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The Rost invariant has trivial kernel for quasi-split groups of low rank

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Abstract. For G a simple simply connected algebraic group defined over a field F, Rost has shown that there exists a canonical map $R_G : H^1(F, G) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$. This includes the Arason invariant for quadratic forms and Rost's mod 3 invariant for exceptional Jordan algebras as special cases. We show that R_G has trivial kernel if G is quasi-split of type E_6 or E_7. A case-by-case analysis shows that it has trivial kernel whenever G is quasi-split of low rank.

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For G a simple simply connected algebraic group over a field F, the set of all natural transformations of functors

$$H^1(?, G) \longrightarrow H^3(?, \mathbb{Q}/\mathbb{Z}(2))$$

is a finite cyclic group [KMRT98, §31] with a canonical generator. (Here $H^i(?, M)$ is the Galois cohomology functor which takes a field extension of the base field F and returns a group if M is abelian and a pointed set otherwise. When F has characteristic 0, $\mathbb{Q}/\mathbb{Z}(2)$ is defined to be $\varinjlim \boldsymbol{\mu}_n^{\otimes 2}$ for $\boldsymbol{\mu}_n$ the algebraic groups of nth roots of unity; see [EKLV98, p. 95] or [Gil00, I.1(b)] for a more complete definition.) This generator is called the *Rost invariant* of G and we denote it by R_G. In an abuse of notation, we also write R_G for the map $H^1(F, G) \longrightarrow H^1(F, \mathbb{Q}/\mathbb{Z}(2))$.

This map provides a useful invariant for algebraic structures classified by $H^1(F, G)$, and an important and typically difficult question is to describe the kernel of R_G. For example, when G is split of type D_n, R_G is essentially the Arason invariant $I^3F \to H^3(F, \mathbb{Z}/2)$ for quadratic forms, where I^nF is as usual the nth power of the ideal IF of even-dimensional quadratic forms in the Witt ring of F. That the kernel of the Arason invariant is precisely I^4F is a quite difficult result due independently to Merkurjev–Suslin [MS91] and Rost. (The proof of the main result of this paper somehow boils down to this one fact.) In general, one doesn't even know if the kernel of R_G is trivial. On the other hand, the question becomes tractable if we assume that G is quasi-split. Generally R_G has nontrivial kernel; we give easy examples where G is split of type D_8 (in 1.9) and B_7 (in 1.6), and quasi-split of type ${}^2A_6$ (in 1.11). It should be mentioned that R_G can have nontrivial kernel when G is split of type E_8 as well; Gille [Gil, Appendix] has produced an example by applying his results from [Gil00] to reduce the question to the same one for a split group of type D_8.

The principal result in this paper is to enlarge the list of quasi-split groups for which the Rost invariant is known to have trivial kernel.

Main Theorem 0.1. *Suppose that G is a quasi-split simply connected group of type E_6 or E_7. Then the Rost invariant R_G has trivial kernel.*

0.2. There are some easy consequences of this theorem that may help the reader place it in context. The first is that as a vastly less powerful corollary, we obtain Serre's "Conjecture II" for quasi-split groups of type E_6 and E_7, in that if F has p-cohomological dimension ≤ 2 for $p = 2, 3$ (see [Ser94, I.3] for a definition), then the main theorem implies that $H^1(F, G)$ is trivial. This conjecture appeared in print back in 1962 [Ser62], and remained open for such groups until the 1990s, when Chernousov (unpublished) and Gille [Gil01] proved it (amongst other cases) independently and by different methods. Here we get it for free from the Main Theorem.

0.3. Another consequence is the following: Suppose that L is a field extension of F of degree relatively prime to 2 and 3 and that G is a group of type E_6 or E_7. Serre asked in [Ser95, p. 233, Q. 1] if the natural map $H^1(F, G) \to H^1(L, G)$ is injective. Our Main Theorem gives the partial answer that it has trivial kernel in the case where G is quasi-split. This result was already known by experts in the area using arguments special to groups of type E_6 and E_7, but as for Conjecture II we get it for free here.

0.4. There is also an application to finite-dimensional algebras. There is a large family of nonassociative algebras with involution called *structurable algebras* which includes central simple associative algebras with involution (as studied in [KMRT98]) and Jordan algebras (with involution the identity), see [All94] for a survey. The simple structurable algebras have all been classified, and they consist (roughly) of the two families already mentioned plus four others. The most poorly understood of these four additional types consists of 56-dimensional algebras all of which are isomorphic over a separably closed field and have automorphism group which is simply connected of type E_6. Call algebras belonging to this class *Brown algebras*. There is a natural equivalence relation defined on the set of structurable algebras called *isotopy* [AH81] which is weaker than isomorphism, and in the case of Jordan algebras is the same as the traditional notion of isotopy. For Albert algebras, it is known that any algebra isotopic to the split one is actually split. (This is equivalent to the cohomological statement that the map $H^1(F, F_4) \to H^1(F, E_6)$ induced by the embedding $F_4 \to E_6$ described in 2.4 has trivial kernel.) The Main Theorem here combined with [Gar01b, 4.16(2), 5.12] shows that an analogous conclusion holds for Brown algebras, i.e., a Brown algebra isotopic to the split one is quasi-split. This was previously unknown. (This has the cohomological interpretation that the map $H^1(F, E_6^K) \to H^1(F, E_7)$ induced by the embedding $E_6^K \to E_7$ described in 3.5 has trivial kernel.)

The material in [KMRT98] is sufficient to show that the kernel of the Rost invariant is trivial for quasi-split groups of type G_2, D_4 (including those of trialitarian type [KMRT98, 40.16]), and F_4, at least away from the "bad primes" 2 and 3. As easy corollaries to results needed for the E_6 and E_7 cases, we get analogous results for groups of type ${}^2A_n$, B_n, and nontrialitarian groups of type D_n with small n in Section 1. Since $H^1(F, G)$ is always trivial for G split of type A_n or C_n, we get the following:

Theorem 0.5. *Suppose that G is a simple simply connected algebraic group. If G is*

- *quasi-split of (absolute) rank ≤ 5;*
- *quasi-split of type B_6, D_6, or E_6; or*
- *split of type D_7 or E_7,*

then the Rost invariant R_G has trivial kernel.

The proofs of these theorems that we will give here and the material in [KMRT98] rely on the ground field having "good" characteristic, meaning for our purposes $\neq 2, 3$. However, it is a consequence of Gille's main theorem in [Gil00] that one only needs to prove that the Rost invariant has trivial kernel for fields of characteristic 0. Consequently, all fields considered here will be assumed to have characteristic $\neq 2, 3$, but our two theorems will still hold for all characteristics. (Of course, in prime characteristic the group $\mathbb{Q}/\mathbb{Z}(2)$ must be defined somewhat differently [Gil00], but this affects neither the statement of the theorems nor our proofs.)

Section 1 dispenses with the classical groups. (Some of that material is useful later.) Sections 2 and 3 contain the material necessary to reduce questions about the Rost invariant for a larger group to a subgroup. That material easily reduces the proof of the main theorem to considering the quasi-split ${}^2E_6$ case, which is treated in the remaining Sections 4 through 7.[1]

Remark 0.6 (Noninjectivity for F_4)**.** We caution the reader that even when the Rost invariant has trivial kernel, it may be far from injective. For example, for F_4 the split group of type F_4, the set $H^1(F, F_4)$ classifies Albert F-algebras. From known facts about Albert algebras, it is easy to show that two classes α_1, α_2 corresponding to isotopic Jordan algebras J_1, J_2 have the same Rost invariant. Since there are many isotopic Albert algebras which are not isomorphic (for example,

[1] After this paper was released as a preprint, Chernousov sent to me a different proof of the ${}^2E_6$ case [Che00], which uses a completely different argument. His proof will be published elsewhere.

over $\mathbb{R}$ there are 3 isomorphism classes of Albert algebras and two of these are isotopic [Jac71, p. 119]), the Rost invariant for F_4 has trivial kernel but is typically not injective.

Notations and conventions

All algebraic groups considered here will be affine. We say that an algebraic group G is *simple* if it has finite center and no noncentral closed normal subgroups defined over an algebraic closure. When we say that a group is "of type T_n", we implicitly mean that it is simple of that type. We will use the standard notations $\mathbb{G}_m$, $\mathbb{G}_a$, and $\boldsymbol{\mu}_n$ for the algebraic groups with F-points F^*, F, and the nth roots of unity in F, and G° will always denote the identity component of an algebraic group G. For a variety X we write $X(F)$ for its F-points.

Our notation for quadratic forms will follow the standard reference [Lam73], with two quirks: We use the Pfister-approved notation for Pfister forms, so $\ll a_1, \dots, a_n \gg := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, and we write $\mathcal{H}$ for the hyperbolic plane $\langle 1, -1\rangle$.

The standard reference for Galois cohomology is [Ser94, §I.5], and for algebras with involution (including the groups $\mathrm{Spin}(A,\sigma)$, $O(A,\sigma)$, and $SO(A,\sigma)$) it is [KMRT98].

1. Quasi-split groups of type A, B, and D

As indicated in the introduction, the Rost invariant "should" have trivial kernel for quasi-split groups of small rank. To prove this for E_6, we will need a result on groups of type D, which also easily settles this question for groups of type A and B. (For the results in this section, our global hypothesis that our fields have characteristic $\neq 3$ is not required; we need only assume characteristic $\neq 2$.) For q a nondegenerate quadratic form over F, there is a short exact sequence of algebraic groups

$$1 \longrightarrow C \longrightarrow \mathrm{Spin}(q) \longrightarrow SO(q) \longrightarrow 1 \tag{1.1}$$

with C isomorphic to $\boldsymbol{\mu}_2$.

Lemma 1.2. *For q a d-dimensional nondegenerate quadratic form with anisotropic part of dimension d_{an} such that $d \geq 5$ and $d + d_{\mathrm{an}} < 16$, the kernel of the Rost invariant of $\mathrm{Spin}(q)$ is precisely the image of $H^1(F, C)$ in $H^1(F, \mathrm{Spin}(q))$.*

The hypothesis $d \geq 5$ ensures that $\mathrm{Spin}(q)$ is simple and simply connected, so that it makes sense to speak of the Rost invariant $R_{\mathrm{Spin}(q)}$.

Proof. The set $H^1(F, SO(q))$ classifies quadratic forms of the same dimension and discriminant as q [KMRT98, 29.29]. For $\alpha \in H^1(F, \mathrm{Spin}(q))$ we set q_α to be the quadratic form corresponding to the image of α in $H^1(F, SO(q))$. Then $q_\alpha - q$ is not only even-dimensional with trivial discriminant (i.e., $q_\alpha - q \in I^2F$), but since q_α comes from $H^1(F, \mathrm{Spin}(q))$, it has the same Clifford invariant as q [KMRT98, 31.11] and so $q_\alpha - q \in I^3F$ by Merkurjev's Theorem. As described in [KMRT98, p. 437], the Rost invariant of α is the Arason invariant $e_3(q_\alpha - q) \in H^3(F, \mathbb{Z}/2)$. (Since $\mathbb{Z}/2 = \boldsymbol{\mu}_2^{\otimes 2}$, we can consider $\mathbb{Z}/2$ to be a subgroup of $\mathbb{Q}/\mathbb{Z}(2)$ and hence $H^3(F, \mathbb{Z}/2)$ is a subgroup of $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$.)

Suppose first that α is in the image of $H^1(F, C)$. Sequence (1.1) induces an exact sequence

$$SO(q)(F) \longrightarrow H^1(F, C) \longrightarrow H^1(F, \mathrm{Spin}(q)) \longrightarrow H^1(F, SO(q)), \tag{1.3}$$

and since the Rost invariant $R_{\mathrm{Spin}(q)}$ "factors through" $H^1(F, SO(q))$, certainly $R_{\mathrm{Spin}(q)}(\alpha)$ is trivial.

Conversely, suppose that α is in the kernel of the Rost invariant. Then $e_3(q_\alpha - q)$ is trivial, but as mentioned in the introduction the kernel of e_3 is precisely I^4F. Since $\dim q_\alpha = \dim q = d$, the hypotheses on q ensure that the dimension of the anisotropic part of $q_\alpha - q$ is strictly less than 16. Since $q_\alpha - q \in I^4F$, it is hyperbolic by the Arason–Pfister Hauptsatz [Lam73, X.3.1]. Thus q_α is isomorphic to q and α is in the kernel of the map $H^1(F, \mathrm{Spin}(q)) \to H^1(F, SO(q))$, which is just the image of $H^1(F, C)$. □

The first map in (1.3) is the spinor norm, which immediately produces the following lemma.

Corollary 1.4. *Suppose that q is as in Lemma 1.2. Then the kernel of the Rost invariant is isomorphic to $F^*/SN(q)F^{*2}$, where $SN(q)$ is the image of the spinor norm map $SO(q)(F) \to F^*/F^{*2}$.* □

1.5. Quasi-split simply connected groups of type B_n are actually split, so of the form $\mathrm{Spin}(q)$ for $q = n\mathcal{H} \perp \langle 1 \rangle$. In terms of the lemma, $d = 2n + 1$ and $d_{\mathrm{an}} = 1$. So q satisfies the hypotheses for $2 \leq n \leq 6$. Since q is isotropic, it has surjective spinor norm, so the Rost invariant for a split group of type B_n has trivial kernel for $2 \leq n \leq 6$.

Example 1.6 (B_7)**.** As just mentioned, the split simply connected group of type B_7 is isomorphic to $\mathrm{Spin}(q)$ for $q = 7\mathcal{H} \perp \langle 1 \rangle$. The Rost invariant $R_{\mathrm{Spin}(q)}$ can have nontrivial kernel. Sequence (1.1) induces an exact sequence

$$H^1(F, \mathrm{Spin}(q)) \longrightarrow H^1(F, SO(q)) \xrightarrow{\ \partial\ } H^2(F, \boldsymbol{\mu}_2) \tag{1.7}$$

where the set $H^1(F, SO(q))$ classifies nondegenerate quadratic forms with the same dimension (15) and discriminant ($1 \cdot F^{*2}$) as q.

Fix a base field F and a nonhyperbolic 4-fold Pfister form φ over F (e.g. $F = \mathbb{R}$,

$\varphi = \langle\langle -1,-1,-1,-1 \rangle\rangle$). Set $q_\alpha = -\varphi'$ for φ' such that $\varphi = \langle 1 \rangle \perp \varphi'$. Then $\operatorname{disc} q_\alpha = (-1)^{\binom{15}{2}} \det(-\varphi') = 1 \cdot F^{*2}$, so there is a unique element of $H^1(F, SO(q))$ corresponding to q_α. The image of q_α under the connecting homomorphism ∂ is $[C_0(q_\alpha - q)]$, which by [Lam73, V.2.10] is the same as $[C(q_\alpha - q)]$ which is trivial since $q_\alpha - q = -\varphi \in I^3 F$. Thus q_α is the image of some α in $H^1(F, \mathrm{Spin}(q))$. But then $R_{\mathrm{Spin}(q)}(\alpha) = e_3(q_\alpha - q) = e_3(-\varphi)$, which is trivial since $\varphi \in I^4 F$.

1.8. An analysis for groups of type D_n similar to the one in 1.5 for B_n shows that the Rost invariant for a simply connected group is trivial for groups of type ${}^1D_n$ with $3 \le n \le 7$ and for groups of type ${}^2D_n$ with $3 \le n \le 6$. As in the B case, we show that one of these bounds is sharp.

Example 1.9 (${}^1D_8$)**.** The situation here is quite similar to the one in Example 1.6, except that $q = 8\mathcal{H}$. Use the same base field F and nonsplit 4-fold Pfister form φ from before. There is a unique element of $H^1(F, SO(q))$ corresponding to φ and since $\varphi = \varphi - q \in I^4 F$, the same reasoning shows that there is a nontrivial class in $H^1(F, \mathrm{Spin}(q))$ which is the inverse image of φ and which has trivial Rost invariant.

Lemma 1.2 easily deals with quasi-split groups of type ${}^2A_n$ of low rank.

Corollary 1.10. *If G is a quasi-split simply connected group of type ${}^2A_n$ with $n \le 5$, the kernel of the Rost invariant R_G is trivial.*

Proof. Set K to be the quadratic field extension of F which splits G and take (V, h^d) to be a "maximally split" $(n+1)$-dimensional hermitian form over K. (See below for a more explicit description.) Then G is $SU(V, h^d)$, the algebraic group with F-points

$$SU(V, h^d)(F) = \left\{ g \in GL(V)(K) \;\middle|\; \begin{array}{c} h(gv, gv') = h(v, v') \\ \text{for all } v, v' \in V \text{ and} \\ \det g = 1 \end{array} \right\}.$$

The *trace form* of h^d is defined to be the quadratic form q^d on V considered as a $2(n+1)$-dimensional vector space over F given by $q^d(v) = h^d(v, v)$. Then

$$h^d = \begin{cases} m\mathcal{H} & \text{if } n+1 = 2m, \\ m\mathcal{H} \perp \langle 1 \rangle & \text{if } n+1 = 2m+1 \end{cases} \quad \text{and } q^d = \begin{cases} 2m\mathcal{H} & \text{if } n+1 = 2m, \\ 2m\mathcal{H} \perp \langle\langle d \rangle\rangle & \text{if } n+1 = 2m+1, \end{cases}$$

where $K = F(\sqrt{d})$ if $n = 2m$ for some integer m, and the $\mathcal{H}$ occurring in the description of h^d is the usual unitary hyperbolic plane as described in [Sch85, 7.7.3].

The set $H^1(F, G)$ classifies nonsingular hermitian forms h on V which have the same dimension and discriminant as h^d [KMRT98, p. 403]. The group G

embeds in $SO(V, q^d)$ in an obvious manner. The corresponding map $H^1(F, G) \to H^1(F, SO(V, q^d))$ sends h to its trace form q, and this map is an injection by [Sch85, 10.1.1(ii)]. Moreover, the Rost invariant $R_G(h)$ is just $e_3(q - q^d)$ by [KMRT98, 31.44]. Since $\dim q^d = 2n + 2 < 13$ and the anisotropic part of q^d has dimension 0 (if $n+1$ is even) and 2 (if $n+1$ is odd), as in the proof of Lemma 1.2, if $R_G(h)$ is trivial, $q \cong q^d$ and so $h \cong h^d$. □

Example 1.11 (${}^2A_6$)**.** Take $F = \mathbb{R}$, $K = \mathbb{C}$, and consider $G = SU(V, h^d)$ for h^d the hermitian form $3\mathcal{H} \perp \langle 1 \rangle$ over K, so that G is simply connected quasi-split of type ${}^2A_6$. Then the hermitian form $h = \langle -1, -1, -1, -1, -1, -1, -1 \rangle$ has trace form $q = -7\langle\langle -1 \rangle\rangle$ which is not hyperbolic, so h corresponds to a (unique) nontrivial class in $H^1(F, G)$. However,

$$q - q^d = -7\langle\langle -1 \rangle\rangle - \langle\langle -1 \rangle\rangle = -\langle\langle -1, -1, -1, -1 \rangle\rangle \in I^4F,$$

so $R_G(h)$ is trivial.

2. Folded root systems

2.1. The Rost multiplier. A *loop* in an arbitrary algebraic group G is a homomorphism $\mathbb{G}_m \to G$. Let G_* be the set of loops in G. As in [KMRT98, p. 432], we set $Q(G)$ to be the abelian group of all integer-valued functions on G_* such that

(1) for ${}^g f$ the loop given by ${}^g f(x) = g f(x) g^{-1}$, $q({}^g f) = q(f)$ for all $g \in G$ and $f \in G_*$; and
(2) for any two loops f and h with commuting images, the function $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $(k, m) \mapsto q(f^k h^m)$ is a quadratic form.

When G is a simple group, $Q(G)$ is cyclic with a canonical generator which is positive definite [KMRT98, 31.27], hence is identified with $\mathbb{Z}$. Now suppose that we have two simple simply connected groups $H \subset G$. The inclusion gives a map $H_* \to G_*$, so we in turn have a map $\mathbb{Z} = Q(G) \to Q(H) = \mathbb{Z}$. Because the canonical generators are positive definite, this map must be multiplication by a positive integer n, which we define to be the *Rost multiplier* of the inclusion.

The naturality of the Rost invariant implies that we have a commutative diagram

$$\begin{array}{ccc} H^1(F, H) & \xrightarrow[R_H]{} & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\ \big\downarrow & & \big\downarrow n\cdot \\ H^1(F, G) & \xrightarrow{R_G} & H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \end{array}$$

where n is the Rost multiplier of the inclusion [KMRT98, 31.34]. This is the motivation for our study of this invariant.

2.2. Luckily, it can be quite easy to compute such a "Rost multiplier". Suppose that G and H are split and contain split maximal tori S and T respectively such that the T lies in S. Since G and H are simply connected, the character groups $X(T)$ and $X(S)$ are identified with the weight lattices, but the character groups are dual to the loop groups S_* and T_* [Bor91, 8.6] and the weight lattices are dual to the lattices generated by the coroots, which we denote by $\Lambda_{c,G}$ and $\Lambda_{c,H}$, respectively. (By a *coroot*, we mean the roots of the dual root system, which are denoted by $\check{\alpha}$ in [Bou68, VI.1] for α a root.) Putting these dualities together, we obtain identifications $S_* = \Lambda_{c,G}$ and $T_* = \Lambda_{c,H}$, so the inclusion $T \subset S$ induces a map $\Lambda_{c,H} \to \Lambda_{c,G}$. Now the dual root systems (whose roots are the coroots) are indeed root systems [Bou68, VI.1.1, Prop. 2] and so they each have a unique minimal Weyl-group invariant positive-definite integer-valued quadratic form [Bou68, VI.1.2, Prop. 7], say q and r (for the forms for G and H respectively). Hence q induces such a form on $\Lambda_{c,H}$, which must be of the form nr for some natural number n. This n is the Rost multiplier of the inclusion.

Criterion (2) in the definition of $Q(G)$ implies that its canonical generator is identified with the positive-definite Weyl-group invariant quadratic form on the dual root system which takes the value 1 on short coroots. (Short roots correspond to long roots, where we adopt the convention that short = long in the event that all roots have the same length. In that case, the quadratic form is very easy to identify, in that its Gram matrix is simply the Cartan matrix of the root system with all entries divided by 2.) So one can simply compute the image of a short coroot from H in the dual root system for G to find the Rost multiplier of the inclusion.

Example 2.3 ($SL_n \to SL_{2n}$)**.** The block diagonal embedding $SL_n \hookrightarrow SL_{2n}$ via $x \mapsto \left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right)$ has Rost multiplier 2. The map given by $x \mapsto \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right)$ has Rost multiplier 1.

Example 2.4 (Folding)**.** The split simply connected group of type E_6 can be realized as the group $\mathrm{Inv}\,(J)$ of invertible linear maps of the split Albert algebra J which preserve the cubic norm form. The algebra J has a nondegenerate symmetric bilinear trace form T given by setting $T(x, y)$ to be the trace of the product $x \cdot y$ [Jac68, p. 240, Thm. 5], and for $\varphi \in \mathrm{Inv}\,(J)(F)$ we define $\varphi^\dagger \in GL(J)(F)$ to be the unique map satisfying $T(\varphi(j), \varphi^\dagger(j')) = T(j, j')$ for all $j, j' \in J$. This defines an outer automorphism of $E_6 = \mathrm{Inv}\,(J)$ [Jac61, p. 76, Prop. 3] and the subgroup of elements fixed by this automorphism is the split group F_4 of F-algebra automorphisms of J.

We would like to compute the Rost multiplier of the inclusion $F_4 \subset E_6$. Fix an F-split maximal torus S in $G := E_6$ which is preserved by the automorphism (such as the one denoted by "S_6" in [Gar01b, pf. of 7.2]) and fix a set of simple roots Δ of G with respect to S. We would like our outer automorphism to leave Δ invariant, although it probably does not do so. However, two things are apparent from the definition of the Rost multiplier: it is not changed by scalar extension nor by modifying the automorphism $\varphi \mapsto \varphi^\dagger$ by an inner automorphism of E_6. So we may assume that the base field is separably closed and so that the F-points of the Weyl group of G with respect to S (i.e., the F-points of $N_G(S)/S$) is the full Weyl group of the root system of G with respect to S. Then we may modify our outer automorphism by an element of the Weyl group so that F_4 is described as the subgroup of E_6 fixed by the automorphism f induced by the automorphism of Δ which is given by the unique nontrivial automorphism of the Dynkin diagram. That is, we set $H := F_4 = G^f$ (= the subgroup of G of elements fixed by f), and $T := (S^f)^\circ$ (= the identity component of $T \cap G^f$) is a maximal torus in H. The restrictions of elements of Δ to T give a root system of H with respect to T [Sch69, p. 108] and the fibers of this restriction map are the orbits of f in Δ [Sch69, 3.5].

Now $\Lambda_{c,G}$ is a free $\mathbb{Z}$-module with basis $\check{\Delta} = \{\check{\delta} \mid \delta \in \Delta\}$ which is permuted by f and $\Lambda_{c,H}$ is the fixed sublattice. So $\Lambda_{c,H}$ has a basis consisting of one element for each orbit of f in $\check{\Delta}$, and this element is given by the sum of the elements in the orbit in $\check{\Delta}$. There is a coroot $\check{\delta} \in \check{\Delta}$ which is fixed by f, hence $\check{\delta}$ is a member of the $\mathbb{Z}$-basis for $\Lambda_{c,H}$. The form q on $\Lambda_{c,G}$ restricts to a positive-definite Weyl-invariant form on $\Lambda_{c,H}$ such that $q(\check{\delta}) = 1$, consequently q restricts to be the minimal such form r. By the discussion in 2.2 the Rost multiplier of the inclusion $F_4 \subset E_6$ is 1.

Remark 2.5. Presumably this same argument also works in the other instances where one obtains a root system by "folding up" another root system all of whose roots have the same length, i.e., $C_{\ell+1} \subset A_{2\ell+1}$, $B_{n-1} \subset D_n$, and $G_2 \subset D_4$. The other root system consisting of roots of the same length, $A_{2\ell}$, folds up to give the smaller root system BC_ℓ, see [Hec84, Table I].

3. Small representations

We say a representation V of an algebraic group G is *small* if G has an open orbit in $\mathbb{P}(V)$. We are interested in small representations in the case where G is simple, which have all been classified as a consequence of the (more general) classification of prehomogeneous vector spaces, see [Kim88] for a survey. These small representations also provide "standard relative sections" in the language of [Pop94, 1.7], and in that sense were classified in [Èla72, Table 1]. Our motivation for studying these representations comes from the following easy lemma, which was pointed out to me by Rost.

Lemma 3.1. *Suppose that G is an algebraic group over a field F such that G has a small representation V, and that F is infinite or G is connected. Let H be a subgroup of G consisting of the elements which stabilize some F-point in the open* *orbit in $\mathbb{P}(V)$. Then the natural map*

$$H^1(F,H) \to H^1(F,G)$$

is surjective.

Proof. If the base field F is finite, then by hypothesis G is connected, and by Lang's Theorem $H^1(F,G)$ is trivial so the lemma holds. So we may assume that F is infinite.

Fix a 1-cocycle $z \in Z^1(F,G)$. It defines a twisted version $\mathbb{P}(V)_z$ of $\mathbb{P}(V)$ which is the same as $\mathbb{P}(V)$ over the separable closure F_{sep} of F but has a different Galois action: For $w \in \mathbb{P}(V)_z(F_{\text{sep}})$ and $\sigma \in \text{Gal}(F_{\text{sep}}/F)$, σ acts by

$$\sigma * w = z_\sigma \sigma w$$

where juxtaposition denotes the usual action. The twisted version U_z of U, defined analogously, is an open subset of $\mathbb{P}(V)_z$.

Since the representation gives a map $G \to GL(V)$, $\mathbb{P}(V)_z$ is F-isomorphic to $\mathbb{P}(V)$. In particular, since F is infinite, $\mathbb{P}(V)_z(F)$ is dense in $\mathbb{P}(V)_z(F_{\text{sep}})$. Since $U_z(F_{\text{sep}})$ is open in $\mathbb{P}(V)_z(F_{\text{sep}})$, the two sets $U_z(F_{\text{sep}})$ and $\mathbb{P}(V)_z(F)$ must meet nontrivially, i.e., U_z has some F-point which we will denote by x_z.

Now let $x \in U(F)$ be the point with stabilizer subgroup H and fix some $g \in G(F_{\text{sep}})$ such that $gx = x_z$. Then for all $\sigma \in \text{Gal}(F_{\text{sep}}/F)$, the element $g^{-1}z_\sigma(\sigma g)$ fixes x and so lies in $H(F_{\text{sep}})$. Thus z is cohomologous to something in the image of $Z^1(F,H)$. □

Example 3.2 ($O_{n-1} \subset O_n$)**.** Write O_n for the orthogonal group of the dot product on F^n. Then the subgroup of O_n which stabilizes $[v] \in \mathbb{P}(F^n)$ where v has nonzero length is just $O_{n-1} \times \boldsymbol{\mu}_2$, where O_{n-1} is the orthogonal group for the $(n-1)$-dimensional space of vectors in F^n which are orthogonal to v. Iterating this process recovers the fact that all nondegenerate quadratic forms are diagonalizable, a.k.a. the Spectral Theorem.

Example 3.3 (Spin_n [Igu70], [GV78], [Pop80])**.** For Spin_n the spin group for an n-dimensional maximally split quadratic form, the spin representation (if n is odd) or the half-spin representation (if n is even) is small for $n \leq 12$ and $n = 14$. In the $n = 14$ case, the stabilizer subgroup is isomorphic to $(G_2 \times G_2) \rtimes \boldsymbol{\mu}_8$, and this leads to structural statements about 14-dimensional forms in I^3F, see [Ros99].

Example 3.4 ($F_4 \times \boldsymbol{\mu}_3 \subset E_6$)**.** We write E_6 for the split group of type E_6 which can be realized as $\text{Inv}\,(J)$ as described in 2.4. By [Jac61, p. 71, Thm. 7], E_6 acts transitively on the subset of J consisting of elements of norm 1, so certainly this is a small representation.

Take H to be the subgroup of E_6 consisting of elements which fix the identity element 1_J of J projectively. Since the norm form is cubic, $\boldsymbol{\mu}_3$ is contained in H and is central (since it consists of scalar endomorphisms), and any element of H differs by an element of $\boldsymbol{\mu}_3$ from something which fixes 1_J absolutely. This subgroup of elements fixing 1_J is well-known — it is the automorphism group F_4 of J [Jac59, p. 186, Thm. 4], which is split of type F_4. So H is isomorphic to $F_4 \times \boldsymbol{\mu}_3$, and the resulting surjective map $H^1(F, F_4 \times \boldsymbol{\mu}_3) \to H^1(F, E_6)$ is the statement that $H^1(F, E_6)$ classifies cubic forms of the form λN for N the norm form on some Albert F-algebra and $\lambda \in F^*$, see [Spr62]. This can also be interpreted in terms of structurable algebras, see [Gar01b, 2.8(1)].

Example 3.5 ($E_6 \rtimes \boldsymbol{\mu}_4 \subset E_7$)**.** Write E_7 for the split simply connected group of type E_7 over F. It is the group of vector space automorphisms of $V = \left(\begin{smallmatrix} F & J \\ J & F \end{smallmatrix}\right)$ which preserve a quartic form q as given in [Bro69, p. 87]. Then E_7 acts transitively on the open subset of $\mathbb{P}(V)$ consisting of points $[v]$ such that $q(v) \neq 0$ by [Fer72, 7.7].

We set H to be the subgroup of E_7 which stabilizes the vector $v = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ projectively. This vector has $q(v) \neq 0$, and so by [Fer72, 3.7] there are two uniquely determined (up to scalar multiples) "strictly regular" elements e_1 and e_2 such that v lies in their span. These are $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Since E_7 preserves the property of being strictly regular, every element of H must projectively stabilize e_1 and e_2 as well, and perhaps interchange them.

Now, the map ω defined by

$$\omega \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} i\beta & ij' \\ ij & i\alpha \end{pmatrix}$$

lies in H, where i is some fixed square root of -1 in the separable closure of F. We would like to describe an arbitrary $h \in H$, which after modification by ω we may assume projectively stabilizes each of e_1 and e_2. Then by [Bro69, p. 96, Lem. 12], h must be of the form

$$h \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \mu^{-1}\alpha & \varphi(j) \\ \varphi^\dagger(j') & \mu\beta \end{pmatrix}$$

where φ is a similarity of the norm form on J with multiplier μ and $\varphi^\dagger$ is as defined in 2.4. Since h also stabilizes v, we must have that $\mu = \pm 1$. In particular, after modifying h by $\omega^2 = -1$, we may assume that h has the form

$$h \begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \varphi(j) \\ \varphi^\dagger(j') & \beta \end{pmatrix}$$

where φ preserves the cubic norm on J and so lies in E_6. We have shown that H is isomorphic to $E_6 \rtimes \boldsymbol{\mu}_4$.

The surjection on Galois cohomology coming from this example will be more useful if we can replace $E_6 \rtimes \boldsymbol{\mu}_4$ with a simple group. For K a quadratic étale F-algebra, we write E_6^K for the simply connected quasi-split group of type E_6 over F which is split by an extension L of F if and only if $L \otimes_F K \cong L \times L$.

Proposition 3.6 (Cf. [Gar01b, 4.14])**.** *For each $\alpha \in H^1(F, E_7)$ there is some quadratic étale F-algebra K such that E_6^K embeds in E_7 with Rost multiplier* 1 *and α is in the image of the induced map $H^1(F, E_6^K) \to H^1(F, E_7)$.*

Proof. Fix some $a \in Z^1(F, E_6 \rtimes \boldsymbol{\mu}_4)$ representing α. The natural projection $E_6 \rtimes \boldsymbol{\mu}_4 \to \boldsymbol{\mu}_4$ has an obvious section given by sending $i \mapsto \omega$; set b to be the image of a given by the map induced by the composition $E_6 \rtimes \boldsymbol{\mu}_4 \to \boldsymbol{\mu}_4 \to E_6 \rtimes \boldsymbol{\mu}_4$. Twist $E_6 \rtimes \boldsymbol{\mu}_4$ by b to obtain a new group $(E_6 \rtimes \boldsymbol{\mu}_4)_b$, with a twisted Galois action $*$ so that

$$\sigma * g = b_\sigma ({}^\sigma g) b_\sigma^{-1},$$

where ${}^\sigma g$ denotes the usual action. There is an isomorphism

$$H^1(F, (E_6 \rtimes \boldsymbol{\mu}_4)_b) \xrightarrow[\tau_b]{\sim} H^1(F, E_6 \rtimes \boldsymbol{\mu}_4)$$

where $\tau_b^{-1}(\alpha)$ is the class of a 1-cocycle given by $\sigma \mapsto a_\sigma b_\sigma^{-1}$ with values in the identity component of the twisted group $(E_6 \rtimes \boldsymbol{\mu}_4)_b$. This identity component is just E_6 twisted by b, and we would like to show that it is isomorphic to E_6^K for some quadratic étale F-algebra K. If σ in $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ has $b_\sigma = \pm 1$, then σ acts in the usual manner upon the twisted E_6. On the other hand, if $b_\sigma = \pm\omega$, then the twisted action is given by

$$(\sigma * h)\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = (\pm\omega)\sigma h \sigma^{-1}(\pm\omega)^{-1}\begin{pmatrix} \alpha & j \\ j' & \beta \end{pmatrix} = \begin{pmatrix} \alpha & \sigma\varphi^\dagger\sigma^{-1}(j') \\ \sigma\varphi\sigma^{-1}(j) & \beta \end{pmatrix}.$$

This is precisely the description of the Galois action on E_6^K given in [Gar01b, 2.4] for K determined by the image of b under the composition $H^1(F, E_6 \rtimes \boldsymbol{\mu}_4) \to H^1(F, \boldsymbol{\mu}_4) \to H^1(F, \boldsymbol{\mu}_2) = F^*/F^{*2}$, so $(E_6)_b$ is isomorphic to E_6^K. To see that E_6^K embeds in E_7, we observe that the 1-cocycle b is trivial in $H^1(F, E_7)$ by [Gar01b, 4.10, 5.10], so we have a map

$$E_6^K \subset (E_6 \rtimes \boldsymbol{\mu}_4)_b \hookrightarrow (E_7)_b \xrightarrow[f]{\sim} E_7$$

where (by a simple computation having nothing to do with E_7) $H^1(f) = \tau_b$. This proves the proposition aside from the claim about the Rost multiplier.

But that claim is easy in the split case (where $K = F \times F$), since the embedding of E_6 in E_7 comes from the obvious embedding of root systems. Since the Rost multiplier is invariant under scalar extension, the embeddings of quasi-split groups of type E_6^K in E_7 given above all have Rost multiplier 1 as well. $\square$

4. ${}^1D_4 \subset {}^2E_6$

For the remainder of the paper we will study the quasi-split group E_6^K of type ${}^2E_6$ defined in 3.5. In this section we introduce a particular subgroup G of E_6^K which is reductive of semisimple type ${}^1D_4$. Defining G will necessitate digging more deeply in to the structure of Cayley and Albert algebras.

Definition 4.1. Fix $\mathfrak{C}$ to be the split Cayley algebra endowed with hyperbolic norm form $\mathfrak{n}$ and canonical involution $\bar{\ }$. (For more information about Cayley algebras, see [KMRT98, §33.C] or [Sch66, Ch. III, §4].) If $t \in GL(\mathfrak{C})(F)$ satisfies $\mathfrak{n}(t(c)) = m\mathfrak{n}(c)$ for some $m \in F^*$ and all $c \in \mathfrak{C}$, we say that m is a *similarity* of $\mathfrak{n}$ with multiplier $\mu(t) := m$. (Note that if $\sigma_\mathfrak{n}$ is the involution on $\mathrm{End}_F(\mathfrak{C})$ which is adjoint for $\mathfrak{n}$ so that $\mathfrak{n}(tc, c') = \mathfrak{n}(c, \sigma_\mathfrak{n}(t)c')$ for all $c, c' \in \mathfrak{C}$, then $\mu(t) = \sigma_\mathfrak{n}(t)t$.) Set $GO^\circ(\mathfrak{C}, \mathfrak{n})$ to be the algebraic group with F-points

$$GO^\circ(\mathfrak{C}, \mathfrak{n})(F) := \left\{ t \in GL(\mathfrak{C})(F) \,\middle|\, \begin{array}{l} t \text{ is a similarity of } \mathfrak{n} \text{ with multiplier} \\ \mu(t) \text{ such that } \det(t) = \mu(t)^4 \end{array} \right\}.$$

We can also define a new, seemingly uglier multiplication $\star$ on $\mathfrak{C}$ by setting $x \star y := \bar{x}\bar{y}$ as in [KMRT98, §34.A]. A *related triple* is a triple (t_0, t_1, t_2) in $GO^\circ(\mathfrak{C}, \mathfrak{n})^{\times 3}$ such that

$$\mu(t_i)^{-1} t_i(x \star y) = t_{i+2}(x) \star t_{i+1}(y)$$

for all $x, y \in \mathfrak{C}$ and $i = 0, 1, 2$ with subscripts taken modulo 3. Write $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ for the algebraic subgroup of $GO^\circ(\mathfrak{C}, \mathfrak{n})^{\times 3}$ consisting of related triples and $\mathrm{Spin}(\mathfrak{n})$ for the subgroup of $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ consisting of triples with multiplier one (i.e., those triples such that $\mu(t_i) = 1$ for all i).

4.2. The vector space underlying the split Albert F-algebra J is the subspace of $M_3(\mathfrak{C})$ consisting of elements fixed by the conjugate transpose $*$ which applies $\bar{\ }$ to each entry and takes the transpose. It is the algebra denoted by $\mathfrak{H}(\mathfrak{C}_3)$ in the notation of [Jac68, §I.5] and has multiplication $a \cdot b := (ab + ba)/2$, where juxtaposition denotes the usual multiplication on $M_3(\mathfrak{C})$. When writing down explicit elements of J, we will use a "$\cdot$" to indicate entries whose values are forced by this symmetry condition. The reductive group $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ embeds in the group $\mathrm{Inv}\,(J)$ of norm isometries of J via the map $\underline{t} \mapsto g_{\underline{t}}$ given by

$$g_{\underline{t}} \begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \mu(t_0)^{-1}\varepsilon_0 & t_2(c_2) & \cdot \\ \cdot & \mu(t_1)^{-1}\varepsilon_1 & t_0(c_0) \\ t_1(c_1) & \cdot & \mu(t_2)^{-1}\varepsilon_2 \end{pmatrix}. \tag{4.3}$$

Let e_i denote the element of J whose only nonzero entry is a 1 in the $(i{+}1, i{+}1)$-position. Any element of $\mathrm{Inv}\,(J)(K)$ which fixes e_1, e_2, and e_3 is of the form $g_{\underline{t}}$ for some $\underline{t} \in \mathrm{Spin}(\mathfrak{n})$ by [Sod66, p. 155, Thm. 1]. This implies that every element of $\mathrm{Inv}\,(J)(F)$ which leaves each of subspaces Fe_i invariant is in the image of $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$.

4.4. Definition of G. Since $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ embeds in $\mathrm{Inv}\,(J)$ over F, it embeds in E_6^K over K. However, we can identify E_6^K with $\mathrm{Inv}\,(J)$ with a different ι-action where ${}^\iota f := \iota f^\dagger \iota$, where ι is the nontrivial F-automorphism of K and juxtaposition denotes the usual action; we fix this identification for the rest of the paper. The map $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n}) \to E_6^K$ is not defined over F: For $\underline{t} = (t_0, t_1, t_2) \in \mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})(K)$ and $g_{\underline{t}} \in E_6^K$, we have ${}^\iota g_{\underline{t}} = g_{\iota\sigma_\mathfrak{n}(\underline{t})^{-1}\iota}$ which is typically not the same as $g_{\iota\underline{t}\iota}$ where $\sigma_\mathfrak{n}(\underline{t})$ means to apply $\sigma_\mathfrak{n}$ to each component of $\underline{t}$. Define G to be the algebraic group over F which is the same as $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ over K but with a different ι-action:

for $\underline{t} \in G(K)$, set ${}^\iota\underline{t} := \iota\sigma_{\mathfrak{n}}(\underline{t})^{-1}\iota$. Then G injects into E_6^K over F via the map g from (4.3).

This group G is reductive with absolute rank 6 and semisimple part $\mathrm{Spin}(\mathfrak{n})$ of type ${}^1D_4$.

4.5. The center P of G. Set N_1 to be the algebraic group with F-points the elements of K^* with norm 1 in F. This group is the same as $\mathbb{G}_m$ over K, but has a different ι-action given by ${}^\iota\lambda = \iota(\lambda)^{-1}$. It is sometimes denoted by $R^{(1)}_{K/F}(\mathbb{G}_{m,K})$.

The center of $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ is the subgroup of $\mathbb{G}_m^{\times 3}$ consisting of triples whose product is one. But we are concerned with G, which has a different ι-action; its center P is isomorphic to the subgroup of $N_1^{\times 3}$ consisting of triples whose product is 1. This rank 2 torus is F-anisotropic and K-split.

The importance of G is given by the following lemma, excavated from a paper by Ferrar:

Ferrar's Lemma 4.6. [Fer69, p. 65, Lem. 3] *The natural map $H^1(K/F, G) \to H^1(K/F, E_6^K)$ is surjective.*

Comments. Ferrar proved this by explicit computations in the Jordan algebra. However, this can also be seen with more algebraic group-theoretic methods, as was pointed out to me by Gille. We must assume that our base field has characteristic 0, which as was observed in the introduction does not harm our main results in any way.

The group Spin_8 is split simply connected of type D_4 and so contains a subgroup which is isogenous to $SL_2^{\times 4}$. Each copy of the group SL_2 contains a rank 1 torus which is anisotropic over F and split over K, and we set T_4 to be the image in Spin_8 of these four tori. Let T be the subtorus of G generated by T_4 and the center P. It is a rank 6 F-anisotropic torus which is split over K. Let B be a Borel subgroup of E_6^K defined over K and containing T. Then ${}^\iota B \cap B = T$, so by [PR94, p. 369, Lem. 6.28] the natural map $H^1(K/F, T) \to H^1(K/F, E_6^K)$ is a surjection. □

Now imagine how the argument for proving the main theorem in the ${}^2E_6$ case must proceed: We apply some simple argumentation and Ferrar's Lemma to show that any class in $H^1(F, E_6^K)$ with trivial Rost invariant must come from $H^1(K/F, G)$. Then we apply some facts about Rost invariants on this smaller group to obtain the theorem. However, G is reductive, so we want to put our class with trivial Rost invariant into a simple subgroup if we hope to apply our results from Section 1. This requires further study of the center of G.

4.7. The group $H^1(K/F, P)$. There is a short exact sequence over K

$$1 \longrightarrow P \longrightarrow N_1^{\times 3} \xrightarrow{\pi} N_1 \longrightarrow 1,$$

where π is the product map, which induces an exact sequence

$$H^1(K/F,P) \longrightarrow H^1(K/F,N_1^{\times 3}) \xrightarrow{H^1(\pi)} H^1(K/F,N_1).$$

The first map is an injection because the product map π is a surjection on F-points. Any 1-cocycle in $Z^1(K/F,N_1)$ is determined by its value at ι, and the condition that it is a 1-cocycle forces that this value lies in F^*. The obvious check shows that two such are cohomologous if and only if they differ by a norm from K^*. So $H^1(K/F,P)$ is isomorphic to the subgroup of $(F^*/N_{K/F}(K^*))^{\times 3}$ consisting of elements with product in $N_{K/F}(K^*)$.

4.8. The map $H^1(K/F,G) \to H^1(K/F,P)$. There is a short exact sequence

$$1 \longrightarrow \mathrm{Spin}(\mathfrak{n}) \longrightarrow G \longrightarrow P \longrightarrow 1$$

where the map $G \to P$ is given by sending each t_i to its multiplier $\mu(t_i) = \sigma_{\mathfrak{n}}(t_i)t_i \in N_1$. This sequence is even exact over K (instead of just over a separable closure of F) because the map $G \to P$ is surjective over K by [KMRT98, 35.4]. A 1-cocycle $\gamma \in Z^1(K/F,G)$ is determined by its value γ_ι at ι, and the image of γ in $H^1(K/F,P)$ is the multiplier of γ_ι.

A natural question is the following: Any 1-cocycle in $Z^1(K/F,E_6^K)$ comes from $H^1(K/F,G)$ by Ferrar's Lemma and so has an image in $H^1(K/F,P)$. Is that image an invariant of the original class in $H^1(K/F,E_6^K)$? The answer is no, as is shown in the following lemma. (Explicit situations where the hypotheses are satisfied nontrivially will be given in 6.6 and 7.10.)

Moving Lemma 4.9. *Let η be a 1-cocycle in $Z^1(K/F,G)$ whose image in $Z^1(K/F,P)$ takes the value $\underline{a}$ at ι. Suppose that there is some $j \in e_0 \times J_K$ such that*

$$j^\# = 0 \qquad \text{and} \qquad T(j,\eta_\iota \iota j) = r \in F^*.$$

Then η is cohomologous in $H^1(K/F,E_6^K)$ to a 1-cocycle coming from $Z^1(K/F,G)$ whose image in $Z^1(K/F,P)$ takes the value $(r^{-1},a_0,a_0^{-1}r)$ at ι.

The hypotheses in the lemma make use of the Freudenthal cross product $\times : J \times J \to J$, which is a commutative bilinear map defined by the relation $6N(j) = T(j, j \times j)$ for all $j \in J$. The map $\#\colon J \to J$ is defined by $2j^\# := j \times j$.

The proof is an adaptation of an argument in [Fer80, p. 277].

Proof. First observe that the three elements j, e_0, and $e_0 \times j'$ for $j' := \eta_\iota \iota j$ all have "rank one", i.e., are sent to zero by the map $x \mapsto x^\#$.

For x and y in J, we have the identity [McC69, (19)]:

$$x \times (x^\# \times y) = N(x)y + T(x,y)x^\#. \tag{4.10}$$

Setting $x = w + z$, we have $x^\# = w^\# + w \times z + z^\#$. The term of $N(x) = N(w+z)$ which has degree 1 in w and degree 2 in z is $T(w, z^\#)y$. Substituting $x = w + z$

into (4.10) and taking the terms on both sides with this degree, we obtain the identity

$$w \times (z^{\#} \times y) + z \times (y \times (z \times w)) = T(w, z^{\#})y + T(z, y)(w \times z) + T(w, y)z^{\#}. \quad (4.11)$$

Since the vector space $e_0 \times J$ is preserved by ι and $G(K)$, we have $j' = e_0 \times w$ for some w. Applying (4.11), we obtain

$$e_0 \times (e_0 \times j') = T(e_0, e_0)(e_0 \times y) = j'. \quad (4.12)$$

For N trilinearized so that $N(x, x, x) = N(x)$, we have

$$6N(e_0, j, e_0 \times j') = T(e_0 \times j, e_0 \times j') = T(j, e_0 \times (e_0 \times j')) = T(j, j') = r \neq 0.$$

(The triple e_0, j, $e_0 \times j'$ is said to be "in general position".) By [SV68, 3.11], this implies that there exist some $f \in \mathrm{Inv}\,(J)(K)$ and $\rho_i \in K^*$ such that

$$f(j) = \rho_0 e_0, \quad f(e_0) = \rho_1 e_1, \text{ and } f(e_0 \times j') = \rho_2 e_2.$$

Since $\mathfrak{n}$ is hyperbolic, there is some $g = g_{\underline{t}} \in \mathrm{Inv}\,(J)(K)$ such that $g(e_0) = \rho_0^{-1} e_0$, $g(e_1) = \rho_1^{-1} e_1$, and $g(e_2) = \rho_0 \rho_1 e_2$. By replacing f with gf, we may assume that $\rho_0 = \rho_1 = 1$. Moreover, f preserves N, and so $\rho_2 = 6N(e_0, e_1, \rho_2 e_2) = r$.

Set $\eta' \in Z^1(K/F, E_6^K)$ to be the cocycle cohomologous to η given by $\eta'_\iota = f^\dagger \eta_\iota\, {}^\iota(f^\dagger)^{-1}$. It is standard that the maps $\dagger$ and inverse commute on $\mathrm{Inv}\,(J)$, hence ${}^\iota(f^\dagger)^{-1} = {}^\iota(f^{-1})^\dagger = \iota f^{-1} \iota$, where the action of ι on E_6^K is as in 4.4. Thus we have

$$\eta'_\iota = f^\dagger \eta_\iota\, \iota f^{-1} \iota.$$

Keeping in mind the facts that $e_i \times e_{i+1} = e_{i+2}$; $f^\dagger(u \times v) = f(u) \times f(v)$ for all $u, v \in J_K$; equation (4.12); and $j \times (e_0 \times j') = re_0$ (as can be verified by examining the explicit formula for $\times$ given in [Jac68, p. 358, (4)], although the reader should be cautioned that our definition of $\times$ — which agrees with the one in [KMRT98] and [McC69] — differs from Jacobson's by a factor of 2), one can now easily calculate that $f^\dagger(e_0) = e_1$ and $f^\dagger(j') = re_0$. It follows that

$$\eta'_\iota(e_0) = re_0, \quad \textit{and} \quad \eta'_\iota(e_1) = a_0^{-1} e_1.$$

Since η is a 1-cocycle, we have $\eta_\iota \iota(u \times v) = (\iota \eta_\iota^{-1} u) \times (\iota \eta_\iota^{-1} v)$. Thus $\eta_\iota \iota(e_0 \times j') = a_0 e_0 \times j$, and we have

$$\eta'_\iota(e_2) = (a_0/r) e_2.$$

Since η'_ι preserves the linear subspaces Ke_i for all i, it belongs to $G(K)$, and we are done. □

5. ${}^2D_5 \subset {}^2E_6$

For the purpose of making computations, we will need to make use of another subgroup of E_6^K, which we define to be the subgroup consisting of elements h such that h and $h^\dagger$ both fix the element $e_0 \in J$. Since the map $h \to h^\dagger$ is a group homomorphism on $\operatorname{Inv}(J)$, it is clear that H is indeed a subgroup of E_6^K over K, and it is preserved by the ι-action so it is even defined over F. Our first task is to describe it explicitly.

5.1. Fix a particular basis $u_1, u_2, \ldots, u_8$ for the split Cayley algebra $\mathfrak{C}$ as given in [Gar98, p. 388]. One important thing for us to know about this basis is that when we bilinearize the norm form $\mathfrak{n}$ so that $\mathfrak{n}(x, x) = 2\mathfrak{n}(x)$, we have

$$\mathfrak{n}(u_i, u_j) = \begin{cases} 1 & \text{if } i + j = 9 \\ 0 & \text{otherwise,} \end{cases}$$

so that the Gram matrix of the symmetric bilinear form with respect to this basis is a matrix we will denote by S_8. It is the 8×8 matrix which has zeroes everywhere except for a line of ones connecting the $(1, 8)$ and the $(8, 1)$ entries. Also, the canonical involution $\bar{\ }$ is given by

$$\overline{u_i} = \begin{cases} -u_i & \text{if } i \neq 4, 5 \\ u_5 & \text{if } i = 4 \\ u_4 & \text{if } i = 5. \end{cases}$$

5.2. Over K, H is isomorphic to $\mathrm{Spin}_{10} \rtimes \boldsymbol{\mu}_2$**.** Let A denote the 10-dimensional subspace $e_0 \times J$ of J, which is $A = \left(\begin{smallmatrix} 0 & 0 & \cdot \\ \cdot & F & \mathfrak{C} \\ 0 & \cdot & F \end{smallmatrix}\right)$. For f in $\operatorname{Inv}(J)(K)$, we have $f(e_0 \times j) = f^\dagger(e_0) \times f^\dagger(j)$, so for $f \in H$ we have $f(A) = A$. The multiplication on J restricts to give A the structure of a central simple Jordan algebra as well, albeit with a different unit element. It has norm form N_A given by

$$N_A \left(\begin{smallmatrix} 0 & 0 & \cdot \\ \cdot & \alpha & c \\ 0 & \cdot & \beta \end{smallmatrix}\right) = \alpha\beta - \mathfrak{n}(c).$$

Extend scalars to $K(t)$ and fix f in $H(K)$. Then $N(te_0 + j) = N(f(te_0 + j)) = N(te_0 + f(j))$. The coefficient of t in this expression is $T(e_0, j^\#) = T(e_0, f(j)^\#)$. For j actually lying in A, $T(e_0, j^\#) = N_A(j)$, so f restricts to preserve the norm on A. Write $O(A)$ for the algebraic subgroup of $GL(A)$ consisting of maps which preserve the norm N_A (i.e., the orthogonal group of the 10-dimensional quadratic form N_A). We have proven that restriction provides a map $H \to O(A)$ which is defined over K.

The map $H \to O(A)$ has kernel of order 2: Anything in H which maps to the identity in $O(A)$ fixes all of the idempotents e_i, and so is of the form $g_{\underline{t}}$ for some $\underline{t} \in \mathrm{Spin}(\mathfrak{n})$. However, t_0 must also be the identity, so $\underline{t} = (1, 1, 1)$ or $(1, -1, -1)$ by [Gar98, 1.5(2)].

We would like to show that the map $H \to O(A)$ is surjective. Note that $O(A)$ is generated by

- the special orthogonal group $SO(B)$ for B the codimension 1 subspace of A spanned by $\mathfrak{C}$ and e_1-e_2 endowed with the quadratic form given by restricting N_A;
- anything in $O(A)$ with determinant -1; and
- anything in $O(A)$ which does not leave B invariant.

Since for $f \in \mathrm{Aut}(J)$, $f^\dagger = f$, the subgroup $\mathrm{Aut}(J/e_0)$ of elements of $\mathrm{Aut}(J)$ which fix e_0 is a subgroup of H. As described in [Jac68, p. 376, Thm. 4], $\mathrm{Aut}(J/e_0) \cong \mathrm{Spin}(B)$ and the restriction to B gives the surjection onto $SO(B)$. The map

$$\begin{pmatrix} \varepsilon_0 & c_2 & \cdot \\ \cdot & \varepsilon_1 & c_0 \\ c_1 & \cdot & \varepsilon_2 \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon_0 & c_1 & \cdot \\ \cdot & \varepsilon_2 & c_0 \\ c_2 & \cdot & \varepsilon_1 \end{pmatrix}$$

lies in $H(K)$ and restricts to have determinant -1 on A. Finally, we consider Freudenthal's maps from [Jac61, p. 74]. For $E_{ij} \in M_3(\mathfrak{C})$ the matrix whose only nonzero entry is a 1 in the (i,j)-position, 1_3 the 3×3 identity matrix, $x \in \mathfrak{C}$, and $a \in J$, he defines a map $\psi_{ij}(x) \in \mathrm{Inv}\,(J)$ given by

$$\psi_{ij}(x)(a) = (1_3 + xE_{ij})a(1_3 + xE_{ij})^*,$$

where juxtaposition denotes the usual multiplication in $M_3(\mathfrak{C})$, not the Jordan multiplication. So $\psi_{ij}(x) \in H(K)$ if $i, j \neq 1$. In particular, $\psi_{32}(u_5)|_A$ is given by

$$\psi_{32}(u_5)|_A \begin{pmatrix} \varepsilon_1 & c_0 \\ \cdot & \varepsilon_2 \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & c_0+\varepsilon_1 u_4 \\ \cdot & \varepsilon_2+\mathfrak{n}(c_0,u_4) \end{pmatrix},$$

which does not leave B invariant.

Finally, we observe that H° is isomorphic to $\mathrm{Spin}(A)$. The inverse image, call it H', of $SO(A)$ maps onto $SO(A)$ with a kernel which is central and of order 2. Consequently, H' is simple and hence must be isomorphic to $\mathrm{Spin}(A)$. Since H' is connected and $[H : H'] = 2$, $H^\circ = H'$.

5.3. Over F, H is isomorphic to $\mathrm{Spin}(4\mathcal{H} \perp \langle -1, k\rangle) \rtimes \boldsymbol{\mu}_2$. To compute the isomorphism class of H over F, we observe that the map $h \mapsto h^\dagger$ restricts to the identity on the kernel of the K-map $H \to O(A)$, so the ι-action on H induces one on $O(A)$, which we will calculate explicitly.

Fix the basis $(u_1, u_2, u_3, u_4, e_1, e_2, u_5, \ldots, u_8)$ for A so that the Gram matrix for the symmetric bilinear form associated with N_A becomes

$$\begin{pmatrix} & & -S_4 \\ & S_2 & \\ -S_4 & & \end{pmatrix},$$

for S_2 and S_4 defined analogously to how S_8 was in 5.1. Then $SO(A)$ is generated by

- a torus T consisting of diagonal matrices with diagonal entries $(d_1, d_2, \ldots, d_5, d_5^{-1}, d_4^{-1}, \ldots, d_1^{-1})$;
- root groups $U_{ij} \colon \mathbb{G}_a \to SO(A)$ given by

  $$U_{ij}(r) = 1_{10} + rE_{ij} - rE_{j^*i^*}$$

  for 1_{10} the 10×10 identity matrix, $i^* := 11 - i$, and $(i,j) = (i, i+1)$ for $i = 1, 2, 3$, and their transposes; and

- root groups $V_{ij}\colon \mathbb{G}_a \to SO(A)$ given by
$$V_{ij}(r) = 1_{10} + r(E_{ij} + E_{j^* i^*})$$
for $(i,j) = (4,5)$ and $(4,6)$, and their transposes. (Note that $V_{45}(r) = \psi_{32}(ru_5)|_A$ and $V_{46}(r) = \psi_{23}(ru_4)|_A$ for $r \in F = \mathbb{G}_a(F)$.)

Since the torus lies in the image of $\mathrm{Rel}\,(\mathfrak{C}, \mathfrak{n})$ and $g_{\underline{t}}^\dagger = g_{\sigma_{\mathfrak{n}}(\underline{t})^{-1}}$, the action on T and on the first kind of root groups is the usual ι-action. However,
$$V_{45}(r)^\dagger = \psi_{32}(ru_5)^\dagger = \psi_{23}(-ru_4) = V_{46}(-r).$$
So the map $h \mapsto h^\dagger$ induces on $SO(A)$ the map $f \mapsto MfM^{-1}$ for
$$M = \begin{pmatrix} 1_4 & & \\ & -S_2 & \\ & & 1_4 \end{pmatrix}.$$
Write η for the 1-cocycle in $Z^1(K/F, O(A))$ given by $\eta_\iota = M$. The K-map $H \to O(A)$ descends to a map over F from H onto the twisted group $O(A)_\eta$, so we wish to describe the group $O(A)_\eta$.

But this is now just a problem of explicitly computing a quadratic form given by descending down a quadratic extension. So we need to find a K-basis of $A \otimes K$ consisting of elements fixed by the map $a \otimes \kappa \mapsto M(a) \otimes \iota(\kappa)$. Then $O(A)_\eta$ is isomorphic to $O(q)$, where q is the restriction of N_A to the F-span of those fixed vectors. Such a K-basis is given by u_i for $1 \le i \le 8$, $e_1 - e_2$, and $\sqrt{k}e_1 + \sqrt{k}e_2$. These vectors give an orthogonal basis for a quadratic form $4\mathcal{H} \perp \langle -1, k\rangle$, which proves the claim.

Following is a little lemma which foreshadows the way we will prove the Main Theorem for quasi-split groups of type ${}^2E_6$.

Lemma 5.4. *The Rost multiplier of the inclusion $H^\circ \subset E_6^K$ is 1. The restriction of the Rost invariant on $H^1(F, E_6^K)$ to the image of $H^1(F, H^\circ)$ has trivial kernel.*

Proof. Since the Rost multiplier is invariant under scalar extension, we may work over K, where this embedding is described in 5.2. Then some of the coroots (identified with copies of $\mathbb{G}_m$ lying in the maximal torus T from 5.3) for H° are the same as those for $\mathrm{Spin}(\mathfrak{n})$ considered as a subgroup of $\mathrm{Inv}\,(J)$ via the map g. Since the inclusion $\mathrm{Spin}(\mathfrak{n}) \hookrightarrow \mathrm{Inv}\,(J)$ has Rost multiplier 1, so does $H^\circ \subset E_6^K$.

Since the quadratic form $q = 4\mathcal{H} \perp \langle -1, k\rangle$ is isotropic, the spinor norm map $SO(q)(F) \to F^*/F^{*2}$ is surjective. The Rost invariant R_{H° has trivial kernel by 1.4, and the second claim follows. □

6. Special cocycles

Definition 6.1. For $\underline{a} = (a_0, a_1, a_2) \in (F^*)^{\times 3}$ with product 1, we define a "special" cocycle $z := z_{K,\underline{a}}$ in $H^1(K/F, G)$. Set $z_\iota = (z_0, z_1, z_2)$ where $z_j = m_j(\underline{a})dP$

for P the permutation matrix giving the map $u_k \mapsto u_{\pi(k)}$ for π the permutation $(1\,2)(3\,6)(4\,5)(7\,8)$, $m_j(\underline{a})$ the diagonal matrix

$$m_j(\underline{a}) := \operatorname{diag}(1, a_j, a_j, a_{j+2}^{-1}, a_{j+1}^{-1}, 1, 1, a_j) \tag{6.2}$$

with subscripts taken modulo 3, and

$$d := \operatorname{diag}(1, 1, -1, 1, 1, -1, 1, 1).$$

The z_j form a related triple by [Gar98, 1.6, 1.7, 1.5(3)], so $z_\iota \in G(K)$. Note that $\sigma_{\mathfrak{n}}(m_j(\underline{a})) = Pm_j(\underline{a})P$, and, since P is an isometry of $\mathfrak{n}$, $\sigma_{\mathfrak{n}}(P) = P^{-1} = P$. We have

$$^{\iota}z_j = \sigma_{\mathfrak{n}}(m_j(\underline{a})dP)^{-1} = Pm_j(\underline{a})^{-1}PdP = z_j^{-1}$$

and so z is indeed in $Z^1(K/F, G)$.

The image of $z_{K,\underline{a}}$ in $H^1(K/F, P)$ is the class of $\underline{a}$.

6.3. Freedom in the definition. Of course, some of these special cocycles are cohomologically equivalent in $H^1(K/F, G)$. If $\underline{a}$ and $\underline{a'}$ are two triples in $(F^*)^{\times 3}$ such that $a_j^{-1}a'_j \in N_{K/F}(K^*)$ for all j, fix $\lambda_j \in K^*$ such that $a_j^{-1}a'_j = \lambda_j\iota(\lambda_j)$. Then for $\underline{\ell} = (\ell_0, \ell_1, \ell_2)$ with $\ell_j = Pm_j(\underline{\lambda})P$, $\underline{\ell}$ is a related triple by [Gar98], so $\underline{\ell} \in G(K)$. We have ${}^{\iota}\underline{\ell}\,(z_{K,\underline{a'}})_\iota\,\underline{\ell}^{-1} = (z_{K,\underline{a}})_\iota$, i.e., the two cocycles $z_{K,\underline{a}}$ and $z_{K,\underline{a'}}$ are cohomologous.

6.4. We will twist by these cocycles to move a cocycle in $H^1(F, G)$ so that it takes values in a semisimple group. For now, we just observe that the semisimple group we get from one of them, $\operatorname{Spin}(\mathfrak{n})_z$, is described in [Gar98, pp. 403, 404]: Let $k \in F^*$ be such that $K = F(\sqrt{k})$ and let Q_i denote the quaternion algebra $(k, a_i)_F$ generated by elements x, y such that $x^2 = k$, $y^2 = a_i$, and $xy = -yx$. The group $\operatorname{Spin}(\mathfrak{n})_z$ is isomorphic to $\operatorname{Spin}(A_i, \sigma_i)$ where A_i is isomorphic to $M_4(Q_i)$, σ_i is an isotropic orthogonal involution with trivial discriminant, and

$$(C_0(A_i, \sigma_i), \underline{\sigma_i}) \cong (A_{i+1}, \sigma_{i+1}) \times (A_{i+2}, \sigma_{i+2}), \tag{6.5}$$

where the subscripts are taken modulo 3. (These properties specify the σ_i up to isomorphism [Gar01a, 2.3].)

The Moving Lemma lets us say something useful about the Rost invariant of our special cocycles.

Corollary 6.6. *The Rost invariant $R_{E_6^K}(z_{K,\underline{a}})$ is trivial if and only if $z_{K,\underline{a}}$ is cohomologically trivial in $H^1(F, E_6^K)$.*

Proof. Consider the element $j = \left(\begin{smallmatrix} 0 & 0 & \cdot \\ \cdot & 0 & c \\ 0 & \cdot & 0 \end{smallmatrix}\right)$ in $e_0 \times J_K$ for $c = u_2/2 + u_8$. Then $\mathfrak{n}(c) = 0$ and, consulting the explicit formula for $j^\#$ in [Jac68, p. 358], we see that

$j^\# = 0$. Moreover, for $z := z_{K,\underline{a}}$, we have $z_\iota \iota j = \begin{pmatrix} 0 & 0 & \cdot \\ \cdot & 0 & c' \\ 0 & \cdot & 0 \end{pmatrix}$ for $c' = u_1/2 + u_7$. Then

$$T(j, j') = c\overline{c'} + \overline{c}c' = \mathfrak{n}(c, c') = 1.$$

Applying the Moving Lemma shows that z is equivalent in $H^1(K/F, E_6^K)$ to some $z' \in Z^1(K/F, G)$ whose image in $H^1(K/F, P)$ is $(1, a_0, a_0^{-1})$. In particular, the 0-component of the triple z'_ι in $GO^\circ(\mathfrak{C}, \mathfrak{n})^{\times 3}$ belongs to $SO(\mathfrak{C}, \mathfrak{n})$ and the 1- and 2-components have multipliers a_0 and a_0^{-1} respectively. Thus the restriction of z'_ι to the 10-dimensional subalgebra A defined in 5.2 has determinant 1 and so lies in H°. If the Rost invariant $R_{E_6^K}(z)$ is trivial, then $z_{K,\underline{a}}$ is trivial in $H^1(F, E_6^K)$ by Lemma 5.4. □

In a special case the value of the Rost invariant of our special cocycles can be computed explicitly.

Lemma 6.7. *For $a, k \in F^*$ such that $K = F(\sqrt{k})$, the Rost invariant of the 1-cocycle $z_{K,(1,a,a^{-1})}$ is $(a) \cup (k) \cup (-1)$ in $H^3(F, \mathbb{Z}/2) \subset H^3(F, \mathbb{Q}/\mathbb{Z}(2))$.*

Proof. The cocycle $z := z_{K,(1,a,a^{-1})}$ takes values in H and restricts to have determinant 1 on the subalgebra A defined in 5.2, so $z \in Z^1(K/F, H^\circ)$. Since the inclusion $H^\circ \subset E_6^K$ has Rost multiplier 1, to compute the Rost invariant of z, we may compute the Rost invariant of z in $H^1(F, H^\circ)$. But recall that H° is isomorphic to $\mathrm{Spin}(q)$ for $q = 4\mathcal{H} \perp \langle -1, k\rangle$ and that $H^1(F, SO(q))$ classifies nondegenerate quadratic forms of the same dimension and discriminant as q. So we can compute the Rost invariant of z by computing the quadratic form q_z corresponding to the image of z in $H^1(F, SO(q))$, which is just the restriction of $\mathfrak{q} \otimes K$ to the vector subspace fixed by the action $a \otimes \kappa \mapsto z_\iota M(a) \otimes \iota(\kappa)$ for M as in 5.3.

We perform the Galois descent calculation by decomposing $A \otimes K$ into 2-dimensional subspaces and calculating the Galois action on those subspaces.

subspace basis	restriction of $z_\iota M$	F-basis for fixed subspace	contribution to q_z
(u_1, u_2)	S_2		totally
(u_7, u_8)	S_2		isotropic
(u_3, u_6)	$-S_2$	$u_3 - u_6, \sqrt{k}u_3 + \sqrt{k}u_6$	$\langle 1, -k\rangle$
(u_4, u_5)	$\begin{pmatrix} & a \\ a^{-1} & \end{pmatrix}$	$au_4 + u_5, -a\sqrt{k}u_4 + \sqrt{k}u_5$	$\langle -a, ak\rangle$
(e_1, e_2)	$\begin{pmatrix} & -a^{-1} \\ -a & \end{pmatrix}$	$-e_1 + ae_2, \sqrt{k}e_1 + a\sqrt{k}e_2$	$\langle -a, ak\rangle$

The first two subspaces form a complementary pair of totally isotropic subspaces, so they contribute two hyperbolic planes to q_z. Thus the image of z is $q_z = 2\mathcal{H} \perp \langle 1, -k, -a, ak, -a, ak\rangle$ and the Rost invariant of z is the Arason invariant of $q_z - q = \langle\langle a, k, -1\rangle\rangle$. □

7. Quasi-split groups of type E_6 and E_7

This section consists solely of a proof of the main theorem, beginning with a nearly trivial lemma.

Lemma 7.1. *Suppose that C is a central subgroup in a simple simply connected group Γ. Then $H^1(F,C)$ acts on $H^1(F,\Gamma)$ and for $\zeta \in H^1(F,C)$ and $\gamma \in H^1(F,\Gamma)$, we have*

$$R_\Gamma(\zeta \cdot \gamma) = R_\Gamma(\zeta) + R_\Gamma(\gamma),$$

where $R_\Gamma(\zeta)$ denotes the image of ζ under the composition $H^1(F,C) \longrightarrow H^1(F,\Gamma) \xrightarrow{R_\Gamma} H^3(F,\mathbb{Q}/\mathbb{Z}(2))$.

Proof. Pick a 1-cocycle $z \in Z^1(F,C)$ which represents ζ. We have a diagram

$$\begin{array}{ccccc} H^1(F,\Gamma) & = & H^1(F,\Gamma_z) & \xrightarrow[\tau_z]{\sim} & H^1(F,\Gamma) \\ {\scriptstyle R_\Gamma}\downarrow & & {\scriptstyle R_{\Gamma_z}}\downarrow & & \downarrow{\scriptstyle R_\Gamma} \\ H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & = & H^3(F,\mathbb{Q}/\mathbb{Z}(2)) & \xrightarrow{\cdot + R_\Gamma(\zeta)} & H^3(F,\mathbb{Q}/\mathbb{Z}(2)). \end{array}$$

Here the group Γ_z is the usual twist of Γ by the cocycle z; it is just the group Γ with a different Galois action so that a member σ of $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ maps $g \mapsto z_\sigma {}^\sigma g z_\sigma^{-1}$. In our case, z_σ is central, so in fact Γ_z is identical to Γ. The map τ_z is the usual twisting map [Ser94, I.5.5], defined by sending $a \in Z^1(F,\Gamma_z)$ to the 1-cocycle $\sigma \mapsto a_\sigma z_\sigma$. The composition of the two maps on the top row is then the action of ζ.

The left-hand box commutes because the Rost invariant is canonical. The right-hand box commutes by [Gil00, p. 76, Lem. 7]. The desired equality is equivalent to the commutativity of the outer rectangle. □

This result has the obvious corollary that the induced map $H^1(F,C) \longrightarrow H^3(F,\mathbb{Q}/\mathbb{Z}(2))$ is a group homomorphism.

7.2. Groups of type ${}^1E_6$. Suppose first that our simply connected quasi-split group of type E_6 is split and denote it simply by E_6. From Example 3.4, we have an embedding $F_4 \times \boldsymbol{\mu}_3 \hookrightarrow E_6$ which induces a surjection on H^1 terms. So for $\varepsilon \in H^1(F,E_6)$, we can find $\phi \in H^1(F,F_4)$ and $\zeta \in H^1(F,\boldsymbol{\mu}_3)$ such that $\phi \oplus \zeta \mapsto \varepsilon$. Since E_6 is split and the image of $\boldsymbol{\mu}_3$ is the center of E_6, the image of $H^1(F,\boldsymbol{\mu}_3) \to H^1(F,E_6)$ is trivial. If ε is in the kernel of the Rost invariant R_{E_6}, by Lemma 7.1 ϕ must be killed by the composition

$$H^1(F,F_4) \to H^1(F,E_6) \xrightarrow{R_{E_6}} H^3(F,\mathbb{Q}/\mathbb{Z}(2)).$$

As described in 2.4, the Rost multiplier of the embedding $F_4 \subset E_6$ is 1, so ϕ lies in the kernel of the Rost invariant R_{F_4}, which is known to be trivial. Thus ε is the image of ζ, which we have already observed is trivial.

Remark 7.3 (Noninjectivity for $^1E_6$)**.** The Rost invariant is typically noninjective for the group E_6. To see this, we can not simply apply Remark 0.6 and the fact that the embedding $F_4 \hookrightarrow E_6$ has Rost multiplier 1, since two isotopic Albert algebras have the same image in $H^1(F, E_6)$.

Instead, fix a ground field F which supports a division (= nonreduced) Albert F-algebra J. Over the field $F(t)$, the norm N of J does not represent t as can be seen by elementary valuation theory [Jac68, p. 417, Lem. 1]. Consequently, N is not isomorphic to tN over $F(t)$, so the images of the two classes $(J) \oplus (1)$ and $(J) \oplus (t)$ under the map $H^1(F, F_4) \times H^1(F, \boldsymbol{\mu}_3) \to H^1(F, E_6)$ are distinct by [Gar01b, 2.8(2)]. However, since the image of $H^1(F, \boldsymbol{\mu}_3) \to H^1(F, E_6)$ is trivial, by Lemma 7.1 the two classes in $H^1(F, E_6)$ have the same Rost invariant.

7.4. Groups of type $^2E_6$. Suppose now that our quasi-split simply connected group of type E_6 is not actually split, so that it only becomes split over some quadratic field extension K of F. Write E_6^K for this group, as we have since Section 4. By the split case, any $\alpha \in H^1(F, E_6^K)$ which is in the kernel of the Rost invariant must become trivial over K and so belongs to $H^1(K/F, E_6^K)$. Applying Ferrar's Lemma 4.6, we have that α is the image of some $\beta \in H^1(K/F, G)$.

7.4. Twisting. Fix a triple $\underline{a} = (a_0, a_1, a_2) \in (F^*)^{\times 3}$ such that $a_0a_1a_2 = 1$ which represents the image of β in $H^1(K/F, P)$. (This makes sense thanks to the description of $H^1(K/F, P)$ in 4.7.) Then we set $z := z_{K,\underline{a}}$ as defined in 6.1, and we can twist E_6^K by z to obtain a diagram

$$\begin{array}{ccccc} H^1(F, G_z) & \longrightarrow & H^1(F, (E_6^K)_z) & \xrightarrow{R_{(E_6^K)_z}} & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\ \tau_z \downarrow & & \tau_z \downarrow & & \downarrow \cdot + R_{E_6^K}(z) \\ H^1(F, G) & \longrightarrow & H^1(F, E_6^K) & \xrightarrow{R_{E_6^K}} & H^3(F, \mathbb{Q}/\mathbb{Z}(2)), \end{array}$$

where the right vertical arrow has the specified value by [Gil00, p. 76, Lem. 7].

7.6. The image of $\tau_z^{-1}(\beta)$ in $H^1(F, SO(A, \sigma))$. We want to say something about what kind of class $\beta' := \tau_z^{-1}(\beta)$ can be. In particular, its image in $H^1(K/F, P_z)$ is trivial, so β' comes from the semisimple part of G_z, which is isomorphic to $\mathrm{Spin}\,(A, \sigma)$ for (A, σ) one of the three algebras A_i described in 6.4.

We may think of β' as lying in $H^1(K/F, \mathrm{Spin}\,(A, \sigma))$ and consider its image in $H^1(K/F, SO(A, \sigma))$. Let L be a generic splitting field of A (e.g., a function field of its Severi–Brauer variety) and consider the image of β' in $H^1(L, SO(A, \sigma))$. Since A is split by L, σ becomes adjoint to the quadratic form $\langle\langle k, a_{i+1} \rangle\rangle \perp 2\mathcal{H}$ [Gar01a, 2.3]. The image of β' determines an 8-dimensional quadratic form q over L, and the Rost invariant of β' is just the class of $q - \langle\langle k, a_{i+1} \rangle\rangle$ in I^3L/I^4L. However, by the twisting argument above, the Rost invariant of β' over F is $-R_{E_6^K}(z)$. Since A is split over L, $a_i \in L^*$ is a norm from KL, so by 6.3 and Lemma 6.7 the Rost invariant becomes $(k) \cup (a_{i+1}) \cup (-1)$ over L.

For $\phi = \ll k, a_{i+1} \gg$, we have that $q - \phi$ lies in I^3L and $q - \phi \equiv \phi \ll -1 \gg$ mod I^4L. But then

$$q + \phi = (q - \phi) + 2\phi \equiv 4\phi \equiv 0 \mod I^4L.$$

So $q + \phi$ is in I^4L. However $\dim(q \perp \phi) = 12 < 16$, so by the Arason–Pfister Hauptsatz, $q \perp \phi$ is hyperbolic and we have $q \cong \langle -1 \rangle \phi \perp 2\mathcal{H}$.

Consequently, the image of β' in $H^1(L, SO(A,\sigma))$ is the same as the image of $-1 \in F^*/F^{*2} = H^1(F, Z(SO(A,\sigma)))$. Since A is Brauer-equivalent to a quaternion algebra, it follows from the material in [Sch85, Ch. 10] that the canonical map $H^1(F, SO(A,\sigma)) \to H^1(L, SO(A,\sigma))$ is injective. (This was shown independently in [PSS] and [Dej01].) Thus the image of β' in $H^1(F, SO(A,\sigma))$ must also be -1.

7.7. More generally, any simply connected group Γ of type ${}^1D_4$ is isomorphic to $\mathrm{Spin}(A_i, \sigma_i)$ for three central simple algebras A_i of degree 8 with $i = 0, 1, 2$ endowed with an orthogonal involution σ_i with trivial discriminant and related as in (6.5).

Each of the three descriptions of Γ comes paired with natural maps $\Gamma \to SO(A_i, \sigma_i) \to P\Gamma$ for $P\Gamma$ the adjoint group associated to Γ. The kernel of the second map is $Z(SO(A_i, \sigma_i)) \cong \boldsymbol{\mu}_2$, and the kernel of the composition is $Z(\Gamma)$, which is isomorphic to the subgroup of $\boldsymbol{\mu}_2^{\times 3}$ of elements with product 1. The group $H^1(F, Z(\Gamma))$ can be identified with the set of triples $\underline{b} = (b_0, b_1, b_2) \in F^*/F^{*2}$ with product 1 [KMRT98, 44.14] and where the map $H^1(F, Z(\Gamma)) \to H^1(F, Z(SO(A_i, \sigma_i)))$ is given by $\underline{b} \mapsto b_i$.

Lemma 7.8. (Notation as in the preceding paragraph.) *Suppose an element $\eta \in H^1(F, \Gamma)$ has the same image in $H^1(F, SO(A_i, \sigma_i))$ as $c_i \in F^*/F^{*2} = H^1(F, Z(SO(A_i, \sigma_i)))$ for $i = 1, 2$. Then η is the image of $((c_1c_2)^{-1}, c_1, c_2)$ coming from $H^1(F, Z(\Gamma))$.*

Proof. The short exact sequence $1 \to Z(SO(A_i, \sigma_i)) \to SO(A_i, \sigma_i) \to P\Gamma \to 1$, gives that η is killed by the composition $H^1(F, \Gamma) \to H^1(F, SO(A_i, \sigma_i)) \to H^1(F, P\Gamma)$ for $i = 1$. Thus η is the image of some class (n_0, n_1, n_2) in $H^1(F, Z(\Gamma))$.

For general Galois-cohomological reasons, the map $H^1(F, Z(\Gamma)) \to H^1(F, \Gamma)$ is a group homomorphism. (Although the second set doesn't have a group structure, the image of the first set does.) The kernel of this map can be described fully by suitably applying [KMRT98, 35.4], but for our purposes it is enough to observe that it contains all elements of the form $(s, s^{-1}, 1)$ for s a spinor norm of an element in $SO(A_2, \sigma_2)(F)$ and symmetrically. Let $G(A_i, \sigma_i)^\circ$ be the algebraic group of *proper similarity factors*, i.e., the group with F-points

$$G(A_i, \sigma_i)^\circ(F) = \left\{ m \in F^* \mid \exists\, f \in A_i^* \text{ such that } m = \sigma_i(f)f \text{ and } \mathrm{Nrd}_{A_i}(f) = m^4 \right\}.$$

For every $m_0 \in G(A_0, \sigma_0)^\circ(F)$, the kernel contains an element of the form (m_0, m_1, m_2) and symmetrically. Conversely, if (b_0, b_1, b_2) is in the kernel, then

$b_i \in G(A_i, \sigma_i)^\circ(F)$ for all i.

It is also the case that the natural map $F^*/F^{*2} = H^1(F, Z(SO(A_i, \sigma_i))) \to H^1(F, SO(A_i, \sigma_i))$ is a group homomorphism, and its kernel is precisely $G(A_i, \sigma_i)^\circ(F)$. Thus we may modify n_2 by an element of $G(A_2, \sigma_2)^\circ(F)$ and so assume that $n_2 = c_2$.

Now consider the middle component of the triple (n_0, n_1, n_2). By hypothesis, $n_1 = m_1 c_1$ for some $m_1 \in G(A_1, \sigma_1)^\circ(F)$. By [Mer96, p. 262, Prop.], the group $\mathrm{SN}(A_2, \sigma_2)(F)$ of spinor norms from $SO(A_2, \sigma_2)(F)$ is the group generated by F^{*2} and the norms from finite field extensions E which split A_2 and make σ_2 isotropic. By [Mer96, p. 263, Prop.], $G(A_1, \sigma_1)^\circ(F)$ is equal to the group generated by the norms from every extension field E which splits A_1 and makes σ_1 hyperbolic. Since the (A_i, σ_i) are related by (6.5), any extension which splits A_1 and makes σ_1 hyperbolic certainly splits A_2 and makes σ_2 isotropic, so $\mathrm{SN}(A_2, \sigma_2)(F) \supseteq G(A_1, \sigma_1)^\circ(F)$. Consequently, the element $(m_1, m_1^{-1}, 1)$ belongs to the kernel of $H^1(F, Z(\Gamma)) \to H^1(F, \Gamma)$.

Thus η is the image of

$$(n_0, n_1, n_2)(m_1, m_1^{-1}, 1) = ((m_1 c_1 c_2)^{-1}, m_1 c_1, c_2)(m_1, m_1^{-1}, 1) = ((c_1 c_2)^{-1}, c_1, c_2)$$

as desired. □

7.9. β' is in the image of $H^1(K/F, Z(\mathrm{Spin}\,(A, \sigma)))$. Let $(A, \sigma) = (A_0, \sigma_0)$ for (A_i, σ_i) as in 6.4. Combining the result from 7.6 with Lemma 7.8, we have that $\beta' \in H^1(F, \mathrm{Spin}\,(A, \sigma))$ is the image of $(1, -1, -1) \in H^1(F, Z(\mathrm{Spin}\,(A, \sigma)))$. However, for $k \in F^*$ such that $K = F(\sqrt{k})$, since K certainly splits A and makes σ hyperbolic and $-k = N_{K/F}(\sqrt{k})$, by Merkurjev's norm principle [Mer96, p. 262, Prop.] there is some element of $SO(A, \sigma)(F)$ with spinor norm $-k$. Then as described in the proof of Lemma 7.8, β' is also the image of $(1, k, k^{-1}) \in H^1(F, Z(\mathrm{Spin}\,(A, \sigma)))$, which itself is in the image of $H^1(K/F, Z(\mathrm{Spin}\,(A, \sigma)))$.

7.10 Consider the 1-cocycle $b = \tau_z(b') \in Z^1(K/F, G)$ for b' the image of $(1, k, k^{-1})$ as above. (Note that b represents the class of β and is the 1-cocycle which takes the value $g_{(1,-1,-1)} z_{K,\underline{a}}$ at ι.) For j and c as in the proof of 6.6, we set $j' := b_\iota \iota j$, so that $j' = \left(\begin{smallmatrix} 0 & 0 & \cdot \\ \cdot & 0 & c' \\ 0 & \cdot & 0 \end{smallmatrix}\right)$ for $c' = u_1/2 + u_7$ and $T(j, j') = \mathfrak{n}(c, c') = 1$. By the Moving Lemma 4.9, we may replace β by a different inverse image of α in $H^1(K/F, G)$ and so assume that $\underline{a} = (1, a_0, a_0^{-1})$.

Any element of G with multiplier $(1, \cdot, \cdot)$ lies in H, and since such an element restricts to have determinant 1 on the subspace A defined in 5.2, it in fact lies in H°. Thus α is in the image of $H^1(F, H^\circ)$. Since the Rost invariant of α is trivial, α must be the trivial class by Lemma 5.4.

7.11. Groups of type E_7. We are left with proving that the Rost invariant has trivial kernel for G split of type E_7, but this follows directly from the same conclusion for quasi-split groups of type E_6, thanks to Proposition 3.6.

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