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Quasihyperbolic boundary conditions and capacity: Hölder continuity of quasiconformal mappings

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Dedicated to Olli Martio on the occasion of his honorary doctorate at the University of Jyväskylä

Abstract. We prove that quasiconformal maps onto domains which satisfy a quasihyperbolic boundary condition are globally Hölder continuous in the internal metric. The primary improvement here over existing results along these lines is that no assumptions are made on the source domain. We reduce the problem to the verification of a capacity estimate in domains satisfing a quasihyperbolic boundary condition, which we establish using a combination of a chaining argument involving the Poincaré inequality on Whitney cubes together with Frostman's theorem.

We also discuss related results where the quasihyperbolic boundary condition is slightly weakened; in this case the Hölder continuity of quasiconformal maps is replaced by uniform continuity with a modulus of continuity which we calculate explicitly.

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1. Introduction

It is well-known that quasiconformal maps are locally well-behaved with respect to distance distortion. If $f:\Omega\to\Omega'$ is a K-quasiconformal mapping between domains $\Omega,\Omega'\subset\mathbb{R}^n$, $n\geq 2$, then f is locally Hölder continuous with exponent $\alpha=\mathrm{K}^{1/(1-n)}$, i.e.

$$|f(x)-f(y)|\leq \mathrm{M}|x-y|^\alpha \tag{1.1}$$

whenever x and y lie in a fixed compact set E in Ω. Here M is a constant depending only on K and E which can in general tend to infinity as the distance

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from E to the boundary of Ω tends to zero. To conclude global Hölder continuity for the map f, that is, to conclude that (1.1) holds for all $x, y \in \Omega$, it is necessary to make some geometric assumptions on the domains Ω and Ω'. An early result along these lines was obtained by Becker and Pommerenke [3], who considered the case of simply connected domains in the plane. If $f : \mathbb{D} \to \Omega' \subset \mathbb{C}$ is a conformal mapping, then f is globally β-Hölder continuous, $0 < \beta \leq 1$, if and only if the hyperbolic metric $\rho_{\Omega'}$ in Ω' satisfies a logarithmic growth condition

$$\rho_{\Omega'}(z_0, z) \leq \frac{1}{\beta} \log \frac{\operatorname{dist}(z_0, \partial\Omega')}{\operatorname{dist}(z, \partial\Omega')} + \mathrm{C}_0, \tag{1.2}$$

where $z_0 = f(0)$ and $\mathrm{C}_0 < \infty$. Here $\operatorname{dist}(\cdot, \partial\Omega')$ denotes the Euclidean distance to the boundary of Ω'.

To extend this result to multiply connected domains and to higher dimensions, Gehring and Martio [5] replaced the hyperbolic metric $\rho_{\Omega'}$ with the quasihyperbolic metric $k_{\Omega'}$ (see section 2 for the definition). By [5, Theorem 3.17], if $f : \Omega \to \Omega'$ is a K-quasiconformal mapping between domains $\Omega, \Omega' \subsetneq \mathbb{R}^n$, $n \geq 2$, and if there exists $0 < \beta \leq 1$ so that the quasihyperbolic metric $k_{\Omega'}$ satisfies a logarithmic growth condition

$$k_{\Omega'}(x_0, x) \leq \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega')}{\operatorname{dist}(x, \partial\Omega')} + \mathrm{C}_0 \tag{1.3}$$

for some (each) $x_0 \in \Omega'$ and a constant $\mathrm{C}_0 = \mathrm{C}_0(x_0) < \infty$, then f is Hölder continuous on each (open) ball $\mathrm{B} \subset \Omega$ with an exponent α and constant M which depend only on n, K, and the constants β and C_0 but are independent of B. If in addition Ω is sufficiently nice [5, p. 204], then f is globally Hölder continuous with exponent α. Here "niceness" of the source domain Ω means that any two points in Ω can be joined by a curve whose length is no more than a fixed constant multiple of the distance between the points, and that stays sufficiently far away from the boundary when measured in a certain averaged sense. To compare this with the result of Becker and Pommerenke in the plane, recall that the hyperbolic and the quasihyperbolic metrics are comparable in simply connected plane domains by the Koebe distortion theorem.

We now state our principal result. In what follows, we denote by $\delta_\Omega(x, y)$ the internal distance between a pair of points $x, y \in \Omega$, i.e., the infimum of the lengths of curves in Ω joining x to y.

Theorem 1.1. *Let $\Omega, \Omega' \subsetneq \mathbb{R}^n$, $n \geq 2$, be domains and assume that Ω' satisfies a quasihyperbolic boundary condition of the form (1.3) for some $\beta \in (0, 1]$. Then any quasiconformal mapping $f : \Omega \to \Omega'$ satisfies the global Hölder condition*

$$|f(x) - f(y)| \leq \mathrm{M}\delta_\Omega(x, y)^\alpha$$

for all $x, y \in \Omega$, where $0 < \alpha \leq 1$ and $\mathrm{M} < \infty$ which depend only on the data.

If Ω is a quasiconvex domain (that is, Ω satisfies the first part of the "niceness" assumption in the previous paragraph: any two points in Ω can be joined by a curve whose length is no more than a fixed constant multiple of the (Euclidean) distance between the two points), then the internal metric δ_Ω and the Euclidean metric in Ω are bi-Lipschitz equivalent. We thus have the following corollary to Theorem 1.1.

Corollary 1.2. *Let Ω, Ω', and f be as in Theorem 1.1 and assume in addition that Ω is quasiconvex. Then f satisfies the global Hölder condition*

$$|f(x)-f(y)| \leq \mathrm{M}|x-y|^{\alpha}$$

for all $x,y \in \Omega$, where $0<\alpha\leq 1$ and $\mathrm{M}<\infty$ which depend only on the data.

We emphasize a fundamental distinction between Theorem 1.1 and the result of Gehring and Martio: in Theorem 1.1 we make no assumptions whatsoever on the initial domain Ω. In Corollary 1.2, quasiconvexity is used only to convert between the internal and the Euclidean metrics in Ω. Our results are new even in the case of conformal maps between planar domains (at least in the infinitely connected case): [1]

Corollary 1.3. *Let $\Omega \subsetneq \mathbb{C}$ be a quasiconvex domain and let $\Omega' \subsetneq \mathbb{C}$ be a conformally equivalent domain which satisfies (1.2). Then any conformal map $f:\Omega\to\Omega'$ is globally α-Hölder continuous for some $0<\alpha\leq 1$ which depends only on the data.*

Our results address the question of global length distortion. Astala and Koskela [2] study the question of global volume distortion, where again the relevant hypothesis is the logarithmic growth condition on the quasihyperbolic metric in the target domain. By Theorem 1.2 of [2], if $f:\Omega\to\Omega'$ is a K-quasiconformal map onto a domain Ω' satisfying (1.3), then $|f'|\in \mathrm{L}^p(\Omega)$ for some $p>n$ depending only on n, K, and the constants in (1.3).

Our proof of Theorem 1.1 relies on certain capacity estimates in domains satisfying the quasihyperbolic boundary condition. Specifically, we establish the following result.

Theorem 1.4. *Let Ω be a proper subdomain of $\mathbb{R}^n$, $n\geq 2$, with diameter one which satisfies (1.3). Let Q_0 denote a fixed Whitney cube containing the basepoint x_0. Then there exists a constant $\mathrm{M}<\infty$ depending only on n, β, and C_0 so that*

$$\operatorname{cap}(\mathrm{E},Q_0;\Omega) \geq \frac{1}{\mathrm{M}}\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{1-n} \tag{1.4}$$

for all continua $\mathrm{E}\subset\Omega$.

Here $\operatorname{cap}(\mathrm{E},\mathrm{F};\Omega)$ denotes the n-capacity between a pair of disjoint continua E and F in the domain Ω, see section 2.

[1] In connection with the global regularity of planar quasiconformal maps, the reader may also be interested in the following recent result of Bishop [4, Theorem 5.1]: there exists an absolute constant $\mathrm{K}_0<\infty$ so that to any simply connected planar domain Ω there corresponds a K_0-quasiconformal mapping $f:\Omega\to\mathbb{D}$ which is Lipschitz in the internal metric on Ω.

We prove Theorem 1.4 by a chaining argument involving the Poincaré inequality on Whitney cubes in Ω. This ingredient in the proof was already used by Herron and Koskela in [9] to prove a special case of Theorem 1.4. To prove the general case, we introduce a new technique in this context: the use of a Frostman measure on the continuum E. In a companion paper [13], we use this technique to verify global Poincaré inequalities in domains satisfying (1.3).

Theorems 1.1 and 1.4 answer in the affirmative Questions 8.4 and 8.3, respectively, in [9] (see also Conjecture 5.2 in [12]).

We briefly outline the structure of the paper. In section 2 we present a number of technical lemmas relating to the geometry of Whitney cubes and quasihyperbolic geodesics which will be of importance in the proof of Theorem 1.4. Section 3 contains the proofs of Theorems 1.1 and 1.4. In section 4 we study domains which satisfy weaker versions of the quasihyperbolic boundary condition (1.3). In this case we can no longer show global Hölder continuity for quasiconformal mappings onto such domains, but we are able to establish global uniform continuity with a modulus of continuity which we calculate explicitly.

1.1. Notations and definitions

We denote by $\mathbb{R}^n$, $n \geq 1$, the Euclidean space of dimension n. For a cube $Q \subset \mathbb{R}^n$ with center x and side length $s(Q)$ and for a factor $\lambda > 0$, we denote by λQ the dilated cube which is again centered at x but has side length $\lambda s(Q)$. We denote the Lebesgue measure in $\mathbb{R}^n$ by m, although we usually abbreviate $dm(x) = dx$. For a domain $\Omega \subset \mathbb{R}^n$, we denote by δ_Ω the internal metric in Ω, i.e., $\delta_\Omega(x,y) = \inf\{\operatorname{diam} \mathrm{E} : \mathrm{E} \text{ a connected set in } \Omega \text{ joining } x \text{ to } y\}$. We say that Ω is *quasiconvex* if the internal metric δ_Ω is bi-Lipschitz equivalent to the Euclidean metric, equivalently, if there exists a constant $\mathrm{L} < \infty$ so that any two points $x, y \in \Omega$ are contained in a connected set E in Ω with $\operatorname{diam} \mathrm{E} \leq \mathrm{L}|x-y|$.

For an increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$, we denote by $\mathcal{H}^\infty_\varphi$ the *Hausdorff* φ*-content*: $\mathcal{H}^\infty_\varphi(\mathrm{E}) = \inf \sum_i \varphi(r_i)$, where the infimum is taken over all coverings of $\mathrm{E} \subset \mathbb{R}^n$ with balls $\mathrm{B}(x_i, r_i)$, $i = 1, 2, \ldots$. When $\varphi(t) = t^s$ for some $0 < s < \infty$ we write $\mathcal{H}^\infty_s = \mathcal{H}^\infty_\varphi$.

For disjoint compact sets E and F in the domain Ω, we denote by $\operatorname{cap}(\mathrm{E},\mathrm{F};\Omega)$ the *conformal* (or n-) *capacity* of the pair (E,F);

$$\operatorname{cap}(\mathrm{E},\mathrm{F};\Omega) = \inf_u \int_\Omega |\nabla u|^n \, dx,$$

where the infimum is taken over all continuous functions u in the Sobolev space $\mathrm{W}^{1,n}(\Omega)$ which satisfy $u(x) \leq 0$ for $x \in \mathrm{E}$ and $u(x) \geq 1$ for $x \in \mathrm{F}$.

For $\mathrm{K} \geq 1$ and Ω, Ω' as above, we say that a homeomorphism $f : \Omega \to \Omega'$ is

K *-quasiconformal* if

$$\frac{1}{\mathrm{K}}\,\mathrm{cap}(\mathrm{E},\mathrm{F};\Omega) \le \mathrm{cap}(\mathrm{E}',\mathrm{F}';\Omega') \le \mathrm{K}\,\mathrm{cap}(\mathrm{E},\mathrm{F};\Omega)$$

whenever E and F are disjoint compact sets in Ω, where $\mathrm{E}' = f(\mathrm{E})$ and $\mathrm{F}' = f(\mathrm{F})$. For the basic theory of quasiconformal maps, we refer the reader to the book [17] of Väisälä.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$. Set $s(\Omega) = n^{-1/2}\operatorname{diam}\Omega$. We denote by $\mathcal{W} = \mathcal{W}(\Omega)$ a *Whitney decomposition* of the domain Ω into *Whitney cubes* Q, i.e., the cubes in $\mathcal{W}$ have pairwise disjoint interiors, $\Omega = \cup_{Q\in\mathcal{W}} Q$, and vertices in the set

$$2^{-\mathbb{N}}s(\Omega)\cdot\mathbb{Z}^n := \{(2^{-j}s(\Omega)l_1,\ldots,2^{-j}s(\Omega)l_n) : j\in\mathbb{N}, l_1,\ldots,l_n\in\mathbb{Z}\}$$

and satisfy $\operatorname{diam} Q \le \operatorname{dist}(Q,\partial\Omega) \le 4\operatorname{diam} Q$ for each $Q\in\mathcal{W}$. For the existence of such a decomposition, we refer to Stein's book [16, VI.1]. For any λ, $1 < \lambda < 5/4$, the expanded collection of cubes $\{\lambda Q : Q\in\mathcal{W}\}$ has bounded overlap, specifically,

$$\sup_{x\in\Omega}\sum_{Q\in\mathcal{W}}\chi_{\lambda Q}(x) \le 12^n < \infty.$$

See, e.g., [16, VI.1.3, Proposition 3]. For $j\in\mathbb{N}$, we let $\mathcal{W}_j$ denote the collection of cubes $Q\in\mathcal{W}$ for which $\operatorname{diam} Q = 2^{-j}\operatorname{diam}\Omega$.

2. Preliminary results on the quasihyperbolic metric

Throughout this section, Ω will denote a proper subdomain in the Euclidean space $\mathbb{R}^n$, $n \ge 2$. Recall that the *quasihyperbolic metric* k_Ω in the domain Ω is defined to be

$$k_\Omega(x,y) = \inf_\gamma k_\Omega - \mathrm{length}(\gamma),$$

where the infimum is taken over all rectifiable curves γ in D which join x to y and

$$k_\Omega - \mathrm{length}(\gamma) = \int_\gamma \frac{ds}{\operatorname{dist}(x,\partial\Omega)}$$

denotes the *quasihyperbolic length* of γ in D. This metric was introduced by Gehring and Palka in [7]. A curve γ joining x to y for which $k_\Omega - \mathrm{length}(\gamma) = k_\Omega(x,y)$ is called a *quasihyperbolic geodesic*. Quasihyperbolic geodesics joining any two points of a proper subdomain of $\mathbb{R}^n$ always exist, see [6, Lemma 1]. If γ is a quasihyperbolic geodesic in Ω and $x',y'\in\gamma$, we denote by $\gamma(x',y')$ the portion of γ which joins x' to y'.

When x and y are sufficiently far apart, $k_\Omega(x,y)$ is roughly equal to the number $N(x,y)$ of Whitney cubes Q that intersect a quasihyperbolic geodesic γ

joining x to y. More precisely,

$$N(x,y)/\mathrm{C} \le k_\Omega(x,y) \le \mathrm{C}N(x,y)$$

for all $x,y \in \Omega$ with $|x-y| \ge \operatorname{dist}(x,\partial\Omega)/2$, where $\mathrm{C} = \mathrm{C}(n)$.

Let $\beta \in (0,1]$ and fix a basepoint $x_0 \in \Omega$. Following Gehring and Martio [5], we say that Ω satisfies a β*-quasihyperbolic boundary condition* if for some (each) $x_0 \in \Omega$ there exists a constant $\mathrm{C}_0 = \mathrm{C}_0(x_0) < \infty$ so that

$$k_\Omega(x_0,x) \le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)} + \mathrm{C}_0 \tag{2.1}$$

for all $x \in \Omega$. Then Ω is bounded, in fact $\operatorname{diam}\Omega \le (2/\beta)e^{\mathrm{C}_0\beta} \operatorname{dist}(x_0,\partial\Omega)$ by [5, Lemma 3.9]. The value of β is necessarily less than or equal to one as a consequence of the following simple estimate (c.f. [7]):

$$k_\Omega(x_0,x) \ge \log \frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)} \tag{2.2}$$

for all $x \in \Omega$.

The following result of Smith and Stegenga [14, Theorem 3] is fundamental to our work. A more general version of this result will be proved below in Lemma 4.6.

Lemma 2.1. *Let $\Omega \subsetneq \mathbb{R}^n$ satisfy the quasihyperbolic boundary condition (2.1). Then there exists a finite constant $\mathrm{C}_1 = \mathrm{C}_1(\beta,\mathrm{C}_0)$ so that for all $x_1 \in \Omega$, we have*

$$k_\Omega(x_0,x) \le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{length}(\gamma(x,x_1))} + \mathrm{C}_1$$

whenever γ is a quasihyperbolic geodesic joining x_0 to x_1 and $x \in \gamma$.

For the remainder of this section, we assume that Ω satisfies the quasihyperbolic boundary condition (2.1) for some $\beta \le 1$. Our first lemma controls the number of Whitney cubes of a given size or larger which can intersect a given quasihyperbolic geodesic.

Lemma 2.2. *Let γ be a quasihyperbolic geodesic in Ω starting at the basepoint x_0. Then there exists a constant $\mathrm{C} = \mathrm{C}(n,\beta,\mathrm{C}_0)$ so that*

$$\operatorname{card}\{Q \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j : Q \cap \gamma \ne \emptyset\} \le \mathrm{C}j$$

for all $j \ge 1$. Here $\operatorname{card}\mathrm{S}$ denotes the cardinality of the set S.

Proof. Assume that we have N Whitney cubes $Q_1,\dots,Q_N$ satisfying $s(Q_i) \ge 2^{-j}\operatorname{diam}\Omega$ and $Q_i \cap \gamma \ne \emptyset$, $i = 1,\dots,N$. Fix $\lambda = \frac{9}{8}$ so that the dilated cubes λQ_i have bounded overlap. If we let γ_i denote the part of the curve γ which lies in the cube λQ_i, then the quasihyperbolic lengths of the curves γ_i are uniformly bounded from below:

$$k_\Omega - \operatorname{length}(\gamma_i) \ge \frac{\operatorname{length}(\gamma \cap \lambda Q_i)}{\sup\{\operatorname{dist}(x,\partial\Omega) : x \in \lambda Q_i\}} \ge \frac{1}{\mathrm{C}(n)} > 0$$

for $i = 1, \ldots, N$.

In order to apply Lemma 2.1, let $x_1 \in Q_N \cap \gamma$. If N is chosen sufficiently large relative to n, then one of the cubes λQ_i, $N/2 \le i \le N$, will be disjoint from λQ_N and hence will satisfy $\operatorname{dist}(Q_i, Q_N) \ge c 2^{-j} \operatorname{diam} \Omega$ for some $c > 0$. Let x denote the terminal point of exit of γ from the cube Q_i. By Lemma 2.1,

$$\begin{aligned} \frac{1}{\mathrm{C}(n)} \frac{N}{2} &\le \sum_{i=1}^{N/2} k_\Omega - \operatorname{length}(\gamma_i) \le k_\Omega(x_0, x) \\ &\le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{length}(\gamma(x, x_1))} + \mathrm{C}_1 \\ &\le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(Q_i, Q_N)} + \mathrm{C}_0 \\ &\le \mathrm{C}(n, \beta, \mathrm{C}_0) j. \end{aligned}$$

The lemma follows. □

We now fix a Whitney cube Q_0 and assume that x_0 is the center of Q_0. For each cube $Q \in \mathcal{W}$, we choose a quasihyperbolic geodesic γ joining x_0 to the center of Q and we let $\mathrm{P}(Q)$ denote the collection of all of the Whitney cubes $Q' \in \mathcal{W}$ which intersect γ. Then we define the *shadow* $\mathrm{S}(Q)$ of the cube Q to be

$$\mathrm{S}(Q) = \bigcup_{\substack{Q_1 \in \mathcal{W} \\ Q \in \mathrm{P}(Q_1)}} Q_1.$$

Shadows of Whitney cubes defined in this manner have been used, for example, to investigate the questions of when Euclidean domains satisfy global Poincaré inequalities [14, §§6-7] and when the boundaries of domains are removable for quasiconformal and/or Sobolev functions [11].

Informally speaking, our next lemma says that the amount of overlap of the shadows of Whitney cubes of a fixed size is bounded.

Lemma 2.3. *There exists a finite constant* $\mathrm{C} = \mathrm{C}(n, \beta, \mathrm{C}_0)$ *so that*

$$\sum_{Q \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j} \chi_{\mathrm{S}(Q)}(x) \le \mathrm{C} j$$

for every $j \ge 1$ *and* $x \in \Omega$.

Proof. Since the Whitney collection $\mathcal{W}$ has bounded overlap, we may without loss of generality work with the (disjoint) interiors of the Whitney cubes. If $Q_1, \ldots, Q_N \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j$ are such that $\mathrm{F} := \mathrm{S}(Q_1) \cap \cdots \cap \mathrm{S}(Q_N)$ is nonempty, then F contains an entire Whitney cube; in particular, it contains its center point x. But then the chosen quasihyperbolic geodesic joining x_0 to x intersects each of the cubes Q_i, $i = 1, \ldots, N$. Then the result follows from Lemma 2.2. □

We now estimate the size of the shadow of a Whitney cube Q in terms of the size of Q.

Lemma 2.4. *There exists* $\mathrm{C} = \mathrm{C}(n, \beta, \mathrm{C}_0)$ *so that*

$$\operatorname{diam} \mathrm{S}(Q) \le \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^{\beta}$$

for all $Q \in \mathcal{W}$ *.*

Proof. We first show that $\operatorname{diam} Q_1 \le \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^{\beta}$ for each cube $Q_1 \subset \mathrm{S}(Q)$. If $Q_1 = Q$ this is obvious so assume $Q_1 \neq Q$. Let x_1 denote the center of Q_1 , let γ be a quasihyperbolic geodesic joining x_0 to x_1 , and let x be any point in $Q \cap \gamma$. It is clear that the (Euclidean) length of that portion of γ which lies in Q_1 is at least $c \operatorname{diam} Q_1$ for some constant $c = c(n) > 0$. We apply Lemma 2.1 together with (2.2) to deduce that

$$\log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} \le k_\Omega(x_0, x) \le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{diam} Q_1} + \mathrm{C}_1.$$

The desired result follows since $\operatorname{dist}(x, \partial\Omega) \approx \operatorname{diam} Q$.

It thus suffices to show that the set Z consisting of all of the centers of cubes contained in $\mathrm{S}(Q)$ satisfies $\operatorname{diam} \mathrm{Z} \le \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^{\beta}$. To this end, let $x_1, x_2 \in \mathrm{Z}$. Choose points x_1' and x_2' in $\gamma_{x_1} \cap Q$ and $\gamma_{x_2} \cap Q$, respectively, where γ_x denotes the chosen quasihyperbolic geodesic joining x to x_0 . Then

$$\begin{aligned}
|x_1 - x_2| &\le \operatorname{length}(\gamma_{x_1}(x_1', x_1)) + \operatorname{diam} Q + \operatorname{length}(\gamma_{x_2}(x_2', x_2)) \\
&\le \operatorname{diam} Q + \mathrm{C} \operatorname{dist}(x_0, \partial\Omega) e^{-\beta k_\Omega(x_0, x_1')} + \mathrm{C} \operatorname{dist}(x_0, \partial\Omega) e^{-\beta k_\Omega(x_0, x_2')} \\
&\le \operatorname{diam} Q + \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} \operatorname{dist}(x_1', \partial\Omega)^{\beta} \\
&\quad + \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} \operatorname{dist}(x_2', \partial\Omega)^{\beta} \\
&\le (\operatorname{diam} \Omega)^{1-\beta} (\operatorname{diam} Q)^{\beta} + \mathrm{C} \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^{\beta}
\end{aligned}$$

by Lemma 2.1 and (2.2). Since $\operatorname{diam} \Omega \le \mathrm{C}(\beta, \mathrm{C}_0) \operatorname{dist}(x_0, \partial\Omega)$, the result follows. □

3. Proofs of Theorems 1.1 and 1.4

We now begin the proofs of our main results. Theorem 1.4 has been proved in Theorem 6.1 of [9] in the special case when E is a closed ball (or cube) in Ω . Our proof makes use of the ideas of the proof in [9] but introduces an important new ingredient: a Frostman measure on the continuum E . We also make use of the lemmas in the preceding section.

Proof of Theorem 1.4. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with diameter one which satisfies (2.1) for some $0 < \beta \le 1$ and let $\mathrm{E} \subset \Omega$ be a continuum. Let $u \in \mathrm{W}^{1,n}(\Omega)$ be a test function for the n-capacity of the pair (Q_0, E) in Ω , i.e., $u : \Omega \to [0, 1]$ is a continuous function and $u(x) = 1$ for $x \in \mathrm{E}$ and $u(x) = 0$ for

$x \in Q_0$. Recall that our goal is to show that

$$\int_\Omega |\nabla u(x)|^n \, dx \ge \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}} \right)^{1-n} .$$

For each $x \in \mathrm{E}$, let $Q(x)$ denote the Whitney cube containing x . Recall that the path $\mathrm{P}(Q(x))$ consists of the collection of all of the Whitney cubes which intersect the quasihyperbolic geodesic joining x_0 to the center of $Q(x)$. We define a subpath $\mathrm{P}'(Q(x)) \subset \mathrm{P}(Q(x))$ as follows: $\mathrm{P}'(Q(x)) = \{Q_s, \dots, Q_f\}$ consists of a chain of Whitney cubes, which begins with the terminal cube $Q_s = Q(x)$ and continues back along the path $\mathrm{P}(Q(x))$ until it reaches the first cube Q_f for which $\operatorname{diam} Q_f \ge \frac{1}{5} \operatorname{diam} \mathrm{E}$. (Note that it is possible that $Q_f = Q_s$.) Since adjacent Whitney cubes Q_1 and Q_2 have $\operatorname{diam} Q_1 \le 5 \operatorname{diam} Q_2$, we must have $\operatorname{diam} Q \le \operatorname{diam} \mathrm{E}$ for all $Q \in \mathrm{P}'(Q(x))$.

We first claim that without loss of generality we may make some initial assumptions regarding the average values of u on the cubes in $\mathrm{P}'(Q(x))$, namely, that $⨍_{Q(x)} u(y)\, dy \ge \frac{1}{2}$ and $⨍_{Q_f} u(x)\, dx \le \frac{1}{2}$. In the following two paragraphs we will briefly indicate why these simplifications can be made, but the short reason is that the other cases are covered by existing results in the literature. The remaining case, which we leave to the end, is where we must make use of a new argument involving a Frostman measure on E .

First, suppose that $⨍_{Q(x)} u(y)\, dy \le \frac{1}{2}$ for some $x \in \mathrm{E}$. Then we can find a subset F of $Q(x)$ whose Hausdorff 1-content $\mathcal{H}_1^\infty(\mathrm{F})$ is comparable to the diameter of $Q(x)$ and for which $u(y) \le \frac{1}{2}$ for all $y \in \mathrm{F}$. Recall that the enlarged cube $\lambda Q(x)$ is a subset of Ω for some $\lambda > 1$ (e.g. $\lambda = \frac{9}{8}$). We divide the proof into two cases, according whether $\mathrm{E} \subset \lambda Q(x)$ or $\mathrm{E} \cap (\mathbb{R}^n \setminus \lambda Q(x)) \ne \emptyset$. In the former case, E and F are subsets of the cube $\lambda Q(x)$ and so

$$\int_\Omega |\nabla u(x)|^n \, dx \ge \int_{\lambda Q(x)} |\nabla u(x)|^n \, dx \ge \operatorname{cap}(\mathrm{E}, \mathrm{F}; \lambda Q(x)) \ge \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}} \right)^{1-n}$$

by a standard estimate for conformal capacity (see [18]). In the latter case, $\mathcal{H}_1^\infty(\mathrm{E} \cap \lambda Q(x)) \ge \operatorname{diam}(\mathrm{E} \cap \lambda Q(x)) \approx \operatorname{diam} Q(x)$. Then we have two compact sets $\mathrm{E} \cap \lambda Q(x)$ and F in the cube $\lambda Q(x)$, both of which have Hausdorff 1-content comparable from below to $\operatorname{diam} Q(x)$. In this situation a straightforward maximal function argument (cf. the proof of Theorem 5.9 in [8]) can be employed to deduce that

$$\int_\Omega |\nabla u(x)|^n \, dx \ge \int_{\lambda Q(x)} |\nabla u(x)|^n \, dx \ge c(n) \ge \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}} \right)^{1-n}$$

since the diameter of E is ≤ 1 .

Next we suppose that the final cube Q_f in the path $\mathrm{P}'(Q(x))$ satisfies $⨍_{Q_f} u(x)\, dx \ge \frac{1}{2}$. Then we have two cubes Q_0 and Q_f in the domain Ω and a continuous $\mathrm{L}^{1,n}$ function u satisfying $u \equiv 0$ on Q_0 and $⨍_{Q_f} u(x)\, dx \ge \frac{1}{2}$. In this case we may invoke an earlier proof of Theorem 1.4 for the special case when E is a closed cube in Ω (see [9, Theorem 6.1]) to deduce that

$$\int_\Omega |\nabla u(x)|^n \, dx \geq \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} Q_f} \right)^{1-n} \geq \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}} \right)^{1-n}$$

since $\operatorname{diam} Q_f \geq \frac{1}{5} \operatorname{diam} \mathrm{E}$.

Thus, as stated above, we assume that $⨍_{Q(x)} u(y)\, dy \geq \frac{1}{2}$ for all $x \in \mathrm{E}$ and that the path $\mathrm{P}'(Q(x))$ consists of Whitney cubes all of which have diameter $\leq \operatorname{diam} \mathrm{E}$ and for which the final cube Q_f satisfies $⨍_{Q_f} u(x)\, dx \leq \frac{1}{2}$. In this situation a straightforward chaining argument involving the Poincaré inequality on the Whitney cubes in the path $\mathrm{P}'(Q(x))$ (c.f. [10, pp. 519-520] or [15, Lemma 8]) yields the estimate

$$1 \leq \mathrm{C} \sum_{Q \in \mathrm{P}'(Q(x))} \operatorname{diam} Q ⨍_Q |\nabla u(y)| \, dy. \tag{3.1}$$

We now choose a Frostman measure μ on the continuum E for the growth function $\varphi(r) = (\log 1/r)^{-n}$, i.e., a Borel measure supported on E satisfying

$$\mu(\mathrm{E} \cap \mathrm{B}(x,r)) \leq (\log 1/r)^{-n} \tag{3.2}$$

for all balls $\mathrm{B}(x,r)$ and

$$\mu(\mathrm{E}) \geq \frac{1}{\mathrm{C}(n)} \mathcal{H}^\infty_\varphi(\mathrm{E}) \geq \frac{1}{\mathrm{C}(n)} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}} \right)^{-n}. \tag{3.3}$$

See, for example, Theorem 5.1.12 in [1].

Integrating (3.1) over the set E with respect to the Frostman measure μ and applying Hölder's inequality, we see that

$$\mu(\mathrm{E}) \leq \mathrm{C} \int_\mathrm{E} \sum_{Q \in \mathrm{P}'(Q(x))} \left(\int_Q |\nabla u(y)|^n \, dy \right)^{1/n} d\mu(x).$$

We now interchange the order of summation and integration to deduce that

$$\mu(\mathrm{E}) \leq \mathrm{C} \sum_{\substack{Q \in \mathcal{W} \\ \operatorname{diam} Q \leq \operatorname{diam} \mathrm{E}}} \mu(\mathrm{S}(Q) \cap \mathrm{E}) \left(\int_Q |\nabla u(y)|^n \, dy \right)^{1/n}.$$

Applying Hölder's inequality again leads to

$$\begin{aligned}\mu(\mathrm{E}) &\le \mathrm{C}\left(\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le \operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap \mathrm{E})^{n/(n-1)}\right)^{1-1/n}\left(\sum_{Q\in\mathcal{W}}\int_Q |\nabla u(y)|^n\,dy\right)^{1/n}\\ &\le \mathrm{C}\left(\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le \operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap \mathrm{E})^{1+1/(n-1)}\right)^{1-1/n}\left(\int_\Omega |\nabla u(y)|^n\,dy\right)^{1/n}.\end{aligned} \tag{3.4}$$

We require an estimate for terms of the form

$$\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le \operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap \mathrm{E})^{1+\delta}$$

for $\delta > 0$, which we give in the following lemma:

Lemma 3.1. *Let Ω be a domain in $\mathbb{R}^n$ with diameter one which satisfies (2.1) and let $\delta > 0$. Suppose that μ is a Borel measure on $\mathbb{R}^n$ which satisfies the growth condition $\mu(\mathrm{B}(x,r)) \le (\log 1/r)^{-a}$ for some $a > 1/\delta$. Then there exists a constant $\mathrm{C} = \mathrm{C}(n,a,\delta,\beta,\mathrm{C}_0)$ so that*

$$\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le \operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap \mathrm{E})^{1+\delta} \le \mathrm{C}\mu(\mathrm{E})\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{1-a\delta}$$

for any set $\mathrm{E}\subset\Omega$.

We defer the proof of this lemma momentarily. To complete the proof of Theorem 1.4, we apply Lemma 3.1 in (3.4) with $\delta = 1/(n-1)$; note that $a = n > 1/\delta$. The measure μ satisfies the requisite growth condition by (3.2) and we see that

$$\mu(\mathrm{E}) \le \mathrm{C}(n,\beta,\mathrm{C}_0)\mu(\mathrm{E})^{1-1/n}\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{-1/n}\left(\int_\Omega |\nabla u(y)|^n\,dy\right)^{1/n}.$$

Thus by (3.3) we see that

$$\int_\Omega |\nabla u(y)|^n\,dy \ge \frac{1}{\mathrm{C}(n,\beta,\mathrm{C}_0)}\mu(\mathrm{E})\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right) \ge \frac{1}{\mathrm{M}}\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{1-n}$$

for some finite constant $\mathrm{M} = \mathrm{M}(n,\beta,\mathrm{C}_0)$. The proof is complete. □

Remark 3.2. The proof of Theorem 1.4 shows that (1.4) holds for some compact sets which are not continua as well. Indeed, the required Frostman measure μ can be found on E whenever E has positive Hausdorff dimension.

Proof of Lemma 3.1. We may choose $j_0 \in \mathbb{N}$ with $j_0 \le \mathrm{C}\log(1/\operatorname{diam}\mathrm{E})$ so that $\operatorname{diam} Q \le \operatorname{diam}\mathrm{E}$ implies $Q \in \mathcal{W}_j$ for some $j \ge j_0$. The growth condition on μ implies that

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le\operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap\mathrm{E})^{1+\delta} &\le \sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j}\mu(\mathrm{S}(Q)\cap\mathrm{E})\mu(\mathrm{S}(Q)\cap\mathrm{E})^{\delta}\\
&\le \sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j}\mu(\mathrm{S}(Q)\cap\mathrm{E})\left(\log\frac{1}{\operatorname{diam}\mathrm{S}(Q)}\right)^{-a\delta}\\
&\le \mathrm{C}(n,\beta,\mathrm{C}_0)\sum_{j=j_0}^{\infty}\sum_{Q\in\mathcal{W}_j}\mu(\mathrm{S}(Q)\cap\mathrm{E})\left(\log\frac{1}{\operatorname{diam} Q}\right)^{-a\delta}\\
&\le \mathrm{C}(n,\beta,\mathrm{C}_0)\sum_{j=j_0}^{\infty} j^{-a\delta}\sum_{Q\in\mathcal{W}_j}\mu(\mathrm{S}(Q)\cap\mathrm{E}).
\end{aligned} \tag{3.5}$$

where the third line follows from Lemma 2.4.

For $j \in \mathbb{Z}$, set $a_j = \sum_{Q\in\mathcal{W}_j}\mu(\mathrm{S}(Q)\cap\mathrm{E})$ and let $\mathrm{A}_j = a_1 + \cdots + a_j$. We apply summation by parts to the right hand side of (3.5) to see that

$$\sum_{j=j_0}^{\infty} j^{-a\delta}a_j \le \mathrm{C}(n,\beta,\mathrm{C}_0,a,\delta)\left[j_0^{-a\delta}\mathrm{A}_{j_0} + \sum_{j=j_0}^{\infty} j^{-1-a\delta}\mathrm{A}_j\right],$$

where we used the estimate $|j^{-a\delta} - (j-1)^{-a\delta}| \le \mathrm{C}(a,\delta)j^{-1-a\delta}$.

By Lemma 2.3, $\mathrm{A}_j \le \mathrm{C}(n,\beta,\mathrm{C}_0)\mu(\mathrm{E})\cdot j$ for each j and so

$$\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le\operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap\mathrm{E})^{1+\delta} \le \mathrm{C}(n,\beta,\mathrm{C}_0,a,\delta)\left[j_0^{-a\delta}\mathrm{A}_{j_0} + \mu(\mathrm{E})\sum_{j=j_0}^{\infty} j^{-a\delta}\right].$$

The sum converges since $a\delta > 1$ and we see that

$$\begin{aligned}
\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le\operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap\mathrm{E})^{1+\delta} &\le \mathrm{C}(n,\beta,\mathrm{C}_0,a,\delta)\mu(\mathrm{E})j_0^{1-a\delta}\\
&\le \mathrm{C}(n,\beta,\mathrm{C}_0,a,\delta)\mu(\mathrm{E})\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{1-a\delta}
\end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem 1.1. Let $f : \Omega \to \Omega'$ be a K-quasiconformal map onto a domain Ω' satisfying (2.1). We may scale the domain Ω' to have diameter one; this introduces a constant into the Hölder coefficient for f which depends only on β, C_0 and $\operatorname{dist}(x_0, \partial\Omega')$. Fix a Whitney cube $\mathrm{F}' = Q_0$ in Ω' with center x_0 and let $\mathrm{F} = f^{-1}(\mathrm{F}')$. Since $\mathrm{F}' = Q_0$ is a Whitney cube, $\frac{3}{2}Q_0 \subset \Omega'$. Let

$\hat{\mathrm{F}} = f^{-1}(\frac{3}{2}Q_0)$. By elementary properties of quasiconformal mappings, there exists $\delta = \delta(n, \mathrm{K}) > 0$ so that the set of points $x \in \mathbb{R}^n$ with $\operatorname{dist}(x, \mathrm{F}) \le \delta \operatorname{diam} \mathrm{F}$ is contained in $\hat{\mathrm{F}}$.

Let $x, y \in \Omega$. Note that f is automatically Hölder continuous as a map from the (compact) subset $\hat{\mathrm{F}} \subset \Omega$ with the Euclidean (hence also the internal) metric into Ω'; the Hölder data depends only on n, K and $\operatorname{dist}(f^{-1}(x_0), \partial\Omega)$. Thus we may assume that either x or y is in $\Omega \setminus \hat{\mathrm{F}}$; without loss of generality let this be the case for x.

Next, note that if $\delta_\Omega(x, y) \ge \frac{1}{4}\delta \operatorname{diam} \mathrm{F}$, then

$$\frac{|f(x) - f(y)|}{\delta_\Omega(x, y)^\alpha} \le \mathrm{C}(n, \mathrm{K}) \frac{\operatorname{diam} \Omega'}{(\operatorname{diam} \mathrm{F})^\alpha} \le \mathrm{C}(n, \mathrm{K}, \beta, \mathrm{C}_0, \alpha, \operatorname{dist}(f^{-1}(x_0), \partial\Omega))$$

for any choice of α. Thus it suffices to verify the Hölder condition in the case $4\delta_\Omega(x, y) < \delta \operatorname{diam} \mathrm{F} \le \operatorname{dist}(x, \mathrm{F})$. Choose a continuum $\mathrm{E} \subset \Omega$ joining x to y with $\operatorname{diam} \mathrm{E} \le 2\delta_\Omega(x, y) < \frac{1}{2}\delta \operatorname{diam} \mathrm{F}$. Then $\operatorname{diam} \mathrm{E} < \frac{1}{2}\delta \operatorname{diam} \mathrm{F} \le \operatorname{dist}(\mathrm{E}, \mathrm{F})$ by a simple calculation. A fundamental property of the conformal capacity (see Fact 3.1(e) of [9]) states that in this case

$$\operatorname{cap}(\mathrm{E}, \mathrm{F}; \Omega) \le \mathrm{C}(n) \left(\log \frac{\operatorname{dist}(\mathrm{E}, \mathrm{F})}{\operatorname{diam} \mathrm{E}} \right)^{1-n}.$$

Set $\mathrm{E}' = f(\mathrm{E})$. By Theorem 1.4,

$$\operatorname{cap}(\mathrm{E}', \mathrm{F}'; \Omega') \ge \frac{1}{\mathrm{M}} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}'} \right)^{1-n}$$

with $\mathrm{M} = \mathrm{M}(n, \beta, \mathrm{C}_0)$. Hence

$$\begin{aligned} \left(\log \frac{1}{\operatorname{diam} \mathrm{E}'} \right)^{1-n} &\le \mathrm{C}(n)\mathrm{KM} \left(\log \frac{\operatorname{dist}(\mathrm{E}, \mathrm{F})}{\operatorname{diam} \mathrm{E}} \right)^{1-n} \\ &\le \mathrm{C}(n)\mathrm{KM} \left(\log \frac{\frac{1}{2}\delta \operatorname{diam} \mathrm{F}}{\operatorname{diam} \mathrm{E}} \right)^{1-n}, \end{aligned}$$

or

$$\operatorname{diam} \mathrm{E}' \le \mathrm{C}(\operatorname{diam} \mathrm{E})^\alpha$$

for some α depending only on n, K, β and C_0 and C depending on these parameters as well as on the values $\operatorname{dist}(x_0, \partial\Omega')$ and $\operatorname{dist}(f^{-1}(x_0), \partial\Omega)$. Since $|x' - y'| \le \operatorname{diam} \mathrm{E}'$ and $\operatorname{diam} \mathrm{E} \le 2\delta_\Omega(x, y)$, the proof of Theorem 1.1 is complete. □

4. Weaker quasihyperbolic boundary conditions and uniform continuity for quasiconformal maps

Our arguments in the previous two sections are robust enough to apply under weaker geometric hypotheses and still yield global regularity properties of quasiconformal maps. In this section, we give a sample of the type of results which may be obtained. It is not clear at precisely what level of generality our technique can be made to apply, see Remark 4.4 and Example 4.5.

We begin with a simple modification of (2.1), replacing the logarithmic growth of the quasihyperbolic metric with growth no more than a power of the logarithm.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ with fixed basepoint $x_0 \in \Omega$ and let $s \geq 1$. We say that Ω satisfies a *quasihyperbolic boundary condition with exponent* s if there exist constants $\beta > 0$ and $\mathrm{C}_0 < \infty$ so that

$$k_\Omega(x_0, x) \leq \frac{1}{\beta}\left(\log^+ \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}\right)^s + \mathrm{C}_0 \tag{4.1}$$

for all $x \in \Omega$. Here $\log^+ t = \max\{\log t, 0\}$.

As before, domains satisfying (4.1) are always bounded with diameter controlled by a constant depending only on s, β, and C_0. However, note that it is no longer the case that a change of the basepoint x_0 will affect only the constant C_0, rather, it may affect the choice of β as well. For this reason we fix once and for all a choice of basepoint x_0 which (as before) we take to be the center of a fixed Whitney cube Q_0.

In this section, we will prove the following analogues of Theorems 1.4 and 1.1 for domains satisfying (4.1).

Theorem 4.2. *Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, with diameter one which satisfies (4.1) for some $s \geq 1$. Then there exists $\mathrm{M} < \infty$ depending only on n, s, β, and C_0 so that*

$$\operatorname{cap}(\mathrm{E}, Q_0; \Omega) \geq \frac{1}{\mathrm{M}}\left(\log \frac{1}{\operatorname{diam} \mathrm{E}}\right)^{s^2(1-n)} \tag{4.2}$$

for all continua $\mathrm{E} \subset \Omega$.

Corollary 4.3. *Let $f : \Omega \to \Omega'$ be a K-quasiconformal between domains $\Omega, \Omega' \subset \mathbb{R}^n$, $n \geq 2$, and assume that Ω' satisfies (4.1) for some $s \geq 1$. Then $f : \Omega \to \Omega'$ is uniformly continuous as a map from (Ω, δ_Ω) to Ω' with modulus of continuity*

$$\omega_f(t) = \mathrm{C} \exp\{-c(\log 1/t)^{1/s^2}\} \tag{4.3}$$

where $\mathrm{C} = \mathrm{C}(n, \mathrm{K}, s, \beta, \mathrm{C}_0, \operatorname{dist}(f^{-1}(x_0, \partial\Omega))) < \infty$ and $c = c(n, \mathrm{K}, s, \beta, \mathrm{C}_0) > 0$.

For any convex increasing function $\psi : [0, \infty) \to [0, \infty)$, we may consider a quasihyperbolic boundary condition of the form

$$k_\Omega(x_0, x) \leq \psi\left(\log^+ \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)}\right), \qquad x \in \Omega, \tag{4.4}$$

and ask when it is the case that quasiconformal maps onto domains satisfying (4.4) are uniformly continuous in the internal metric. By considering the situation for conformal maps of simply connected planar domains (see the following remark), we can derive an integral condition sufficient for global uniform continuity. It is reasonable to conjecture that the integral condition (4.7) remains sufficient, even in higher dimensions and for quasiconformal maps.

Remark 4.4. Suppose that $f : \mathbb{D} \to \Omega'$ is a conformal map onto a planar domain satisfying a growth condition on the hyperbolic metric of the form

$$\rho_{\Omega'}(f(0), f(z)) \le \psi\left(\log^+ \frac{\operatorname{dist}(f(0), \partial\Omega')}{\operatorname{dist}(f(z), \partial\Omega')}\right) \tag{4.5}$$

for $z \in \mathbb{D}$, where ψ is as above. A sufficient condition for global uniform continuity of f is that there exist a function φ, integrable over the interval $[0,1)$, for which

$$\sup_{\theta\in[0,2\pi]} |f'(re^{i\theta})| \le \varphi(r). \tag{4.6}$$

By combining (4.5) with the Koebe distortion theorem $(1-|z|)|f'(z)| \approx \operatorname{dist}(f(z), \partial\Omega')$ and using the inequality $\rho_{\Omega'}(f(0), f(z)) = \rho_{\mathbb{D}}(0,z) \ge \log \frac{1}{1-|z|}$, we see that (4.6) holds with

$$\varphi(r) = \mathrm{C}\frac{1}{1-r}\exp\{-\psi^{-1}(\log\frac{1}{1-r})\}$$

for some absolute constant $\mathrm{C} < \infty$. Thus the integral condition

$$\int_0^1 \exp\{-\psi^{-1}(\log\frac{1}{1-t})\}\,\frac{dt}{1-t} < \infty,$$

equivalently

$$\int_0^\infty e^{-\psi^{-1}(s)}\,ds < \infty, \tag{4.7}$$

is sufficient for global uniform continuity of f. Note that (4.7) allows for growth functions ψ significantly larger than those considered in Definition 4.1.

Example 4.5. The following example shows that (4.7) is essentially the sharp integral condition on ψ for global uniform continuity of f. Suppose that ψ is a growth function as above for which

$$\int_0^\infty e^{-\psi^{-1}(s)}\,ds = \infty. \tag{4.8}$$

Let $\mathrm{M} = \mathrm{M}(x)$ be the solution to the differential equation

$$\mathrm{M}'(x) = e^{\psi^{-1}(\mathrm{M}(x))}, \qquad \mathrm{M}(0) = 0.$$

The divergence of the integral in (4.8) guarantees that $\mathrm{M}(x)$ is finite for all $0 \le x < \infty$. Set $g(x) = \exp\{-\psi^{-1}(\mathrm{M}(|x|))\}$ and

$$\Omega' = \{z = x + iy \in \mathbb{C} : |y| < g(x)\}$$

and let f be a conformal map of $\mathbb{D}$ onto Ω' satisfying $f(0) = 0$. Note that Ω' is unbounded and so f is not uniformly continuous. However, we claim that the quasihyperbolic metric in Ω' satisfies the growth condition

$$k_{\Omega'}(0,z) \le \mathrm{C}_1\psi\left(\log^+\frac{\operatorname{dist}(0,\partial\Omega')}{\operatorname{dist}(z,\partial\Omega')} + \mathrm{C}_2\right)$$

for some constants C_1 and C_2.

To see this, note that $\operatorname{dist}(z,\partial\Omega') \approx g(x) - |y| \le g(x) \le 1$ for all $z = x+iy \in \Omega'$. Thus

$$\begin{aligned} k_{\Omega'}(0,z) &\le \int_0^{|x|} \frac{dt}{\operatorname{dist}(t,\partial\Omega')} + \int_0^{|y|} \frac{dt}{\operatorname{dist}(x+it,\partial\Omega')} \\ &\le C\int_0^{|x|} \frac{dt}{g(t)} + C\int_0^{|y|} \frac{dt}{g(x)-t} \\ &= C\int_0^{|x|} e^{\psi^{-1}(M(t))}\,dt + C\log\frac{g(x)}{g(x)-|y|} \\ &= CM(|x|) + C\log\frac{g(x)}{g(x)-|y|} \\ &\le C\psi\left(\log^+\frac{g(0)}{g(x)}\right) + C\log^+\frac{\operatorname{dist}(0,\partial\Omega')}{\operatorname{dist}(z,\partial\Omega')} + C \\ &\le C_1\psi\left(\log^+\frac{\operatorname{dist}(0,\partial\Omega')}{\operatorname{dist}(z,\partial\Omega')} + C_2\right). \end{aligned}$$

We turn now to the proofs of Theorem 4.2 and Corollary 4.3, beginning with an analog of the result of Smith and Stegenga [14, Theorem 3] which appears in Lemma 2.1.

Lemma 4.6. *Let Ω satisfy (4.1). Then there exists $C_1 = C_1(s,\beta,C_0) < \infty$ so that for all $x_1 \in \Omega$, we have*

$$k_\Omega(x_0,x) \le \frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{length}(\gamma(x,x_1))}\right)^s + C_1 \tag{4.9}$$

whenever γ is a quasihyperbolic geodesic joining x_0 to x_1 and $x \in \gamma$.

Proof. Fix $x_1 \in \Omega$ and a quasihyperbolic geodesic γ joining x_0 to x_1 in Ω. Thus γ is a rectifiable arc in Ω and

$$k_\Omega(y_1,y_2) = \int_{\gamma(y_1,y_2)} \frac{ds}{\operatorname{dist}(x,\partial\Omega)}$$

for each pair of points $y_1, y_2 \in \Omega$. Assume that (4.9) is false, then for every $\tilde{C} \ge \frac{1}{2\beta} + C_0$ there exists a point $y_0 \in \gamma$ so that

$$\frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{length}(\gamma(y_0,x_1))}\right)^s + \tilde{C} < k_\Omega(x_0,y_0). \tag{4.10}$$

Let $L := \operatorname{length}(\gamma(y_0,x_1))$. Define recursively $y_k \in \gamma(y_{k-1},x_1)$ so that $\operatorname{length}(\gamma(y_{k-1},y_k)) = 2^{-k}L$ for $k \in \mathbb{N}$. For $k = 0,1,2,\ldots$, let $\delta_k = \sup\{\operatorname{dist}(x,\partial\Omega) : x \in \gamma(y_k,x_1)\}$.

Combining (4.10) and (4.1) and using the relation $(A+B)^s \le 2^{s-1}(A^s+B^s)$, valid for $A, B \ge 0$ and $s \ge 1$, we see that for all $x \in \gamma(y_0,x_1)$ the following chain of inequalities holds:

$$\begin{aligned}
\frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\mathrm{L}}\right)^s+\tilde{\mathrm{C}} &< k_\Omega(x_0,y_0)\le k_\Omega(x_0,x)\\
&\le\frac{1}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)}\right)^s+\mathrm{C}_0\\
&\le\frac{1}{\beta}\left(\log^+\frac{\mathrm{L}}{\operatorname{dist}(x,\partial\Omega)}+\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\mathrm{L}}\right)^s+\mathrm{C}_0\\
&\le\frac{2^{s-1}}{\beta}\left(\log^+\frac{\mathrm{L}}{\operatorname{dist}(x,\partial\Omega)}\right)^s+\frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\mathrm{L}}\right)^s+\mathrm{C}_0.
\end{aligned}$$

Thus $\delta_0/\mathrm{L}\le\exp\{-2^{-1+1/s}\beta^{1/s}(\tilde{\mathrm{C}}-\mathrm{C}_0)^{1/s}\}\le e^{-\frac12}$. Now we can choose $\tilde{\mathrm{C}}\ge\frac{1}{2\beta}+\mathrm{C}_0$ so that the ratio L/δ_0 is so large that

$$\left(\log^+(\mathrm{L}/\delta_0)^{k+1}\right)^s\le\frac{\beta}{2^{s-1}}2^{-k}(\mathrm{L}/\delta_0)^k \tag{4.11}$$

for all $k\in\mathbb{N}$. We will prove by induction that $\delta_{k-1}/\mathrm{L}\le(\delta_0/\mathrm{L})^k$ for all $k\in\mathbb{N}$. This is trivially true when $k=1$; assume it holds for some $k\ge1$. Combining (4.10), the induction hypothesis, and (4.11), we see that for all $x\in\gamma(y_k,x_1)$ we have

$$\begin{aligned}
&\frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\mathrm{L}}\right)^s+\tilde{\mathrm{C}}+\frac{2^{s-1}}{\beta}\left(\log^+(\mathrm{L}/\delta_0)^{k+1}\right)^s\\
&\quad< k_\Omega(x_0,y_0)+2^{-k}(\mathrm{L}/\delta_0)^k\le k_\Omega(x_0,y_0)+2^{-k}\mathrm{L}/\delta_{k-1}\\
&\quad\le k_\Omega(x_0,y_0)+k_\Omega(y_{k-1},y_k)\le k_\Omega(x_0,x)\le\frac{1}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\operatorname{dist}(x,\partial\Omega)}\right)^s+\mathrm{C}_0\\
&\quad\le\frac{2^{s-1}}{\beta}\left(\log^+\frac{\mathrm{L}}{\operatorname{dist}(x,\partial\Omega)}\right)^s+\frac{2^{s-1}}{\beta}\left(\log^+\frac{\operatorname{dist}(x_0,\partial\Omega)}{\mathrm{L}}\right)^s+\mathrm{C}_0
\end{aligned}$$

and so $\delta_k/\mathrm{L}\le(\delta_0/\mathrm{L})^{k+1}$ which completes the proof of the induction.

Since

$$0<\operatorname{dist}(x_1,\partial\Omega)\le\delta_k\le\mathrm{L}(\delta_0/\mathrm{L})^{k+1}\le\mathrm{L}(e^{-\frac12})^{k+1}$$

for all $k\ge1$, we have a contradiction and thus the lemma is proved. □

Armed with this lemma, we can prove Theorem 4.2 and Corollary 4.3 in much the same way as in the previous section. For the sake of brevity we only sketch the main ideas, indicating along the way how the various lemmas must be modified. Recall that in Theorem 4.2 we assume that the diameter of Ω is one.

First, Lemmas 2.2 and 2.3 take the following form: if Ω satisfies (4.1) for some $s\ge1$, then

$$\operatorname{card}\{Q\in\mathcal{W}_1\cup\cdots\cup\mathcal{W}_j : Q\cap\gamma\ne\emptyset\}\le\mathrm{C}j^s$$

for all j and all quasihyperbolic geodesics γ which start at x_0, furthermore,

$$\sum_{Q\in\mathcal{W}_1\cup\cdots\cup\mathcal{W}_j} \chi_{\mathrm{S}(Q)}(x) \le \mathrm{C} j^s$$

for every $j\in\mathbb{N}$ and $x\in\Omega$.

Next, in Lemma 2.4, the Hölder-type bound for the diameter of the shadow of a Whitney cube Q in terms of the diameter of Q is replaced by the estimate

$$\operatorname{diam}\mathrm{S}(Q) \le \psi(\operatorname{diam} Q),$$

where $\psi(t) = \mathrm{C}\operatorname{dist}(x_0,\partial\Omega)\exp\{-\beta^{1/s}(\log\frac{\operatorname{dist}(x_0,\partial\Omega)}{t})^{1/s}\}$.

Finally, Lemma 3.1 reads as follows: if $\delta>0$ and if μ is a Borel measure on $\mathbb{R}^n$ which satisfies the growth condition [2]

$$\mu(\mathrm{B}(x,r)) \le (\log 1/r)^{-s^2/\delta}(\log\log 1/r)^{-b}$$

for some $b>1/\delta$, then there exists a constant $\mathrm{C}=\mathrm{C}(n,\delta,s,\beta,\mathrm{C}_0)$ so that

$$\sum_{\substack{Q\in\mathcal{W}\\ \operatorname{diam} Q\le\operatorname{diam}\mathrm{E}}} \mu(\mathrm{S}(Q)\cap\mathrm{E})^{1+\delta} \le \mathrm{C}\mu(\mathrm{E})\left(\log\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{-b\delta}.$$

Now Theorem 4.2 follows by repeating the proof of Theorem 1.4. The Frostman measure μ is now chosen to satisfy

$$\mu(\mathrm{E}\cap\mathrm{B}(x,r)) \le (\log 1/r)^{-s^2/\delta}(\log\log 1/r)^{-n}$$

for all balls $\mathrm{B}(x,r)$ and

$$\mu(\mathrm{E}) \ge \frac{1}{\mathrm{C}(n)}\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{-s^2/\delta}\left(\log\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{-n}.$$

Estimating the n-capacity as before, we find that

$$\operatorname{cap}_n(\mathrm{E},Q_0;\Omega) \ge \frac{1}{\mathrm{M}}\left(\log\frac{1}{\operatorname{diam}\mathrm{E}}\right)^{s^2(1-n)}.$$

The proof of Corollary 4.3 follows the argument used to prove Theorem 1.1.

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[2] The use of the $\log\log$ term in this growth condition is strictly speaking not necessary if we are just interested in obtaining uniform continuity, but it leads to a slightly sharper modulus of continuity. If we instead require that $\mu(\mathrm{B}(x,r)) \le (\log 1/r)^{-as^2}$ for some $a>1/\delta=n-1$, then we can show Theorem 4.2 with (4.2) replaced by

$$\operatorname{cap}(\mathrm{E},Q_0;\Omega) \ge (1/\mathrm{M})(\log 1/\operatorname{diam}\mathrm{E})^{s(1-n)-as(s-1)}$$

and Corollary 4.3 with (4.3) replaced by

$$\omega_f(t) = \mathrm{C}\exp\{-c(\log 1/t)^{(s+as(s-1)/(n-1))^{-1}}\}.$$

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