β have type N, we see as above that $G/H = P_\alpha Y^0 = P_\beta Y^0$. Thus, α and β are orthogonal by Lemma 4. Using Lemma 3, we may assume that $G = \text{PGL}(2) \times \text{PGL}(2)$ and that H contains a copy of $\text{PGL}(2)$. Then H is conjugate to $\text{PGL}(2)$ embedded diagonally in G . But then both α and β have type T, a contradiction.

Finally, if α has type N and β has type T, then there exists $y \in Y^0$ such that $(P_\beta)_y$ is contained in $R(P_\beta)T$. Since the homogeneous spaces $P_\beta/R(P_\beta)T$ and $R(P_\beta)T/(P_\beta)_y$ are affine, the same holds for $P_\beta/(P_\beta)_y \cong P_\beta Y^0$. It follows that $G/H - P_\beta Y^0$ is pure of codimension 1 in G/H . But $P_\beta Y^0$ meets both B-orbits of codimension 1 in G/H , so that $P_\beta Y^0 = G/H$. This case is excluded as above. Thus, type N does not occur. \square

We next study oriented paths in $\Gamma(G/H)$. Let γ be such a path, with source Y and target Y' . Let $(a_1, a_2, \dots, a_\ell)$ be the sequence of labels of edges of γ , where $\ell = \ell(\gamma)$ is the length of the path. Let $\ell_U(\gamma)$ (resp. $\ell_T(\gamma)$, $\ell_N(\gamma)$) be the number of edges of type U (resp. T, N) in γ . Then

$$\ell_U(\gamma) + \ell_T(\gamma) + \ell_N(\gamma) = \ell(\gamma) = \dim(Y') - \dim(Y).$$

Define an element $w(\gamma)$ of W by $w(\gamma) = s_{\alpha_\ell} \cdots s_{\alpha_2} s_{\alpha_1}$.

Lemma 5. (i) $s_{\alpha_\ell} \cdots s_{\alpha_2} s_{\alpha_1}$ is a reduced decomposition of $w(\gamma)$; equivalently, $\ell(w(\gamma)) = \ell$.

(ii) $\ell_T(\gamma) + \ell_N(\gamma) = r(Y') - r(Y)$. In particular, $\ell_T(\gamma) + \ell_N(\gamma)$ and $\ell_U(\gamma)$ depend only on the endpoints of γ .

(iii) The morphism $G \times^B Y \rightarrow G/H : (g, y)B \rightarrow gy$ restricts to a morphism $\overline{B}w(\gamma)\overline{B} \times^B Y \rightarrow Y'$ that is surjective and generically finite of degree $2^{\ell_N(\gamma)}$. In particular, $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of γ and on $w(\gamma)$. Moreover, $w(\gamma)$ is in $W^{\Delta(Y)}$, and $w(\gamma)^{-1}$ is in $W^{\Delta(Y')}$.

(iv) If H is contained in a Borel subgroup of G , then $\ell_N(\gamma) = 0$ so that $\ell_T(\gamma)$ depends only on the endpoints of γ .

Proof. (i) Observe that $Bs_{\alpha_1}Y$ is dense in $P_{\alpha_1}Y$, as α_1 raises Y . By induction, it follows that $Bs_{\alpha_\ell}B \cdots s_{\alpha_2}Bs_{\alpha_1}Y$ is dense in Y' . Because $\dim(Y') = \dim(Y) + \ell$,

we must have $\dim(\overline{Bs_{\alpha_\ell}B \cdots s_{\alpha_2}Bs_{\alpha_1}B}/B) = \ell$, whence $\ell(s_{\alpha_\ell} \cdots s_{\alpha_2}s_{\alpha_1}) = \ell$.

(ii) follows from the fact that $r(Y') = r(Y)$ (resp. $r(Y) + 1$) if Y is the source of an edge with target Y' and type U (resp. T, N).

(iii) By (i), the product maps

$$P_{\alpha_i} \times^B \cdots \times^B P_{\alpha_2} \times^B P_{\alpha_1} \rightarrow \overline{Bs_{\alpha_i} \cdots s_{\alpha_2}s_{\alpha_1}B}$$

are birational for $1 \leq i \leq \ell$. It follows that the morphism $\overline{Bw(\gamma)B} \times^B Y \rightarrow G/H$ has image Y' ; moreover, its degree is the product of the degrees of the

$$\pi_i : P_{\alpha_i} \times^B (P_{\alpha_{i-1}} \cdots P_{\alpha_1} Y) \rightarrow P_{\alpha_i} P_{\alpha_{i-1}} \cdots P_{\alpha_1} Y,$$

that is, $2^{\ell_N(\gamma)}$.

Let $w = w(\gamma)$. We show that $w^{-1} \in W^{\Delta(Y')}$. Otherwise, there exists $\alpha \in \Delta(Y')$ such that $\ell(s_\alpha w) = \ell(w) - 1$. Thus, $BwB = Bs_\alpha Bs_\alpha wB$, and $Y' = \overline{BwY} = \overline{Bs_\alpha Bs_\alpha wY}$. Let $Y'' = \overline{Bs_\alpha wY}$, then α raises Y'' to Y' . This contradicts the assumption that $\alpha \in \Delta(Y')$. A similar argument shows that $w \in W^{\Delta(Y)}$.

(iv) If $\ell_N(\gamma) > 0$, then there exists a point $x \in G/H$, a simple root α and a surjective group homomorphism $(P_\alpha)_x \rightarrow N$ where N is the normalizer of a torus in $\text{PGL}(2)$. Since N consists of semi-simple elements, it is a quotient of $(P_\alpha)_x/R_u(P_\alpha)_x$. By assumption, the latter is isomorphic to a subgroup of $B/U = T$. Thus, N is abelian, a contradiction. \square

Corollary 1. *Let $Y, Y' \in \mathcal{B}(G/H)$, then $Y \preceq Y'$ if and only if there exists $w \in W$ such that $Y' = \overline{BwY}$.*

Proof. Recall that \overline{BwB} (closure in G) is a product of minimal parabolic subgroups. Thus, $Y \preceq \overline{BwBY} = \overline{BwY}$. The converse has just been proved. \square

For later use, we study the behavior of $\Gamma(G/H)$ under parabolic induction in the following sense (see [6] 1.2.) Let $P = P_I$ be a standard parabolic subgroup with Levi subgroup $L = L_I$ and let H' be a spherical subgroup of L , then the induced homogeneous space is $G/R_u(P)H'$. In other words, G/H is the total space of the homogeneous bundle over G/P with fiber L/H' . By [loc. cit.], each $Y \in \mathcal{B}(G/H)$ can be written uniquely as $\overline{BwY'}$ for $w \in W^I$ and $Y' \in \mathcal{B}(L/H')$; then $r(Y) = r(Y')$. We thus identify $\mathcal{B}(G/H)$ to $W^I \times \mathcal{B}(L/H')$. The next result describes the edges of $\Gamma(G/H)$ in terms of those of $\Gamma(L/H')$.

Lemma 6. *Let $\alpha \in \Delta$, $w \in W^I$ and $Y' \in \mathcal{B}(L/H')$; let $\beta = w^{-1}(\alpha)$. Then the edges of $\Gamma(G/H)$ with source (w, Y') and label α are as follows:*

- (i) *If $\beta \in \Phi^+ - I$, join (w, Y') to $(s_\alpha w, Y')$ by an edge of type U .*
- (ii) *If $\beta \in I$ and β raises Y' , join (w, Y') to $(w, (P_\beta \cap L)Y')$ by an edge of the same type as the edge from Y' to $(P_\beta \cap L)Y'$.*

Proof. Since $w \in W^I$, we have $s_\alpha w \in W^I$ if and only if $\beta \notin I$. In that case, α

raises Y if and only if $\ell(s_\alpha w) = \ell(w) + 1$, that is, $\beta \in \Phi^+$. Then $P_\alpha Y = \overline{Bs_\alpha w Y'}$ and the map $\pi_{Y,\alpha}$ is the pull-back of $\pi_{\overline{BwP}/P,\alpha}$ under the map $\overline{BwY'} \rightarrow \overline{BwP}/P$. This yields case (i).

But if $\beta \in I$, then $s_\alpha w = ws_\beta$ has length $\ell(w) + 1$, so that

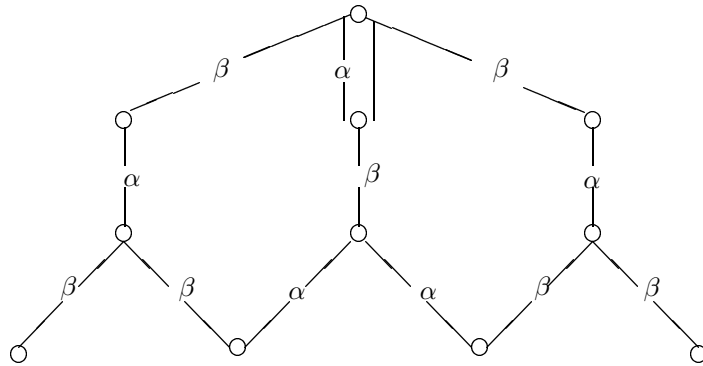
$$P_\alpha Y = \overline{Bs_\alpha BwY'} = \overline{Bs_\alpha wY'} = \overline{Bws_\beta Y'} = \overline{BwBs_\beta Y'} = \overline{Bw(P_\beta \cap L)Y'}.$$

Thus, α raises Y if and only if β raises Y' . Then, as $s_\alpha w = ws_\beta$, we can join Y' to $P_\alpha Y$ by two paths: one beginning with $\ell(w)$ edges of type U followed by an edge from Y to $P_\alpha Y$, and another one beginning with an edge from Y' to $(P_\beta \cap L)Y'$ followed by $\ell(w)$ edges of type U . Using Lemma 5, this yields case (ii). \square

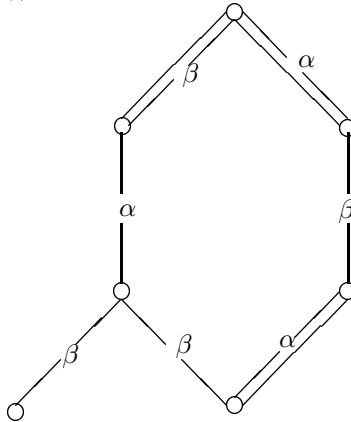
For instance, Example 1 is obtained from $SL(2)/N$ by parabolic induction.

Returning to the general case, we shall see in Proposition 5 that the numbers $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of the oriented path γ , if G is simply-laced (that is, if all roots have the same length for an appropriate choice of the W -invariant scalar product on X ; equivalently, Φ is a product of irreducible root systems of type A , D or E .) This assumption cannot be omitted, as shown by

Example 4. Let $G = SP(4)$ be the subgroup of $GL(4)$ preserving a non-degenerate symplectic form, and let $H = GL(2)$ be the subgroup of G preserving two complementary lagrangian planes. The normalizer $N_G(H)$ contains H as a subgroup of index 2. The graph $\Gamma(G/H)$ is as follows:



And here is $\Gamma(G/N_G(H))$:



Using parabolic induction, one constructs similar examples for Φ of type B, C or F.

To proceed, we need the following definition taken from [6]:

Definition. For $Y \in \mathcal{B}(G/H)$, let $W(Y)$ be the set of all $w \in W$ such that the morphism $\pi_{Y,w} : \overline{BwB} \times^B Y \rightarrow GY$ is surjective and generically finite. For $w \in W(Y)$, let $d(Y, w)$ be the degree of $\pi_{Y,w}$.

By Lemma 5, $W(Y)$ consists of all $w(\gamma)$ where γ is an oriented path from Y to GY ; moreover, $d(Y, w(\gamma)) = 2^{\ell_N(\gamma)}$, and $w^{-1} \in W^{\Delta(G/H)}$ for all $w \in W(Y)$.

We now introduce a notion of neighbors in $W(Y)$, and we show that any two elements of that set are connected by a chain of neighbors. Let α, β be distinct simple roots and let m be a positive integer. Let

$$(s_\alpha s_\beta)^{(m)} = \cdots s_\beta s_\alpha s_\beta s_\alpha \quad (m \text{ terms.})$$

Then we have the braid relation $(s_\alpha s_\beta)^{(m(\alpha,\beta))} = (s_\beta s_\alpha)^{(m(\alpha,\beta))}$, where $m(\alpha, \beta)$ denotes the order of $s_\alpha s_\beta$ in W .

Definition. Two elements u and v of W are *neighbors* if there exist x, y in W together with distinct α, β in Δ and a positive integer $m < m(\alpha, \beta)$ such that

$$u = x(s_\alpha s_\beta)^{(m)}y, v = x(s_\beta s_\alpha)^{(m)}y, \quad \text{and} \quad \ell(u) = \ell(x) + m + \ell(y) = \ell(v).$$

For example, any two simple reflections are neighbors, since $s_\alpha = (s_\alpha s_\beta)^{(1)}$.

Proposition 2. Let $Y \in \mathcal{B}(G/H)$ and let u, v be distinct elements of $W(Y)$. Then there exists a sequence $(u = u_0, u_1, \dots, u_n = v)$ in $W(Y)$ such that each u_{i+1} is a neighbor of u_i .

Proof. By induction on $\ell(u) = \ell(v) = \ell$, the case where $\ell = 1$ being evident.

If there exists $\alpha \in \Delta$ such that $\ell(us_\alpha) = \ell(vs_\alpha) = \ell - 1$, then α raises Y , and us_α, vs_α are in $W(P_\alpha Y)$. Now the induction assumption for $P_\alpha Y$ concludes the proof in this case.

Otherwise, we can find distinct $\alpha, \beta \in \Delta$ such that $\ell(us_\alpha) = \ell(vs_\beta) = \ell - 1$. Then P_α and P_β raise Y to subvarieties of $P_{\alpha,\beta}Y$. Let m be the common codimension of $P_\alpha Y$ and $P_\beta Y$ in $P_{\alpha,\beta}Y$, then we have

$$P_{\alpha,\beta}Y = \cdots P_\alpha P_\beta P_\alpha Y = \overline{B \cdots s_\alpha s_\beta s_\alpha Y} \quad (m \text{ terms})$$

Choose $x \in W(P_{\alpha,\beta}Y)$, then $W(Y)$ contains $x(s_\alpha s_\beta)^{(m)}$ and, similarly, $x(s_\beta s_\alpha)^{(m)}$, as neighbors. Moreover, $W(P_\alpha Y)$ contains us_α and $x(s_\beta s_\alpha)^{(m-1)}$, whereas $W(P_\beta Y)$ contains $x(s_\beta s_\alpha)^{(m-1)}$ and vs_β . Now we conclude by the induction assumption for $P_\alpha Y$ and $P_\beta Y$. \square

Neighbors in W are also close to each other for the Bruhat-Chevalley order \leq :

Proposition 3. *For any neighbors $u, v \in W$ such that $u^{-1}, v^{-1} \in W^{\Delta(G/H)}$, there exists $w \in W$ such that $u \leq w, v \leq w, w^{-1} \in W^{\Delta(G/H)}$ and $\ell(w) = \ell(u) + 1 = \ell(v) + 1$.*

Proof. Write $u = x(s_\alpha s_\beta)^{(m)}y$ and $v = x(s_\beta s_\alpha)^{(m)}y$. Let

$$w = x(s_\alpha s_\beta)^{(m)}s_\beta y.$$

We claim that $\ell(w)$ equals $\ell(x) + m + 1 + \ell(y) = \ell(u) + 1 = \ell(v) + 1$. Otherwise, $\ell(w) \leq \ell(x) + \ell(y) + m - 1 < \ell(u)$ and $w = uy^{-1}s_\beta y = us_{y^{-1}(\beta)}$. By the strong exchange condition ([13] Theorem 5.8 applied to u), one of the following cases occurs:

- (i) $w = x'(s_\alpha s_\beta)^{(m)}y$ where $\ell(x') = \ell(x) - 1$. Comparing both expressions for w , we obtain $x'(s_\alpha s_\beta)^{(m)} = x(s_\alpha s_\beta)^{(m)}s_\beta$. Thus, there exists $\gamma \in \Phi_{\alpha,\beta}^+$ such that $x' = xs_\gamma$. But $\ell(xs_\alpha) = \ell(xs_\beta) = \ell(x) + 1$, for $\ell(x(s_\alpha s_\beta)^{(m)}y) = \ell(x(s_\beta s_\alpha)^{(m)}y) = \ell(x) + m + \ell(y)$. It follows that $x(\alpha)$ and $x(\beta)$ are in Φ^+ . Thus, $x \in W^{\alpha,\beta}$. Since $s_\gamma \in W_{\alpha,\beta}$, we have $\ell(x') = \ell(x) + \ell(s_\gamma) \geq \ell(x)$, a contradiction.
- (ii) $w = xzy$ where z is obtained from $(s_\alpha s_\beta)^{(m)}$ by deleting a simple reflection. Then the equality $z = (s_\alpha s_\beta)^{(m)}s_\beta$ leads to a braid relation of length at most $m < m(\alpha, \beta)$, a contradiction.
- (iii) $w = x(s_\alpha s_\beta)^{(m)}y'$ where $\ell(y') = \ell(y) - 1$. Then $y' = s_\beta y$. But $\ell(s_\beta y) = \ell(y) + 1$, for $\ell(v) = \ell(x) + m + \ell(y)$; a contradiction.

By the claim and [13] Theorem 5.10, we have $u \leq w$ and $v \leq w$. Write $w = w''w'$ where $w'' \in W_{\Delta(G/H)}$ and $(w')^{-1} \in W^{\Delta(G/H)}$; then $\ell(w) = \ell(w') + \ell(w'')$. Since $u^{-1} \leq w^{-1}$ and $u^{-1} \in W^{\Delta(G/H)}$, it follows that $u^{-1} \leq (w')^{-1}$ by [10] Lemma 3.5. Thus, $u \leq w'$ and $v \leq w'$. Since $u \neq v$ and $\ell(u) = \ell(v) = \ell(w) - 1 \geq \ell(w') - 1$, we must have $w = w'$, so that $w^{-1} \in W^{\Delta(G/H)}$. \square

Recall that $r(Y) \leq r(G/H)$ for any $Y \in \mathcal{B}(G/H)$. If equality holds, then neighbors in $W(Y)$ have a very simple form:

Proposition 4. *Let $Y \in \mathcal{B}(G/H)$ such that $r(Y) = r(G/H)$; let $u, v \in W(Y)$ be neighbors. Then $u = xs_\alpha y$ and $v = xs_\beta y$ where $x, y \in W$ and α, β are orthogonal simple roots such that $\ell(u) = \ell(v) = \ell(x) + \ell(y) + 1$. Moreover, $X(G/H)$ contains $x(\alpha + \beta)$.*

Proof. Write $u = x(s_\alpha s_\beta)^{(m)}y$ and $v = x(s_\beta s_\alpha)^{(m)}y$ as in the definition of neighbors. Then $x(s_\alpha s_\beta)^{(m)}$ and $x(s_\beta s_\alpha)^{(m)}$ are neighbors in $W(\overline{ByY})$. Moreover, $r(\overline{ByY}) \geq r(Y)$, whence $r(\overline{ByY}) = r(G/H)$. Thus, we may assume that $y = 1$.

Let $Y' = \overline{B(s_\alpha s_\beta)^{(m)}Y}$ and $Y'' = \overline{B(s_\beta s_\alpha)^{(m)}Y}$, then we obtain similarly: $r(Y') = r(Y'') = r(G/H)$ and $x \in W(Y') \cap W(Y'')$. If $x \neq 1$, write $x = s_\gamma x'$ where $\gamma \in \Delta$ and $\ell(x) = \ell(x') + 1$. Then $\overline{Bx'Y'}$ and $\overline{Bx'Y''}$ have rank $r(G/H)$ and are raised to G/H by γ . Thus, γ has type U, so that $\overline{Bx'Y'} = \overline{Bx'Y''}$. By induction on $\ell(x)$, we obtain $Y' = Y''$. This subvariety is stable under $P_{\alpha, \beta}$. Applying Lemmas 2 and 3, we may assume that $Y' = G/H$ (i.e., $x = 1$), $\Delta = \{\alpha, \beta\}$, the center of G is trivial, and H has finite index in its normalizer. Moreover, we have $P(G/H) = B$, for P_α and P_β do not stabilize $(G/H)^0$.

We claim that any $Z \in \mathcal{B}(G/H)$ can be written as

$$\overline{B(s_\alpha s_\beta)^{(n)}Y} = \cdots P_\beta P_\alpha Y \quad \text{or} \quad \overline{B(s_\beta s_\alpha)^{(n)}Y} = \cdots P_\alpha P_\beta Y \quad (n \text{ terms}),$$

where $n = \dim(Z) - \dim(Y)$ satisfies $0 \leq n \leq m$. For this, we argue by induction on the codimension of Z in G/H . We may assume that α raises Z . By the induction assumption, we have

$$P_\alpha Z = P_\beta P_\alpha \cdots Y \quad \text{or} \quad P_\alpha Z = P_\alpha P_\beta \cdots Y \quad (n + 1 \text{ terms}).$$

In the latter case, let $Z' = P_\beta \cdots Y$ (n terms). Since $P_\alpha Z = P_\alpha Z'$ and $r(Z) = r(Z') = r(P_\alpha Z) = r(Y)$, it follows that $Z = Z'$. In the former case, $P_\alpha Z$ is stable under G and hence equal to G/H ; in particular, Z has codimension 1 in G/H . Now $G/H = P_\alpha P_\beta \cdots Y$ (m terms), so that we are in the previous case.

By the claim, all B -orbit closures in G/H have the same rank, and Y^0 is the unique closed B -orbit. Let $y \in Y^0$; we may assume that $H = G_y$. Since the H -orbit in G/B corresponding to the B -orbit Y^0 in G/H is closed, the connected isotropy group B_y^0 is a Borel subgroup of H^0 . It follows that $r(Y) = r(B) - r(B_y) = 2 - r(H)$. On the other hand, $r(Y) = r(G/H)$ by assumption. Thus, $r(G/H) = 2 - r(H)$.

If $r(G/H) = 0$ then H is a parabolic subgroup of G (in fact, a Borel subgroup as $P(G/H) = B$.) Moreover, Y is the B -fixed point in G/H . But then $W(Y)$ consists of a unique element (of maximal length in W), a contradiction.

If $r(G/H) = 1$ then $r(H) = 1$ as well. Using the classification of homogeneous spaces of rank 1 under semi-simple groups of rank 2 (see e.g. Table 1 of [26]), this forces $G = \text{PGL}(2) \times \text{PGL}(2)$ and $H = \text{PGL}(2)$ embedded diagonally in G .

As a consequence, the simple roots α and β are orthogonal, and $X(G/H)$ is generated by $\alpha + \beta$.

If $r(G/H) = 2$ then $r(H) = 0$, that is, H^0 is unipotent. Since G/H is spherical, H^0 is a maximal unipotent subgroup of G . This contradicts the assumption that H has finite index in its normalizer. \square

Proposition 5. *If G is simply-laced, then*

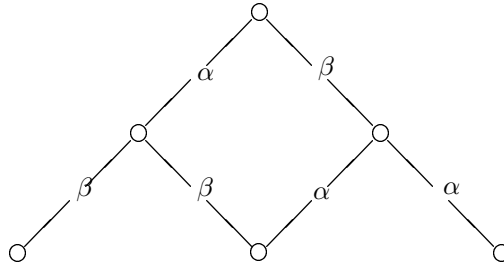
(i) *for any oriented path γ in $\Gamma(G/H)$, both $\ell_T(\gamma)$ and $\ell_N(\gamma)$ depend only on the endpoints of γ .*

(ii) *for any $Y \in \mathcal{B}(G/H)$, there exists an oriented path γ joining Y to G/H through a sequence of simple edges followed by a sequence of double edges.*

Proof. (i) Let Y (resp. Y') be the source (resp. target) of γ , and let δ be another oriented path from Y to Y' . By Lemma 5, it suffices to show that $\ell_N(\gamma) = \ell_N(\delta)$. Joining Y' to G/H by an oriented path, we reduce to the case where $Y' = G/H$; then $w(\gamma)$ and $w(\delta)$ are in $W(Y)$. By Proposition 2, we may assume moreover that $w(\gamma)$ and $w(\delta)$ are neighbors. Using Lemmas 2 and 3, we reduce to the case where the center of G is trivial, $\Delta = \{\alpha, \beta\}$, H has finite index in its normalizer, $w(\gamma) = (s_\alpha s_\beta)^{(m)}$ and $w(\delta) = (s_\beta s_\alpha)^{(m)}$ for some $m < m(\alpha, \beta)$.

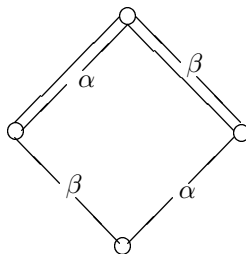
Since G is simply-laced, we have either $G = \text{PGL}(2) \times \text{PGL}(2)$ and $m(\alpha, \beta) = 2$, or $G = \text{PGL}(3)$ and $m(\alpha, \beta) = 3$. In particular, $m \leq 2$. If $m = 1$ then $\ell_N(\gamma) = \ell_N(\delta) = 0$ by Proposition 1. If $m = 2$ then $G = \text{PGL}(3)$. Using Lemma 5 (iv), we may assume moreover that H is not contained in any Borel subgroup. Then we see by inspection that H is conjugate to $\text{PO}(3)$ or to $\text{GL}(2)$.

In the latter case, here is $\Gamma(G/H)$:



Thus, $\ell_N(\gamma) = \ell_N(\delta) = 0$.

In the former case, we have $\ell_N(\gamma) = \ell_N(\delta) = 1$, since $\Gamma(G/H)$ is as follows:



(ii) Let γ be an oriented path joining Y to G/H . We may assume that γ contains double edges. Consider the lowest maximal subpath δ of γ that consists only of double edges; we may assume that the endpoint of δ is not G/H . Let Y' be the source of the top edge of δ , and let α (resp. β) be the label of that edge (resp. of the next edge of γ , a simple edge by assumption.) We claim that there exists an oriented path γ' joining Y' to G/H and beginning with a simple edge; then assertion (ii) will follow by induction on $\ell(\delta) + \text{codim}_{G/H}(Y')$.

To check the claim, it suffices to join Y' to $P_{\alpha\beta}Y'$ by an oriented path γ' beginning with a simple edge. As above, we reduce to the case where G equals $\text{PGL}(2) \times \text{PGL}(2)$ or $\text{PGL}(3)$, and H is not contained in a Borel subgroup of G ; moreover, H has finite index in its normalizer. Using the fact that $\Gamma(G/H)$ contains a double edge followed by a simple edge, one checks that H is a product of subgroups of $\text{PGL}(2)$ if $G = \text{PGL}(2) \times \text{PGL}(2)$; and if $G = \text{PGL}(3)$, then H is conjugate to the subgroup of Example 1, or to its transpose. The path γ' exists in all of these cases. \square

From Proposition 5, we will deduce a criterion for the graph of G/H to contain only simple edges. To formulate it, we need more notation and a preliminary result.

Let $\mathcal{D}(G/H)$ be the subset of $\mathcal{B}(G/H)$ consisting of all irreducible B -stable divisors. The elements of $\mathcal{D}(G/H)$ are called *colors*; they play an important role in the classification of embeddings of G/H , see [15]. Let $D \in \mathcal{D}(G/H)$, with preimage \tilde{D} in G . Replacing G by a finite cover, we may assume that \tilde{D} is the divisor of a regular function f_D on G . Then f_D is an eigenvector of B acting by left multiplication; let ω_D be its weight. Since f_D is uniquely defined up to multiplication by a regular invertible function on G , then ω_D is unique up to addition of a character of G . In particular, for any $\alpha \in \Delta$, the number $\langle \omega_D, \tilde{\alpha} \rangle$ is a non-negative integer depending only on D and α .

Lemma 7. *The degree $d(D, \alpha)$ of the morphism $\pi_{D, \alpha} : P_\alpha \times^B D \rightarrow G/H$ equals $\langle \omega_D, \tilde{\alpha} \rangle$ if $\pi_{D, \alpha}$ is generically finite; otherwise, $\langle \omega_D, \tilde{\alpha} \rangle = 0$.*

Proof. Note that D is P_α -stable if and only if f_D is an eigenvector of P_α , that

is, ω_D extends to a character of that group. This amounts to: $\langle \omega_D, \check{\alpha} \rangle = 0$.

Let V be the H -stable divisor in G/B corresponding to the B -stable divisor D in G/H . Then V is the zero scheme of a section of the homogeneous line bundle on G/B associated with the character ω_D of B . Let $p : G/B \rightarrow G/P_\alpha$ be the natural map, then $d(D, \alpha)$ equals the degree of the restriction $p_V : V \rightarrow G/P_\alpha$. The latter degree is the intersection number of V with a fiber of p , that is, $\langle \omega_D, \check{\alpha} \rangle$. \square

A direct consequence of Lemma 7 and Proposition 5 is

Corollary 2. *For simply-laced G , the following conditions are equivalent:*

- (i) *Each edge of $\Gamma(G/H)$ is simple.*
- (ii) *For any $D \in \mathcal{D}(G/H)$ and $\alpha \in \Delta$, we have $\langle \omega_D, \check{\alpha} \rangle \leq 1$.*

It follows e.g. that the graphs of the symmetric spaces $GL(p+q)/GL(p) \times GL(q)$, $SL(2n)/SP(2n)$, $SO(2n)/GL(n)$ and E_6/F_4 contain only simple edges. For this, one uses the explicit description of colors of symmetric spaces given in [25].

Note that Corollary 2 does not extend to multiply-laced groups G . Consider, for example, $G = SO(2n+1)$ and its subgroup $H = O(2n)$, the stabilizer of a non-degenerate line in \mathbb{C}^{2n+1} . Then G/H is symmetric of rank 1 and its graph consists of a unique oriented path: a double edge followed by $n-1$ simple edges.

2. Orbit closures in regular completions

Recall from [2] that a variety X with an action of G is called *regular* if it satisfies the following three conditions:

- (i) X is smooth and contains a dense G -orbit whose complement is a union of irreducible smooth divisors (the *boundary divisors*) with normal crossings.
- (ii) Any G -orbit closure in X is the transversal intersection of those boundary divisors that contain it.
- (iii) For any $x \in X$, the normal space to the orbit Gx contains a dense orbit of the isotropy group of x .

If moreover X is complete and its dense G -orbit is isomorphic to G/H , we call X a *regular completion* of that homogeneous space. Recall from [3] that G/H admits a regular completion if and only if it is spherical; then any equivariant completion of G/H is the image of a regular one by an equivariant morphism.

We fix a regular completion X and we denote by $\mathcal{B}(X)$ (resp. $\mathcal{B}(G/H, X)$) the set of all B -orbit closures in X (resp. of those that meet G/H .) By [6] 1.4, any $Y \in \mathcal{B}(G/H, X)$ meets all G -orbit closures properly. Moreover, for any closed G -orbit Z , the irreducible components of $Y \cap Z$ are the Schubert varieties $\overline{Bw^{-1}z}$ where $w \in W(Y)$ and z is the unique T -fixed point of Z such that $\overline{Bz} = Z$.

This point z will be called the *base point* of Z ; its isotropy group is opposed to $P(G/H)$, see e.g. [3] 2.2. The intersection multiplicity of Y and Z along $\overline{Bw^{-1}z}$ equals $d(Y, w)$.

We shall generalize this to intersections of Y with arbitrary G -orbit closures. For this, we recall the local structure of X (see e.g. [3] 2.3). Set $P = P(G/H)$ and $L = L(G/H)$. Let X_0 be the set of all $x \in X$ such that Bx is open in Gx . Then X_0 is an open P -stable subset of X . Moreover, there exists an L -stable subvariety S of X_0 , fixed pointwise by $[L, L]$, such that the map

$$\begin{aligned} R_u(P) \times S &\rightarrow X_0 \\ (g, x) &\mapsto gx \end{aligned}$$

is an isomorphism. As a consequence, S is a smooth toric variety (for a quotient of T) of dimension $r(G/H)$, the rank of G/H ; moreover, S meets each G -orbit along a unique T -orbit. Let $\varphi : X_0 \cong R_u(P) \times S \rightarrow S$ be the second projection, then φ is L -equivariant; it can be seen as the quotient map by the action of $R_u(P)$.

Choose $w \in W(Y)$, then wY meets X_0 (because $\overline{BwY} = X$), and the intersection $X_0 \cap wY$ is stable by the group $P \cap wBw^{-1}$. The latter contains $R_u(P) \cap wUw^{-1}$ as a normal subgroup. We shall see that $R_u(P) \cap wUw^{-1}$ acts freely on $wY \cap X_0$, with section

$$S_{Y,w} = wY \cap (U \cap wU^{-1}w^{-1})S.$$

Note that $U \cap wU^{-1}w^{-1}$ is contained in $R_u(P)$, because $w^{-1} \in W^P$. Thus, $S_{Y,w}$ is a closed T -stable subvariety of $wY \cap X_0$. Let

$$\varphi_{Y,w} : S_{Y,w} \rightarrow S$$

be the restriction of $\varphi : X_0 \rightarrow S$, then $\varphi_{Y,w}$ is T -equivariant.

Proposition 6. *Keep notation as above.*

(i) *The map*

$$\begin{aligned} (R_u(P) \cap wUw^{-1}) \times S_{Y,w} &\rightarrow wY \cap X_0 \\ (g, x) &\mapsto gx \end{aligned}$$

is an isomorphism.

(ii) *The variety $S_{Y,w}$ is irreducible and meets each G -orbit along a unique T -orbit. In particular, $S_{Y,w} \cap G/H$ is a unique T -orbit, dense in $S_{Y,w}$ and contained in wY^0 ; and $S_{Y,w} \cap Z = \{z\}$ for any closed G -orbit Z with base point z .*

(iii) *The morphism $\varphi_{Y,w}$ is finite surjective of degree $d(Y, w)$. As a consequence, the dimension of $S_{Y,w}$ is the rank of X .*

Proof. (i) The product map $(R_u(P) \cap wUw^{-1}) \times (R_u(P) \cap wU^{-1}w^{-1}) \rightarrow R_u(P)$ is an isomorphism; moreover, $R_u(P) \cap wU^{-1}w^{-1} = U \cap wU^{-1}w^{-1}$. Therefore, the

product map

$$(R_u(P) \cap wUw^{-1}) \times (U \cap wU^{-1}w^{-1})S \rightarrow X_0$$

is an isomorphism. The assertion follows by intersecting with wY .

(ii) and (iii) The union of all G -orbits in X that contain Z in their closure is a G -stable open subset of X . Thus, we may assume that Z is the unique closed G -orbit in X . Let D_1, \dots, D_r be the boundary divisors, then $r = r(X)$. Moreover, S is isomorphic to affine space \mathbb{A}^r with coordinate functions x_1, \dots, x_r , equations of $D_1 \cap S, \dots, D_r \cap S$. The compositions $f_1 = x_1 \circ \varphi, \dots, f_r = x_r \circ \varphi$ are equations of $D_1 \cap X_0, \dots, D_r \cap X_0$; they generate the ideal of $Z \cap X_0 = Bz$ in X_0 . The map $\varphi : X_0 \rightarrow S$ identifies to $(f_1, \dots, f_r) : X_0 \rightarrow \mathbb{A}^r$. The intersections of G -orbit closures with X_0 are the pull-backs of coordinate subspaces of \mathbb{A}^r .

By (i), $S_{Y,w}$, is irreducible. We check that $S_{Y,w} \cap Z = \{z\}$. For this, note that the product map

$$(R_u(P) \cap wUw^{-1}) \times (S_{Y,w} \cap Z) \rightarrow wY \cap X_0 \cap Z = wY \cap Bz$$

is an isomorphism. Moreover, since Y meets Z properly, with $\overline{Bw^{-1}z}$ as an irreducible component, it follows that $wY \cap Bz$ is equidimensional, with

$$\overline{wBw^{-1}z} \cap Bz = (B \cap wBw^{-1})z$$

as an irreducible component. The latter is isomorphic to $R_u(P) \cap wUw^{-1}$. Thus, the T -stable set $S_{Y,w} \cap Z$ is finite, so that it consists of T -fixed points. Since z is the unique T -fixed point in Bz , our assertion follows.

The map $\varphi_{Y,w} : S_{Y,w} \rightarrow S$ identifies with $(f_1, \dots, f_r) : S_{Y,w} \rightarrow \mathbb{A}^r$. We just saw that the set-theoretical fiber of 0 is $\{z\}$. Since 0 is the unique closed T -orbit in \mathbb{A}^r , all fibers of $\varphi_{Y,w}$ are finite. Thus, $S_{Y,w}$ contains a dense T -orbit. Since $S_{Y,w}$ is affine and contains a T -fixed point z , it follows that $\varphi_{Y,w}$ is finite and that the pull-back of any T -orbit in S is a unique T -orbit. This implies (ii).

Finally, we check that the degree of $\varphi_{Y,w}$ equals $d(Y, w)$, that is, the degree of the natural map $\overline{BwB} \times^B Y \rightarrow X$. For this, note that the map

$$U \cap wU^{-1}w^{-1} \rightarrow \overline{BwB}/B, \quad g \rightarrow gwB/B$$

is an open immersion. Thus, $d(Y, w)$ is the degree of the product map $(U \cap wU^{-1}w^{-1}) \times wY \rightarrow X$, or, equivalently, of its restriction

$$p : (U \cap wU^{-1}w^{-1}) \times (wY \cap X_0) \rightarrow X_0.$$

The latter map fits into a commutative diagram

$$\begin{array}{ccc} (U \cap wU^{-1}w^{-1}) \times (wY \cap X_0) & \rightarrow & X_0 \\ \downarrow & & \downarrow \\ S_{Y,w} & \rightarrow & S, \end{array}$$

where the bottom horizontal map is $\varphi_{Y,w}$; indeed,

$$(U \cap wU^{-1}w^{-1}) \times (wY \cap X_0) \cong (R_u(P) \cap wU^{-1}w^{-1}) \times (R_u(P) \cap wUw^{-1}) \times S_{Y,w}$$

by (i). Moreover, the fibers of the right (resp. left) vertical map are isomorphic to $R_u(P)$ (resp. to $(R_u(P) \cap wU^-w^{-1}) \times (R_u(P) \cap wUw^{-1}) \cong R_u(P)$.) Thus, the diagram is cartesian, and the degree of p equals the degree of $\varphi_{Y,w}$. \square

Thus, we can view $S_{Y,w}$ as a “slice” in wY to $wBw^{-1}z = (R_u(P) \cap wUw^{-1})z$ at z . But $S_{Y,w}$ may be non transversal to Z at z : indeed, the intersection multiplicity of $S_{Y,w}$ and Z at z equals the intersection multiplicity of Y and Z along $\overline{Bw^{-1}z}$, and the latter equals $d(Y, w)$ by [6] 1.4 (alternatively, this can be deduced from Proposition 6 (iii).) On the other hand, it is not clear whether $S_{Y,w}$ is smooth, that is, $Y \cap w^{-1}X_0$ consists of smooth points of Y ; see Corollary 3 below for a partial answer to this question.

We now relate the “slices” associated with both endpoints of an edge in $\Gamma(G/H)$. Let $Y \in \mathcal{B}(G/H, X)$ and let $\alpha \in \Delta$ raising Y . Choose $v \in W(P_\alpha Y)$, then $w = vs_\alpha$ is in $W(Y)$, and $\ell(w) = \ell(v) + 1$. Thus, $v(\alpha) \in \Phi^+ \cap w(\Phi^-)$. Let $U_{v(\alpha)}$ be the corresponding unipotent subgroup of dimension 1, then $U_{v(\alpha)}$ is contained in $R_u(P) \cap vUv^{-1}$.

Proposition 7. *With notation as above, $S_{Y,w}$ is contained in $U_{v(\alpha)}S_{P_\alpha Y, v}$ and the latter is isomorphic to $U_{v(\alpha)} \times S_{P_\alpha Y, \tau}$. Denoting by*

$$\varphi_{Y, \alpha} : S_{Y,w} \rightarrow S_{P_\alpha Y, v}$$

the corresponding projection, then $\varphi_{Y,w} = \varphi_{P_\alpha Y, v} \circ \varphi_{Y, \alpha}$. Moreover, $\varphi_{Y, \alpha}$ is finite surjective of degree $d(Y, \alpha)$.

Proof. We have

$$\begin{aligned} S_{Y,w} &= wY \cap (U \cap wU^-w^{-1})S = wY \cap U_{v(\alpha)}(U \cap vU^-v^{-1})S \\ &\subseteq vP_\alpha Y \cap U_{v(\alpha)}(U \cap vU^-v^{-1})S = U_{v(\alpha)}(vP_\alpha Y \cap (U \cap vU^-v^{-1})S) = \\ &= U_{v(\alpha)}S_{P_\alpha Y, v}. \end{aligned}$$

Moreover, since $U_{v(\alpha)} \subseteq R_u(P) \cap vUv^{-1}$, the product map $U_{v(\alpha)} \times S_{P_\alpha Y, v} \rightarrow U_{v(\alpha)}S_{P_\alpha Y, v}$ is an isomorphism. Now the equality $\varphi_{Y,w} = \varphi_{P_\alpha Y, v} \circ \varphi_{Y, \alpha}$ follows from the definitions. Together with Proposition 6 (iii), it implies that $\varphi_{Y, \alpha}$ is finite surjective of degree $d(Y, w)d(P_\alpha Y, v)^{-1} = d(Y, \alpha)$. \square

Next we describe the intersection of a B -orbit closure with an arbitrary G -orbit closure, generalizing [6] Theorem 1.4.

Theorem 1. *Let X be a regular completion of G/H , let $Y \in \mathcal{B}(G/H, X)$ and let X' be a G -orbit closure in X . Then $W(Y)$ is the disjoint union of the $W(Y')$ where Y' runs over all irreducible components of $Y \cap X'$. Moreover, for any such Y' and $w \in W(Y')$, we have*

$$d(Y, w) = d(Y', w)i(Y', Y \cdot X'; X)$$

where $i(Y', Y \cdot X'; X)$ denotes the intersection multiplicity of Y' and X' along Y' in X . As a consequence, this multiplicity is a power of 2.

Proof. By [6] Lemma 1.3, $W(Y)$ is the union of the $W(Y')$. Choose Y' and $w \in W(Y')$, then $Y' \cap w^{-1}X_0$ is an irreducible component of $Y \cap w^{-1}X_0 \cap X'$. The latter is isomorphic to $(U \cap w^{-1}R_u(P)) \times w^{-1}(S_{Y,w} \cap X')$, and $S_{Y,w} \cap X'$ is a unique T -orbit, by Proposition 6. It follows that $Y \cap w^{-1}X_0 \cap X' = Y' \cap w^{-1}X_0$ is irreducible, so that Y' is uniquely determined by w . Equivalently, the $W(Y')$ are pairwise disjoint.

Let Z be a closed G -orbit in X' , then

$$d(Y, w) = i(\overline{Bw^{-1}z}, Y \cdot Z; X) = i(\overline{Bw^{-1}z} \cap w^{-1}X_0, (Y \cap w^{-1}X_0) \cdot (Z \cap w^{-1}X_0); w^{-1}X_0),$$

where the former equality follows from [6] 1.4, and the latter from [12] 8.2. Moreover, we have by Proposition 6: $\overline{Bw^{-1}z} \cap w^{-1}X_0 = Bw^{-1}z$ and $Z \cap w^{-1}X_0 = w^{-1}Bz$. Thus,

$$d(Y, w) = i(Bw^{-1}z, (Y \cap w^{-1}X_0) \cdot w^{-1}Bz, w^{-1}X_0).$$

Using the fact that $Y \cap w^{-1}X_0 \cap X' = Y' \cap w^{-1}X_0$ is irreducible, together with associativity of intersection multiplicities (see [12] 7.1.8), we obtain

$$\begin{aligned} d(Y, w) &= i(Bw^{-1}z, (Y' \cap w^{-1}X_0) \cdot w^{-1}Bz; w^{-1}X_0 \cap X') i(Y', Y \cdot X'; X) \\ &= i(\overline{Bw^{-1}z}, Y' \cdot Z; X') i(Y', Y \cdot X'; X) = d(Y', w) i(Y', Y \cdot X'; X). \end{aligned}$$

□

These results motivate the following

Definition. A B -orbit closure Y in G/H is *multiplicity-free* if $d(Y, w) = 1$ for all $w \in W(Y)$. Equivalently, all oriented paths in $\Gamma(G/H)$ with source Y contain only simple edges.

For example, Y is multiplicity-free if $r(Y) = r(G/H)$, or if H is contained in a Borel subgroup of G (this follows from Lemma 5.)

Other examples of multiplicity-free orbit closures arise from parabolic induction: with notation as in Lemma 6, Y is multiplicity-free if and only if Y' is.

If G is simply-laced, then Y is multiplicity-free if and only if it can be joined to X by an oriented path with only simple edges (Proposition 5).

Corollary 3. *With notation as in Theorem 1, if Y is multiplicity-free then all irreducible components of $Y \cap X'$ are multiplicity-free as well, and the corresponding intersection multiplicities equal 1. Moreover, for any $w \in W(Y)$, the map $\varphi_{Y,w} : S_{Y,w} \rightarrow S$ is an isomorphism. As a consequence, $Y \cap w^{-1}X_0$ consists of smooth points of Y .*

Proof. The first assertion follows from Theorem 1. By Proposition 6, $\varphi_{Y,w}$ is

finite surjective of degree 1, hence an isomorphism because S is smooth. \square

We next characterize those B -orbit closures $Y \in \mathcal{B}(G/H, X)$ that are multiplicity-free, in terms of the intersection numbers $\int_X [Y] \cdot [Y']$ where $Y' \in \mathcal{B}(X)$. Here $\int_X [Y] \cdot [Y']$ denotes the degree of the product of the classes of Y, Y' in the integral cohomology ring of X ; the latter is generated as an abelian group by classes of B -stable subvarieties.

Corollary 4. *With notation as in Theorem 1, the numbers $\int_X [Y] \cdot [Y']$ are 0 or powers of 2, for all $Y' \in \mathcal{B}(X)$. Moreover, Y is multiplicity-free if and only if $\int_X [Y] \cdot [Y']$ equals 0 or 1, for any $Y' \in \mathcal{B}(X)$.*

Proof. Let $Y' \in \mathcal{B}(X)$. By [6] 1.4 Corollary, $\int_X [Y] \cdot [Y'] \neq 0$ if and only if: $\dim(Y) + \dim(Y') = \dim(X)$, and Y meets $w_0 Y'$ (where w_0 denotes the longest element of W .) Under these hypotheses, $Y \cap w_0 Y'$ consists of a unique point y , fixed by T . Moreover, the proof of [*loc. cit.*] shows that $w_0 y \in Y'^0$. Thus, \overline{By} and $\overline{B^{-1}y} = w_0 \overline{Bw_0 y} = w_0 Y'$ meet transversally at y in $\overline{Gy} = GY'$. As a consequence, we have

$$\dim(By) = \dim(GY') - \dim(w_0 Y') = \dim(GY') + \dim(Y) - \dim(X) = \dim(Y \cap GY').$$

Since $Y \cap GY'$ is equidimensional and B -stable, it follows that its unique irreducible component through y is \overline{By} .

Using the projection formula, we obtain

$$\begin{aligned} \int_X [Y] \cdot [Y'] &= \int_X [Y] \cdot [w_0 Y'] = \int_{GY'} ([Y] \cdot [GY']) \cdot [w_0 Y'] \\ &= i(\overline{By}, Y \cdot GY'; X) \int_{GY'} \overline{By} \cdot [w_0 Y'] = i(\overline{By}, Y \cdot GY'; X). \end{aligned}$$

Thus, by Theorem 1, $\int_X [Y] \cdot [Y']$ is a power of 2; if moreover Y is multiplicity-free, then $\int_X [Y] \cdot [Y'] = 1$.

Conversely, assume that $\int_X [Y] \cdot [Y']$ equals 0 or 1 for all $Y' \in \mathcal{B}(X)$. Let then $w \in W(Y)$; choose a closed G -orbit Z with base point z and consider $Y' = \overline{Bw_0 w^{-1}z}$. Then $w_0 Y'$ is transversal to $\overline{Bw^{-1}z}$ at z in Z , so that $\dim(Y') = \text{codim}_Z(\overline{Bw^{-1}z})$. But $\overline{Bw^{-1}z}$ is an irreducible component of $Y \cap Z$, and this intersection is proper. Thus, $\dim(Y') = \text{codim}_X(Y)$. Since Y meets $w_0 Y'$ at $w^{-1}z$, and $GY' = Z$, we have

$$\int_X [Y] \cdot [Y'] = i(\overline{Bw^{-1}z}, Y \cdot Z; X);$$

and the latter equals $d(Y, w)$, by [6] 1.4. It follows that Y is multiplicity-free. \square

We now show that the intersections of B -orbit closures with G -orbit closures in regular completions satisfy Hartshorne's connectedness theorem, see [11] 18.2.

That theorem is proved there for schemes of depth at least 2; but B-orbit closures may have depth 1 at some points, see Example 6 in Section 4.

Theorem 2. *Consider a regular completion X of G/H , a B-orbit closure Y and a G-orbit closure X' in X . Then $Y \cap X'$ is connected in codimension 1 (that is, the complement in $Y \cap X'$ of any closed subset of codimension at least 2 is connected.)*

Proof. We may assume that $Y \in \mathcal{B}(G/H, X)$. If $X' = Z$ is a closed G-orbit, then the irreducible components of $Y \cap Z$ are the $\overline{Bu^{-1}z}$, where $u \in W(Y)$. Consider neighbors $u, v \in W(Y)$ and let \overline{w} be as in Proposition 3. Then $\overline{Bw^{-1}z}$ is a divisor in both $\overline{Bu^{-1}z}$ and $\overline{Bv^{-1}z}$. Since any two elements of $W(Y)$ are connected by a chain of neighbors by Proposition 2, the assertion follows.

For arbitrary X' , let Z be a closed G-orbit in X' . Let Y'_1, Y'_2 be unions of irreducible components of $Y \cap X'$ such that $Y \cap X' = Y'_1 \cup Y'_2$. Then $Y'_1 \cap Z$ and $Y'_2 \cap Z$ are unions of irreducible components of $Y' \cap Z$ (for any irreducible component of $Y \cap X'$ meets Z properly in X'); Moreover, their intersection has codimension 1 in $Y'_1 \cap Z$ and $Y'_2 \cap Z$, by the first step of the proof. It follows that $Y'_1 \cap Y'_2$ has codimension 1 in both Y'_1 and Y'_2 . \square

3. Orbit closures of maximal rank

Let $\mathcal{B}(G/H)_{max}$, be the set of all $Y \in \mathcal{B}(G/H)$ such that $r(Y) = r(G/H)$, that is, the set of all B-orbit closures of maximal rank. Recall that all such orbit closures are multiplicity-free. Here is another characterization of them that generalizes a well known property of Schubert varieties.

Proposition 8. (i) *For any $Y \in \mathcal{B}(G/H)_{max}$, and $w \in W(Y)$, we have: $BwY^0 = (G/H)^0$ and $w^{-1} \in W^{\Delta(G/H)}$. Moreover, $W(Y)$ is disjoint from all $W(Y')$ where $Y' \in \mathcal{B}(G/H)$ and $Y' \neq Y$.*

(ii) *Conversely, if $Y \in \mathcal{B}(G/H)$ and there exists $w \in W$ such that $BwY^0 = (G/H)^0$, then Y has maximal rank. If moreover $w^{-1} \in W^{\Delta(G/H)}$, then $w \in W(Y)$, and $\Delta(Y)$ consists of those $\alpha \in \Delta$ such that $w(\alpha) \in \Delta(G/H)$.*

Proof. (i) We prove that $BwY^0 = (G/H)^0$ by induction over $\ell(w)$, the case where $\ell(w) = 0$ being evident. If $\ell(w) \geq 1$, we can write $w = w's_\alpha$ for some simple root α and some $w' \in W$ such that $\ell(w') = \ell(w) - 1$; then $BwB = Bw'Bs_\alpha B$. Then $G/H = \overline{BwY} = \overline{Bw'P_\alpha Y}$. Since $\ell(w) = \text{codim}_{G/H}(Y)$, it follows that α raises Y and that $w' \in W(P_\alpha Y)$. Because Y has maximal rank, $P_\alpha Y^0$ consists of two B-orbits, both of maximal rank. But $P_\alpha Y^0 = Y^0 \cup Bs_\alpha Y^0$ so that $Bs_\alpha Y^0$ is a unique B-orbit of maximal rank and of codimension $\ell(w')$ in G/H . By the induction assumption, we have $Bw'Bs_\alpha Y^0 = (G/H)^0$, that is, $BwY^0 = (G/H)^0$.

If moreover $w \in W(Y')$ for some $Y' \in \mathcal{B}(G/H)$, then a similar induction shows that $Y' = Y$.

If $w^{-1} \notin W^{\Delta(G/H)}$ then there exists $\beta \in \Delta(G/H)$ such that $\ell(s_\alpha w) = \ell(w) - 1$. Thus, $BwB = Bs_\beta Bs_\beta wB$, so that $s_\beta Bs_\beta wY^0$ is contained in $(G/H)^0$. But $s_\beta(G/H)^0 = (G/H)^0$; therefore, $Bs_\beta wY^0 = (G/H)^0$, and $\overline{Bs_\beta wY} = G/H$. It follows that $\text{codim}_{G/H}(Y) \leq \ell(s_\beta w) = \ell(w) - 1$, a contradiction.

(ii) Let \dot{w} be a representative of w in the normalizer of T . By assumption, the map

$$\begin{aligned} U \times Y^0 &\rightarrow (G/H)^0 \\ (u, y) &\mapsto u\dot{w}y \end{aligned}$$

is surjective. Thus, it induces an injective homomorphism from the ring $\mathbb{C}[(G/H)^0]$ of regular functions on $(G/H)^0$, to $\mathbb{C}[U \times Y^0]$. The group of invertible regular functions $\mathbb{C}[(G/H)^0]^*$ is mapped into $\mathbb{C}[U \times Y^0]^* = \mathbb{C}[Y^0]^*$. Quotienting by \mathbb{C}^* and taking ranks, we obtain $r(G/H) \leq r(Y)$ by Lemma 1, whence $r(Y) = r(G/H)$.

If moreover $w^{-1} \in W^{\Delta(G/H)}$, we show that $w \in W(Y)$ by induction over $\ell(w)$; we may assume that $w \neq 1$. Then we can write $w = w's_\alpha$ where $w' \in W$, $\alpha \in \Delta$ and $\ell(w) = \ell(w') + 1$. It follows that $w(\alpha) \in \Phi^-$.

We begin by checking that $s_\alpha Y^0 \neq Y^0$. Otherwise, by Lemma 1, there exists $y \in Y^0$ fixed by $[L_\alpha, L_\alpha]$. Thus, $\dot{w}y \in (G/H)^0$ is fixed by $w[L_\alpha, L_\alpha]w^{-1}$. Since the unipotent radical of $P(G/H)$ acts freely on $(G/H)^0$ by [17], it follows that $w(\alpha) \in \Phi_{\Delta(G/H)}^-$. Then $\alpha \in \Delta \cap w^{-1}(\Phi_{\Delta(G/H)}^-)$ which contradicts the assumption that $w^{-1} \in W^{\Delta(G/H)}$.

As above, it follows that $Bs_\alpha Y^0$ is a B -orbit of maximal rank and of dimension $\dim(Y) + 1$; moreover, $Bw'Bs_\alpha Y^0 = (G/H)^0$. We can write $w' = uv$ where $u \in W_{\Delta(G/H)}$, $v^{-1} \in W^{\Delta(G/H)}$, and $\ell(w') = \ell(u) + \ell(v)$. Thus, $BwB = BuBvBs_\alpha B$, and $BvBs_\alpha Y^0 = (G/H)^0$ as $u^{-1}(G/H)^0 = (G/H)^0$. By the induction assumption, $v \in W(\overline{Bs_\alpha Y})$. Moreover, $\ell(vs_\alpha) = \ell(v) + 1$, for $w = uvs_\alpha$ and $\ell(w) = \ell(u) + \ell(v) + 1$. It follows that $vs_\alpha \in W(Y)$; in particular, $s_\alpha v^{-1} \in W^{\Delta(G/H)}$. But $w^{-1} = s_\alpha v^{-1}u^{-1}$ is in $W^{\Delta(G/H)}$ as well. Thus, $u = 1$ and $w^{-1} \in W(Y)$.

Let α be a simple root of Y . Then we see as above that $w(\alpha) \in \Phi_{\Delta(G/H)}$. We have $ws_\alpha = s_{w(\alpha)}w$ with $s_{w(\alpha)} \in W_{\Delta(G/H)}$ and $w^{-1} \in W^{\Delta(G/H)}$. Thus, $\ell(ws_\alpha) = \ell(s_{w(\alpha)}) + \ell(w)$ which forces $w(\alpha) \in \Phi^+$ (as $\ell(s_\alpha w) = \ell(w) + 1$) and $w(\alpha) \in \Delta$ (as $\ell(s_{w(\alpha)}) = 1$.) We conclude that $w(\alpha)$ is a simple root of G/H .

Conversely, let $\alpha \in \Delta$ such that $w(\alpha)$ is a simple root of G/H . Then $\ell(ws_\alpha) = \ell(w) + 1$, whence

$$BwBs_\alpha Y^0 = Bws_\alpha Y^0 = Bs_{w(\alpha)}wY^0 = Bs_{w(\alpha)}BwY^0 = Bs_{w(\alpha)}(G/H)^0 = (G/H)^0.$$

Let \mathcal{O} be a B -orbit in $Bs_\alpha Y^0$. Then $Bw\mathcal{O} = (G/H)^0$. By (i), we have $\mathcal{O} = Y^0$, whence $s_\alpha Y^0 = Y^0$ and $\alpha \in \Delta(Y)$. □

This preliminary result, combined with those of Section 2, implies a structure theorem for orbit closures of maximal rank in a regular completion X . We denote by $\mathcal{B}(X)_{max}$ the set of these orbit closures, a subset of $\mathcal{B}(G/H, X)$.

Theorem 3. *Let $Y \in \mathcal{B}(X)_{max}$ and $w \in W(Y)$. Choose a “slice” $S_{Y,w}$ as in Proposition 6, so that the product map*

$$(U \cap w^{-1}R_u(P)w) \times w^{-1}S_{Y,w} \rightarrow Y \cap w^{-1}X_0$$

is an isomorphism. Then $w^{-1}S_{Y,w}$ is fixed pointwise by $[L(Y), L(Y)]$. Moreover, $Y \cap w^{-1}X_0$ is $P(Y)$ -stable and meets each G -orbit along a unique B -orbit, of maximal rank in this G -orbit. In particular, there exists $y \in Y^0$ fixed by $[L(Y), L(Y)]$ such that the product map $(U \cap w^{-1}R_u(P)w) \times Ty \rightarrow Y^0$ is an isomorphism.

As a consequence, for each G -orbit closure X' in X , all irreducible components of $Y \cap X'$ have maximal rank in X' . Moreover, a given $Y' \in \mathcal{B}(X')$ is an irreducible component of $Y \cap X'$ if and only if $W(Y')$ is contained in $W(Y)$.

Proof. With notation as in Section 2, recall that

$$w^{-1}S_{Y,w} = Y \cap (U^- \cap w^{-1}Uw)w^{-1}S$$

and that $[L(X), L(X)]$ acts trivially on S . Together with Proposition 8, it follows that $[L(Y), L(Y)]$ fixes pointwise S and normalizes $U^- \cap w^{-1}Uw$. Thus, $[L(Y), L(Y)]$ stabilizes $w^{-1}S_{Y,w}$. Moreover, intersecting that space with those boundary divisors that contain a given closed G -orbit, we obtain $[L(Y), L(Y)]$ -stable hypersurfaces meeting transversally at a fixed point. Therefore, $[L(Y), L(Y)]$ fixes pointwise $w^{-1}S_{Y,w}$.

By Proposition 6, $w^{-1}S_{Y,w}$ meets each G -orbit along a unique T -orbit. As a consequence, the intersection of $Y \cap w^{-1}X_0$ with each G -orbit is contained in a unique B -orbit. We apply this to GY^0 , the open G -orbit in X . Since $Y \cap w^{-1}X_0 \cap GY^0 = Y \cap w^{-1}X^0$ equals Y^0 by Proposition 8, we see that the product map

$$(U \cap w^{-1}R_u(P)w) \times (w^{-1}S_{Y,w} \cap Y^0) \rightarrow Y^0$$

is an isomorphism. Moreover, $w^{-1}S_{Y,w} \cap Y^0$ is a unique T -orbit of dimension equal to the rank of X .

It follows that each U -orbit in Y^0 is a unique orbit of $U \cap w^{-1}R_u(P)w$. Indeed, any U -orbit is isomorphic to some affine space, and its projection to $w^{-1}S_{Y,w} \cap Y^0$ is a morphism to a torus, hence is constant.

Choose $y_0 \in Y^0$ and let $y \in Y \cap w^{-1}X_0$. Since $By_0 = Y^0$ is dense in $Y \cap w^{-1}X_0$, we have $\dim(Uy) \leq \dim(Uy_0)$. The latter equals $\dim(U \cap w^{-1}R_u(P)w)$ by the previous step. Because $U \cap w^{-1}R_u(P)w$ acts freely on $Y \cap w^{-1}X_0$, it follows that $(U \cap w^{-1}R_u(P)w)y$ is open in Uy . But both are affine spaces, so that they are equal. Thus, $Y \cap w^{-1}X_0$ is B -stable. It is even $P(Y)$ -stable, because $P(Y) \subseteq w^{-1}Pw$ by Proposition 8.

Since $w^{-1}S_{Y,w}$ meets each G -orbit along a unique T -orbit, $Y \cap w^{-1}X_0$ meets each G -orbit along a unique B -orbit. Let $y \in Y \cap w^{-1}X_0$, then $wBy \subseteq X_0$ and, therefore, $wBy \subseteq (Gy)^0$. By Proposition 8 again, we have $r(By) = r(Gy)$.

The remaining assertions follow from Theorem 1 together with Proposition 8. □

As a consequence, we determine all B -orbit closures Y' such that $\int_X [Y] \cdot [Y'] \neq 0$; by Corollary 4, this amounts to $\int_X [Y] \cdot [Y'] = 1$.

Corollary 5. *Let Y be a B -orbit closure of maximal rank in a regular completion X and let $Y' \in \mathcal{B}(X)$. Then the intersection number $\int_X [Y] \cdot [Y']$ equals 1 if and only if $Y' = \overline{Bw_0w^{-1}z}$ for some $w \in W(Y)$ and some closed G -orbit Z with base point z .*

Proof. If $\int_X [Y] \cdot [Y'] = 1$, then $Y \cap w_0Y'$ consists of a unique T -fixed point $y \in w_0Y'^0$, and \overline{By} is an irreducible component of $Y \cap \overline{GY'}$, by the proof of Corollary 4. Therefore, \overline{By} has maximal rank in $\overline{GY'} = \overline{Gy'}$. But $r(\overline{By}) = 0$ because y is fixed by T . Thus Gy , being a G -orbit of rank 0, is closed in X . Let z be its base point, then $y = w^{-1}z$ for some $w \in W(Y)$, so that $Y' = \overline{Bw_0y} = \overline{Bw_0w^{-1}z}$. The converse is clear. □

We now describe the intersections of B -orbit closures of maximal rank with G -orbit closures, in terms of Knop's action of the Weyl group W on the set $\mathcal{B}(X)$. This action can be defined as follows.

Let $\alpha \in \Delta$ and $Y \in \mathcal{B}(X)$, then s_α fixes Y , except in the following cases:

- Type U: $P_\alpha Y^0 = Y^0 \cup Z^0$ for $Z \in \mathcal{B}(X)$ with $r(Z) = r(Y)$. Then s_α exchanges Y and Z .
- Type T: $P_\alpha Y^0 = Y^0 \cup Y_-^0 \cup Z^0$ for $Z \in \mathcal{B}(X)$ with $r(Y) = r(Y_-) = r(Z) - 1$. Then s_α exchanges Y and Y_- .

By [18, §4], this defines indeed a W -action (that is, the braid relations hold); moreover, $X(w(Y)) = w(X(Y))$ for all $w \in W$. In particular, this action preserves the rank (and, obviously, the subset $\mathcal{B}(G/H, X)$). If moreover G is simply-laced, then this action preserves multiplicity-freeness by Proposition 5; but this does not extend to multiply-laced G , as shown by Example 4.

For $Y \in \mathcal{B}(X)_{max}$ and $w \in W(Y)$, we have $w(Y) = X$. Thus, $\mathcal{B}(X)_{max}$ is the W -orbit of X in $\mathcal{B}(X)$. The isotropy group of X depends only on G/H ; denote it by $W(G/H)$. This group acts on $X(G/H)$ and contains $W_{\Delta(G/H)}$. The latter acts trivially on $X(G/H)$ by Lemma 1. In fact, $W_{(G/H)}$ stabilizes $\Phi_{\Delta(G/H)}$ (indeed, $\Phi_{\Delta(G/H)}$ consists of all roots that are orthogonal to $X(G/H)$, if G/H is non-degenerate in the sense of [17]; and the general case reduces to that one, by [17] §5.)

The normalizer of $\Phi_{\Delta(G/H)}$ in W is the semi-direct product of $W_{\Delta(G/H)}$ with the normalizer of $\Delta(G/H)$. Therefore, $W_{(G/H)}$ is the semi-direct product

of $W_{\Delta(G/H)}$ with

$$W_{G/H} = \{w \in W \mid w(G/H) = G/H \text{ and } w(\Delta(G/H)) = \Delta(G/H)\}.$$

The latter identifies to the image of $W_{(G/H)}$ in $\text{Aut } X(G/H)$, that is, to the “Weyl group of G/H ”, see [18] Theorem 6.2.

In fact, $W_{G/H}$ is the set of all $w \in W_{(G/H)}$ such that $w(\rho) - \rho \in X(G/H)$, where ρ denotes the half sum of positive roots (see [16] 6.5); we shall not need this result.

Let

$$W^{(G/H)} = \{w \in W \mid \ell(wu) \geq \ell(w) \quad \forall u \in W_{(G/H)}\},$$

the set of all elements of minimal length in their right $W_{(G/H)}$ -coset.

Proposition 9. *With notation as above, we have*

$$W^{(G/H)} = \left\{ w \in W^{\Delta(G/H)} \mid \ell(wu) \geq \ell(w) \quad \forall u \in W_{(G/H)} \right\},$$

and, for any $w \in W$,

$$W(w(G/H)) = \left\{ v \in W \mid v^{-1} \in W^{(G/H)} \cap wW_{(G/H)} \right\}.$$

As a consequence, all elements of minimal length in a given left $W_{(G/H)}$ -coset have the same length and are contained in a left $W_{G/H}$ -coset. Moreover, the subsets $W(Y)$, $Y \in \mathcal{B}(G/H)_{max}$, are exactly the subsets of all elements of minimal length in a given left $W_{(G/H)}$ -coset.

For any G -orbit closure X' in a regular completion X , we have

$$w(X) \cap X' = \bigcup_{w' \in W^{(G/H)} \cap wW_{(G/H)}} w'(X').$$

Proof. Clearly, $W^{(G/H)}$ is contained in $W^{\Delta(G/H)}$. And since $W_{G/H}$ stabilizes $\Delta(G/H)$, the set $W^{\Delta(G/H)}$ is stable under right multiplication by $W_{G/H}$. This implies the first assertion.

Let $Y = w(G/H)$ and observe that $\text{codim}_{G/H}(Y) \leq \ell(w)$ with equality if and only if $w^{-1} \in W(Y)$ (indeed, a reduced decomposition of w defines a non-oriented path in $\Gamma(G/H)$ with endpoints Y and G/H).

Let $v \in W(Y)$. Since $v(Y) = G/H$, we have $v^{-1} \in wW_{(G/H)}$. Moreover, $\ell(v^{-1}) = \ell(v) = \text{codim}_X(Y) \leq \ell(w)$. Since we can change w in its right $W_{(X)}$ -coset, it follows that $v^{-1} \in W^{(G/H)}$.

Conversely, let $u \in W$ such that $u^{-1} \in W^{(G/H)} \cap wW_{(G/H)}$. Then $u(Y) = G/H$, whence $\ell(u) \geq \ell(v)$ and $u \in W_{(G/H)}v$. Since $u^{-1} \in W^{(G/H)}$, this forces $\ell(u) = \ell(v)$ and then $u \in W(Y)$. This proves the first assertion. Together with Theorem 3, this implies the second assertion. \square

Example 5. Let \mathbf{G} be a connected reductive group. Consider the group $G = \mathbf{G} \times \mathbf{G}$ acting on \mathbf{G} by $(x, y) \cdot z = xzy^{-1}$. Then \mathbf{G} is a spherical homogeneous

space: consider the Borel subgroup $B = \mathbf{B} \times \mathbf{B}^-$ of G , where \mathbf{B} and \mathbf{B}^- are opposed Borel subgroups of G . With evident notation, the B -orbits in G are the $\mathbf{B}w\mathbf{B}^-$, $w \in \mathbf{W}$. This identifies $\mathcal{B}(G)$ to \mathbf{W} . Moreover, all B -orbits have maximal rank, and the Weyl group $W = \mathbf{W} \times \mathbf{W}$ acts on \mathbf{W} by $(u, v)w = u w v^{-1}$. Thus, $\Delta(G)$ is empty, $W_{(G)}$ is the diagonal in $\mathbf{W} \times \mathbf{W}$, and $\mathbf{W} \times \{1\}$ is a system of representatives of $W/W_{(G)}$. One checks that

$$W^{(G)} = \{(u, v) \in \mathbf{W} \times \mathbf{W} \mid \ell(u) + \ell(v) = \ell(uv^{-1})\}.$$

In particular, $(w, 1) \in W^{(G)}$ for all $w \in \mathbf{W}$. Moreover,

$$W^{(G)} \cap (w, 1)W_{(G)} = \{(u, v) \in \mathbf{W} \times \mathbf{W} \mid uv^{-1} = w \text{ and } \ell(u) + \ell(v) = \ell(w)\}.$$

This identifies $W^{(G)} \cap (w, 1)W_{(G)}$ to the set of all $u \in \mathbf{W}$ such that $u \preceq w$ for the right weak order on \mathbf{W} . Now the last statement of Proposition 9 gives back [6] Theorem 2.1.

Returning to arbitrary G/H , we shall deduce from Proposition 4 the following

Theorem 4. *The group $W_{(G/H)}$ is generated by reflections s_α where α is a root such that $\alpha \in \Phi_{\Delta(G/H)}$ or that $2\alpha \in X(G/H)$, and by products $s_\alpha s_\beta$ where α, β are orthogonal roots such that $\alpha + \beta \in X(G/H)$.*

Proof. Let $w \in W_{(G/H)}$. We choose a reduced decomposition $w = s_{\alpha_\ell} \cdots s_{\alpha_2} s_{\alpha_1}$ and we argue by induction on the length ℓ .

If $\alpha_1 \in \Delta(G/H)$ then s_{α_1} is a reflection in $W_{(G/H)}$, so that $s_{\alpha_\ell} \cdots s_{\alpha_2} \in W_{(G/H)}$. Now we conclude by the induction assumption.

If $\alpha_1 \notin \Delta(G/H)$ then $s_{\alpha_1}(G/H)$ has codimension 1 in G/H . Let i be the largest integer such that $\text{codim}_{G/H} s_{\alpha_i} \cdots s_{\alpha_1}(G/H) = i$. Let $Y = s_{\alpha_i} \cdots s_{\alpha_1}(G/H) = i$, then $Y \in \mathcal{B}(G/H)_{max}$ and $s_{\alpha_1} \cdots s_{\alpha_i} \in W(Y)$.

If $P_{\alpha_{i+1}}Y = Y$ then $s_{\alpha_{i+1}}(Y) = Y$ by definition of the W -action and maximality of i . Let $\alpha = s_{\alpha_1} \cdots s_{\alpha_i}(\alpha_{i+1})$. Then s_α is a reflection of $W_{(G/H)}$, and $w = s_{\alpha_\ell} \cdots s_{\alpha_{i+2}} s_{\alpha_i} \cdots s_{\alpha_1} s_\alpha$. If $\alpha_{i+1} \in \Delta(Y)$, then $\alpha \in \Delta(G/H)$ by Proposition 8. Otherwise, $P_{\alpha_{i+1}}Y^0/R(P_{\alpha_{i+1}})$ is isomorphic to $\text{PGL}(2)/T$ or to $\text{PGL}(2)/N$; it follows that $2\alpha_{i+1} \in X(Y)$, and that $2\alpha \in X(G/H)$. Now we conclude by the induction assumption.

If $P_{\alpha_{i+1}}Y \neq Y$ then α_{i+1} raises Y to (say) Y' . Choose $u \in W(Y')$, then $\ell(u) = i - 1$ and $us_{\alpha_{i+1}} \in W(Y)$. Moreover, $us_{\alpha_{i+1}}s_{\alpha_i} \cdots s_{\alpha_1} \in W_{(G/H)}$. We have $w = vus_{\alpha_{i+1}}s_{\alpha_i} \cdots s_{\alpha_1}$ for some $v \in W_{(G/H)}$ such that $\ell(vu) = \ell(u) - 1$. Thus, $\ell(v) \leq \ell(vu) + \ell(u) = \ell(u) - 1$. Therefore, we may assume that there exist $Y \in \mathcal{B}(G/H)_{max}$ and $w_1, w_2 \in W(Y)$ such that $w = w_2 w_1^{-1}$. By Proposition 2, we may assume moreover that w_1 and w_2 are neighbors. Then we conclude by Proposition 4. \square

As a direct consequence, we recover the following result of Knop, see [17] and [18].

Corollary 6. *The image of $W_{G/H}$ in $\text{Aut } \mathcal{X}(G/H)$ is generated by reflections.*

4. Singularities of orbit closures

We begin by recalling the notion of rational singularities, see e.g. [14] p. 50.

Let Y be a variety. Choose a resolution of singularities $\varphi : Z \rightarrow Y$, that is, Z is smooth and φ is proper and birational. Then the sheaves $R^n \varphi_* \mathcal{O}_Z$ ($n \geq 0$) are independent of the choice of Z . The singularities of Y are rational if $R^n \varphi_* \mathcal{O}_Z = 0$ for all $n \geq 1$ and $\varphi_* \mathcal{O}_Z = \mathcal{O}_Y$; the latter condition is equivalent to normality of Y . Varieties with rational singularities are Cohen-Macaulay.

Let now X be a spherical variety and Y a B -stable subvariety. If Y is G -stable, then its singularities are rational, see e.g. [5]. But this does not extend to arbitrary Y : generalizing Example 1 in Section 1, we shall construct examples of B -orbit closures in a spherical homogeneous space, of arbitrary dimension but of depth 1 at some points. In particular, such orbit closures are neither normal nor Cohen-Macaulay.

Example 6. Let r, n be integers such that $0 \leq r \leq n-1$. Let X be the space of smooth quadrics of dimension r in projective space \mathbb{P}^n : points of X are pairs (Q, L) where L is a linear subspace of \mathbb{P}^n of dimension $r+1$, and Q is a smooth hypersurface of degree 2 in L . The group $G = \text{GL}(n+1)$ acts transitively on X ; an isotropy group H consists of all matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A is a scalar multiple of an orthogonal $(r+1) \times (r+1)$ matrix, B is a $(r+1) \times n$ matrix, and C is an invertible $(n-r) \times (n-r)$ matrix. Thus, X is obtained from the symmetric space $\text{GL}(r+1)/\text{O}(r+1)\mathbb{C}^*$ by parabolic induction. It follows that X is spherical of rank $r+1$.

Let m be an integer such that $1 \leq m \leq n-r-1$ and let \mathbb{P}^m be a linear subspace of \mathbb{P}^n of dimension m . Consider the set

$$Y_m = \{(Q, L) \in X \mid Q \text{ meets } \mathbb{P}^m\},$$

a subvariety of X of codimension $n-m-r$. The stabilizer P_m of \mathbb{P}^m in G , a maximal parabolic subgroup, stabilizes Y_m as well; in fact, Y_m contains an open P_m -orbit, the subset of all (Q, L) such that Q meets \mathbb{P}^m in a unique point. In particular, Y_m is an orbit closure of a Borel subgroup of G ; one checks that $r(Y_m) = r$. The corresponding H -orbit closure V_m in the flag variety of G is the set of all complete flags whose m -dimensional subspace meets the H -fixed quadric. Thus, V_m is irreducible for $r \geq 1$, whereas it has two irreducible components for $r = 0$ (then the quadric consists of two points).

We claim that the non-normal locus of Y_m equals the complement Y'_m of its open P_m -orbit. For this, note that the set of all (Q, L) in X such that Q meets

\mathbb{P}^m in two distinct points is a unique P_m -orbit, dense in Y'_m . Let

$$\tilde{Y}_m = \{(x, Q, L) \in \mathbb{P}^n \times X \mid x \in Q \cap \mathbb{P}^m\}$$

Then \tilde{Y}_m , is a P_m -variety, and the map $\nu : \tilde{Y}_m \rightarrow Y_m, (x, Q, L) \mapsto (Q, L)$ is an isomorphism over the open P_m -orbit in Y_m but has degree 2 over Y'_m . This implies our claim. One checks that Y'_m has codimension $n - m - r$ in Y_m and that $r(Y'_m) = r + 1$.

If $m = n - r - 1$ then Y_m is a divisor in X and hence is Cohen-Macaulay (for $n = 2, r = 0$ and $m = 1$, we recover Example 1 in Section 1.) But if $m \leq n - r - 2$ then Y_m has depth 1 along Y'_m by Serre's criterion, see [11] 18.3. In particular, Y_m is not Cohen-Macaulay.

Let $\alpha_1, \dots, \alpha_n$ be the simple roots of G . Then $P_{\alpha_m} Y_m = Y_{m+1}$, and α_m is the unique simple root raising Y_m . The corresponding edge in $\Gamma(X)$ is simple, except for $m = n - r - 1$. Thus, Y_m is the source of a unique oriented path with target X , and the top edge of this path is double. In particular, Y_m is not multiplicity-free.

Such examples of bad singularities do not occur for multiplicity-free orbit closures:

Theorem 5. *Let X be a spherical variety and let Y be a multiplicity-free B -orbit closure. Then the singularities of Y are rational. If in addition X is regular, then for any G -orbit closure X' in X , the scheme-theoretical intersection $Y \cap X'$ is reduced. Moreover, the local equations of the boundary divisors containing X' at any point of $Y \cap X'$ are a regular sequence in the local ring of Y at that point.*

Proof. The main point is rationality of singularities, proved by induction on the codimension of Y . Here is an outline of the argument: choose a resolution of singularities $\varphi : Z \rightarrow Y$ which is equivariant for the normalizer $N_G(Y)$, a standard parabolic subgroup of G . This yields $N_G(Y)$ -linearized coherent sheaves on Y :

$$\mathcal{C}_0 = (\varphi_* \mathcal{O}_Z) / \mathcal{O}_Y \quad \text{and} \quad \mathcal{C}_n = R^n \varphi_* \mathcal{O}_Z \quad \text{for} \quad n \geq 1$$

For every standard parabolic subgroup P , we can induce these sheaves to P -linearized coherent sheaves $P \times^B \mathcal{C}_n$ on $P \times^B Y$. Choose for P a minimal parabolic subgroup raising Y and consider the natural map $\pi : P \times^B Y \rightarrow PY$, then π is birational, and the singularities of PY are rational by the induction assumption. Using this, we show that $\pi_*(P \times^B \mathcal{C}_n) = 0$ for all n , and we deduce that the sheaves \mathcal{C}_n are trivial.

We begin the detailed argument by fixing notation. Let P be a minimal parabolic subgroup that raises Y . Let

$$f : P \times^B Y \rightarrow P/B \cong \mathbb{P}^1$$

be the locally trivial fiber bundle with fiber the B -variety Y , and let

$$\pi : P \times^B Y \rightarrow PY$$

be the natural morphism, then π is birational and P -equivariant. The B -equivariant resolution $\varphi : Z \rightarrow Y$ induces a P -equivariant resolution

$$\rho : P \times^B Z \rightarrow P \times^B Y.$$

Composing ρ with π , we obtain a P -equivariant resolution

$$\tilde{\pi} : P \times^B Z \rightarrow PY.$$

We have an exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \varphi_* \mathcal{O}_Z \rightarrow \mathcal{C}_0 \rightarrow 0$$

of $N_G(Y)$ -linearized coherent sheaves on Y . Moreover, every B -linearized (coherent) sheaf \mathcal{F} on Y corresponds to a P -linearized sheaf $P \times^B \mathcal{F}$ on $P \times^B Y$, uniquely defined by imposing that the restriction of $P \times^B \mathcal{F}$ to $B \times^B Y \cong Y$ is \mathcal{F} ; and $P \times^B \mathcal{O}_Y = \mathcal{O}_{P \times^B Y}$, whereas $P \times^B \varphi_* \mathcal{O}_Z = \rho_* \mathcal{O}_{P \times^B Z}$. Thus, we have an exact sequence of P -linearized sheaves on $P \times^B Y$:

$$0 \rightarrow \mathcal{O}_{P \times^B Y} \rightarrow \rho_* \mathcal{O}_{P \times^B Z} \rightarrow P \times^B \mathcal{C}_0 \rightarrow 0.$$

Applying π_* , we obtain an exact sequence of P -linearized sheaves on PY :

$$0 \rightarrow \pi_* \mathcal{O}_{P \times^B Y} \rightarrow \tilde{\pi}_* \mathcal{O}_{P \times^B Z} \rightarrow \pi_*(P \times^B \mathcal{C}_0) \rightarrow R^1 \pi_* \mathcal{O}_{P \times^B Y}.$$

Moreover, $\pi_* \mathcal{O}_{P \times^B Y} = \tilde{\pi}_* \mathcal{O}_{P \times^B Z} = \mathcal{O}_{PY}$, since PY is normal. Thus, $\pi_*(P \times^B \mathcal{C}_0)$ injects into $R^1 \pi_* \mathcal{O}_{P \times^B Y}$.

We claim that $R^1 \pi_* \mathcal{O}_{P \times^B Y} = 0$. For this, we factor π as the closed immersion

$$i = (f, \pi) : P \times^B Y \rightarrow P/B \times PY$$

followed by the projection

$$p : P/B \times PY \rightarrow P/B$$

with fibers \mathbb{P}^1 . Then $R^1 \pi_* \mathcal{O}_{P \times^B Y} = R^1 p_*(i_* \mathcal{O}_{P \times^B Y})$. Moreover, we have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{P/B \times PY} \rightarrow i_* \mathcal{O}_{P \times^B Y} \rightarrow 0$$

where \mathcal{I} is the ideal sheaf of $P \times^B Y$. This yields an exact sequence

$$R^1 p_* \mathcal{O}_{P/B \times PY} \rightarrow R^1 p_*(i_* \mathcal{O}_{P \times^B Y}) \rightarrow R^2 p_* \mathcal{I}.$$

But $R^2 p_* \mathcal{I} = 0$, as p is proper with fibers of dimension 1; and $R^1 p_* \mathcal{O}_{P/B \times PY} = 0$, as $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. This proves our claim; it results that $\pi_*(P \times^B \mathcal{C}_0) = 0$.

Next we claim that $\pi_*(P \times^B \mathcal{C}_n) = 0$ for every $n \geq 1$, where $\mathcal{C}_n = R^n \varphi_* \mathcal{O}_Z$. For this, we use the Leray spectral sequence associated with $\tilde{\pi} = \pi \circ \rho$. Since $\tilde{\pi}$ is a resolution of PY , the induction assumption yields that $R^q \tilde{\pi}_* \mathcal{O}_{P \times^B Z} = 0$ for all $q \geq 1$. On the other hand, $R^p \pi_* \mathcal{F} = 0$ for any $p \geq 2$ and for any coherent sheaf \mathcal{F} on $P \times^B Y$, as π is proper with fibers of dimension ≤ 1 . It follows that the spectral sequence

$$R^p \pi_*(R^q \rho_* \mathcal{O}_{P \times^B Z}) \Rightarrow R^{p+q} \tilde{\pi}_* \mathcal{O}_{P \times^B Z}$$

degenerates at E_2 : then $\pi_*(R^n \rho_* \mathcal{O}_{P \times^B Z})$ is a quotient of $R^n \tilde{\pi}_* \mathcal{O}_{P \times^B Z}$, and hence is trivial. Moreover, $R^n \rho_* \mathcal{O}_{P \times^B Z} = P \times^B (R^n \varphi_* \mathcal{O}_Z) = P \times^B \mathcal{C}_n$, since $\rho : P \times^B Z \rightarrow P \times^B Y$ is induced from $\varphi : Z \rightarrow Y$. This proves the claim.

We now need the following

Lemma 8. *Let \mathcal{F} be a coherent $N_G(Y)$ -linearized sheaf on Y such that $\pi_*(P \times^B \mathcal{F})$ is trivial for any minimal parabolic subgroup P that raises Y . Then the support of \mathcal{F} is G -stable.*

Proof. Let Y' be an irreducible component of the support of \mathcal{F} , with ideal sheaf $\mathcal{I}_{Y'}$. Define a subsheaf \mathcal{F}' of \mathcal{F} by

$$\mathcal{F}'(V) = \{s \in \mathcal{F}(V) \mid \mathcal{I}_{Y'}(V)s = 0\}$$

for every open subset V of Y . Observe that \mathcal{F}' is a coherent sheaf of $\mathcal{O}_{Y'}$ -modules, with support precisely Y' (indeed, if V is affine, then the ideal $\mathcal{I}_{Y'}(V)$ is a minimal prime of the support of $\mathcal{F}(V)$; thus, this ideal is an associated prime of $\mathcal{F}(V)$.) Note that $N_G(Y)$ stabilizes Y' and acts on \mathcal{F}' . Moreover, $\pi_*(P \times^B \mathcal{F}')$ is trivial, since $\pi_*(P \times^B \mathcal{F})$ is.

Assume that Y' is not G -stable and let P be a minimal parabolic subgroup raising Y' ; then P raises Y , for Y' is stable under $N_G(Y)$. Let $\pi' : P \times^B Y' \rightarrow PY'$ be the restriction of π to $P \times^B Y'$. Then π' is generically finite (as P raises Y'), and the support of $P \times^B \mathcal{F}'$ is $P \times^B Y'$ (as the support of \mathcal{F}' is Y'). Thus, the support of $\pi'_*(P \times^B \mathcal{F}')$ is PY' . But this contradicts the fact that $\pi'_*(P \times^B \mathcal{F}') = \pi_*(P \times^B \mathcal{F}')$ is trivial. \square

Thus, the support of each \mathcal{C}_n is G -stable. On the other hand, we may assume that Y contains no G -orbit (to see this, replace G (resp. X) by $\tilde{G} = G \times \text{PGL}(2)$ (resp. $\tilde{X} = X \times \mathbb{P}^1$ with the natural \tilde{G} -action). Then \tilde{X} is a spherical \tilde{G} -variety, and $\tilde{Y} = Y \times \{\infty\} \simeq Y$ is multiplicity-free and contains no \tilde{G} -orbit). It then follows that each \mathcal{C}_n is trivial. We have proved that the singularities of Y are rational.

For the remaining assertions, let D_1, \dots, D_r be the boundary divisors containing X' , then the local equations of D_1, \dots, D_r at any point $x \in X'$ are a regular sequence in $\mathcal{O}_{X,x}$. Moreover, the scheme-theoretical intersection $Y \cap X'$ is equidimensional of codimension r and generically reduced. Since Y is Cohen-Macaulay, then $Y \cap X'$ is reduced and the local equations of D_1, \dots, D_r at any point $y \in Y \cap X'$ are a regular sequence in $\mathcal{O}_{Y,y}$. \square

We now apply these results to H -orbit closures in the flag variety of G . For this, we recall a construction from [6] 1.5. Let V be a H -orbit closure in G/B and let \hat{V} be the corresponding B -orbit closure in G/H . Let Y be the closure of \hat{V} in a regular completion X , then $Y \in \mathcal{B}(G/H, X)$. Consider the natural

morphism

$$\pi : G \times^B Y \rightarrow X$$

and the projection

$$f : G \times^B Y \rightarrow G/B.$$

The fibers of π identify to closed subschemes of G/B via f_* . Let x be the image in X of the base point of G/H , then $\pi^{-1}(x)$ identifies to V . On the other hand, let Z be a closed G -orbit in X with B -fixed point z , then the set $f(\pi^{-1}(z))$ equals

$$V_0 = \bigcup_{w \in W(Y)} \overline{Bw_0wB}/B$$

where w_0 denotes the longest element of W . Moreover, we have in the integral cohomology ring of G/B :

$$[V] = \sum_{w \in W(Y)} d(Y, w) [\overline{Bw_0wB}/B].$$

Now Theorem 2 and Proposition 5 imply the following

Corollary 7. *With notation as above, V_0 is connected in codimension 1. If moreover G is simply-laced, then $[V] = 2^{\ell_N(\gamma)} [V_0]$ where γ is any oriented path in $\Gamma(X)$ joining Y to X .*

We shall call V multiplicity-free if Y is. Equivalently, the cohomology class of V decomposes as a sum of Schubert classes with coefficients 0 or 1.

Note that any multiplicity-free H -orbit closure V is irreducible, even if H is not connected. Indeed, H acts transitively on the set of all irreducible components of V , so that any two such components have the same cohomology class; but the class of V is indivisible in the integral cohomology of G/B .

Theorem 6. *With notation as above, if V is multiplicity-free then its singularities are rational, the morphism π is flat, and its fibers are reduced. As a consequence, the fibers of π realize a degeneration of V to the reduced subscheme V_0 of G/B .*

Proof. Note that the singularities of Y are rational by Theorem 5; thus, the same holds for $\hat{V} = Y \cap G/H$. Let $\varphi : Z \rightarrow \hat{V}$ be a resolution of singularities; consider the quotient map $q_H : G \rightarrow G/H$, the preimage $V' = q_H^{-1}(\hat{V})$ in G , and the fiber product $Z' = Z \times_{\hat{V}} V'$. Then Z' is smooth, since Z and q_H are; the projection $\varphi' : Z' \rightarrow V'$ is proper and birational, since φ is; and $R^n \varphi'_* \mathcal{O}_{Z'} = 0$ for $n \geq 1$, since cohomology commutes with flat base extension. Therefore the singularities of V' are rational. Now $V' = q_B^{-1}(V)$ and q_B is a locally trivial fibration, so that the singularities of V are rational as well.

For the second assertion, we identify Y to its image $B \times^B Y$ in $G \times^B Y$. Since π is G -equivariant, it is enough to check the statement at $y \in Y$. Let D_1, \dots, D_r be the boundary divisors containing y , with local equations f_1, \dots, f_r in $\mathcal{O}_{X,y}$. It follows from Theorem 5 that the pull-backs $\pi^* f_1, \dots, \pi^* f_r$ are a regular sequence in $\mathcal{O}_{G \times^B Y, y}$ and generate the ideal of $\pi^{-1}(Gy)$. Moreover, the restriction of π to $\pi^{-1}(Gy)$ is flat with reduced fibers, as π is G -equivariant. Now we conclude by a local flatness criterion, see [11] Corollary 6.9. \square

A direct consequence is the following

Corollary 8. *Consider a a multiplicity-free H -orbit closure V in G/B and an effective line bundle L on G/B . Then the restriction map $H^0(G/B, L) \rightarrow H^0(V, L)$ is surjective, and $H^n(V, L) = 0$ for all $n \geq 1$.*

Indeed, this holds with V replaced by V_0 , a union of Schubert varieties (see [19].) The result follows by semicontinuity of cohomology in a flat family.

We now obtain a partial converse to Corollary 8:

Proposition 10. *Let V be a H -orbit closure in G/B with corresponding B -orbit closure Y in G/H . If Y is the source of a double edge of $\Gamma(G/H)$, then there exists an effective line bundle L on G/B such that the restriction $H^0(G/B, L) \rightarrow H^0(V, L)$ is not surjective.*

Proof. Let α be the label of a double edge with source Y . Denote by $p : G/B \rightarrow G/P_\alpha$ the natural map and by $p_V : V \rightarrow \pi(V)$ its restriction to V , then p is a projective line bundle and p_V is generically finite of degree 2. Choose an ample line bundle L on G/P_α , then p^*L is an effective line bundle on G/B . Now it suffices to check the following claim: the restriction map

$$r_n : H^0(p^{-1}p(V), p^*(L^{\otimes n})) \rightarrow H^0(V, p^*(L^{\otimes n}))$$

is not surjective for large n .

For this, note that

$$\begin{aligned} H^0(p^{-1}p(V), p^*(L^{\otimes n})) &= H^0(p(V), L^{\otimes n}), \\ H^0(V, p^*(L^{\otimes n})) &= H^0(p(V), L^{\otimes n} \otimes p_{V*} \mathcal{O}_V) \end{aligned}$$

by the projection formula. Thus, r_n identifies with the map

$$H^0(p(V), L^{\otimes n}) \rightarrow H^0(p(V), L^{\otimes n} \otimes p_{V*} \mathcal{O}_V)$$

defined by the inclusion of $\mathcal{O}_{p(V)}$ into $p_{V*} \mathcal{O}_V$. Since p_V has degree 2, the quotient $\mathcal{F} = p_{V*} \mathcal{O}_V / \mathcal{O}_{p(V)}$ has rank 1 as a sheaf of $\mathcal{O}_{p(V)}$ -modules. Moreover, since L is ample, the cokernel of r_n is isomorphic to $H^0(p(V), \mathcal{F} \otimes L^{\otimes n})$ for large n . This implies the claim. \square

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Michel Brion
Université de Grenoble I
Département de Mathématiques
Institute Fourier
UMR 5582 du CNRS
F-38402 Saint-Martin d'Hères Cedex
France
e-mail: Michel.Brion@ujf-grenoble.fr

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