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Transitively twisted flows of 3-manifolds

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Abstract. A non-singular C^1 vector field X of a closed 3-manifold M generating a flow φ_t induces a flow of the bundle NX orthogonal to X. This flow further induces a flow $P\varphi_t$ of the projectivized bundle of NX. In this paper, we assume that the projectivized bundle is a trivial bundle, and study the lift $\angle\varphi_t$ of $P\varphi_t$ to the infinite cyclic covering $M \times \mathbb{R}$. We prove that the flow $\angle\varphi_t$ is not minimal, and construct an example of φ_t such that $\angle\varphi_t$ has a dense orbit. If φ_t is almost periodic and minimal, then $\angle\varphi_t$ is shown to be classified into three cases: (1) All the orbits of $\angle\varphi_t$ are bounded. (2) All the orbits of $\angle\varphi_t$ are proper. (3) $\angle\varphi_t$ is transitive.

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1. Introduction

Let M be a closed 3-dimensional manifold, and X, a non-singular C^1 vector field of M. Denote by φ_t the flow generated by X. Let TM denote the tangent bundle of M and let NX denote the quotient bundle of TM by the 1-dimensional bundle tangent to X. For any t, the derivative $D\varphi_t$ of φ_t induces a flow on NX, denoted by $N\varphi_t$, which is called the *infinitesimal flow* of φ_t. Let PX denote the projectivized bundle $\bigcup_{z\in M}((N_zX - 0)/v \sim kv)$ $(v \in N_zX - 0, k \in \mathbb{R} - 0)$, where N_zX is the fiber of NX at z. Then $N\varphi_t$ also induces a flow on PX, which is denoted by $P\varphi_t$. The flow $P\varphi_t$ represents the angular part of $N\varphi_t$.

In this paper, we assume that PX is a trivial bundle (in particular, if $H^2(M) = 0$). We parametrize PX by $M\times\mathbb{R}/\mathbb{Z}$, i.e. each fiber is the 1-dimensional projective space $\mathbb{P}^1$, which is identified with S^1 and is parametrized by $\mathbb{R}/\mathbb{Z}$. Denote by $[s]$ the element of $\mathbb{P}^1$ represented by $s \in \mathbb{R}$ and by $\pi : M \times \mathbb{R} \to PX$ the projection $(\pi(z, s) = (z, [s]))$. Then there is a unique flow $\angle\varphi_t$ of $M \times \mathbb{R}$ which is a lift of $P\varphi_t$ (See §2). We call it the angular flow of φ_t.

In this paper, we are concerned with dense orbits of $\angle\varphi_t$. It will be shown

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in §3 (Corollary 3) that there is a C^∞ flow whose angular flow has a dense orbit (i.e. *transitive*). However it is impossible that all the orbits of $\angle\varphi_t$ are dense (i.e. *minimal*), which will be shown in §2 (Corollary 1). We will further prove in §2 that, if φ_t is almost periodic and minimal, then $\angle\varphi_t$ is classified into the following three cases (Corollary 2).

(1) All the orbits of $\angle\varphi_t$ are bounded.
(2) All the orbits of $\angle\varphi_t$ are proper.
(3) $\angle\varphi_t$ is transitive.

The author wishes to thank the referee of this paper, who informed the author of the brilliant history of this subject as follows: In 1976 in an international conference on Dynamical systems at IMPA, Rio de Janeiro, Brazil, Alberto Verjovsky delivered a conference in which he constructed a flow, (which he called the "inductance" flow) canonically associated to a smooth nonsingular flow, $\varphi_t : M \to M$, defined on a closed, smooth 3-manifold and corresponding to the vector field X. This flow was constructed with the specific purpose to be applied to the so-called Gottschalk conjecture which states that no minimal flow exists on the 3- sphere. If $G_2(M)$ denotes the Grassmannian bundle of oriented 2-planes tangent to M and if $N \subset G_2(M)$ denotes the subset of $G_2(M)$ of two planes which contain the line field generated by the nonsingular vector field X then N is the total space of a locally trivial fibre bundle $\pi : N \to M$ which, in particular, is the trivial bundle, $N = M \times S^1$ if $H^2(M, \mathbb{Z}) = 0$. Via the action of $D\varphi_t$, on N we obtain a flow $g_t : N \to N$, which is an extension of φ_t, i. e. $\pi \circ g_t = \varphi_t \circ \pi$. This is the "inductance" flow of Verjovsky. The name is an obvious reference to Ampère's Law. The flow g_t preserves the circles which are the fibres and send each circle onto its image by projective transformations of $S^1 = \mathbb{P}^1$. At that conference attended, in particular, Dennis Sullivan, the late Michael Herman and Etienne Ghys who after the conference gave some ideas related to the talk. In particular, Etienne Ghys could prove that no minimal transversely conformal flow could exist in the 3-sphere. In this case the action of g_t on the fibres is, after conjugation, by rotations. A natural question that arose from that conference was: Under what conditions is a smooth minimal flow on a 3-manifold tangent to a foliation, such is the case of the horocycle flow on $PSL(2, \mathbb{R})/\Gamma$ where Γ is a co-compact discrete subgroup. Such manifolds can be homology 3-spheres (such is the case of Brieskorn manifolds $V_{p,q,r}$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$). Some months later after the conference cited before, Michael Herman constructed an example of a smooth diffeomorphism of the 2-torus T^2, such that its differential acts minimally on the space of lines tangent to the torus and it is isotopic to the identity. Therefore, the flow which is the suspension of this diffeomorphism, and defined on the 3-torus T^3 can never be tangent to a foliation. Michael Herman never published his result, however, he wrote an excellent paper, in collaboration with Albert Fathi, in which they proved, in particular, that if a compact manifold S admits a smooth locally-free action of the torus T^n $(n > 1)$ then S admits a smooth minimal action of $\mathbb{R}^{n-1}$. For $n = 1$, S admits a minimal diffeomorphism. Actually this was previously given by Anosov and Katok ([1]).

In this paper, we use the idea of the above construction of the minimal flows to construct a flow whose angular flow is transitive but not minimal. The author would like to express his gratitude to Shigenori Matsumoto, who communicated to the author the construction of Alberto Verjovsky which inspired of this modification from the minimality to the transitivity.

2. Dense orbits of angular flows

First we give a precise definition of the angular flow $\angle\varphi_t$. We define $\Phi : PX \times \mathbb{R} \to PX$ by $\Phi(z, [s], t) = P\varphi_t(z, [s])$ for $z \in M$ and $[s] \in \mathbb{P}^1$. Then there is a map $\Psi : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ satisfying $\pi \circ \Psi = \Phi \circ (\pi \times \mathrm{id})$ and $\Psi(z, 0, 0) = (z, 0)$ for some $z \in M$.

$$\begin{array}{ccc} M \times \mathbb{R} \times \mathbb{R} & \xrightarrow{\Psi} & M \times \mathbb{R} \\ \pi \times \mathrm{id} \downarrow & \circlearrowright & \downarrow \pi \\ PX \times \mathbb{R} & \xrightarrow{\Phi} & PX \end{array}$$

Then we have $\Psi(z, s, 0) = (z, s)$ for any z and s. We define the angular flow $\angle\varphi_t : M \times \mathbb{R} \to M \times \mathbb{R}$ by $\angle\varphi_t(z, s) = \Psi(z, s, t)$. Then $\angle\varphi_t$ is a lift of $P\varphi_t$ by definition, i.e. $\pi \circ \angle\varphi_t = P\varphi_t \circ \pi$. For any u $(0 \leqq u \leqq 1)$, we obtain $\pi\Psi(z, s, u(t_1 + t_2)) = \pi\Psi(\Psi(z, s, ut_1), ut_2)$. Hence we have $\Psi(z, s, t_1 + t_2) = \Psi(\Psi(z, s, t_1), t_2)$, which implies that $\angle\varphi_t$ is a flow of $M \times \mathbb{R}$. Conversely, it can be shown that a flow of $M \times \mathbb{R}$ which is a lift of $P\varphi_t$ is $\angle\varphi_t$.

Remark. If $P\varphi_t$ is generated by a vector field, then $\angle\varphi_t$ is generated by its lift on $M \times \mathbb{R}$. We define $\angle\varphi_t$ in terms of isotopy as above because φ_t is assumed to be of C^1 and, furthermore, we will construct a flow with the transitive angular flow in §3 by using these isotopies.

Let $O(z, s)$ denote the orbit of $\angle\varphi_t$ passing through $(z, s) \in M \times \mathbb{R}$, and let $O_+(z, s)$ (resp. $O_-(z, s)$) denote the positive (resp. negative) semiorbit $\{\angle\varphi_t(z, s); t \geqq 0\}$ (resp. $\{\angle\varphi_t(z, s); t \leqq 0\}$). Denote by p_i $(i = 1, 2)$ the i-th projection of $M \times \mathbb{R}$ and $M \times \mathbb{P}^1$. The orbit $O(z, s)$ is called *upper bounded* (resp. *lower bounded*) if $\{p_2\angle\varphi_t(z, s)\,;\, t \in \mathbb{R}\}$ is upper (resp. lower) bounded. The upper (resp. lower) bounded semiorbit is defined in the same way.

Next we show two general properties concerning with lifts needed later.

Lemma 1. *There is $C > 0$ such that, if $|p_2\angle\varphi_{t_2}(z, s) - p_2\angle\varphi_{t_1}(z, s)| \geqq 1$, then*

$$|p_2\angle\varphi_{t_2}(z, s) - p_2\angle\varphi_{t_1}(z, s)| < C|t_2 - t_1|$$

for any $(z, s) \in M \times \mathbb{R}$, $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$.

Proof. Without loss of generality, we can assume that t_2 is greater than t_1. Let d denote the metric of $\mathbb{P}^1$ induced from the natural metric of $\mathbb{R}/\mathbb{Z}$. Since $P\varphi_t$ is a continuous flow of a compact manifold, there is $C > 0$ such that $d(p_2 P\varphi_t(z,[s]),[s]) < 1/2$ for any $z \in M$, $[s] \in \mathbb{P}^1$ and $|t| < 1/C$. Hence $|p_2\angle\varphi_t(z,s) - s| = 1/2$ implies that $|t| \geqq 1/C$ for any $z \in M$ and $s \in \mathbb{R}$. Here we assume that $p_2\angle\varphi_{t_1}(z,s)$ is less than $p_2\angle\varphi_{t_2}(z,s)$. Let n denote the largest integer smaller than or equal to $p_2\angle\varphi_{t_2}(z,s) - p_2\angle\varphi_{t_1}(z,s)$. Then n is greater than or equal to 1 by assumption. We take a finite sequence $\{b_j\}_{j=0,1,\cdots,2n}$ such that $t_1 = b_0 < b_1 < \cdots < b_{2n} \leqq t_2$ and $p_2\angle\varphi_{b_j}(z_0,s_0) = p_2\angle\varphi_{t_1}(z_0,s_0) + j/2$ $(j = 1,2,\cdots,2n)$. Then $|p_2\angle\varphi_{b_{j+1}}(z_0,s_0) - p_2\angle\varphi_{b_j}(z_0,s_0)| = 1/2$. Hence we have $b_{j+1} - b_j \geqq 1/C$. Thus we obtain that $t_2 - t_1 \geqq 2n/C \geqq (n+1)/C > |p_2\angle\varphi_{t_2}(z,s) - p_2\angle\varphi_{t_1}(z,s)|/C$. We can show the lemma in the same way in case where $p_2\angle\varphi_{t_1}(z,s)$ is greater than $p_2\angle\varphi_{t_2}(z,s)$. □

Lemma 2. *Let $\tau : M\times\mathbb{R} \to M\times\mathbb{R}$ denote the shift defined by $\tau(z,s) = (z,s+1)$. Then $\angle\varphi_t$ commutes with τ.*

Proof. By definition, we have

$$\begin{aligned} &\pi\tau^{-1}\angle\varphi_t\tau(z,s) \\ &= \pi\angle\varphi_t(z,s+1) \\ &= P\varphi_t\ \pi(z,s+1) \\ &= P\varphi_t\ \pi(z,s) = \Phi(\pi(z,s),t). \end{aligned}$$

Therefore, $(z,s,t) \mapsto \tau^{-1}\angle\varphi_t\tau(z,s)$ is a lift of Φ satisfying $\tau^{-1}\angle\varphi_0\tau(z,0) = (z,0)$. Thus we obtain $\tau^{-1}\angle\varphi_t\tau = \angle\varphi_t$. □

Let Y be a topological space. For a homeomorphism $g : Y \to Y$ and a continuous function $h : Y \to \mathbb{R}$, we define a homeomorphism ψ of $Y\times\mathbb{R}$ by $\psi(z,s) = (g(z), s+h(z))$, which is called a *cylinder homeomorphism*. Gottschalk and Hedlund ([3]) studied cylinder homeomorphisms and showed several important properties. Though the angular flow $\angle\varphi_t$ does not satisfy that $p_2\angle\varphi_t(z,s_2) - p_2\angle\varphi_t(z,s_1) = s_2 - s_1$, the following argument similar to that of Gottschalk and Hedlund is valid by Lemma 2.

Lemma 3. *Let (z_0,s_0) be a point of $M\times\mathbb{R}$. If there are sequences $\{u_n\}_{n=1,2,\cdots} \subset \mathbb{R}$ and $\{v_n\}_{n=1,2,\cdots} \subset \mathbb{R}$ such that $u_n < v_n$ and*

$$\begin{aligned} &\max\{p_2\angle\varphi_{u_n}(z_0,s_0), p_2\angle\varphi_{v_n}(z_0,s_0)\} + n \\ &\qquad\leqq \max\{p_2\angle\varphi_t(z_0,s_0)\,;\, u_n \leqq t \leqq v_n\} \\ (\textit{resp.}\ \ &\min\{p_2\angle\varphi_{u_n}(z_0,s_0), p_2\angle\varphi_{v_n}(z_0,s_0)\} - n \\ &\qquad\geqq \min\{p_2\angle\varphi_t(z_0,s_0)\,;\, u_n \leqq t \leqq v_n\}), \end{aligned}$$

then there is an orbit of $\angle\varphi_t$ which is upper (resp. lower) bounded.

Proof. We only prove the existence of upper bounded orbit of $\angle\varphi_t$ in case where $\max\{p_2\angle\varphi_{u_n}(z_0,s_0), p_2\angle\varphi_{v_n}(z_0,s_0)\} + n \leqq \max\{p_2\angle\varphi_t(z_0,s_0)\,;\, u_n \leqq t \leqq v_n\}$.

Let w_n be the time between u_n and v_n such that $p_2\angle\varphi_{w_n}(z_0, s_0)$ is the maximum of $\{p_2\angle\varphi_t(z_0, s_0)\,;\, u_n \leqq t \leqq v_n\}$. By lemma 1, there is $C > 0$ such that $w_n - u_n > n/C$ and $v_n - w_n > n/C$. Hence we have $\lim_{n\to\infty} w_n - u_n = \infty$ and $\lim_{n\to\infty} v_n - w_n = \infty$.

Let $(z_n, s_n) = \angle\varphi_{w_n}(z_0, s_0)$. Denote by $\lfloor s\rfloor$ the largest integer smaller than or equal to s. For any t satisfying $u_n - w_n \leqq t \leqq v_n - w_n$, we have

$$\begin{aligned}
&p_2\angle\varphi_t(z_n, s_n - \lfloor s_n\rfloor)\\
&= p_2\angle\varphi_t\tau^{-\lfloor s_n\rfloor}\angle\varphi_{w_n}(z_0, s_0)\\
&= p_2\angle\varphi_{t+w_n}(z_0, s_0) - \lfloor s_n\rfloor\\
&\leqq p_2\angle\varphi_{w_n}(z_0, s_0) - \lfloor s_n\rfloor, \text{because } u_n \leqq t + w_n \leqq v_n\\
&= s_n - \lfloor s_n\rfloor\\
&< 1
\end{aligned}$$

By taking a subsequence, we can assume that $(z_n, s_n - \lfloor s_n\rfloor)$ converges to some point (z_∞, s_∞) as $n \to \infty$. Then we can show $p_2\angle\varphi_t(z_\infty, s_\infty) \leqq 1$ for any $t \in \mathbb{R}$ as follows: Suppose on the contrary that there is $t\in\mathbb{R}$ such that $p_2\angle\varphi_t(z_\infty, s_\infty) > 1$. Then there is a neighborhood U of (z_∞, s_∞) such that $p_2\angle\varphi_t(z, s) > 1$ for any $(z, s) \in U$. For a sufficiently large n, $(z_n, s_n - \lfloor s_n\rfloor)$ is contained in U and $u_n - w_n \leqq t \leqq v_n - w_n$. However this contradicts the above consideration that $p_2\angle\varphi_t(z_n, s_n - \lfloor s_n\rfloor)$ is less than 1. Thus $O(z_\infty, s_\infty)$ is an upper bounded orbit. □

Lemma 4. *If $\angle\varphi_t$ has a bounded positive (negative) semiorbit, then there is a bounded orbit of $\angle\varphi_t$. Moreover, if φ_t is minimal (i.e. all the orbits of φ_t are dense), then all the orbits of $\angle\varphi_t$ are bounded.*

Proof. If $\angle\varphi_t$ has a bounded positive (resp. negative) semiorbit, then its ω-limit (resp. α-limit) set K is a non-empty compact invariant set. Then an orbit contained in K is bounded in the positive and negative time. For any (z_0, s_0) of K, the orbit $O(z_0, s_0 + n)$ is bounded for any $n \in \mathbb{Z}$, because $\angle\varphi_t$ commutes with τ. Hence, $O(z_0, s)$ is also bounded for any $s \in \mathbb{R}$. If φ_t is further assumed to be minimal, then $\pi(K)$ is the whole manifold M. Therefore, all the orbits of $\angle\varphi_t$ are bounded. □

The orbit $O(z, s)$ is called *proper* if $\lim_{t\to+\infty} p_2\angle\varphi_t(z, s) = +\infty$ or $-\infty$ and $\lim_{t\to-\infty} p_2\angle\varphi_t(z, s) = +\infty$ or $-\infty$. Then $O(z, s)$ is a closed set of $M \times \mathbb{R}$.

Theorem 1. *If $\angle\varphi_t$ has an orbit which is not proper, then there are an upper bounded orbit and a lower bounded orbit.*

Proof. We will only show that $O(z, s)$ is proper for any $(z, s) \in M \times \mathbb{R}$ if no orbits of $\angle\varphi_t$ are upper bounded. We can prove in the same way that $O(z, s)$ is proper for any $(z, s) \in M \times \mathbb{R}$ if no orbits of $\angle\varphi_t$ are lower bounded.

Suppose that there is a point (z, s) of $M \times \mathbb{R}$ such that $O_+(z, s)$ (resp. $O_-(z, s)$) is not upper bounded and $\lim_{t\to+\infty} p_2\angle\varphi_t(z, s) \neq +\infty$ (resp. $\lim_{t\to-\infty} p_2\angle\varphi_t(z, s) \neq +\infty$). Then there exist a number C and a sequence $\{t_n\}_{n=1,2,\cdots} \subset \mathbb{R}$ such that $\lim_{n\to\infty} t_n = +\infty$ (resp. $-\infty$) and $p_2\angle\varphi_{t_n}(z, s) \leqq C$. Since $O_+(z, s)$ (resp. $O_-(z, s)$) is assumed not to be upper bounded, there is a sequence $\{u_n\}_{n=1,2,\cdots} \subset \mathbb{R}$ such that $p_2\angle\varphi_{u_n}(z, s) > C + n$ and $u_n > t_n$ (resp. $u_n < t_n$). Thus there is an increasing (resp. decreasing) sequence $\{v_n\}_{n=1,2,\cdots} \subset \mathbb{R}$ such that $p_2\angle\varphi_{v_{2n-1}}(z, s) \leqq C$ and $p_2\angle\varphi_{v_{2n}}(z, s) > C + n$ $(n = 1, 2, \cdots)$. By Lemma 3, there is an upper bounded orbit, which contradicts the assumption. Thus we conclude that $\lim_{t\to+\infty} p_2\angle\varphi_t(z, s) = +\infty$ (resp. $\lim_{t\to-\infty} p_2\angle\varphi_t(z, s) = +\infty$) if $O_+(z, s)$ (resp. $O_-(z, s)$) is not upper bounded. We can prove in the same way that $\lim_{t\to+\infty} p_2\angle\varphi_t(z, s) = -\infty$ (resp. $\lim_{t\to-\infty} p_2\angle\varphi_t(z, s) = -\infty$) if $O_+(z, s)$ (resp. $O_-(z, s)$) is not lower bounded.

If no orbits of $\angle\varphi_t$ are upper bounded, then no positive (resp. negative) semiorbits are bounded by Lemma 4, which implies $\lim_{t\to+\infty} p_2\angle\varphi_t(z, s) = \pm\infty$ (resp. $\lim_{t\to-\infty} p_2\angle\varphi_t(z, s) = \pm\infty$) for any $(z, s) \in M \times \mathbb{R}$ by the above consideration. Therefore the orbit passing through (z, s) is proper. □

Corollary 1. *$\angle\varphi_t$ is not minimal.*

A subset A of $\mathbb{R}$ is called *syndetic* if $\mathbb{R} = \{a + k\,;\, a \in A,\, k \in K\}$ for some compact set K of $\mathbb{R}$, and a flow φ_t of M is called *almost periodic* if, for any $\varepsilon > 0$, there is a syndetic set A such that $d(z, \varphi_a(z)) < \varepsilon$ for any $z \in M$ and $a \in A$, where d is a metric of M. In this case, we can further analyze the orbit structure of $\angle\varphi_t$ as follows.

Theorem 2. *Let φ_t be an almost periodic minimal flow. If no orbits of $\angle\varphi_t$ are bounded and $\angle\varphi_t$ has an upper bounded positive semiorbit and a lower bounded positive semiorbit, then $\angle\varphi_t$ is transitive.*

Proof. Let W_1 and W_2 be arbitrary open sets of $M \times \mathbb{R}$. We have only to show that $(\bigcup_{t\in\mathbb{R}} \angle\varphi_t(W_2)) \cap W_1 \neq \emptyset$ (Theorem 9.20 of [3]). There are open sets U_1 and U_2 of M and open intervals $I_1 = (a_1, b_1)$ and $I_2 = (a_2, b_2)$ satisfying $U_1 \times I_1 \subset W_1$ and $U_2 \times I_2 \subset W_2$. Then it is enough to show that $(\bigcup_{t\in\mathbb{R}} \angle\varphi_t(U_2 \times I_2)) \cap (U_1 \times I_1) \neq \emptyset$.

First claim that there are a connected open set V_2 contained in U_2 and a syndetic set A such that $\varphi_a(V_2)$ is contained in U_1 for any $a \in A$. Let x_1 be a point of U_1. Then there is $\varepsilon > 0$ such that the ε-ball $B_\varepsilon(x_1)$ with center x_1 is contained in U_1. By the minimality of φ_t, there is $t_1 \in \mathbb{R}$ such that $\varphi_{t_1}(x_1)$ is contained in U_2. Since φ_t is almost periodic, there is a syndetic set A' such that $d(\varphi_a(x), x) < \varepsilon/2$ for any $x \in M$ and $a \in A'$. Let V_2 be a connected component of $U_2 \cap \varphi_{t_1}(B_{\varepsilon/2}(x_1))$. For any $y \in V_2$, we have $d(\varphi_{-t_1}(y), x_1) < \varepsilon/2$. Furthermore, we obtain $d(\varphi_{a-t_1}(y), \varphi_{-t_1}(y)) < \varepsilon/2$ for any $a \in A'$. Hence $d(\varphi_{a-t_1}(y), x_1) < \varepsilon$, which implies that $\varphi_{a-t_1}(y) \in U_1$. Since $\{a - t_1\,;\, a \in A'\}$ is also syndetic, we conclude the claim. In particular, there is $C_1 > 0$ such that, for any $t \in \mathbb{R}$, there is u satisfying $-C_1/2 \leqq u \leqq C_1/2$ and $\varphi_{t+u}(V_2) \subset U_1$. Hence, for any $t \in \mathbb{R}$, there is v $(0 \leqq v \leqq C_1)$ satisfying $\varphi_{t+v}(V_2) \subset U_1$ by the above consideration for $t + (C_1/2)$.

The set $\{z\,; O_+(z,s)$ is lower bounded for any $s \in \mathbb{R}\}$ is a nonempty invariant set of M. Hence it is dense in M by the minimality of φ_t. Furthermore, the set $\{z\,; O_+(z,s)$ is upper bounded for any $s \in \mathbb{R}\}$ is also dense in M. Thus there are points (z_1,s_1) and (z_2,s_2) of $V_2 \times I_2$ such that $O_+(z_1,s_1)$ is lower bounded and $O_+(z_2,s_2)$ is upper bounded. By Lemma 4, $O_+(z_1,s_1)$ is not upper bounded and $O_+(z_2,s_2)$ is not lower bounded.

Denote by C_2 the minimal of $\{p_2\angle\varphi_t(z_1,s_1)\,; t \geqq 0\}$. By Lemma 1, there is $C_3 > 1/C_1$ such that, if $|p_2\angle\varphi_t(z,s) - p_2\angle\varphi_u(z,s)| \geqq 1$, then $|p_2\angle\varphi_t(z,s) - p_2\angle\varphi_u(z,s)| < C_3|t-u|$ for any $z \in M$, $s \in \mathbb{R}$, $t \in \mathbb{R}$ and $u \in \mathbb{R}$.

We claim that there exists $C_4 > 0$ such that $\max\{p_2\angle\varphi_u(z_1,s_1)\,; t \leqq u \leqq t+C_4\} > b_1 + C_1C_3$ for any $t \geqq 0$. If not, there is a positive sequence $\{t_n{}'\}_{n=1,2,\cdots}$ such that $C_2 \leqq p_2\angle\varphi_{t_n{}'+u}(z_1,s_1) \leqq b_1 + C_1C_3$ for $0 \leqq u \leqq n$. Let (z_0,s_0) be an accumulating point of $\{\varphi_{t_n{}'}(z_1,s_1)\}_{n=1,2,\ldots}$. Then the positive semiorbit starting from (z_0,s_0) is bounded, which contradicts the assumption by Lemma 4.

We choose $t_2 \geqq 0$ such that $p_2\angle\varphi_{t_2}(z_2,s_2)$ is less than $a_1 - C_3(C_1 + C_4)$. Since $C_3(C_1 + C_4) > 1$, we have $p_2\angle\varphi_{t_2+t}(z_2,s_2) < a_1$ for $0 \leqq t \leqq C_1 + C_4$ by the choice of C_3. By the choice of C_4, there is t_3 such that $0 \leqq t_3 \leqq C_4$ and $p_2\angle\varphi_{t_2+t_3}(z_1,s_1) > b_1 + C_1C_3$. Furthermore, by the choice of C_1, there is t_4 $(0 \leqq t_4 \leqq C_1)$ such that $\varphi_{t_2+t_3+t_4}(V_2) \subset U_1$. Here we have $p_2\angle\varphi_{t_2+t_3+t_4}(z_1,s_1) > b_1$ because $C_1C_3 > 1$. On the other hand, we have $p_2\angle\varphi_{t_2+t_3+t_4}(z_2,s_2) < a_1$ because $0 \leqq t_3 + t_4 \leqq C_1 + C_4$. In consequence, we obtain that $\varphi_{t_2+t_3+t_4}(V_2) \subset U_1$, $p_2\angle\varphi_{t_2+t_3+t_4}(z_1,s_1) > b_1$ and $p_2\angle\varphi_{t_2+t_3+t_4}(z_2,s_2) < a_1$. This implies that $\angle\varphi_{t_2+t_3+t_4}(V_2 \times I_2)$ intersects $U_1 \times I_1$ □

Corollary 2. *Let φ_t be an almost periodic minimal flow. Then $\angle\varphi_t$ is classified into three cases:*

(1) *All the orbits of $\angle\varphi_t$ are bounded.*
(2) *All the orbits of $\angle\varphi_t$ are proper.*
(3) *$\angle\varphi_t$ is transitive.*

Proof. If $\angle\varphi_t$ has a bounded orbit, then all the orbits of $\angle\varphi_t$ are bounded by Lemma 4. If $\angle\varphi_t$ has no bounded orbits and there is an orbit which is not proper, then there are an upper bounded orbit and a lower bounded orbit by Theorem 1, and furthermore $\angle\varphi_t$ is transitive by Theorem 2. □

3. Construction of transitive angular flows

In this section, we will construct a C^∞ flow of T^3 whose angular flow is transitive by using a suspension of a toral diffeomorphism.

Let TT^2 denote the tangent bundle of T^2. Denote by PT^2 the projectivized bundle $\bigcup_{z\in T^2}((T_zT^2-0)/v\sim kv)$ $(k\in\mathbb{R}-0)$, where T_zT^2 is the tangent space of T^2 at z. Then PT^2 is a trivial bundle. We parametrize PT^2 by $T^2\times\mathbb{P}^1=T^2\times\mathbb{R}/\mathbb{Z}$. Denote by $\pi: T^2\times\mathbb{R}\to PT^2$ the natural projection $(\pi(z,s)=(z,[s]))$.

Let G denote the set of C^∞ isotopies of T^2 from id obtained by identifying homotopic ones (i.e. $G=\{F: T^2\times I\to T^2\,;\, F$ is a C^∞ map, $F|(T^2\times\{t\})$ is a C^∞ diffeomorphism for any $t\in I=[0,1]$ and $F|(T^2\times\{0\})=\mathrm{id}\,\}/\sim$, where two isotopies are identified if they are homotopic with the end points fixed in the sense of paths of diffeomorphisms). Then G is a complete metrizable space with respect to the induced C^∞ topology ([2]). Let $F_t=F|(T^2\times\{t\})$ for $F\in G$ and $t\in I$. Then DF_t induces a bundle map $PF_t: PT^2\to PT^2$. We define $PF: PT^2\times I\to PT^2$ by $PF(z,[s],t)=PF_t(z,[s])$. Then there is a lift $\widetilde{F}: T^2\times\mathbb{R}\times I\to T^2\times\mathbb{R}$ of PF such that $\widetilde{F}(z,0,0)=(z,0)$ for some $z\in T^2$.

$$
\begin{tikzcd}
T^2\times\mathbb{R}\times I \arrow[r, "\widetilde{F}"] \arrow[d, "\pi\times\mathrm{id}"'] \arrow[dr, phantom, "\circlearrowright"] & T^2\times\mathbb{R} \arrow[d, "\pi"] \\
PT^2\times I \arrow[r, "PF"] & PT^2
\end{tikzcd}
$$

Here we remark that $\widetilde{F}(z,s,0)=(z,s)$ for any $z\in T^2$ and $s\in\mathbb{R}$ by the property of lifts. We define the *angular lift* $\angle F: T^2\times\mathbb{R}\to T^2\times\mathbb{R}$ by $\angle F(z,s)=\widetilde{F}(z,s,1)$. Then $\angle F$ is a lift of PF_1.

Let F and F' be isotopies of G. Denote by FF' the isotopy $\{F_t\circ F'_t\}_{0\leqq t\leqq 1}$ and by F^{-1} the isotopy $\{F_t{}^{-1}\}_{0\leqq t\leqq 1}$. Then the following properties hold.

Proposition 1.

$$(1)\qquad \angle F\angle F'=\angle(FF')$$
$$(2)\qquad \angle(F^{-1})=(\angle F)^{-1}.$$

Let F be an isotopy of G. Now T^3 is obtained by identifying $T^2\times\{0\}$ and $T^2\times\{1\}$ of $T^2\times I$ by id. We define a flow φ_t of $T^3=\{(z,u)\,;\, z\in T^2,\ u\in S^1\}$ by $\varphi_t(z,u)=(F_{u+t}F_u^{-1}(z),u+t)$ (In order to construct a C^∞ flow of T^3, we need some modification along $T^2\times\{0\}$ and $T^2\times\{1\}$). The flow φ_t is the image of the flow $(z,u)\mapsto(z,u+t)$ by $(z,u)\mapsto(F_u(z),u)$. Then φ_t is the suspension flow of F_1. Denote by X the vector field generating φ_t. Then the projectivized bundle PX can be identified with $(\bigcup_{z\in T^2}(T_zT^2-0/v\sim kv))\times S^1$ $(k\neq 0)$. By construction of $\angle F$, the time one map of the angular flow $\angle\varphi_t$ restricted to $T^2\times\{0\}$ coincides with $\angle F$ (This is the reason why we take the lift $\widetilde{F}$ satisfying $\widetilde{F}(z,s,0)=(z,s)$ for any $z\in T^2$ and $s\in\mathbb{R}$). Thus it is enough to construct an isotopy F of G whose angular lift is transitive in order to construct a flow of T^3 whose angular flow is transitive.

Theorem 3. *There is an isotopy $F: T^2\times I\to T^2$ of G whose angular lift $\angle F$ is transitive.*

Let $\{U_i\}_{i=1,2,\cdots}$ be a countable base of $T^2 \times \mathbb{R}$. Let

$$\mathfrak{M}_{ij} = \Big\{ F \in G\,;\ \Big(\bigcup_{n\in\mathbb{Z}} \angle F^n(U_i) \Big) \cap U_j \neq \emptyset \Big\}.$$

In order to prove that there is an isotopy F of G whose angular lift $\angle F$ is transitive, we have only to show that $\mathfrak{M} = \bigcap_{i,j} \mathfrak{M}_{ij}$ is not empty ([3]).

Let $R_\theta : T^2 \times I \to T^2$ ($\theta \in S^1 = \mathbb{R}/\mathbb{Z}$) denote the isotopy defined by $R_\theta(x, y, t) = (x + t\theta, y)$ where $x, y \in S^1 = \mathbb{R}/\mathbb{Z}$ and $t \in I = [0, 1]$. Then the angular lift $\angle R_\theta$ satisfies $\angle R_\theta(x, y, s) = (x + \theta, y, s)$. Let H denote the subset $\{FR_\theta F^{-1}\,;\ F \in G,\ \theta \in S^1\}$ of G, where $FR_\theta F^{-1} = \{F_t(R_\theta)_t {F_t}^{-1}\}_{0\leqq t\leqq 1}$. Denote by $\overline{H}$ its closure in G. Since G is a complete metrizable space, $\overline{H}$ is also a complete metrizable space, and is a Baire space. In order to prove $\overline{H} \cap \mathfrak{M}$ is not empty (in particular, $\mathfrak{M} \neq \emptyset$), we have only to show $\overline{H} \cap \mathfrak{M}_{ij}$ is an open dense set in $\overline{H}$ by Baire's category theorem.

First remark that $\mathfrak{M}_{ij}$ is an open set in $\overline{H}$ if $\mathfrak{M}_{ij}$ is open in G. On the other hand, if there is $n \in \mathbb{Z}$ such that $\angle F^n(U_i) \cap U_j$ is not empty, then $(\angle F')^n(U_i) \cap U_j$ is also nonempty for any F' of G sufficiently near F. Thus $\mathfrak{M}_{ij}$ is open in G.

The remaining part of the proof of Theorem 3 is to show that $\mathfrak{M}_{ij}$ is dense in $\overline{H}$, whose key lemma is the following.

Lemma 5. *For any open sets U and V of $T^2 \times \mathbb{R}$, there is an isotopy F of G such that $(\bigcup_{\theta\in S^1} \angle R_\theta \angle F(U)) \cap \angle F(V) \neq \emptyset$.*

Proof. We take sufficiently small cubes $C_1 = (x_1, x_2) \times (y_1, y_2) \times (s_1, s_2)$ contained in U and $C_2 = (x'_1, x'_2) \times (y'_1, y'_2) \times (s'_1, s'_2)$ contained in V satisfying $(x_1, x_2) \cap (x'_1, x'_2) = \emptyset$ (See Figure 1). By moving and twisting id, we obtain an isotopy F of G such that

$$F_t|_{\{(x,y);\ x_1<x<x_2,\ y\in S^1\}} = \mathrm{id} \quad (t \in [0, 1]), \tag{1}$$

$$\angle F(C_2) \cap \{(x, y, s);\ x \in S^1,\ y_1 < y < y_2,\ s_1 < s < s_2\} \neq \emptyset. \tag{2}$$

Since $\bigcup_{\theta\in S^1} \angle R_\theta(C_1) = \{(x, y, s)\,;\ x \in S^1, y_1 < y < y_2, s_1 < s < s_2\}$ and $\angle F(C_1) = C_1$, it is concluded that $\angle F(C_2)$ intersects $\bigcup_{\theta\in S^1} \angle R_\theta(\angle F(C_1))$. Hence we have $(\bigcup_{\theta\in S^1} \angle R_\theta \angle F(U)) \cap \angle F(V)$ is not empty. □

Remark. If we can take an isotopy F of Lemma 5 such that $\angle F(V)$ intersects every circle $\{(x, y, s)\,;\ x \in S^1\}$ instead of (2), then there is an isotopy F such that $\angle F$ is minimal ([1],[2]). However this is impossible, which is closely related to the non-minimality of $\angle\varphi_t$ (Corollary 1).

The following proof is the modification of the proof for the minimality, which was given by Anosov and Katok ([1]). They showed that a compact manifold admits a smooth minimal flow if it admits a smooth, locally-free action of the torus T^2 (see also [2] in the higher dimensional case).

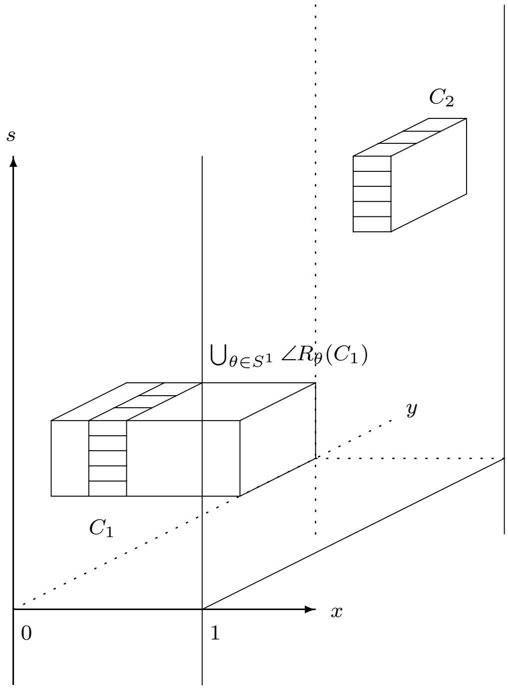

Figure 1

Let $r_\theta : T^2 \to T^2$ $(\theta \in S^1)$ denote the rotation of T^2 defined by $r_\theta(x, y) = (x + \theta, y)$ for $x, y \in S^1$.

Lemma 6. *Let U and V be open sets of $T^2 \times \mathbb{R}$. For any rational number ρ, there is an isotopy F of G such that*

(1) $r_\rho F_t {r_\rho}^{-1} = F_t$ *for any* $t \in I$,

(2) $(\bigcup_{\theta\in S^1} \angle R_\theta \angle F(U)) \cap \angle F(V) \neq \emptyset$.

Proof. Denote by $\eta : T^2 \times \mathbb{R} \to T^2/r_\rho \times \mathbb{R}$ the natural projection. By Lemma 5, there is an isotopy $F' : T^2/r_\rho \times I \to T^2/r_\rho$ from id such that $(\bigcup_{\theta\in S^1} \angle R_\theta \angle F'\eta(U)) \cap \angle F'\eta(V) \neq \emptyset$. Let z_0' be a point of T^2/r_ρ, and let $z_0 \in T^2$ be a lift of z_0'. Then there is a lift $F : T^2 \times I \to T^2$ of F' satisfying $F(z_0, 0) = z_0$. By construction, F is also an isotopy from id.

$$\begin{array}{ccc} T^2 \times I & \xrightarrow{F} & T^2 \\ \downarrow & \circlearrowright & \downarrow \\ T^2/r_\rho \times I & \xrightarrow{F'} & T^2/r_\rho \end{array}$$

Since the isotopy $(z, t) \mapsto r_\rho F_t {r_\rho}^{-1}(z)$ is also a lift of F' satisfying $r_\rho F_0 {r_\rho}^{-1}(z_0) =$

z_0, we obtain $r_\rho F_t {r_\rho}^{-1} = F_t$ for any $t \in I$.

Since $\bigcup_{\theta\in S^1} \angle R_\theta \angle F'\eta(U) \cap \angle F'\eta(V)$ is not empty, there are $\theta \in S^1$, $(z_1, s_1) \in U$ and $(z_2, s_2) \in V$ satisfying $\angle R_\theta \angle F'\eta(z_1, s_1) = \angle F'\eta(z_2, s_2)$. On the other hand, we have $\eta\circ\widetilde{F} = \widetilde{F}'\circ(\eta \times \mathrm{id}\,)$. Hence, $\eta\angle F = \angle F'\eta$. Since $\angle R_\theta \eta \angle F(z_1, s_1) = \eta\angle F(z_2, s_2)$, there is $\theta' \in S^1$ such that $\angle R_{\theta'} \angle F(z_1, s_1) = \angle F(z_2, s_2)$. □

Let $S^2 = \{(x, y, z) \in \mathbb{R}^3\, ;\, x^2 + y^2 + z^2 = 1\}$. For the θ-rotation along the z-axis, the argument similar to Lemmas 5 and 6 are valid because it is enough to consider sufficiently small open sets U and V. Thus we can construct such a flow of S^3 from the Hopf fibration.

Lemma 7. *For any rational number ρ, R_ρ is contained in $\overline{\mathfrak{M}_{ij} \cap H}$ for any i and j.*

Proof. Let ρ be an arbitrary rational number. By Lemma 6, there is an isotopy F of G such that $r_\rho F_t {r_\rho}^{-1} = F_t$ $(t \in I)$ and $(\bigcup_{\theta\in S^1} \angle R_\theta \angle F^{-1}(U_i)) \cap \angle F^{-1}(U_j) \neq \emptyset$. Hence the subset $\{\theta \in S^1\, ;\, \angle F \angle R_\theta \angle F^{-1}(U_i) \cap U_j \neq \emptyset\}$ of S^1 is a non-empty open set. Let $\{\alpha_k\}_{k=1,2,\dots}$ be a sequence of irrational numbers converging to ρ. For any k, there is an integer n_k such that $\angle F \angle R_{n_k\alpha_k} \angle F^{-1}(U_i) \cap U_j \neq \emptyset$. Thus $(\bigcup_{n\in\mathbb{Z}} (\angle F \angle R_{\alpha_k} \angle F^{-1})^n(U_i)) \cap U_j$ is not empty, which implies that $F R_{\alpha_k} F^{-1}$ is contained in $\mathfrak{M}_{ij} \cap H$ and $F R_\rho F^{-1} \in \overline{\mathfrak{M}_{ij} \cap H}$.

Let $\Lambda_s : T^2 \times I \to T^2$ $(s \in [0, 1])$ denote the 1- parameter family of the isotopies defined by

$$\Lambda_s(z, t) = \begin{cases} r_{t\rho}(z) & 0 \leqq t \leqq \dfrac{1-s}{1+s} \\ F_{\frac{(1+s)t-1+s}{1+s}}\, r_{t\rho}\, {F_{\frac{(1+s)t-1+s}{1+s}}}^{-1}(z) & \dfrac{1-s}{1+s} \leqq t \leqq 1 \end{cases}.$$

Then we have

$$\Lambda_s(z, 1) = F_{\frac{2s}{1+s}}\, r_\rho\, {F_{\frac{2s}{1+s}}}^{-1}(z) = r_\rho,$$

$$\Lambda_0(z, t) = r_{t\rho}(z)$$

and

$$\Lambda_1(z, t) = F_t\ r_{t\rho}\ {F_t}^{-1}(z).$$

Thus R_ρ is identified with $F R_\rho F^{-1}$ in G, which is contained in $\overline{\mathfrak{M}_{ij} \cap H}$. □

Lemma 8. *$\mathfrak{M}_{ij} \cap \overline{H}$ is dense in $\overline{H}$ for any i and j.*

Proof. We have only to show that $F R_\rho F^{-1}$ is an element of $\overline{\mathfrak{M}_{ij} \cap H}$ for any rational number ρ and $F \in G$ because this implies that $H \subset \overline{\mathfrak{M}_{ij} \cap H}$. We choose open sets U_k and U_l of $T^2 \times \mathbb{R}$ from the countable base $\{U_i\}$ so that U_k is contained in $\angle F^{-1}(U_i)$ and U_l is contained in $\angle F^{-1}(U_j)$. By Lemma 7, R_ρ is an element of $\overline{\mathfrak{M}_{kl} \cap H}$. Hence there is an isotopy F' of $\mathfrak{M}_{kl} \cap H$ sufficiently near R_ρ. By definition, we have

$$\begin{aligned} &\bigcup_{n\in\mathbb{Z}} (\angle F \angle F' \angle F^{-1})^n (U_i) \cap U_j \\ &= \angle F(\bigcup_{n\in\mathbb{Z}} (\angle F')^n \angle F^{-1}(U_i) \cap \angle F^{-1}(U_j)) \\ &\supset \angle F(\bigcup_{n\in\mathbb{Z}} (\angle F')^n (U_k) \cap U_l) \neq \emptyset. \end{aligned}$$

Thus we conclude that $FF'F^{-1}$ is contained in $\mathfrak{M}_{ij} \cap H$, and $FR_\rho F^{-1} \in \overline{\mathfrak{M}_{ij} \cap H}$. $\square$

Corollary 3. *There is a C^∞ flow φ_t of T^3 such that $\angle\varphi_t$ is transitive.*

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